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Applications of Mathematics, Vol. 55 (2010), No. 5, 405–418

Persistent URL: http://dml.cz/dmlcz/140710

55 (2010) APPLICATIONS OF MATHEMATICS No. 5, 405–418

INFINITELY MANY SOLUTIONS OF A SECOND-ORDER p-LAPLACIAN PROBLEM WITH IMPULSIVE CONDITION

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(Received June 2, 2008)

Abstract. Using the critical point theory and the method of lower and upper solutions, we present a new approach to obtain the existence of solutions to a p-Laplacian impulsive problem. As applications, we get unbounded sequences of solutions and sequences of arbitrarily small positive solutions of the p-Laplacian impulsive problem.

Keywords: critical point theory, lower and upper solutions, impulsive, p-Laplacian

MSC 2010: 34B37, 47H15

1. Introduction

The p-Laplacian operator appears in many research areas. For instance, in the study of torsional creep (elastic for $p = 2$, plastic as $p \to \infty$), flow through porous media ($p = 3/2$) or glacial sliding ($p \in (1, 4/3]$), see [2]. The existence of multiple solutions of the p-Laplacian problem was considered in many papers, see for example [1], [3], [5], [6] and the references therein. Most of them treated the problem under conditions on f which imply some sort of oscillations between a sublinear and a superlinear behaviour.

In [3], a contribution was made for the case when $pF(x,s)/|s|^p$ interacts asymptotically with the first eigenvalue. In [5], the existence of at least one solution was obtained when the nonlinearity $pF(x,s)/|s|^p$ stays asymptotically between the first two eigenvalues of the p-Laplacian operator. More recently, the authors of [6] obtained the existence of multiple nontrivial solutions for the case $\lim\limits_{s\to\infty} pF(x,s)/|s|^p < \lambda_1$.

Motivated by the above works, in this paper we consider a p-Laplacian problem

$$(\Phi_p(u'))' + f(t,u) = 0, \quad t \in (0,1) \setminus \{t_1, \ldots, t_m\}, \tag{1.1}$$

$$u'(0) - \mu_0 u(0) = 0, \quad u'(1) + \mu_1 u(1) = 0, \tag{1.2}$$

with an impulsive condition

$$\Delta\Phi_p(u'(t_k)) - \sigma_k u(t_k) = 0, \quad k \in \{1, 2, \ldots, m\}, \tag{1.3}$$

where $p > 1$, $0 = t_0 < t_1 < t_2 < \ldots < t_m < t_{m+1} = 1$ are fixed points, $\Phi_p(s) = |s|^{p-2}s$, $\Delta\Phi_p(u'(t_k)) = \Phi_p(u'(t_k^+)) - \Phi_p(u'(t_k^-))$, $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\mu_0 \geqslant 0$, $\mu_1 > 0$, $\sigma_k \geqslant 0$, $k = 1, 2, \ldots, m$.

Define the functional $\varphi\colon W^{1,p}(0,1) \to \mathbb{R}$ by

$$\varphi(u) = \int_0^1 \Big[\frac{|u'|^p}{p} - F(t,u)\Big]\,\mathrm{d}t + \sum_{k=1}^{m} \frac{|\sigma_k u(t_k)|^2}{2} + \frac{|\mu_0 u(0)|^p}{p} + \frac{|\mu_1 u(1)|^p}{p}, \tag{1.4}$$

where $F(t,u) = \int_0^u f(t,s)\,\mathrm{d}s$ and $W^{1,p}(0,1)$ is the usual Sobolev space endowed with the norm $\|u\| = \big(\int_0^1 (|u(t)|^p + |u'(t)|^p)\,\mathrm{d}t\big)^{1/p}$.

We say that $u \in W^{1,p}(0,1)$ is a solution of BVP (1.1)–(1.3) if it satisfies (1.1)–(1.2) and for every $k = 0, 1, \ldots, m$, $u_j = u|_{(t_j, t_{j+1})}$ is such that $u_j \in W^{2,p}(t_j, t_{j+1})$. For $k = 1, 2, \ldots, m$, the limits $u'(t_k^+)$, $u'(t_k^-)$ exist, $u'(t_k^-) = u'(t_k)$ and (1.3) holds.

Definition 1.1. A function $\alpha \in W^{1,p}(0,1)$ is called a lower solution of (1.1)–(1.3) if it satisfies

$$\begin{gathered}(\Phi_p(\alpha'(t)))' + f(t, \alpha(t)) \geqslant 0, \quad t \in (0,1) \setminus \{t_1, \ldots, t_m\},\\ \Delta\Phi_p(\alpha'(t_k)) - \sigma_k \alpha(t_k) \geqslant 0, \quad k \in \{1, 2, \ldots, m\},\\ \alpha'(0) - \mu_0 \alpha(0) \geqslant 0, \quad \alpha'(1) + \mu_1 \alpha(1) \leqslant 0.\end{gathered}$$

A function $\beta \in W^{1,p}(0,1)$ is called an upper solution of (1.1)–(1.3) if it satisfies the reversed inequalities.

