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INFINITESIMAL AUTOMORPHISMS AND SECOND VARIATION OF THE ENERGY FOR HARMONIC FOLLATIONS*

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Introduction. In [KT2] [KT3] we considered an energy functional for foliations $\mathcal{F}$ on a smooth compact oriented manifold, defined with respect to a Riemannian metric g_M on M. Harmonic Riemannian foliations were then characterized as critical foliations for this functional under an appropriate class of so called special variations (the relevant concepts are repeated in Section 2 of the present paper). The obvious analogy with the harmonic map theory of Eells and Sampson [ES] was the guiding principle.

In this paper we discuss the effect of curvature properties in the normal bundle of a Riemannian foliation $\mathcal{F}$ on:

(A) the existence of deformations of $\mathcal{F}$ through harmonic foliations;

(B) the existence of infinitesimal metric automorphisms of $\mathcal{F}$.

The curvature properties in the normal bundle of a Riemannian foliation (R-foliation) are described as follows. There is a unique metric and torsionfree connection ∇ in the normal bundle. Thus its curvature operator R_∇ is canonically attached to $\mathcal{F}$. The natural vanishing properties of R_∇ allow to view it as a skew-symmetric operator $R_\nabla(\mu, \nu)$ on sections $\mu, \nu \in \Gamma Q$ with values in the bundle End(Q). If $\mathcal{F}$ is viewed as modelled on a Riemannian manifold N by a Haefliger cocycle with isometric transition functions, then R_∇ is the curvature of the canonical connection on N pulled back by the local submersions with target N defining $\mathcal{F}$. The Ricci operator $\rho_\nabla: Q \to Q$ of $\mathcal{F}$ is then defined in terms of R_∇ by the usual formula (see (1.6) or (1.7) below).

We prove the following result.

THEOREM A. *Let $\mathcal{F}$ be a Riemannian foliation on a compact and oriented manifold M. Let g_M be a bundle-like metric on M in the sense of Reinhart* [R], *and assume $\mathcal{F}$ to be harmonic with respect to g_M. Assume the Ricci operator ρ_∇ of $\mathcal{F}$ to be $\leqq 0$ everywhere, and < 0 for at least one point $x \in M$. Then*:

(i) *there is no special variation of $\mathcal{F}$ through harmonic foliations*;

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(ii) *$\mathscr{F}$ realizes a local minimum of the energy functional under special variations.*

Next we describe infinitesimal metric automorphisms of an R- foliation. These are vector fields on M preserving the foliation $\mathscr{F}$ as well as the metric g_Q on the normal bundle Q. The canonical projection of such a vector field on M to a section of Q is called a transverse Killing field of $\mathscr{F}$ (see Molino [M1] and (3.6) below).

We prove the following result.

THEOREM B. *Let $\mathscr{F}$ be a Riemannian foliation on a compact and oriented manifold M. Assume the Ricci operator ρ_∇ of $\mathscr{F}$ to be $\leqq 0$ everywhere, and < 0 for at least one point $x \in M$. Then every transverse Killing field of $\mathscr{F}$ is trivial, or equivalently every infinitesimal metric automorphism of $\mathscr{F}$ is tangential to $\mathscr{F}$.*

The statement is that a flow φ_t of metric automorphisms of $\mathscr{F}$ under the above curvature assumptions maps each leaf of $\mathscr{F}$ into itself. Thus the local submersions defining $\mathscr{F}$ are unaffected by composition with φ_t.

Even for the point foliation $\mathscr{F}$ on M with normal bundle $Q = TM$ the statement is non-trivial. The assumption on the metric $g_Q = g_M$ is then that the usual Ricci curvature operator is $\leqq 0$ everywhere, and <0 for at least one point. The conclusion is the theorem of Bochner on the vanishing of Killing vector fields on (M, g_M) [BO].

It is interesting to observe that in our context the compactness assumption is made on M, whereas the curvature assumption is made on Q, i.e. the model space of the R-foliation.

Theorems A and B are both consequences of the non-existence of Jacobi fields in the normal bundle of an R-foliation under the specified conditions on ρ_∇. The Jacobi operator $J_\nabla: \Gamma Q \to \Gamma Q$ of an R-foliation is defined in (2.3) below. It is the operator occuring in the following fundamental formula, which will be proved in the appendix.

SECOND VARIATION FORMULA. *Let M be a compact oriented manifold, and $\mathscr{F}$ a Riemannian and harmonic foliation with respect to a bundle-like metric g_M. Consider the 2-parameter family $\mathscr{F}_{s,t}$ of special variations of $\mathscr{F} = \mathscr{F}_{0,0}$ defined by two sections μ, ν of the normal bundle Q. Then for the second derivative of the energy we have*

$$(\partial^2/\partial s\partial t)E(\mathscr{F}_{s,t})|_{s=0,t=0} = \langle(\Delta - \rho_\nabla)\mu, \nu\rangle$$

To explain the RHS, observe that the canonical connection ∇ in Q

defines an exterior differential d_∇ and codifferential d_∇^* on Q-valued forms on M. Then $\Delta = d_\nabla^* d_\nabla$ is the Laplacian on the space of sections on Q, and the scalar product of sections $\mu, \nu \in \Gamma Q$ is given by

$$\langle \mu, \nu \rangle = \int_M g_Q(\mu, \nu) \cdot v_M$$

in terms of the canonical metric g_Q on Q and the volume form v_M associated to the metric g_M on M.

