Filomat 32:14 (2018), 5003–5010
https://doi.org/10.2298/FIL1814003M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Infinitely Many Solutions for a Fourth Order Singular Elliptic Problem

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Abstract. Here, a fourth order singular elliptic problem involving p-biharmonic operator with Dirichlet boundary condition is established where the exponent in the singular term is different from that in the p-biharmonic operator. The existence of infinitely many solutions is proved by the variational methods in Sobolev spaces and the critical points principle of Ricceri. Finally, an example is presented.

1. Introduction

Nonlinear singular elliptic problems have been studied intensively in recent years and arise in some parts of science such that boundary layer phenomena for viscous fluids, chemical heterogeneous catalysts, chemical catalyst kinetics, and theory of heat conduction in electrically conducting materials.

Motivated by such interest, Huang and Liu in [5] studied the following singular problem.

$$\begin{cases} \Delta_p^2 u - \frac{\mu}{|x|^{2p}}|u|^{p-2}u = f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Where $\Omega \subset \mathrm{R}^N$ is a bounded domain and $1 < p < \frac{N}{2}$.

Also we can mention the following problem with Hardy potential

$$\begin{cases} -\Delta_p u = \mu \frac{|u|^{p-2}u}{|x|^p} + \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

which is studied in [4].
In [15] the following problem is studied.

$$\begin{cases} M\left(\int_\Omega |\Delta u|^p \, dx\right)\Delta_p^2 u - \frac{a}{|x|^{2p}}|u|^{p-2}u = \lambda f(x,u) + \mu g(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < \frac{N}{2}$ and $M : [0, +\infty[\to \mathbb{R}$ is a continuous function.

2010 *Mathematics Subject Classification*. 35J40; 34B16; 35J30.
Keywords. Infinitely many solutions; singular problem; p-biharmonic operator; variational methods; Rellich's inequality.
Received: 03 March 2018; Accepted: 02 April 2018
Communicated by Maria Alessandra Ragusa
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Motivated by such works on this topic, we concern the existence of weak solutions for the following problem

$$\begin{cases} \Delta_p^2 u + a(x)\frac{|u|^{q-2}}{|x|^{2q}} = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u)$ is the p-biharmonic operator, $a \in L^\infty(\Omega)$ with $\text{ess}_\Omega \inf a(x) > 0$, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, containing the origin and with smooth boundary $\partial\Omega$, λ is a positive parameter, and $1 < q < \frac{N}{2} < p$. Also $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function. As we see, in this article $\text{ess}_\Omega \inf a(x) > 0$, and the exponent in the singular term is different from p in the p-biharmonic operator. Also, the problem is studied where $\frac{N}{2} < p$ therefore we have the compact embedding $X \hookrightarrow C(\bar{\Omega})$, which makes our results different from those in [15].

We can refer to [2, 3, 7, 8, 12] as some general references on subject considered in this paper.

Definition 1.1. *The function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is said to be an L^1-Carathéodory function, if*

(A_1) *the function $x \mapsto f(x,t)$ is measurable for every $t \in \mathbb{R}$;*

(A_2) *the function $t \mapsto f(x,t)$ is continuous for a.e. $x \in \Omega$;*

(A_3) *for every $h > 0$ there exists a function $\ell_h \in L^1(\Omega)$ such that*

$$\sup_{|t|\leq h} |f(x,t)| \leq \ell_h(x), \text{ for a.e. } x \in \Omega.$$

Let $X := W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$, endowed with the norm

$$\|u\| := \left(\int_\Omega |\Delta u(x)|^p \,\mathrm{d}x \right)^{1/p},$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ containing the origin and with smooth boundary $\partial\Omega$. Moreover, $\|\cdot\|_1$ denotes the norm of $L^1(\Omega)$; i.e.,

$$\|u\|_1 := \int_\Omega |u(x)| \,\mathrm{d}x.$$

Remark 1.2. *The power p can be considered as a function i.e. $p : \Omega \to [1, +\infty)$. In this case, one can define the space $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. This approach is usable to extend the set of the research problems answering a wide variety of applications, for example, studying the problem involving $p(x)$-Laplacian $\Delta_{p(x)}u$ as a nonlinear and nonhomogeneous operator,(see [13, 14]) or studying $p(x)$-energy functionals for a non-standard growth problem (see [10, 11]) and etc.*