Combining the lower and upper solutions and the critical point theory, we prove in Theorem 2.1 that for the impulsive problem (1.1)–(1.3) between the well-ordered lower and upper solutions, the related functional has a minimum u. Furthermore, u is a solution of (1.1)–(1.3).

As applications of Theorem 2.1, in Section 3 we prove that problem (1.1)–(1.3) has two unbounded sequences of solutions, which are respectively characterized as local minimizers of φ, assuming that $-\infty < \liminf\limits_{u\to\pm\infty} F(t,u)/|u|^p \leqslant 0$ and $\limsup\limits_{u\to\pm\infty} F(t,u)/|u|^p = \infty$, uniformly in $t \in [0,1]$. We also prove in Section 4 that problem (1.1)–(1.3) has a sequence of arbitrarily small positive solutions, assuming that $\liminf\limits_{u\to 0^+} F(t,u)/|u|^p = 0$ and $\limsup\limits_{u\to 0^+} F(t,u)/|u|^p = \infty$, uniformly in $t \in [0,1]$.

2. Main results

Theorem 2.1. *Let α, β be lower and upper solutions of* (1.1)–(1.3) *with $\alpha \leqslant \beta$ on $[0,1]$ and assume $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function. Then the functional φ defined by* (1.4) *has a minimum on $[\alpha,\beta]$, i.e. there exists u with $\alpha\leqslant u\leqslant\beta$ on $[0,1]$ such that*

$$\varphi(u)=\min\{\varphi(v)\colon\ v\in W^{1,p}(0,1),\ \alpha\leqslant v\leqslant\beta\}.$$

Furthermore, u is a solution of BVP (1.1)–(1.3).

P r o o f. Let us consider the modified problem

$$(\Phi_p(u'))'+f(t,q(t,u))=0,\quad t\in(0,1)\setminus\{t_1,\dots,t_m\},\tag{2.1}$$

$$u'(0)-\mu_0u(0)=0,\quad u'(1)+\mu_1u(1)=0,\tag{2.2}$$

$$\Delta\Phi_p(u'(t_k))-\sigma_ku(t_k)=0,\quad k\in\{1,2,\dots,m\},\tag{2.3}$$

where $q(t,u)=\max\{\alpha(t),\min\{u,\beta(t)\}\}$.

Define the functional $\bar\varphi\colon W^{1,p}(0,1)\to\mathbb{R}$ by

$$\bar\varphi(u)=\int_0^1\Big[\frac{|u'|^p}{p}-\overline{F}(t,u)\Big]\,\mathrm{d}t+\sum_{k=1}^m\frac{|\sigma_ku(t_k)|^2}{2}+\frac{|\mu_0u(0)|^p}{p}+\frac{|\mu_1u(1)|^p}{p},\tag{2.4}$$

where $\overline{F}(t,u)=\int_0^u f(t,q(t,s))\,\mathrm{d}s$.

We can see that $\bar\varphi$ is coercive and weakly lower semicontinuous, hence it can achieve a minimum $u\in W^{1,p}(0,1)$.

We shall complete the proof in two steps.

Step 1. The critical point u of $\bar\varphi$ is a solution of (2.1)–(2.3).

For each $v\in W^{1,p}(0,1)$ we have

$$\begin{aligned}0=&\int_0^1[\Phi_p(u')v'-f(t,q(t,u))v]\,\mathrm{d}t+\sum_{k=1}^m\sigma_ku(t_k)v(t_k)\\&+\Phi_p(\mu_0u(0))v(0)+\Phi_p(\mu_1u(1))v(1).\end{aligned}\tag{2.5}$$

When $v\in W_0^{1,p}(0,1)$, we have

$$0=\int_0^1[\Phi_p(u')v'-f(t,q(t,u))v]\,\mathrm{d}t+\sum_{k=1}^m\sigma_ku(t_k)v(t_k).$$

For $k\in\{1,2,\dots,m\}$, select $v\in W_0^{1,p}(0,1)$ with $v(t)=0$ for every $t\in[0,t_k]\cup[t_{k+1},1]$. Then

$$0=\int_{t_k}^{t_{k+1}}[\Phi_p(u')v'-f(t,q(t,u))v]\,\mathrm{d}t.$$

So u satisfies

$$(\Phi_p(u'))' + f(t, q(t,u)) = 0, \quad t \in (t_k, t_{k+1}).$$

Hence, u satisfies (2.1).