The paper is organized as follows: In Section 1 we discuss curvature properties in directions normal to the leaves of an R-foliation. Section 2 is devoted to the Jacobi operator and its spectrum. In Section 3 we discuss infinitesimal automorphisms of R-foliations and show that they give rise to Jacobi fields. The proof of Theorems A and B is completed in Section 4. The second variation formula is proved in the appendix.

We expect the spectrum, index and nullity of an R-foliation to be of significance in future investigations. A hint in this direction is Hurwitz's 84(g-1) theorem [HU], of which Theorem B generalizes only the finiteness statement to the context of foliations. One can expect an index formula equivariant under the action of the foliation automorphisms, which will involve the characteristic invariants of the normal bundle.

1. **Curvature.** Let M be a manifold and $\mathscr{F}$ a foliation given by an integrable subbundle $L \subset TM$. $\mathscr{F}$ is called Riemannian (or an R-foliation), if the normal bundle is equipped with a holonomy invariant fiber metric g_Q. This condition can be expressed in terms of the Bott connection $\mathring{\nabla}$ (cf. Section 3) by $\mathring{\nabla}_X g_Q = 0$ for $X \in \Gamma L$. If $\mathscr{F}$ is an R-foliation, there is a unique metric and torsionfree connection ∇ in the normal bundle $Q = TM/L$ [KT 3, Theorem 1.11]. Its curvature

$$R_\nabla(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \quad \text{for} \quad X, Y \in \Gamma TM$$

is a 2-form on M with values in End(Q). R_∇ is a basic form in the sense that

$$i(X) R_\nabla = 0 \quad \text{for} \quad X \in \Gamma L\,. \tag{1.1}$$

The geometric interpretation of this property is that ∇ can be thought of as the pull-back of the Riemannian connection in the model space via the local submersions defining $\mathscr{F}$. As a consequence we have the following fact.

PROPOSITION 1.2. *For $\mu, \nu \in \Gamma Q$ the operator $R_\nabla(\mu, \nu)\colon Q \to Q$ is a well-defined endomorphism.*

Note that R_∇ is homogeneous with respect to multiplication with functions on M in all variables.

The usual definitions of geometric quantities arising from a curvature tensor can now be made for R_∇. This is in spirit similar to Kulkarni's introduction of curvature structures [KU], even if technically this does not quite fit into the same framework. Let $x \in M$ and $\sigma \subset Q_x$ a 2-plane in the normal bundle spanned by 2 normal vectors μ_x, ν_x. Then the sectional curvature of $(\mathscr{F}, g_Q)$ at x in directions of σ is defined by

$$K_\nabla(\sigma) = g_Q(R_\nabla(\mu_x, \nu_x)\nu_x, \mu_x)/\{g_Q(\mu_x, \mu_x)g_Q(\nu_x, \nu_x) - g_Q(\mu_x, \nu_x)^2\} . \tag{1.3}$$

Here g_Q denotes the holonomy invariant fiber metric on Q.

For the following formulas we introduce at a point $x \in M$ an orthonormal basis $e_{p+1}, \cdots, e_n$ of Q_x. Here $n = \dim M$, $q = \operatorname{codim} \mathscr{F} = \dim Q$ and $p = n - q = \dim \mathscr{F} = \dim L$. The Ricci curvature S_∇ is then the symmetric bilinear form on Q given at x by

$$S_\nabla(\mu, v)_x = \sum_{\alpha=p+1}^{n} g_Q(R_\nabla(\mu, e_\alpha)e_\alpha, \nu) \tag{1.4}$$

for $\mu, \nu \in \Gamma Q$ (only μ_x, ν_x enter into the definition). A basis free formulation is obtained by introducing $P_\nabla(\mu, \nu) \in \operatorname{End} Q$ via $P_\nabla(\mu, \nu)s = -R_\nabla(\mu, s)\nu$. Then

$$\begin{aligned} S_\nabla(\mu, \nu) &= -\sum_{\alpha=p+1}^{n} g_Q(e_\alpha, R_\nabla(\mu, e_\alpha)\nu) = \sum_{\alpha=p+1}^{n} g_Q(e_\alpha, P_\nabla(\mu, \nu)e_\alpha) \\ &= \operatorname{Trace} P_\nabla(\mu, \nu) . \end{aligned} \tag{1.5}$$

The Ricci operator $\rho_\nabla \colon Q \to Q$ is the corresponding self-adjoint operator given by

$$S_\nabla(\mu, \nu) = g_Q(\rho_\nabla \mu, \nu) \tag{1.6}$$

for $\mu, \nu \in \Gamma Q$. In terms of an orthonormal basis of Q_x as above we have by (1.4)

$$(\rho_\nabla \mu)_x = \sum_{\alpha=p+1}^{n} R_\nabla(\mu, e_\alpha)e_\alpha . \tag{1.7}$$

The scalar curvature σ_∇ finally is given by

$$\sigma_\nabla = \operatorname{Trace} \rho_\nabla . \tag{1.8}$$

All these geometric quantities should be thought of as the corresponding curvature properties of a Riemannian manifold serving as model space for $\mathscr{F}$. In the present paper we are only concerned with the Ricci operator ρ_∇.

We would like to contrast this point of view with the frequently adopted point of view where curvature assumptions are made in the directions tangential to the leaves of a foliation. Here we are concerned with curvature properties in normal directions.