We recall Rellich's inequality [9], which says that

$$\int_\Omega \frac{|u(x)|^q}{|x|^{2q}} \,\mathrm{d}x \leq \frac{1}{H} \int_\Omega |\Delta u(x)|^q \,\mathrm{d}x, \text{ for all } u \in X, \tag{2}$$

where $H := (\frac{N(q-1)(N-2q)}{q^2})^q$. We define $F(x,t) := \int_0^t f(x,\xi)\,\mathrm{d}\xi$, for every (x,t) in $\Omega \times \mathbb{R}$. Also we consider the functional $I_\lambda : X \to \mathbb{R}$ as

$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u),$$

associated with (1), for every $u \in X$, where

$$\Phi(u) := \frac{1}{p} \int_\Omega |\Delta u(x)|^p \,\mathrm{d}x + \frac{1}{q} \int_\Omega \frac{|u(x)|^q}{|x|^{2q}} a(x) \,\mathrm{d}x,$$

and

$$\Psi(u) := \int_\Omega F(x, u(x))\,\mathrm{d}x,$$

for every $u \in X$. By (2) implies

$$\frac{\|u\|^p}{p} \le \Phi(u) \le \frac{\|u\|^p}{p} + \frac{\|a\|_\infty}{qH} \int_\Omega |\Delta u(x)|^q \mathrm{d}x, \tag{3}$$

for every $u \in X$, so Φ is well defined, coercive and Gâteaux differentiable and its Gâteaux derivative is the functional $\Phi' \in X^*$ given by

$$\Phi'(u)(v) = \int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)\,\mathrm{d}x + \int_\Omega \frac{|u(x)|^{q-2}u(x)v(x)}{|x|^{2q}}a(x)\,\mathrm{d}x,$$

for every $v \in X$. Moreover, Φ is sequentially weakly lower semicontinuous and strongly continuous. On the other hand, by standard arguments, Ψ is well defined and continuously Gâteaux differentiable functional whose derivative

$$\Psi'(u)(v) = \int_\Omega f(x, u(x))v(x)\,\mathrm{d}x,$$

is a compact operator for every $v \in X$.

Definition 1.3. *For fixed real parameter λ, it is said that a function $u : \Omega \to \mathrm{R}$ is a weak solution of* (1) *if $u \in X$ and*

$$\int_\Omega |\Delta u(x)|^{p-2}\Delta u(x)\Delta v(x)\,\mathrm{d}x + \int_\Omega \frac{|u(x)|^{q-2}u(x)v(x)}{|x|^{2q}}a(x)\,\mathrm{d}x - \lambda \int_\Omega f(x, u(x))v(x)\,\mathrm{d}x = 0,$$

for every $v \in X$, which shows that the critical points of I_λ are exactly the weak solutions of (1).

Our main tool in order to prove the existence of solutions for the problem (1) is the following theorem [1, Theorem 2.1] which is a version of Ricceri's principle [12, Theorem 2.5].

Theorem 1.4. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathrm{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, put*

$$\varphi(r) := \inf_{\Phi(u)<r} \frac{\sup_{\Phi(v)<r} \Psi(v) - \Psi(u)}{r - \Phi(u)},$$
$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \text{and} \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then the following properties hold:

(a) for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional

$$I_\lambda := \Phi - \lambda\Psi$$

to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) if $\gamma < +\infty$, then for each $\lambda \in]0, \frac{1}{\gamma}[$, the following alternative holds either,

(b_1) I_λ possesses a global minimum, or

(b_2) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty$$

(c) if $\delta < +\infty$, then for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds either:

(c_1) there is a global minimum of Φ which is a local minimum of I_λ. or,

(c_2) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ, with $\lim_{n\to+\infty} \Phi(u_n) = \inf_{u\in X} \Phi(u)$.

2. Multiple solutions

For fixed $x_0 \in \Omega$, set $D > 0$ such that $B(x_0, D) \subseteq \Omega$, where $B(x_0, D)$ denotes the ball with center at x_0 and radius D and $\overline{B(x_0, D)}$ not containing the origin.

The compact embedding $X \hookrightarrow C(\bar{\Omega})$ implies

$$k := \sup_{\substack{u \in X \\ u \neq 0}} \left(\frac{\max_{x \in \bar{\Omega}} |u(x)|}{\|u\|} \right) < \infty. \tag{4}$$

Set

$$L_i := \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \int_{\frac{D}{2}}^{D} |\frac{12(N+1)}{D^3} r - \frac{24N}{D^2} + \frac{9(N-1)}{D} \frac{1}{r}|^i r^{N-1} \mathrm{d}r, \quad (i = p, q) \tag{5}$$

where Γ is the Gamma function and defined by

$$\Gamma(t) := \int_0^{+\infty} z^{t-1} e^{-z} \, \mathrm{d}z, \text{ for all } t > 0,$$

and h is a constant such that $h > 1$.