Now multiplying by $v \in W_0^{1,p}(0,1)$ and integrating between 0 and 1, we have

$$\sum_{k=1}^{m} \Delta\Phi_p(u'(t_k))v(t_k) = \sum_{k=1}^{m} \sigma_k u(t_k)v(t_k).$$

Therefore, $\Delta\Phi_p(u'(t_k)) = \sigma_k u(t_k)$ for every $k \in \{1, 2, \ldots, m\}$.

Applying integration by parts to (2.1), since u satisfies (2.1) and (2.3), we get

$$(2.6) \qquad [-\Phi_p(u'(0)) + \Phi_p(\mu_0 u(0))]v(0) + [\Phi_p(u'(1)) + \Phi_p(\mu_1 u(1))]v(1) = 0.$$

Next we prove that u satisfies (2.2). Without loss of generality, we assume $u'(0) - \mu_0 u(0) > 0$, then let $v(t) = 1 - t \in C_0^\infty$ and we get the contradiction

$$0 = -\Phi_p(u'(0)) + \Phi_p(\mu_0 u(0)) > 0.$$

So $u'(0) - \mu_0 u(0) = 0$. In a similar way we get $u'(1) + \mu_1 u(1) = 0$.

Thus u is a solution of (2.1)–(2.3).

Step 2. The function u is also a solution of (1.1)–(1.3). It is enough to prove $\alpha \leqslant u \leqslant \beta$ on $[0,1]$.

Defining $(u-\beta)^+ := \max\{u - \beta, 0\}$ and integrating by parts, we have

$$\begin{aligned}
0 &\leqslant \int_0^1 [(\Phi_p(u') - \Phi_p(\beta'))' - (f(t, q(t,u)) - f(t,\beta))](u-\beta)^+ \,\mathrm{d}t \\
&= \sum_{k=0}^{m} \int_{t_k}^{t_{k+1}} [(\Phi_p(u') - \Phi_p(\beta'))' - (f(t, q(t,u)) - f(t,\beta))](u-\beta)^+ \,\mathrm{d}t \\
&= -\sum_{k=1}^{m} [\Delta\Phi_p(u'(t_k)) - \Delta\Phi_p(\beta'(t_k))](u(t_k) - \beta(t_k))^+ \\
&\quad + [\Phi_p(u'(1)) - \Phi_p(\beta'(1))](u(1) - \beta(1))^+ \\
&\quad - [\Phi_p(u'(0)) - \Phi_p(\beta'(0))](u(0) - \beta(0))^+ \\
&\quad - \int_{[0,1]^+} [\Phi_p(u') - \Phi_p(\beta')](u' - \beta') \,\mathrm{d}t,
\end{aligned}$$

where $[0,1]^+ = \{t \in [0,1] \colon \ u(t) - \beta(t) > 0\}$.

From the monotonicity of Φ_p we obtain

$$-\int_{[0,1]^+} [\Phi_p(u') - \Phi_p(\beta')](u' - \beta') \,\mathrm{d}t \leqslant 0.$$

If $u(0)-\beta(0) \leqslant 0$, we have

$$[\Phi_p(u'(0)) - \Phi_p(\beta'(0))](u(0)-\beta(0))^+ = 0.$$

If $u(0)-\beta(0) > 0$, the definition of upper solutions of (1.1)–(1.3) implies

$$u'(0)-\beta'(0) \geqslant \mu_0(u(0)-\beta(0)) \geqslant 0.$$

Then we have $\Phi_p(u'(0)) - \Phi_p(\beta'(0)) > 0$ and

$$-[\Phi_p(u'(0)) - \Phi_p(\beta'(0))](u(0)-\beta(0))^+ \leqslant 0.$$

In a similiar way, we can prove that

$$[\Phi_p(u'(1)) - \Phi_p(\beta'(1))](u(1)-\beta(1))^+ \leqslant 0.$$

Thus we have

$$\begin{aligned} 0 &\leqslant \int_0^1 [(\Phi_p(u') - \Phi_p(\beta'))' - (f(t,q(t,u)) - f(t,\beta))](u-\beta)^+ \, \mathrm{d}t \\ &\leqslant -\sum_{k=1}^m [\Delta\Phi_p(u'(t_k)) - \Delta\Phi_p(\beta'(t_k))](u(t_k)-\beta(t_k))^+ \\ &\leqslant -\sum_{k=1}^m \sigma_k (u(t_k)-\beta(t_k))^2 \leqslant 0. \end{aligned}$$

Thus $u(t) \leqslant \beta(t)$, $t \in [0,1]$. In a similar way we can prove $u(t) \geqslant \alpha(t)$, $t \in [0,1]$.