2. Jacobi operator. For an R-foliation $\mathscr{F}$ with metric g_Q and canonical connection ∇ on Q the usual calculus for Q-valued forms on M applies. In particular there are exterior differentials and codifferentials

$$d_\nabla\colon \Omega^r(M, Q) \to \Omega^{r+1}(M, Q)\ , \quad r \geqq 0\ ; \quad d_\nabla^*\colon \Omega^r(M, Q) \to \Omega^{r-1}(M, Q)\ , \quad r > 0$$

and the Laplacian $\Delta = d_\nabla d_\nabla^* + d_\nabla^* d_\nabla$.

In the following let M be compact and oriented. With respect to a metric g_M on M the energy of $\mathscr{F}$ is defined by [KT2] [KT3]

$$E(\mathscr{F}) = (1/2)\|\pi\|^2 = (1/2)\int_M g_Q(\pi \wedge *\pi)$$

where $\pi\colon TM \to Q$ is viewed as a Q-valued 1-form. Now assume g_M to be bundle-like. A section $\nu \in \Gamma Q$ defines then a special variation $\mathscr{F}_t$ of $\mathscr{F}_0 = \mathscr{F}$ through Riemannian foliations by patching the local data

$$\Phi_t^\alpha(x) = \exp_{f^\alpha(x)}(t\nu^\alpha(x))\ . \tag{2.1}$$

Here f^α is a local submersion defining $\mathscr{F}$ in an open set U^α. Φ_t^α is then the local submersion defining $\mathscr{F}_t$ for $|t| \leqq \varepsilon$, where the RHS in (2.1) denotes the endpoint of the geodesic segment starting at $f^\alpha(x)$ and determined by $t\nu^\alpha(x)$. One finds that $(d/dt)|_{t=0}E(\mathscr{F}_t) = \langle \nu, \tau \rangle$ where τ is the tension field of $\mathscr{F}$ [KT3], and it follows that $\mathscr{F}$ is critical for E under all such variations iff $\tau = 0$, which is the harmonicity condition on $\mathscr{F}$. For details we refer to [KT3].

If $\mu, \nu \in \Gamma Q$, then a 2-parameter special variation $\mathscr{F}_{s,t}$ of $\mathscr{F}_{0,0} = \mathscr{F}$ through Riemannian foliations is defined by patching the local definitions

$$\Phi_{s,t}^\alpha(x) = \exp_{f^\alpha(x)}(s\mu^\alpha(x) + t\nu^\alpha(x))\ . \tag{2.2}$$

Calculating the second derivative of $E(\mathscr{F}_{s,t})$ at $s = 0, t = 0$ in case of a harmonic foliation $\mathscr{F}$, one finds the formula stated in the introduction. Details will be given in the appendix. This leads to the following concept.

DEFINITION 2.3. The *Jacobi operator* of an R-foliation $\mathscr{F}$ is given by

$$J_\nabla \nu = (\Delta - \rho_\nabla)\nu \quad \text{for} \quad \nu \in \Gamma Q\ ,$$

where $\Delta = d_\nabla^* d_\nabla$ on sections of Q.

With respect to the natural scalar product on ΓQ the operator $J_\nabla\colon \Gamma Q \to \Gamma Q$ is self-adjoint and strongly elliptic of second order. It follows that J_∇ has an eigenspace decomposition with real eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots \to +\infty \quad \text{for} \quad i \to \infty .$$

The dimension of each eigenspace is finite (see [BR] [S]).

The spectrum of J_∇ is attached to any R-foliation $\mathscr{F}$ and furnishes an important set of invariants. The index (the sum of the dimensions of the eigenspaces corresponding to negative eigenvalues) is discussed for a special case in [KT4].

DEFINITION 2.4. $\nu \in \ker J_\nabla$ is called a *Jacobi field* of $\mathscr{F}$.

Note that if $\mathscr{F}_t$ is a special variation of $\mathscr{F}_0 = \mathscr{F}$ through harmonic foliations, then the generating $\nu \in \Gamma Q$ is a Jacobi field of $\mathscr{F}$. Thus the statement in part (i) of Theorem A is implied by the non-existence of Jacobi fields under the stated curvature assumptions. We prove this in Section 4.

3. Infinitesimal automorphisms. A vector field Y on M is an infinitesimal automorphism of a foliation $\mathscr{F}$ if $[Y, Z] \in \Gamma L$ for every $Z \in \Gamma L$. This means that the flow generated by Y is a flow of automorphisms of $\mathscr{F}$, i.e., maps leaves into leaves.

The exact sequence

$$0 \to L \to TM \xrightarrow{\pi} Q \to 0 \tag{3.1}$$

defines for every $Y \in \Gamma TM$ a section $\nu = \pi(Y) \in \Gamma Q$. Molino [M1] [M2] calls an infinitesimal automorphism Y a foliated vector field and its projection $\nu = \pi(Y)$ the associated transverse field.

