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Infinitesimal automorphisms on the tangent bundle

By

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1. Introduction

Let M be an n-dimensional differentiable manifold of class C^∞ and TM its tangent bundle, which is a $2n$-dimensional differentiable manifold. The prolongation of tensor fields and connections from M to TM has been studied in [3] by Yano and Kobayashi. In particular, for an affine connection ∇ on M, its complete lift ∇^C is an affine connection on TM. If g is a pseudo-Riemannian metric on M, its complete lift g^C is a pseudo-Riemannian metric on TM with n positive and n negative signs.

In [4], Yano and Kobayashi have tried to determine the form of an infinitesimal affine transformation on (TM, ∇^C). However, their work is incomplete because they have determined essentially only the fibre-preserving infinitesimal affine transformations. In the same paper, they have also tried to determine the form of an infinitesimal isometry on (TM, g^C). But their result turned out to be incorrect as was pointed out by Tanno [1], who in turn gave a complete solution on the form of an infinitesimal isometry on (TM, g^C).

In this paper, we shall use the method of adapted frames to determine the most general form of an infinitesimal affine transformation on (TM, ∇^C), without any extra assumption on the infinitesimal affine transformation itself. For the case of fibre-preserving transformations, our result is an improvement over that given in [4]. As an application of our results and further illustration of our method, we shall give an alternative proof of the result of Tanno on infinitesimal isometries on (TM, g^C) mentioned earlier.

2. Preliminaries

In this section, we shall summarize all the basic definitions and results that are needed later. Most of them are well-known, and details can be found in Yano [2] and Yano and Kobayashi [3, 4]. Indices $a, b, c, \ldots; h, i, j, \ldots$ have range in $\{1,\dots,n\}$, while indices $A, B, C,\dots; \lambda, \mu, \nu,\dots$ have range in $\{1,\dots,n; n+1,\dots,2n\}$. We put $\bar{i}=n+i$. Summation over repeated indices is always implied.

Coordinate systems in M are denoted by (U, x^h), where U is the coordinate neighbourhood and x^h the coordinate functions. Components in (U, x^h) of geometric objects on M will be referred to simply as components. We denote the partial differentiation $\frac{\partial}{\partial x^h}$ by ∂_h.

Let ∇ be an affine connection on M with components Γ^h_{ji}. Its covariant differentiation will again be denoted by the same symbol ∇. The curvature tensor R and the torsion tensor T of ∇ have components $R_{kji}{}^h$ and T^h_{ji} respectively. The opposite connection $\hat{\nabla}$ of ∇ has components $\hat{\Gamma}^h_{ji}=\Gamma^h_{ij}$. The covariant differentiation, curvature tensor and torsion tensor of $\hat{\nabla}$ will be denoted by $\hat{\nabla}$, $\hat{R}$ and $\hat{T}$ respectively.

A vector field X on M with components X^h is an *infinitesimal affine transformation* of ∇ if

$$\partial_j\partial_i X^h + X^a\partial_a\Gamma^h_{ji} - \Gamma^a_{ji}\partial_a X^h + \Gamma^h_{ai}\partial_j X^a + \Gamma^h_{ja}\partial_i X^a = 0. \tag{2.1}$$

The left hand side of (2.1) are the components

$$\pounds_X\Gamma^h_{ji} = \nabla_j\hat{\nabla}_i X^h + R_{kji}{}^h X^k \tag{2.2}$$

of the Lie derivative $\pounds_X\nabla$ of ∇ with respect to X. The Lie derivative $\pounds_X R$ of the curvature tensor R is given by

$$\pounds_X R_{kji}{}^h = X^a\nabla_a R_{kji}{}^h - R_{kji}{}^a\hat{\nabla}_a X^h + R_{aji}{}^h\hat{\nabla}_k X^a + R_{kai}{}^h\hat{\nabla}_j X^a + R_{kja}{}^h\hat{\nabla}_i X^a. \tag{2.3}$$

It is known that

$$\pounds_X R_{kji}{}^h = \nabla_k\pounds_X\Gamma^h_{ji} - \nabla_j\pounds_X\Gamma^h_{ki} + T^a_{kj}\pounds_X\Gamma^h_{ai}. \tag{2.4}$$

Hence $\pounds_X R=0$ if X is an infinitesimal affine transformation.