Hence, u is a solution of (1.1)–(1.3).

Notice that the function $\bar{\varphi}(u) - \varphi(u)$ is constant on $\{u \in W^{1,p}(0,1)\colon\ \alpha \leqslant u \leqslant \beta\}$. Consequently, both the functions $\bar{\varphi}$ and φ have the same minimum point between α and β so that the theorem follows. □

3. Unbounded solutions

Theorem 3.1. *Let $f\colon [0,1]\times\mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(A_1) $\inf\limits_{(t,u)\in[0,1]\times[0,\infty)} f(t,u) > -\infty, \quad \sup\limits_{(t,u)\in[0,1]\times(-\infty,0]} f(t,u) < \infty,$

(A_2) $-\infty < \liminf\limits_{u\to\pm\infty} F(t,u)/|u|^p \leqslant 0, \quad \limsup\limits_{u\to\pm\infty} F(t,u)/|u|^p = \infty,$ *uniformly in* $[0,1]$.

Then the impulsive problem (1.1)–(1.3) *has two infinite sequences of solutions* $\{u_n\}$ *and* $\{v_n\}$ *satisfying*

$$\ldots \leqslant v_{n+1} \leqslant v_n \leqslant \ldots \leqslant v_1 \leqslant u_1 \leqslant \ldots \leqslant u_n \leqslant u_{n+1} \leqslant \ldots$$

and

$$\lim_{n\to\infty} \max_{t\in[0,1]} u_n(t) = \infty, \qquad \lim_{n\to\infty} \min_{t\in[0,1]} v_n(t) = -\infty.$$

P r o o f. We will complete the proof in five steps.

Step 1. For every $M \geqslant 0$ there exists β, an upper solution of BVP (1.1)–(1.3), with $\beta \geqslant M$ on $[0,1]$.

By (A_1) there exists $K \geqslant 0$ such that

$$f(t,u) + K \geqslant 0 \quad \text{on } [0,1]\times[0,\infty) \tag{3.1}$$

and

$$F(t,u) + Ku \geqslant 0 \quad \text{on } [0,1]\times[0,\infty). \tag{3.2}$$

Given $M > 0$ and a fixed $0 < \varepsilon < \min\{\mu_1/2, 1/2\}$, select $d > 0$ such that

$$\frac{F(t,d)}{d^p} + \frac{K}{d^{p-1}} \leqslant \frac{p-1}{p}\varepsilon^p \quad \text{and} \quad \varepsilon d > M. \tag{3.3}$$

We define β to be a solution of the problem

$$(\Phi_p(u'))' + f(t,u) + K = 0, \quad u(0) = d, \quad u'(0) = 0. \tag{3.4}$$

Assume that there exists $t_0 \in (0,1]$ such that $\beta(t) > d/2$ on $[0,t_0)$ and $\beta(t_0) = d/2$. From (3.1) and (3.2) we know that $\beta'(t) \leqslant 0$ and $F(t,\beta(t)) + K\beta(t) \geqslant 0$ on $[0,t_0]$. From the conservation of energy for (3.4) we obtain

$$\begin{aligned}\frac{p-1}{p}|\beta'(t)|^p &\leqslant \frac{p-1}{p}|\beta'(t)|^p + F(t,\beta(t)) + K\beta(t)\\ &= F(t,d) + Kd \leqslant \frac{p-1}{p}(\varepsilon d)^p,\end{aligned}$$

i.e.

$$0 \leqslant -\beta'(t) \leqslant \varepsilon d.$$

It follows that for any $t \in [0,t_0]$,

$$d - \beta(t) \leqslant \varepsilon d t_0 \leqslant \varepsilon d,$$

which leads to the contradiction $\beta(t_0) \geqslant (1-\varepsilon)d > d/2$. Then for $t \in [0,1]$ we have $\beta(t) > d/2 > M$ and $\beta'(t) \geqslant -\varepsilon d$. So we can see that

$$\beta'(0) + \mu_0\beta(0) \geqslant 0$$

and

$$\beta'(1) + \mu_1\beta(1) \geqslant -\varepsilon d + \mu_1 \frac{d}{2} > 0.$$

Hence, the conclusion follows.

Step 2. For every $M \geqslant 0$ there exists α, a lower solution of BVP (1.1)–(1.3), with $\alpha \leqslant -M$ on $[0,1]$.