Recall that Q is equipped with the partial Bott connection given by

$$\mathring{\nabla}_X \nu = \pi[X, Y] \quad \text{for} \quad X \in \Gamma L . \tag{3.2}$$

LEMMA 3.3. (i) *If $Y \in \Gamma TM$ is an infinitesimal automorphism, then its associated transverse field $\nu = \pi(Y)$ satisfies*

$$\mathring{\nabla}_X \nu = 0 \quad \textit{for every} \quad X \in \Gamma L . \tag{$*$}$$

(ii) *If $\nu \in \Gamma Q$ conversely satisfies $(*)$, then every $Y \in \Gamma TM$ with $\pi(Y) = \nu$ is an infinitesimal automorphism.*

PROOF. This is obvious from (3.2). ∎

Next we define for an infinitesimal automorphism $Y \in \Gamma TM$

$$\Theta(Y)\nu = \pi[Y, Y_\nu] \quad \text{for} \quad \nu \in \Gamma Q,\ Y_\nu \in \Gamma TM \quad \text{with} \quad \pi(Y_\nu) = \nu . \tag{3.4}$$

The RHS is independent of the choice of the representative Y_ν of ν. Note that for $X \in \Gamma L$ this definition coincides with (3.2). Thus it is appropriate to denote by $\Gamma Q^L \equiv \Gamma(Q^L)$ the sections of Q satisfying $(*)$ in (3.3) (L-invariant sections). We have then an exact sequence of Lie algebras

$$0 \to \Gamma L \to V(\mathscr{F}) \to \Gamma Q^L \to 0 \tag{3.5}$$

where the middle term denotes the infinitesimal automorphisms. The bracket in ΓQ^L is induced from the bracket in $V(\mathscr{F})$. If the elements of ΓL are viewed as interior infinitesimal automorphisms of $\mathscr{F}$, then the elements of ΓQ^L are to be viewed as outer infinitesimal automorphisms of $\mathscr{F}$.

From now on let $\mathscr{F}$ be an R-foliation with metric g_Q on Q (its holonomy invariance is the condition $\mathring{\nabla}_X g_Q = 0$ for $X \in \Gamma L$).

DEFINITION 3.6. $Y \in V(\mathscr{F})$ is *metric*, if $\Theta(Y)g_Q = 0$. If this holds, then $\pi(Y) \in \Gamma Q^L$ is called a *transverse Killing field* of $\mathscr{F}$ (Molino [M1] [M2]).

The explicit meaning of this condition is given by (3.4) and the identity

$$(\Theta(Y)g_Q)(s, t) = Yg_Q(s, t) - g_Q(\Theta(Y)s, t) - g_Q(s, \Theta(Y)t)$$

for $s, t \in \Gamma Q$. The RHS is well defined for $Y \in V(\mathscr{F})$ (not for arbitrary vector fields on M). If $K(\mathscr{F})$ denotes the metric infinitesimal automorphisms, and $\bar{K}(\mathscr{F})$ the transverse Killing fields, then together with (3.5) we have the following diagram displaying the relationship

$$0 \to \Gamma L \begin{array}{l} \nearrow V(\mathscr{F}) \to \Gamma Q^L \to 0 \\ \qquad \cup \qquad\quad \cup \\ \searrow K(\mathscr{F}) \to \bar{K}(\mathscr{F}) \to 0 \end{array} \tag{3.7}$$

REMARK 3.8. For the point foliation $\mathscr{F}$ given by $L = 0$, $Q = TM$ and $g_Q = g_M$ any Riemannian metric on M, (3.6) is the definition of a Killing vector field on (M, g_M). Thus it might be appropriate to call the infinitesimal metric automorphisms of a Riemannian foliation simply Killing fields for $\mathscr{F}$. We avoid this terminology, because in view of the following remark it might be confusing.

REMARK 3.9. If g_M is a bundle-like metric on $(M, \mathscr{F})$ (inducing g_Q on Q), and $Y \in \Gamma TM$ a Killing vector field for (M, g_M), then Y is metric for g_Q. But the converse is not necessarily true: $Y \in V(\mathscr{F})$ may satisfy $\Theta(Y)g_Q = 0$ without satisfying $\Theta(Y)g_M = 0$.

In the next definition we make again use of the unique metric and torsionfree connection ∇ defined in Q. It is an adapted connection, i.e., extends the partial Bott connection $\mathring{\nabla}$ given by (3.2) to a genuine connection [KT3, 1.11]. We define then for $Y \in V(\mathscr{F})$, $s \in \Gamma Q$

$$A_\nabla(Y)s = \theta(Y)s - \nabla_Y s\ . \tag{3.10}$$

Let $Y_s \in \Gamma TM$ with $\pi(Y_s) = s$. Then by (3.4) and the torsionfreeness of ∇ [KT3, 1.5]

$$A_\nabla(Y)s = \pi[Y, Y_s] - \nabla_Y \pi(Y_s) = -\nabla_{Y_s}\pi(Y)\ . \tag{3.11}$$

This formula shows two things: (i) $A_\nabla(Y)$ depends in fact only on $\nu = \pi(Y)$; (ii) $A_\nabla(\nu)$ is a linear operator $Q \to Q$. Thus (3.10) defines

$$A_\nabla(\nu)\colon Q \to Q \quad \text{for} \quad \nu \in \Gamma Q^L\ . \tag{3.12}$$

We define $A_\nabla(\nu)f = 0$ for $f \in \Omega^\circ(M)$. Then $A_\nabla(\nu)$ extends in an obvious way to tensors of any type on Q.