Also set

$$A := \liminf_{\xi \to +\infty} \frac{\|\ell_\xi\|_1}{\xi^{p-1}},$$

and

$$B := \limsup_{\xi \to +\infty} \frac{\int_{B(x_0, \frac{D}{2})} F(x, \frac{\xi}{h}) \, \mathrm{d}x}{\xi^p},$$

where $\ell_\xi \in L^1(\Omega)$ satisfies condition (A_3) on $f(x, t)$ for every $\xi > 0$.

The next theorem shows that the problem (1) admits an unbounded sequence of weak solutions.

Theorem 2.1. *Assume that $f : \Omega \times \mathrm{R} \to \mathrm{R}$ be an L^1-Carathéodory function such that*

(i) $F(x, t) \geq 0$ for every $(x, t) \in \Omega \times \mathrm{R}^+$;

(ii) $A < \frac{h^p}{L_p k^p} B$, where k and L_p are given by (4) and (5).

Then for every $\lambda \in \Lambda :=]\frac{L_p}{ph^p B}, \frac{1}{pk^p A}[$ the problem (1) admits an unbounded sequence of weak solutions in X.

Proof. Let $X := W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$ and Φ and Ψ be the functionals introduced in Section 1.

Fix $\lambda \in]\frac{L_p}{ph^p B}, \frac{1}{pk^p A}[$. As seen in Section 1 the functionals Φ and Ψ satisfy the regularity assumptions requested in Theorem 1.4.

Take $\{\xi_n\} \subset \mathrm{R}^+$ such that $\lim_{n \to +\infty} \xi_n = +\infty$, and $\lim_{n \to +\infty} \frac{\|\ell_{\xi_n}\|_1}{\xi_n^{p-1}} = A$. Thus (4) shows

$$\max_{x \in \bar{\Omega}} |u(x)| \leq k\|u\|. \tag{6}$$

Also from (3) and (6), we obtain

$$\begin{aligned} \Phi^{-1}(]-\infty, r[) &= \{u \in X; \Phi(u) < r\} \\ &\subset \left\{ u \in X; \frac{1}{p}\|u\|^p < r \right\} \\ &\subset \left\{ u \in X; \max_{x \in \bar{\Omega}} |u(x)| < k(pr)^{\frac{1}{p}} \right\}. \end{aligned}$$

Set $r_n := \frac{\xi_n^p}{pk^p}$, for all $n \in N$. Since $\Phi(0) = \Psi(0) = 0$, we have

$$\begin{aligned}
\varphi(r_n) &= \inf_{\Phi(v)<r_n} \frac{(\sup_{\Phi(u)<r_n} \Psi(u)) - \Psi(v)}{r_n - \Phi(v)} \\
&\leq \frac{\sup_{\Phi(u)<r_n} \int_\Omega F(x, u(x))\,\mathrm{d}x}{r_n} \\
&\leq \frac{\sup_{|u(x)|<\xi_n} \int_\Omega F(x, u(x))\,\mathrm{d}x}{r_n} \\
&\leq \frac{\xi_n \|\ell_{\xi_n}\|_1}{\frac{\xi_n^p}{pk^p}}.
\end{aligned}$$

Hence, it follows that

$$\gamma \leq \liminf_{n\to+\infty} \varphi(r_n) \leq pk^p \liminf_{n\to+\infty} \frac{\|\ell_{\xi_n}\|_1}{\xi_n^{p-1}} = pk^p A < +\infty.$$

Now, we show that the functional I_λ is unbounded from below. Indeed, since $\frac{1}{\lambda} < \frac{Bph^p}{L_p}$, there exist $\eta > 0$ and a sequence $\{\tau_n\}$ of positive numbers such that $\tau_n \to +\infty$ and

$$\frac{1}{\lambda} < \eta < \frac{ph^p}{L_p} \frac{\int_{B(x_0,D/2)} F(x, \frac{\tau_n}{h})\,\mathrm{d}x}{\tau_n^p}, \tag{7}$$

for n large enough.