By (A_1) there exists $L \geqslant 0$ such that

$$f(t,u) - L \leqslant 0 \quad \text{on } [0,1] \times (-\infty, 0] \tag{3.5}$$

and

$$F(t,u) - Lu \geqslant 0 \quad \text{on } [0,1] \times (-\infty, 0]. \tag{3.6}$$

Given $M > 0$ and a fixed $0 < \delta < \min\{\mu_1/2, 1/2\}$, select $d > 0$ such that

$$\frac{F(t,-d)}{d^p} + \frac{L}{d^{p-1}} \leqslant \frac{p-1}{p}\delta^p \quad \text{and} \quad \delta d > M. \tag{3.7}$$

We define β to be a solution of the problem

$$(\Phi_p(u'))' + f(t,u) - L = 0, \quad u(0) = -d, \quad u'(0) = 0. \tag{3.8}$$

Assume that there exists $t_0 \in (0,1]$ such that $\alpha(t) < -d/2$ on $[0,t_0)$ and $\beta(t_0) = -d/2$. From (3.5) and (3.6) we know that $\alpha'(t) \geqslant 0$ and $F(t,\alpha(t)) - L\alpha(t) \geqslant 0$ on $[0,t_0]$. From the conservation of energy for (3.4) we obtain

$$\begin{aligned} \frac{p-1}{p}|\alpha'(t)|^p &\leqslant \frac{p-1}{p}|\alpha'(t)|^p + F(t,\alpha(t)) - L\alpha(t) \\ &= F(t,-d) + Ld \leqslant \frac{p-1}{p}(\delta d)^p, \end{aligned}$$

i.e.

$$0 \leqslant \alpha'(t) \leqslant \delta d.$$

It follows that for any $t \in [0,t_0]$,

$$\alpha(t) + d \leqslant \delta d t_0 \leqslant \delta d,$$

which leads to the contradiction $\alpha(t_0) \leqslant (\delta - 1)d < -d/2$. Then for $t \in [0,1]$ we have $\alpha(t) < -d/2 < -M$ and $\alpha'(t) \leqslant \delta d$. So we can see that

$$\alpha'(0) + \mu_0 \alpha(0) \leqslant 0$$

and

$$\alpha'(1) + \mu_1 \alpha(1) \leqslant \delta d - \mu_1 \frac{d}{2} < 0.$$

Hence, the conclusion follows.

Step 3. There exists a sequence of positive real numbers $\{s_n\}$ with $s_n \to \infty$ such that $\varphi(s_n) \to -\infty$, where φ is defined by (1.4).

Choose a sequence $\{s_n\}$ of positive real numbers with

$$s_n \to \infty \quad \text{and} \quad \min_{t\in[0,1]} \frac{F(t,s_n)}{|s_n|^p} \to \infty.$$

We have

$$\begin{aligned} \varphi(s_n) = &-\int_0^1 F(t,s_n)\,\mathrm{d}t + \sum_{k=1}^m \frac{|\sigma_k s_n|^2}{2} + \frac{|\mu_0 s_n|^p}{p} + \frac{|\mu_1 s_n|^p}{p} \\ \leqslant &-\min_{t\in[0,1]} F(t,s_n) + |s_n|^2 \sum_{k=1}^m \frac{|\sigma_k|^2}{2} + |s_n|^p \frac{|\mu_0|^p + |\mu_1|^p}{p}. \end{aligned}$$

It follows that $\varphi(s_n) \to -\infty$.

Step 4. There exists a sequence of negative real numbers $\{t_n\}$ with $t_n \to -\infty$ such that $\varphi(t_n) \to -\infty$.

Choose a sequence $\{t_n\}$ of positive real numbers with

$$t_n \to -\infty \quad \text{and} \quad \min_{t\in[0,1]} \frac{F(t,t_n)}{|t_n|^p} \to \infty.$$

We have

$$\begin{aligned} \varphi(t_n) = &-\int_0^1 F(t,t_n)\,\mathrm{d}t + \sum_{k=1}^m \frac{|\sigma_k t_n|^2}{2} + \frac{|\mu_0 t_n|^p}{p} + \frac{|\mu_1 t_n|^p}{p} \\ \leqslant &-\min_{t\in[0,1]} F(t,t_n) + |t_n|^2 \sum_{k=1}^m \frac{|\sigma_k|^2}{2} + |t_n|^p \frac{|\mu_0|^p + |\mu_1|^p}{p}. \end{aligned}$$

It follows that $\varphi(t_n) \to -\infty$.

Step 5. By Step 1 and Step 2, there exist lower and upper solutions α_1, β_1 of (1.1)–(1.3) with $\alpha_1 \leqslant \beta_1$. Hence, from Theorem 2.1 we can get a solution u_1 of (1.1)–(1.3) such that $\alpha_1 \leqslant u_1 \leqslant \beta_1$.