PROPOSITION 3.13. *Let $\mathscr{F}$ be an R-foliation, $Y \in V(\mathscr{F})$ with $\pi(Y) = \nu \in \Gamma Q^L$. Then the following conditions are equivalent*:

(i) $Y \in K(\mathscr{F})$;

(ii) $A_\nabla(\nu)g_Q = 0$;

(iii) $g_Q(A_\nabla(\nu)s, t) + g_Q(s, A_\nabla(\nu)t) = 0$ *for* $s, t \in \Gamma Q$.

PROOF. Since ∇ is a metric connection in Q, $\nabla_Y g_Q = 0$. Thus $\theta(Y)g_Q = 0 \Leftrightarrow A_\nabla(\nu)g_Q = 0$. Further

$$(A_\nabla(\nu)g_Q)(s, t) = A_\nabla(\nu)g_Q(s, t) - g_Q(A_\nabla(\nu)s, t) - g_Q(s, A_\nabla(\nu)t)$$

This proves the equivalence of (i), (ii) and (iii). ∎

COROLLARY 3.14. *The property of an infinitesimal automorphism Y of $\mathscr{F}$ to be metric depends only on the transverse field $\nu = \pi(Y) \in \Gamma Q^L$.*

The Killing property of the transverse field ν is characterized by the skew-symmetry of $A_\nabla(\nu)$ with respect to g_Q. This gives a direct definition of $\bar{K}(\mathscr{F})$ in (3.7).

We further note that (3.7) is a diagram of Lie algebras and Lie homomorphisms. It suffices to verify that $K(\mathscr{F})$ is a subalgebra of $V(\mathscr{F})$ with the usual bracket. But for the bracket of $X, Y \in V(\mathscr{F})$ one finds the formula

$$(\Theta[X, Y])g_Q = \theta(X)\Theta(Y)g_Q - \Theta(Y)\Theta(X)g_Q\ .$$

Thus for $X, Y \in K(\mathscr{F})$ the RHS vanishes, and therefore so does the LHS, i.e., $[X, Y] \in K(\mathscr{F})$.

The relationship of this concept with the ideas discussed in the first two sections is given by the following result.

PROPOSITION 3.15. *Let $\mathscr{F}$ be an R-foliation on a manifold M. If $\nu \in \Gamma Q^L$ is a transverse Killing field of $\mathscr{F}$, then ν is a Jacobi field of $\mathscr{F}$.*

The proof will proceed by showing that an infinitesimal metric automorphism preserves the connection ∇ in Q, and that such an infinitesimal automorphism projects necessarily to a Jacobi field.

We begin by introducing the following concept.

DEFINITION 3.16. $Y \in V(\mathscr{F})$ preserves the connection ∇ in Q if $\Theta(Y)\nabla = 0$, where

$$(\Theta(Y)\nabla_X)s = \Theta(Y)\nabla_X s - \nabla_X \Theta(Y)s - \nabla_{[Y,X]}s$$

for $X \in \Gamma TM$, $s \in \Gamma Q$.

For $X \in \Gamma L$, it follows from (3.4) that the RHS of this identity equals

$$\pi[Y, [X, Y_s]] - \pi[X, [Y, Y_s]] - \pi[[Y, X], Y_s]$$

and thus vanishes by the Jacobi identity. It follows that the vanishing of $(\theta(Y)\nabla_X)s$ has only to be tested for $X = Y_\mu = Y_{\sigma(\mu)} \in \Gamma\sigma Q$, $\mu \in \Gamma Q$. Here $\sigma\colon Q \to TM$ denotes the splitting of the exact sequence $0 \to L \to TM \xrightarrow{\pi} Q \to 0$ given by the orthogonal decomposition $TM \cong L \oplus L^\perp$.

PROPOSITION 3.17. *Let $\mathscr{F}$ be an R-foliation, $Y \in V(\mathscr{F})$ with $\pi(Y) = \nu \in \Gamma Q^L$. Then the following conditions are equivalent:*

(i) *Y preserves ∇;*

(ii) *$\nabla_{Y_\mu} A_\nabla(\nu) = R_\nabla(\nu, \mu)$ for all $\mu \in \Gamma Q$.*

For the case of the point foliation given by $L = 0$, $Q = TM$ this is Proposition 2.2 in [KO].

PROOF. From (3.16) we get by subtraction of identical terms on both sides

$$(\Theta(Y)\nabla_{Y_\mu})s - [\nabla_Y, \nabla_{Y_\mu}]s = (\Theta(Y) - \nabla_Y)\nabla_{Y_\mu}s - \nabla_{Y_\mu}(\Theta(Y) - \nabla_Y)s - \nabla_{[Y,Y_\mu]}s\ .$$

By the definition of the curvature R_∇ and (3.10) therefore

$$\Theta(Y)\nabla_{Y_\mu} = R_\nabla(Y, Y_\mu) + A_\nabla(Y)\nabla_{Y_\mu} - \nabla_{Y_\mu}A_\nabla(Y) = R_\nabla(\nu, \mu) - \nabla_{Y_\mu}A_\nabla(\nu)\ .$$

which proves the desired result. ∎

COROLLARY 3.18. *The property of an infinitesimal automorphism Y of $\mathscr{F}$ to be connection preserving depends only on the transverse field $\nu = \pi(Y) \in \Gamma Q^L$.*

ν is then appropriately called a *transverse affine field.*

LEMMA 3.19. *Let $\mathscr{F}$ be a R-foliation. A metric infinitesimal automorphism preserves the connection ∇ (equivalently: a transverse Killing field is affine).*