Let g be a pseudo-Riemannian metric on M with components g_{ji}. As usual, $\left\{ {h \atop ji} \right\}$ is the Christoffel symbol of g_{ji} and $[g^{ji}]$ is the inverse of the matrix $[g_{ji}]$. A vector field X on M with components X^h is an *infinitesimal isometry* on (M, g) if

$$X^a\partial_a g_{ji} + g_{aj}\partial_i X^a + g_{ai}\partial_j X^a = 0. \tag{2.5}$$

The left hand side of (2.5) are the components $\pounds_X g_{ji}$ of the Lie derivative $\pounds_X g$ of g with respect to X. In terms of the covariant differentiation in (M, g), we have

$$\text{(2.6)} \qquad \pounds_X g_{ji} = g_{aj} \nabla_i X^a + g_{ai} \nabla_j X^a.$$

It is also known that

$$\text{(2.7)} \qquad \pounds_X \left\{ {h \atop ji} \right\} = \tfrac{1}{2} g^{ha} (\nabla_j \pounds_X g_{ia} + \nabla_i \pounds_X g_{ja} - \nabla_a \pounds_X g_{ji}).$$

Thus an infinitesimal isometry is an infinitesimal affine transformation with respect to the Riemannian connection of g.

We shall be using the following identities in (M, ∇):

$$\text{(2.8)} \qquad \mathscr{C} R_{kji}{}^h = \mathscr{C} T^h_{ak} T^a_{ji} + \mathscr{C} \nabla_k T^h_{ji},$$

$$\text{(2.9)} \qquad \mathscr{C} \nabla_k R_{jil}{}^h = \mathscr{C} T^a_{kj} R_{ial}{}^h,$$

$$\text{(2.10)} \qquad \nabla_k \nabla_j B^h - \nabla_j \nabla_k B^h = R_{kja}{}^h B^a - T^a_{kj} \nabla_a B^h,$$

$$\text{(2.11)} \qquad \nabla_k \nabla_j A^h_i - \nabla_j \nabla_k A^h_i = R_{kja}{}^h A^a_i - R_{kji}{}^a A^h_a - T^a_{kj} \nabla_a A^h_i.$$

Here, $\mathscr{C}$ denotes the cyclic sum of terms in the indices k, j, i. (2.8), (2.9) are respectively the first and second Bianchi identity. (2.10) and (2.11) are the Ricci identities for a vector field with components B^h and a (1,1) tensor with components A^h_i.

Let $\pi: TM \to M$ be the canonical projection of TM onto M. The coordinate system (U, x^h) in M induces in a natural way a coordinate system $\{\pi^{-1}(U), (x^h, y^h)\}$ in TM, which we call the induced coordinate system. We sometimes write y^h as $x^{\bar{h}}$ and (x^h, y^h) as (x^A). Components in $\{\pi^{-1}(U), x^A\}$ of geometric objects on TM will be referred to simply as components. We denote $\frac{\partial}{\partial y^h}$ and $\frac{\partial}{\partial x^A}$ by $\partial_{\bar{h}}$, ∂_A respectively. For a vector field X on M with components X^h, the vertical lift X^V and the complete lift X^C of X are vector fields on TM with components

$$\text{(2.12)} \qquad X^V: \begin{bmatrix} 0 \\ X^h \end{bmatrix}, \qquad X^C: \begin{bmatrix} X^h \\ y^i \partial_i X^h \end{bmatrix}.$$