Let $h > 1$ be as before, we define $\{v_n\} \subset X$ given by

$$v_n(x) := \begin{cases} 0 & \bar{\Omega} \setminus B(x_0, D) \\ \frac{\tau_n}{h}\left(\frac{4}{D^3}\rho^3 - \frac{12}{D^2}\rho^2 + \frac{9}{D}\rho - 1\right) & B(x_0, D) \setminus B(x_0, \frac{D}{2}) \\ \frac{\tau_n}{h} & B(x_0, \frac{D}{2}) \end{cases}$$

where $\rho = \operatorname{dist}(x, x_0)$. Clearly $v_n \in X$ and

$$\begin{aligned}
\int_\Omega |\Delta v_n(x)|^i\,\mathrm{d}x &= \left(\frac{\tau_n}{h}\right)^i \frac{2\Pi^{N/2}}{\Gamma(\frac{N}{2})} \int_{\frac{D}{2}}^{D} \left|\frac{12(N+1)}{D^3} r - \frac{24N}{D^2} + \frac{9(N-1)}{D}\frac{1}{r}\right|^i r^{N-1}\,\mathrm{d}r \\
&= \frac{L_i}{h^i}\tau_n^i, \quad (i = p, q).
\end{aligned}$$

Thus

$$\begin{aligned}
\Phi(v_n) &= \frac{1}{p}\int_\Omega |\Delta v_n(x)|^p\,\mathrm{d}x + \frac{1}{q}\int_\Omega \frac{|v_n(x)|^q}{|x|^{2q}} \cdot a(x)\,\mathrm{d}x \\
&\leq \frac{L_p}{h^p p}\tau_n^p + \frac{\|a\|_\infty}{qH}\int_\Omega |\Delta v_n(x)|^q\,\mathrm{d}x, \\
&= \frac{L_p}{ph^p}\tau_n^p + \frac{\|a\|_\infty L_q}{qh^q H}\tau_n^q.
\end{aligned}$$

Condition (i) and (7) show

$$\Psi(v_n) = \int_\Omega F(x, v_n(x))\,\mathrm{d}x \geq \int_{B(x_0,\frac{D}{2})} F\left(x, \frac{\tau_n}{h}\right)\mathrm{d}x \geq \frac{L_p \eta \tau_n^p}{ph^p},$$

therefore

$$I_\lambda(v_n) = \Phi(v_n) - \lambda\Psi(v_n) \le \frac{L_p}{ph^p}\tau_n^p + \frac{\|a\|_\infty L_q}{qh^q H}\tau_n^q - \frac{\lambda L_p \eta}{ph^p}\tau_n^p$$
$$= \frac{L_p}{ph^p}(1-\lambda\eta)\tau_n^p + \frac{\|a\|_\infty L_q}{qh^q H}\tau_n^q.$$

for every $n \in N$ large enough, which follows that

$$\lim_{n\to+\infty} I_\lambda(v_n) = -\infty$$

as $q < p$ and $(1-\lambda\eta) < 0$. Now, Theorem 1.4 (*b*) implies, the functional I_λ admits an unbounded sequence $\{u_n\} \subset X$ of critical points which are weak solutions for the problem (1). □

Here we prove that the problem (1) has a sequence of weak solutions, which converges strongly to zero.

Theorem 2.2. *Let $f : \Omega \times R \to R$ be an L^1-Carathéodory function such that*

(I) $F(x,t) \ge 0$, for every $(x,t) \in \Omega \times R^+$;

(II) $A' < \frac{h^p}{L_p k^p} B'$, where L_p and k are as in Theorem 2.1 and

$$A' := \liminf_{\xi\to 0^+} \frac{\|\ell_\xi\|_1}{\xi^{p-1}}, \quad B' := \limsup_{\xi\to 0^+} \frac{\int_{B(x_0,D/2)} F(x, \frac{\xi}{h})\, dx}{\xi^p}.$$

Then for every $\lambda \in \Lambda' :=]\frac{L_p}{ph^p B'}, \frac{1}{pk^p A'}[$ the problem (1) *has a sequence of weak solutions, which converges strongly to zero in X.*

Proof. We consider Φ, Ψ and I_λ as in Section 1. Fix $\lambda \in \Lambda'$. Knowing that the functionals Φ and Ψ satisfy the regularity assumptions in Theorem 1.4, we show that $\lambda < \frac{1}{\delta}$.