PROOF. First let $Y \in \Gamma TM$ be an arbitrary infinitesimal automorphism of $\mathscr{F}$. The (local) flow φ_t generated by Y maps leaves into leaves. Let Φ_t be the induced flow on TM. Then Φ_t maps L into itself, and thus induces a (local) flow $\bar{\Phi}_t$ of bundle maps of Q over φ_t, i.e., making the diagram

$$\begin{array}{ccc} Q & \xrightarrow{\bar{\Phi}_t} & Q \\ \downarrow & & \downarrow \\ M & \xrightarrow{\varphi_t} & M \end{array}$$

commutative. Now assume Y to be metric. Then $\bar{\Phi}_t$ is a flow of isometric bundle maps, i.e., $\bar{\Phi}_t^* g_Q = g_Q$ for all t. By the uniqueness theorem for the metric and torsion free connection of a Riemannian foliation [KT3, 1.11], the connections associated to g_Q and $\bar{\Phi}_t^* g_Q$ are the same. This proves that Y is connection preserving. ■

PROOF OF PROPOSITION 3.15. Let g_M be a bundle-like metric and $E_1, \cdots, E_n$ an orthonormal local frame of TM on a neighborhood of $x \in M$, such that $E_1, \cdots, E_p \in \Gamma L$, $E_{p+1}, \cdots, E_n \in \Gamma\sigma Q$, and let $(E_i)_x = e_i$. Then for any $\nu \in \Gamma Q$

$$(\Delta\nu)_x = (d_\nabla^* d_\nabla \nu)_x = -\sum_{i=1}^{n} (\nabla_{e_i}(d_\nabla \nu))(e_i) = -\sum_{i=1}^{n} (\nabla_{e_i}\nabla_{E_i}\nu - \nabla_{\nabla^M_{e_i}E_i}\nu) .$$

In the last expression ∇^M refers to the canonical Riemannian connection attached to g_M.

On the other hand we have by (3.11) for any infinitesimal automorphism Y of $\mathscr{F}$ with $\pi(Y) = \nu$, $A_\nabla(\nu)E_i = -\nabla_{E_i}\nu$. It follows that

$$(\nabla_{E_i}A_\nabla(\nu))(E_j) = \nabla_{E_i}A_\nabla(\nu)E_j - A_\nabla(\nu)\nabla^M_{E_i}E_j = -\nabla_{E_i}\nabla_{E_j}\nu + \nabla_{\nabla^M_{E_i}E_j}\nu .$$

This implies in a neighborhood of x, $\Delta\nu = \sum_{i=1}^{n} (\nabla_{E_i}A_\nabla(\nu))(E_i)$. If ν is a transverse Killing field, it is transverse affine by Lemma 3.19. Using Proposition 3.17 we get then $\Delta\nu = \sum_{i=1}^{n} R_\nabla(\nu, E_i)E_i$. But these terms vanish by (1.1) for $i = 1, \cdots, p$, so that $\Delta\nu = \sum_{\alpha=p+1}^{n} R_\nabla(\nu, E_\alpha)E_\alpha$. By (1.7) this shows that $\Delta\nu = \rho_\nabla\nu$, and ν is indeed a Jacobi field of the foliation. ■

4. **Proof of Theorems A and B.** We are now ready to complete the argument leading to these statements. We suppose that $\rho_\nabla \leqq 0$. Since always $\Delta \geqq 0$, it follows that

$$J_\nabla \geqq 0 \qquad (4.1)$$

for the Jacobi operator (2.3). We will show that for a compact and oriented M the additional assumption $\rho_F < 0$ for at least some point $x \in M$ implies that $\ker J_F = 0$, i.e. that in fact $J_F > 0$. If $\mathcal{F}$ is Riemannian and harmonic, this will imply Theorem A by the second variation formula for the energy stated in the introduction. If $\mathcal{F}$ is Riemannian, this will imply Theorem B by Proposition 3.15. Thus it remains to show that every Jacobi field $\nu \in \Gamma Q$ is trivial.

Let $\nu \in \Gamma Q$ be any section. First we verify the classical identity

$$-(1/2)\Delta g_Q(\nu, \nu) = g_\Omega(\nabla\nu, \nabla\nu) - g_Q(\Delta\nu, \nu) . \tag{4.2}$$

The Laplacian on the LHS is the ordinary Laplacian d^*d of the function $g_Q(\nu, \nu)$ on M. The first term on the RHS is the induced norm square on $\nabla\nu \in \Omega^1(M, Q)$.

Let $x \in M$ and $e_1, \cdots, e_n \in T_xM$ an orthonormal frame. Let $E_1, \cdots, E_n$ be an extension of $e_1, \cdots, e_n$ to an orthonormal frame of TM in a neighborhood of x, and satisfying $\nabla^M_{e_i}E_j = 0$, i.e. the value at x of $\nabla^M_X E_j$ equals 0 for any vector field X such that $X_x = e_i$ (no particular relation of this frame to the foliation is needed). With these notations

$$\begin{aligned}
-(1/2)\cdot\Delta g_Q(\nu, \nu)_x &= -(1/2)(d^*dg_Q(\nu, \nu))_x = \sum_i (\nabla^M_{e_i} dg_Q(\nu, \nu))(e_i)/2 \\
&= \sum_i [\nabla^M_{e_i}(dg_Q(\nu, \nu)(e_i)) - dg_Q(\nu, \nu)(\nabla^M_{e_i}E_i)]/2 \\
&= \sum_i \nabla^M_{e_i}(E_i g_Q(\nu, \nu))/2 = \sum_i e_i(g_Q(\nabla_{E_i}\nu, \nu)) \\
&= \sum_i g_Q(\nabla_{e_i}\nabla_{E_i}\nu, \nu) + \sum_i g_Q(\nabla_{e_i}\nu, \nabla_{e_i}\nu) .
\end{aligned}$$