A (1,1) tensor field C on M with components C^h_i induces a vector field ιC on TM whose components are

$$\text{(2.13)} \qquad \iota C: \begin{bmatrix} 0 \\ y^i C^h_i \end{bmatrix}.$$

On the other hand, similar to Lemma 2.2 of Tanno [1], we can prove that a (1,1) tensor field A on M with components A^h_i induces a vector field A^* on TM

whose components are

$$(2.14)\qquad A^*:\begin{bmatrix} A^h_i y^i \\ -\frac{1}{2}(\Gamma^h_{ai}A^a_j+\Gamma^h_{ja}A^a_i)y^j y^i \end{bmatrix}.$$

For the affine connection ∇ on M, its complete lift ∇^C is an affine connection on TM whose components $\tilde{\Gamma}^A_{CB}$ are:

$$(2.15)\qquad \begin{aligned} &\tilde{\Gamma}^h_{ji}=\Gamma^h_{ji}, &&\tilde{\Gamma}^h_{j\bar{i}}=0, &&\tilde{\Gamma}^h_{\bar{j}i}=0, &&\tilde{\Gamma}^h_{\bar{j}\bar{i}}=0, \\ &\tilde{\Gamma}^{\bar{h}}_{ji}=y^k\partial_k\Gamma^h_{ji}, &&\tilde{\Gamma}^{\bar{h}}_{j\bar{i}}=\Gamma^h_{ji}, &&\tilde{\Gamma}^{\bar{h}}_{\bar{j}i}=\Gamma^h_{ji}, &&\tilde{\Gamma}^{\bar{h}}_{\bar{j}\bar{i}}=0. \end{aligned}$$

3. Adapted frames in *TM*

When an affine connection ∇ with components Γ^h_{ji} is given on M, we can introduce on each induced coordinate neighbourhood $\pi^{-1}(U)$ of TM a frame field which is very useful in our computations. It is called the adapted frame on $\pi^{-1}(U)$ and consists of the following $2n$ linearly independent vector fields on $\pi^{-1}(U)$:

$$(3.1)\qquad D_i=\frac{\partial}{\partial x^i}-y^j\Gamma^h_{ji}\frac{\partial}{\partial y^h},\quad D_{\bar{i}}=\frac{\partial}{\partial y^i}.$$

The non-holonomic objects $\Omega_{\lambda\mu}{}^{\nu}$ of the adapted frame $\{D_\mu\}=\{D_i, D_{\bar{i}}\}$ are defined by

$$(3.2)\qquad [D_\lambda, D_\mu]=\Omega_{\lambda\mu}{}^{\nu}D_\nu.$$

If we denote by $[L^A_\mu]$ the matrix of components of $\{D_\mu\}$, namely

$$[L^A_\mu]=\begin{bmatrix} \delta^h_i & 0 \\ -y^j\Gamma^h_{ji} & \delta^h_i \end{bmatrix}$$

and $[L^\nu_A]$ the inverse matrix of $[L^A_\mu]$, (3.2) becomes

$$(3.3)\qquad \Omega_{\lambda\mu}{}^{\nu}=[D_\lambda(L^A_\mu)-D_\mu(L^A_\lambda)]L^\nu_A.$$

When working out the details in (3.3), we find that the only non-zero components of the non-holonomic objects are

$$(3.4)\qquad \Omega_{ji}{}^{\bar{h}}=-\hat{R}_{jia}{}^{h}y^a,\quad \Omega_{j\bar{i}}{}^{\bar{h}}=-\Omega_{\bar{i}j}{}^{\bar{h}}=\hat{\Gamma}^h_{ji}.$$

In what follows, we shall often consider the components of tensors on

TM with respect to the adapted frame on $\pi^{-1}(U)$. We call such components the *frame components* to distinguish them from the ordinary components of §2. By using (3.1), it is not difficult to show that the frame components of the vector fields X^V and ιC are the same as the ordinary components, namely

$$X^V : \begin{bmatrix} 0 \\ X^h \end{bmatrix}, \quad \iota C : \begin{bmatrix} 0 \\ y^i C_i^h \end{bmatrix}, \tag{3.5}$$

while the frame components of X^C and A^* are

$$X^C : \begin{bmatrix} X^h \\ y^i \nabla_i X^h \end{bmatrix}, \quad A^* : \begin{bmatrix} A_i^h y^i \\ -\frac{1}{2} T^h_{lj} A^l_i y^j y^i \end{bmatrix}. \tag{3.6}$$