Let $\{t_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} t_n = 0$ and $\lim_{n\to+\infty} \frac{\|\ell_{t_n}\|_1}{t_n^{p-1}} = A'$. We know that $\inf_X \Phi = 0$, so $\delta := \liminf_{r\to 0^+} \varphi(r)$. Set $r_n := \frac{t_n^p}{pk^p}$, for all $n \in N$. Taking (4) into account, one has $\|u\|_\infty < t_n$, for all $u \in X$ with $\Phi(u) < r_n$, (similar to the proof of Theorem 4).

Thus

$$\varphi(r_n) \le \frac{\sup_{\Phi(u)<r_n} \Psi(u)}{r_n} \le pk^p \frac{\|\ell_{t_n}\|_1}{t_n^{p-1}}.$$

Therefore

$$\delta \le \liminf_{n\to+\infty} \varphi(r_n) \le pk^p \liminf_{n\to+\infty} \frac{\|\ell_{t_n}\|_1}{t_n^{p-1}} = pk^p A' < +\infty.$$

Hence, $\Lambda' \subset]0, \frac{1}{\delta}[$.

Now, we show that zero is not a local minimum of I_λ. Let $\{\xi_n\}$ be a sequence of positive numbers such that $\lim_{n\to+\infty} \xi_n = 0$ and

$$B' = \lim_{n\to+\infty} \frac{\int_{B(x_0,D/2)} F(x, \frac{\xi_n}{h})\, dx}{\xi_n^p}.$$

Also consider $\{v_n\} \subset X$ similar to the proof of Theorem 2.1, defined by $\{\xi_n\}$ above. By the same argument, we obtain that $I_\lambda(v_n) < 0$ for n large enough. Thus zero is not a local minimum of I_λ as $\lim_{n\to+\infty} I_\lambda(v_n) < I_\lambda(0) = 0$. Therefore, there exists a sequence $\{u_n\} \subset X$ of critical points of I_λ which converges strongly to zero in X as $\lim_{n\to+\infty} \Phi(u_n) = 0$. □

Here we present an example as follows:

Example 2.3. Let $p = \dfrac{7}{3}, N = 3$, and $\rho > 0$ be a real number and $\{t_n\}$, $\{s_n\}$ be two strictly increasing sequences of real numbers that defined by induction. For $n = 1$,

$$t_1 = \rho,\ s_1 = 2\rho$$

and for $n \geq 1$,

$$\begin{aligned} t_{2n} &= \left(2^{2n+1} - 1\right) t_{2n-1}, \qquad t_{2n+1} = \left(2 - \tfrac{1}{2^{2n+1}}\right) t_{2n}, \\ s_{2n} &= \tfrac{t_{2n}}{2^n} = \left(2 - \tfrac{1}{2^{2n}}\right) s_{2n-1}, \quad s_{2n+1} = 2^{n+1} t_{2n+1} = \left(2^{2n+2} - 1\right) s_{2n}. \end{aligned}$$

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x,t) := \begin{cases} 2w(x)t, & (x,t) \in \Omega \times [0, t_1], \\ w(x)\left(s_{n-1} + \frac{s_n - s_{n-1}}{t_n - t_{n-1}}(t - t_{n-1})\right), & (x,t) \in \Omega \times [t_{n-1}, t_n] \text{ for some } n > 1, \end{cases}$$

where $w : \Omega \to \mathbb{R}$ is a positive continuous function with $0 < w_0 \leq w(x) \leq w^0$. Obviously $f(x,t)$ is strictly increasing with respect to t for every $x \in \Omega$, the function $l_\xi(x) := f(x, \xi)$ satisfies in condition (A_3) on f; i.e.,

$$\sup_{|t| \leq \xi} |f(x,t)| \leq l_\xi(x), \quad \text{for a.e. } x \in \Omega,$$

therefore f is an L^1-Carathéodory function. Arguing as in [6], we have

$$\limsup_{\xi \to +\infty} \frac{\int_{B(x_0, \frac{D}{2})} F(x, \frac{\xi}{h}) dx}{\xi^{\frac{7}{3}}} = +\infty, \qquad \liminf_{\xi \to +\infty} \frac{\|l_\xi\|_1}{\xi^{\frac{4}{3}}} = 0,$$

for every $x_0 \in \Omega$, where Ω is a bounded domain in $\mathbb{R}^3$ containing the origin and with smooth boundary $\partial\Omega$, also $D > 0$ such that $B(x_0, D) \subset \Omega$ and $\overline{B(x_0, D)}$ not containing the origin. Hence the function f satisfies the conditions mentioned in Theorem 2.1.

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