From Step 3 we have s_1 such that $s_1 \geqslant u_1$ and $\varphi(s_1) < \varphi(u_1)$. Moreover, Step 1 provides the existence of an upper solution β_2 with $u_1 \leqslant s_1 \leqslant \beta_2$. From Theorem 2.1 we have a solution u_2 of (1.1)–(1.3) satisfying

$$u_1 \leqslant u_2 \leqslant \beta_2$$

and

$$\varphi(u_2) = \min_{v \in W^{1,p}(0,1),\, u_1 \leqslant v \leqslant \beta_2} \varphi(v) \leqslant \varphi(s_1) < \varphi(u_1).$$

It follows that $u_1 \neq u_2$.

Iterating this argument, we obtain

$$u_n \leqslant u_{n+1} \leqslant \beta_{n+1} \quad \text{and} \quad \varphi(u_{n+1}) \leqslant \varphi(s_n) < \varphi(u_n).$$

From $\varphi(s_n) \to -\infty$ we have $\varphi(u_n) \to -\infty$. It follows that

$$\lim_{n\to\infty} \max_{t\in[0,1]} u_n(t) = \infty.$$

Reproducing it in the negative part, we prove the result. □

In a similar way we can get the following results.

Corollary 3.1. *Let $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(B$_1$) $\inf\limits_{(t,u)\in[0,1]\times[0,\infty)} f(t,u) > -\infty$ *and* $f(t,0) \geqslant 0\ \forall\, t \in [0,1]$,

(B$_2$) $-\infty < \liminf\limits_{u\to\infty} F(t,u)/|u|^p \leqslant 0$ *and* $\limsup\limits_{u\to\infty} F(t,u)/|u|^p = \infty$, *uniformly in* $t \in [0,1]$.

Then the impulsive problem (1.1)–(1.3) *has an infinite sequence of nonnegative solutions* $\{u_n\}$ *satisfying*

$$0 \leqslant u_1 \leqslant \ldots \leqslant u_n \leqslant u_{n+1} \leqslant \ldots$$

and

$$\lim_{n\to\infty} \max_{t\in[0,1]} u_n(t) = \infty.$$

Corollary 3.2. *Let $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that*

(C$_1$) $\inf\limits_{(t,u)\in[0,1]\times[-\infty,0)} f(t,u) < \infty$ *and* $f(t,0) \leqslant 0\ \forall\, t \in [0,1]$,

(C$_2$) $-\infty < \liminf\limits_{u\to-\infty} F(t,u)/|u|^p \leqslant 0$ *and* $\limsup\limits_{u\to-\infty} F(t,u)/|u|^p = \infty$, *uniformly in* $[0,1]$.

Then the impulsive problem (1.1)–(1.3) *has an infinite sequence of nonpositive solutions* $\{v_n\}$ *satisfying*

$$\ldots \leqslant v_{n+1} \leqslant v_n \leqslant \ldots \leqslant v_1 \leqslant 0$$

and

$$\lim_{n\to\infty} \min_{t\in[0,1]} v_n(t) = -\infty.$$

4. Arbitrarily small solutions

Theorem 4.1. *Let* $f\colon [0,1]\times[0,\infty)\to\mathbb{R}$ *be a continuous function. Assume that*

$$\liminf_{u\to 0^+} \frac{F(t,u)}{|u|^p} = 0, \quad \limsup_{u\to 0^+} \frac{F(t,u)}{|u|^p} = \infty, \qquad \text{uniformly in } [0,1]. \tag{4.1}$$

If one of the conditions

(D$_1$) $\inf\{u>0\colon \max\limits_{t\in[0,1]} f(t,u)\leqslant 0\} = 0$,

(D$_2$) *There exists* $\delta>0$ *such that*

$$f(t,u) > 0 \quad \text{on } [0,1]\times[0,\delta]$$

holds, then the impulsive problem (1.1)–(1.3) *has an infinite decreasing sequence of positive solutions* $\{u_n\}$ *satisfying*

$$\lim_{n\to\infty} \max_{t\in[0,1]} u_n(t) = 0.$$

P r o o f. We will complete the proof in four steps.