Let $\tilde{\nabla}$ be an arbitrary affine connection on TM (not necessarily ∇^C). The frame components $\tilde{\Gamma}^\nu_{\lambda\mu}$ of $\tilde{\nabla}$ on $\pi^{-1}(U)$ are defined by

$$\tilde{\Gamma}^\nu_{\lambda\mu} = [D_\lambda(L^A_\mu) + \tilde{\Gamma}^A_{CB} L^C_\lambda L^B_\mu] L^\nu_A, \tag{3.7}$$

where $\tilde{\Gamma}^A_{CB}$ are the ordinary components of $\tilde{\nabla}$. For a vector field $\tilde{X}$ on TM with frame components $\tilde{X}^\nu$, it can be shown that

$$\tilde{\nabla}_\lambda \tilde{X}^\nu = D_\lambda(\tilde{X}^\nu) + \tilde{\Gamma}^\nu_{\lambda\mu} \tilde{X}^\mu \tag{3.8}$$

are exactly the frame components of the covariant derivative $\tilde{\nabla}\tilde{X}$ of $\tilde{X}$. There are formulas analogous to (3.8) for tensor fields of other types. The frame components of the curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ are given by

$$\tilde{R}_{\omega\lambda\mu}{}^\nu = D_\omega(\tilde{\Gamma}^\nu_{\lambda\mu}) - D_\lambda(\tilde{\Gamma}^\nu_{\omega\mu}) + \tilde{\Gamma}^\nu_{\omega\tau}\tilde{\Gamma}^\tau_{\lambda\mu} - \tilde{\Gamma}^\nu_{\lambda\tau}\tilde{\Gamma}^\tau_{\omega\mu} - \Omega_{\omega\lambda}{}^\tau \tilde{\Gamma}^\nu_{\tau\mu}. \tag{3.9}$$

From the components of ∇^C given in (2.15), we can use (3.7) to show that the possibly non-zero frame components of the complete lift ∇^C are

$$\tilde{\Gamma}^h_{ji} = \Gamma^h_{ji}, \quad \tilde{\Gamma}^{\bar{h}}_{ji} = y^k R_{kji}{}^h, \quad \tilde{\Gamma}^{\bar{h}}_{j\bar{i}} = \Gamma^h_{ji}. \tag{3.10}$$

It then follows from (3.9) that the possibly non-zero frame components of the curvature tensor $\tilde{R}$ of ∇^C are

$$\begin{aligned} &\tilde{R}_{kji}{}^h = R_{kji}{}^h, \quad \tilde{R}_{kji}{}^{\bar{h}} = y^a \nabla_a R_{kji}{}^h, \\ &\tilde{R}_{\bar{k}ji}{}^{\bar{h}} = \tilde{R}_{k\bar{j}i}{}^{\bar{h}} = \tilde{R}_{kj\bar{i}}{}^{\bar{h}} = R_{kji}{}^h. \end{aligned} \tag{3.11}$$

The computation leading to (3.11) is straightforward except for $\tilde{R}_{kji}{}^{\bar{h}}$, where we have to use Bianchi's second identity (2.9).