The first sum equals $-g_Q(\Delta\nu, \nu)$. This is the calculation at the beginning of the proof of Proposition 3.15, taking into account that the terms $\nabla^M_{e_i}E_i$ vanish. Note that for that portion of the proof the properties $E_1, \cdots, E_p \in \Gamma L$ and $E_{p+1}, \cdots, E_n \in \Gamma\sigma Q$ were not yet relevant. The second sum equals the pointwise square of the norm of $\nabla\nu$ in the space $\Omega^1(M, Q)$. This establishes (4.2).

For a Jacobi field ν we have $\Delta\nu = \rho_F\nu$, so that (4.2) reads

$$-(1/2)\cdot\Delta g_Q(\nu, \nu) = g_\Omega(\nabla\nu, \nabla\nu) - g_Q(\rho_F\nu, \nu) . \tag{4.3}$$

With the assumption $\rho_F \leqq 0$, the RHS is $\geqq 0$, and so is therefore the LHS. Thus $-g_Q(\nu, \nu)$ is a subharmonic function. Its integral over the compact and oriented manifold M vanishes by Green's theorem. Therefore $g_Q(\nu, \nu)$ must be harmonic and hence constant: $g_Q(\nu, \nu) = c$.

On the RHS of (4.3) we have $g_\Omega(\nabla\nu, \nabla\nu) \geqq 0$ and $-g_Q(\rho_F\nu, \nu) \geqq 0$, hence both terms vanish. This shows in particular that for a Jacobi field ν

$$(4.4) \qquad g_Q(\rho_r\nu, \nu) = 0 .$$

By assumption there is at least one $x \in M$ for which $\rho_r(x): Q_x \to Q_x$ is a a strictly negative operator. This is compatible with (4.4) only if $\nu_x = 0$ for that $x \in M$. Since $g_Q(\nu, \nu) =$ constant, it follows that $g_Q(\nu, \nu) = 0$. Hence $\nu = 0$. ■

Appendix. Second variation formula. We need the concept of a special variation of a Riemannian foliation $\mathscr{F}$ on M through Riemannian foliations, with fixed normal bundle, and defined by a section $\nu \in \Gamma Q$ [KT 2, 3]. M is assumed compact, $\mathfrak{U} = (U_\alpha)$ an open covering, $f^\alpha: U_\alpha \to V_\alpha \subset N$ are submersions onto open submanifolds V_α of a model Riemannian manifold N, related on $U_{\alpha\beta} = U_\alpha \cap U_\beta$ by $f^\alpha = \gamma^{\alpha\beta} \circ f^\beta$ with local isometries $\gamma^{\alpha\beta}$ of N. The data $\{U_\alpha, f^\alpha, \gamma^{\alpha\beta}\}$ are a Haefliger cocycle for $\mathscr{F}$. On U_α, $Q = (f^\alpha)^* TN$ and $g_Q = (f^\alpha)^* g_N$.

For $\nu \in \Gamma Q$, $\nu^\alpha = \nu / U_\alpha$, one obtains locally a variation Φ_t^α of $f^\alpha = \Phi_0^\alpha$ by setting [KT 3, 4.9]

$$(A.1) \qquad \Phi_t^\alpha(x) = \exp_{f^\alpha(x)}(t\nu^\alpha(x)) \quad \text{for} \quad x \in U_\alpha,\ |t| \leqq \varepsilon$$

where $\varepsilon > 0$ is sufficiently small. The RHS is the endpoint of the geodesic segment in $V_\alpha \subset N$ starting at $f^\alpha(x)$ and determined by $t\nu^\alpha(x) \in T_{f^\alpha(x)}N$. Clearly

$$(A.2) \qquad \nu^\alpha(x) = \frac{d}{dt}\Big|_{t=0} \Phi_t^\alpha(x) .$$

Next we prove the formula

$$(A.3) \qquad \nabla_{\partial/\partial t}((\Phi_t^\alpha)_*(X)) = \nabla_X(\Phi_*^\alpha(\partial/\partial t)) \equiv \nabla_X((\partial/\partial t)\Phi_t^\alpha)$$

for a vector field $X \in \Gamma TM$. For this purpose we evaluate the torsion

$$T_r(\Phi_*^\alpha(\partial/\partial t), \Phi_*^\alpha X) = \nabla_{\partial/\partial t}((\Phi_t^\alpha)_*(X)) - \nabla_X(\Phi_*^\alpha(\partial/\partial t)) - \Phi_*^\alpha[(\partial/\partial t), X]$$

This expression vanishes, since ∇ is torsionfree. But $[\partial/\partial t, X] = 0$, so (A.3) follows.