Step 1. There exist upper solutions $\{\beta_n\}$ of BVP (1.1)–(1.3) with

$$\min_{t\in[0,1]} \beta_n(t) > 0 \quad \text{and} \quad \max_{t\in[0,1]} \beta_n(t) \to 0. \tag{4.2}$$

If (D$_1$) holds, then there exists a sequence $\{\beta_n\}$ of upper solutions satisfying (4.2). If (D$_2$) holds, then we have

$$F(t,u) > 0 \quad \text{on } [0,1]\times(0,\delta]. \tag{4.3}$$

By virtue of (4.1), we can select a decreasing sequence $\{d_n\}\subset(0,\frac{1}{2}\delta)$ such that

$$\frac{F(t,d_n)}{d_n^p} < \frac{p-1}{p}\frac{1}{3^p}, \quad \text{uniformly in } t\in[0,1] \text{ and } d_n\to 0. \tag{4.4}$$

We define β_n to be a sequence of solutions of the problem

$$(\Phi_p(u'))' + f(t,u) = 0, \quad u(0) = d_n, \quad u'(0) = 0. \tag{4.5}$$

Next we prove that $\frac{1}{2}d_n < \beta_n \leqslant d_n$ on $[0,1]$.

From (D_2), it is obvious that $\beta_n \leqslant d_n$ on $[0,1]$. Assume that there exists $t_0 \in (0,1]$ such that $\frac{1}{2}d_n < \beta(t) \leqslant d_n$ on $[0,t_0)$ and $\beta(t_0) = \frac{1}{2}d_n$. From (D_1) and (4.1), we know that $\beta'(t) \leqslant 0$ and $F(t,\beta(t)) \geqslant 0$ on $[0,t_0]$. From the conservation of energy for (3.4) we obtain

$$\begin{aligned}\frac{p-1}{p}|\beta_n'(t)|^p \leqslant \frac{p-1}{p}|\beta_n'(t)|^p + F(t,\beta_n(t)) \\ = F(t,d_n) \leqslant \frac{p-1}{p}\Big(\frac{1}{3}d_n\Big)^p,\end{aligned}$$

i.e.

$$0 \leqslant -\beta_n'(t) \leqslant \frac{1}{3}d_n.$$

It follows that for any $t \in [0,t_0]$,

$$d_n - \beta_n(t) \leqslant \frac{1}{3}d_n t_0 \leqslant \frac{1}{3}d_n,$$

which leads to the contradiction $\beta(t_0) \geqslant \frac{2}{3}d_n > \frac{1}{2}d_n$. Hence, the conclusion follows.

Step 2. From $\liminf\limits_{u\to 0^+} F(t,u)/|u|^p = 0$, uniformly in $t \in [0,1]$, we have $f(t,0) = 0$ for $t \in [0,1]$. Hence, the function 0 is a lower solution of problem (1.1)–(1.3).

Step 3. There exists a sequence of positive real numbers $\{s_n\}$ with $s_n \to 0^+$ such that $\varphi(s_n) < 0$, where φ is defined by (1.4). Choose a sequence $\{s_n\}$ of positive real numbers with

$$s_n \to 0^+ \quad \text{and} \quad \min_{t\in[0,1]} \frac{F(t,s_n)}{|s_n|^p} \to \infty.$$

We have

$$\begin{aligned}\varphi(s_n) = & -\int_0^1 F(t,s_n)\,\mathrm{d}t + \sum_{k=1}^m \frac{|\sigma_k s_n|^2}{2} + \frac{|\mu_0 s_n|^p}{p} + \frac{|\mu_1 s_n|^p}{p} \\ \leqslant & -\min_{t\in[0,1]} F(t,s_n) + |s_n|^2 \sum_{k=1}^m \frac{|\sigma_k|^2}{2} + |s_n|^p \frac{|\mu_0|^p + |\mu_1|^p}{p}.\end{aligned}$$

It follows that $\varphi(s_n) < 0$.

Step 4. By Step 1 and Step 2, there exist upper solutions β_1 of (1.1)–(1.3) with $\beta_1 \geqslant 0$. Hence, from Theorem 2.1 we get a solution u_1 of (1.1)–(1.3) such that $0 \leqslant u_1 \leqslant \beta_1$.

From Step 3 we get s_1 such that $0 \leqslant s_1 \leqslant \min_{t\in[0,1]} \beta_1$ and $\varphi(s_1) < 0$. According to Theorem 2.1 we have

$$\varphi(u_1) = \min_{v\in W^{1,p}(0,1),\, 0\leqslant v\leqslant\beta_1} \varphi(v) \leqslant \varphi(s_1) < 0.$$

Thus, $u_1 \not\equiv 0$ and it is a positive solution of (1.1)–(1.3) satisfying $\max_{t\in[0,1]} u_1(t) \leqslant d_1$.

Moreover, Step 1 provides the existence of an upper solution β_2 with $\max_{t\in[0,1]} \beta_2 < \max_{t\in[0,1]} u_1$. From Theorem 2.1 we have a solution u_2 of (1.1)–(1.3) satisfying $0 \leqslant u_2 \leqslant \beta_2$ and $u_1 \neq u_2$.