Let us denote the opposite connection of ∇^C by ∇^*. From (3.7) and (3.3), it follows that the frame components $\overset{*}{\Gamma}{}^{\nu}_{\lambda\mu}$ of ∇^* are related to the frame components $\tilde{\Gamma}^{\nu}_{\lambda\mu}$ of ∇^C by

$$\overset{*}{\Gamma}{}^{\nu}_{\mu\lambda}=\tilde{\Gamma}^{\nu}_{\lambda\mu}-\Omega_{\lambda\mu}{}^{\nu}. \tag{3.12}$$

In particular,

$$\overset{*}{\Gamma}{}^{\bar{h}}_{ji}=\tilde{\Gamma}^{\bar{h}}_{ij}-\Omega_{ij}{}^{\bar{h}}=y^k R_{kij}{}^h+\hat{R}_{ijk}{}^h y^k.$$

It can be shown that

$$\hat{R}_{ijk}{}^h=R_{ijk}{}^h+\nabla_i T^h_{kj}-\nabla_j T^h_{ki}+T^a_{ij}T^h_{ka}+T^h_{ai}T^a_{kj}-T^h_{aj}T^a_{ki}.$$

Putting this into the expression for $\overset{*}{\Gamma}{}^{\bar{h}}_{ji}$, and using the first Bianchi's identity, we get

$$\overset{*}{\Gamma}{}^{\bar{h}}_{ji}=y^k(R_{kji}{}^h-\nabla_k T^h_{ji}).$$

Working similarly with other sets of indices, we can show that the possibly non-zero frame components of the opposite connection ∇^* of ∇^C are

$$\begin{aligned}&\overset{*}{\Gamma}{}^{h}_{ji}=\Gamma^h_{ij}, \quad \overset{*}{\Gamma}{}^{\bar{h}}_{ji}=y^k(R_{kji}{}^h-\nabla_k T^h_{ji}),\\ &\overset{*}{\Gamma}{}^{\bar{h}}_{j\bar{i}}=\Gamma^h_{ij}, \quad \overset{*}{\Gamma}{}^{\bar{h}}_{\bar{j}i}=T^h_{ij}.\end{aligned} \tag{3.13}$$

4. Decomposition of infinitesimal affine transformation

We begin our determination of infinitesimal affine transformation on (TM, ∇^C) by refining the discussions given in §4 of Yano and Kobayashi [4]. Thus, let $\tilde{\Gamma}^A_{CB}$ be the components of ∇^C given by (2.15) and $\tilde{X}$ an infinitesimal affine transformation on (TM, ∇^C). By (2.1), the components $\tilde{X}^A$ of $\tilde{X}$ satisfies

$$\partial_C\partial_B\tilde{X}^A+\tilde{X}^E\partial_E\tilde{\Gamma}^A_{CB}-\tilde{\Gamma}^E_{CB}\partial_E\tilde{X}^A+\tilde{\Gamma}^A_{EB}\partial_C\tilde{X}^E+\tilde{\Gamma}^A_{CE}\partial_B\tilde{X}^E=0. \tag{4.1}$$

Let us put $(ACB)=(h\bar{j}\bar{i})$ in (4.1). We get $\partial_{\bar{j}}\partial_{\bar{i}}\tilde{X}^h=0$ and so

$$\tilde{X}^h=A^h_k y^k+B^h \tag{4.2}$$

where A^h_k, B^h are functions of x^h only. It is easy to see that A^h_k and B^h are respectively the components of a (1,1) tensor A and a vector field B on M.

If we put $(ACB)=(hj\bar{i})$ in (4.1) and use (4.2), we get

(4.3) $$\partial_j A_i^h - \Gamma_{ji}^a A_a^h + \Gamma_{ja}^h A_i^a = 0,$$

i.e., $\nabla A = 0$.

If we put $(ACB) = (h\bar{j}i)$ in (4.1) and use (4.2), we get

(4.4) $$\partial_j A_i^h - \Gamma_{ij}^a A_a^h + \Gamma_{aj}^h A_i^a = 0.$$

In the presence of (4.3), (4.4) is equivalent to

(4.5) $$T_{ja}^h A_i^a = T_{ji}^a A_a^h.$$

If we put $(ACB) = (hji)$ in (4.1) and use (4.2), considering only those terms not involving y^h, we get

$$\partial_j \partial_i B^h + B^a \partial_a \Gamma_{ji}^h - \Gamma_{ji}^a \partial_a B^h + \Gamma_{ai}^h \partial_j B^a + \Gamma_{ja}^h \partial_i B^a = 0,$$

which is the condition for B to be an infinitesimal affine transformation.