Using (A.2) we obtain in particular

$$(A.4) \qquad \nabla_{\partial/\partial t|t=0}(\Phi_t^\alpha)_*(X) = \nabla_X \nu^\alpha .$$

In the present context we need a 2-parameter special variation $\mathscr{F}_{s,t}$ of $\mathscr{F} = \mathscr{F}_{0,0}$ defined for $\mu, \nu \in \Gamma Q$ by the local formula on U_α

$$(A.5) \qquad \Phi_{s,t}^\alpha(x) = \exp_{f^\alpha(x)}(s\mu^\alpha(x) + t\nu^\alpha(x))$$

Exactly as before we obtain the formulas

$$\text{(A.6)} \qquad \begin{cases} \nabla_{(\partial/\partial s)|_{s=0,t=0}}(\Phi^\alpha_{s,t})_*(X) = \nabla_X \mu^\alpha \\ \nabla_{(\partial/\partial t)|_{s=0,t=0}}(\Phi^\alpha_{s,t})_*(X) = \nabla_X \nu^\alpha \end{cases}$$

where now we use $[X, \partial/\partial s] = 0$ and $[X, \partial/\partial t] = 0$.

We further need the formula (for $s = 0$, $t = 0$)

$$\text{(A.7)} \qquad \nabla_{\partial/\partial s}(\nabla_{\partial/\partial t}\Phi^\alpha_*(X)) = \nabla_X(\nabla_{\partial/\partial s}\nu^\alpha) + R_\nabla(\mu, \Phi^\alpha_*(X))\nu^\alpha$$

where R_∇ is the curvature of the canonical metric and torsionfree connection in Q (Section 1). Proof of (A.7): By (A.6) we have for $s = 0$, $t = 0$ $\nabla_{\partial/\partial s}(\nabla_{\partial/\partial t}\Phi^\alpha_*(X)) = \nabla_{\partial/\partial s}(\nabla_X \nu^\alpha)$. By the definition of R_∇ and (1.1) we have

$$R_\nabla(\Phi^\alpha_*(\partial/\partial s), \Phi^\alpha_* X)\nu^\alpha = \nabla_{\partial/\partial s}(\nabla_X \nu^\alpha) - \nabla_X(\nabla_{\partial/\partial s}\nu^\alpha) - \nabla_{[\partial/\partial s, X]}\nu^\alpha$$

But $[\partial/\partial s, X] = 0$, so that the resulting formula for $\nabla_{\partial/\partial s}(\nabla_X \nu^\alpha)$ yields precisely the RHS of A.7.

The projection π of the foliation $\mathscr{F}_{s,t}$ is given locally by $\pi_{s,t} = (\Phi^\alpha_{s,t})_*$. For the energy we have then

$$E(\mathscr{F}_{s,t}) = \langle \pi, \pi \rangle / 2 = (1/2)\int_M \pi \wedge * \pi \ .$$

We obtain then

$$(\partial^2/\partial s \partial t)E(\mathscr{F}_{s,t})|_{s=0,t=0} = (1/2)(\partial/\partial s)((\partial/\partial t)\langle \pi, \pi \rangle) = (\partial/\partial s) < \tilde{\nabla}_{\partial/\partial t}\pi, \pi \rangle$$

where $\tilde{\nabla}$ denotes the induced connection in $\Omega^1(M, Q)$. Using (A.6) we get

$$(\partial^2/\partial s \partial t)(E(\mathscr{F}_{s,t})|_{s=0,t=0} = (\partial/\partial s)\langle \nabla \nu, \pi \rangle = \langle (\partial/\partial s)\nabla \nu, \pi \rangle + \langle \nabla \nu, \nabla \mu \rangle \ .$$

By (A.7) the first term can be replaced by

$$\langle \nabla(\nabla_{\partial/\partial s}\nu), \pi \rangle + \langle R_\nabla(\mu, \pi(-))\nu, \pi(-) \rangle = \langle \nabla_{\partial/\partial s}\nu, d^*_\nabla \pi \rangle + \langle R_\nabla(\mu, \pi(-))\nu, \pi(-) \rangle$$

If we now assume $\mathscr{F}$ to be harmonic, then $d^*_\nabla \pi = 0$ [KT3]. The other term is a scalar product in $\Omega^1(M, Q)$. Its integrand is therefore evaluated at $x \in M$, and for an orthonormal frame e_α of Q_x ($\alpha = p + 1, \cdots, n$) by the formula

$$\sum_{\alpha=p+1}^{n} g_Q(R_\nabla(\mu, e_\alpha))\nu, e_\alpha) = -\sum_{\alpha=p+1}^{n} g_Q(R_\nabla(\mu, e_\alpha)e_\alpha, \nu) \ .$$

This is by (1.4) and (1.6) equal to $-S_\nabla(\mu, \nu) = -g_Q(\rho_\nabla \mu, \nu)$ in terms of the Ricci curvature S_∇ and Ricci operator ρ_∇ of ∇. Collecting these results, we obtain therefore

$$\begin{aligned}(\partial^2/\partial s \partial t)E(\mathscr{F}_{s,t})|_{s=0,t=0} &= \langle \nabla \mu, \nabla \nu \rangle - \langle \rho_\nabla \mu, \nu \rangle = \langle \Delta \mu, \nu \rangle - \langle \rho_\nabla \mu, \gamma \rangle \\ &= \langle (\Delta - \rho_\nabla)\mu, \nu \rangle \ .\end{aligned}$$

This completes the proof of the second variation formula for a Riemannian and harmonic foliation on a compact manifold.

For the calculation of the second variation for harmonic maps on compact manifolds, see [ES] and [MA].

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