From Step 3 we can get s_2 such that $0 \leqslant s_2 \leqslant \min_{t\in[0,1]} \beta_2$ and $\varphi(s_2) < 0$. From Theorem 2.1 we have

$$\varphi(u_2) = \min_{v\in W^{1,p}(0,1),\, 0\leqslant v\leqslant\beta_2} \varphi(v) \leqslant \varphi(s_2) < 0.$$

Hence, $u_2 \not\equiv 0$ and it is a positive solution of (1.1)–(1.3) satisfying $\max_{t\in[0,1]} u_2(t) \leqslant d_2$.

Iterating this argument, we build a sequence of distinct positive solutions $\{u_n\}$ satisfying $\lim_{n\to\infty} \max_{t\in[0,1]} u_n(t) = 0$. □

5. Examples

Example 5.1. Consider the problem

$$(\Phi_p(u'))' + f(u) = 0, \tag{5.1}$$

$$u'(0) - \mu_0 u(0) = 0, \quad u'(1) + \mu_1 u(1) = 0, \tag{5.2}$$

$$\Delta\Phi_p(u'(t_k)) - \sigma_k u(t_k) = 0, \quad k \in \{1, 2, \ldots, m\}, \tag{5.3}$$

where $p > 1$, $\mu_0 \geqslant 0$, $\mu_1 > 0$, $\sigma_k \geqslant 0$, $k = 1, 2, \ldots, m$. The continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$f(u) = \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}u}\Big\{(|u|+\lambda)^p \log(\log(|u|+\lambda)) \sin^2(\log(\log(\log(|u|+\lambda)))) \\ \quad + \dfrac{p(|u|+\lambda)^p}{\log(|u|+\lambda)}\Big\} + \operatorname{sgn}(u) p\lambda^{p-1} \dfrac{1 - p\log\lambda}{(\log\lambda)^2}, \quad u \neq 0, \\ 0, \quad u = 0 \end{cases} \tag{5.4}$$

with $\lambda = \mathrm{e}^{\mathrm{e}}$. Then it is easy to prove that

$$\inf_{u\in[0,\infty)} f(u) > -\infty, \qquad \sup_{u\in(-\infty,0]} f(u) < \infty,$$

$$\liminf_{u\to\pm\infty} \frac{\int_0^u f(s)\,\mathrm{d}s}{|u|^p} = 0, \quad \limsup_{u\to\pm\infty} \frac{\int_0^u f(s)\,\mathrm{d}s}{|u|^p} = \infty.$$

Thus by Theorem 3.1, problem (5.1)–(5.3) has two infinite sequences of solutions $\{u_n\}$ and $\{v_n\}$, satisfying

$$\ldots \leqslant v_{n+1} \leqslant v_n \leqslant \ldots \leqslant v_1 \leqslant u_1 \leqslant \ldots \leqslant u_n \leqslant u_{n+1} \leqslant \ldots$$

and

$$\lim_{n\to\infty} \max_{t\in[0,1]} u_n(t) = \infty, \quad \lim_{n\to\infty} \min_{t\in[0,1]} v_n(t) = -\infty.$$

Example 5.2. Consider the problem

$$(\Phi_p(u'))' + g(u) = 0, \tag{5.5}$$

$$u'(0) - \mu_0 u(0) = 0, \quad u'(1) + \mu_1 u(1) = 0, \tag{5.6}$$

$$\Delta\Phi_p(u'(t_k)) - \sigma_k u(t_k) = 0, \quad k \in \{1, 2, \ldots, m\}, \tag{5.7}$$

where $p > 1$, $\mu_0 \geqslant 0$, $\mu_1 > 0$, $\sigma_k \geqslant 0$, $k = 1, 2, \ldots, m$. The continuous function $g\colon [0,\infty) \to \mathbb{R}$ is defined by

$$g(u) = \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}u}\Big\{\Big(u^p \log\log\dfrac{1}{u}\Big)\sin^2\Big(\log\log\log\dfrac{1}{u}\Big) + \dfrac{pu^p}{\log 1/u}\Big\}, & u > 0, \\ 0, \quad u = 0. \end{cases} \tag{5.8}$$

Then it is easy to prove that

$$\liminf_{u\to 0^+} \frac{\int_0^u g(s)\,\mathrm{d}s}{u^p} = 0, \quad \limsup_{u\to 0^+} \frac{\int_0^u g(s)\,\mathrm{d}s}{u^p} = \infty$$

and

$$\inf\{u > 0\colon\ g(u) \leqslant 0\} = 0.$$

Thus by Theorem 4.1, problem (5.5)–(5.7) has an infinite decreasing sequence of positive solutions $\{u_n\}$ satisfying

$$\lim_{n\to\infty} \max_{t\in[0,1]} u_n(t) = 0.$$

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