Let us now put $(ACB) = (\bar{h}\bar{j}\bar{i})$ in (4.1) and use (4.2). We get

(4.6) $$\partial_{\bar{j}} \partial_i \tilde{X}^{\bar{h}} + \Gamma_{ai}^h A_j^a + \Gamma_{ja}^h A_i^a = 0.$$

(4.6) implies that $\Gamma_{ai}^h A_j^a + \Gamma_{ja}^h A_i^a$ is symmetric in j, i, i.e.,

$$\Gamma_{ai}^h A_j^a + \Gamma_{ja}^h A_i^a = \Gamma_{aj}^h A_i^a + \Gamma_{ia}^h A_j^a.$$

In other words,

(4.7) $$T_{ia}^h A_j^a = T_{ja}^h A_i^a.$$

Since $T_{ja}^h A_i^a$ is skew-symmetric in j, i by (4.5) and symmetric in j, i by (4.7), we see that (4.5) and (4.7) are equivalent to

(4.8) $$T_{ja}^h A_i^a = T_{ji}^a A_a^h = 0.$$

On the other hand, from (4.6), we have

(4.9) $$\tilde{X}^{\bar{h}} = -\tfrac{1}{2}(\Gamma_{ai}^h A_j^a + \Gamma_{ja}^h A_i^a) y^j y^i + E_i^h y^i + F^h,$$

where E_i^h, F^h are functions of x^h only.

It now follows that $\begin{bmatrix} B^h \\ E_i^h y^i + F^h \end{bmatrix}$ are the components of the vector field $\tilde{X} - A^*$. By looking at its transformation law, it is easy to see that F^h are the components of a vector field F on M. Thus, $\begin{bmatrix} B^h \\ E_i^h y^i \end{bmatrix}$ are the components of the vector field $\tilde{X}-A^*-F^V$. Guided by the form of the components for B^C (cf. (2.12)), we put $E_i^h=\partial_i B^h+C_i^h$. Then $\begin{bmatrix} 0 \\ C_i^h y^i \end{bmatrix}$ are the components of the vector field $\tilde{X}-A^*-F^V-B^C$. By looking at its transformation law, we see that C_i^h are the components of a (1,1) tensor C on M and that

$$\tilde{X}-A^*-F^V-B^C=\iota C.$$

Let us obtain further information on C. We put $(ACB)=(\bar{h}j\bar{i})$, use (4.2) and (4.9) and consider only those terms not involving y^h. What we get is

$$\pounds_B \Gamma_{ji}^h+\nabla_j C_i^h=0.$$

As B is an infinitesimal affine transformation, we see that $\nabla C=0$.

On the other hand, if we put $(ACB)=(\bar{h}\bar{j}i)$, use (4.2) and (4.9) and consider only those terms not involving y^h, what we get in the presence of $\pounds_B\nabla=0$ is

$$\partial_i C_j^h-\Gamma_{ji}^a C_a^h+\Gamma_{ai}^h C_j^a=0. \tag{4.10}$$

By $\nabla C=0$, we also have

$$\partial_i C_j^h-\Gamma_{ij}^a C_a^h+\Gamma_{ia}^h C_j^a=0. \tag{4.11}$$

(4.10) and (4.11) together gives

$$T_{ij}^a C_a^h+T_{ai}^h C_j^a=0. \tag{4.12}$$

We summarize what we obtain so far in

Proposition 4.1. *Let ∇^C be the complete lift of an affine connection ∇ on M to TM. An infinitesimal affine transformation $\tilde{X}$ on (TM, ∇^C) can be expressed uniquely in the form*

$$\tilde{X}=A^*+B^C+\iota C+F^V,$$

where A is a (1,1) tensor field on M satisfying

$$\nabla A=0, \quad T_{ja}^h A_i^a=T_{ji}^a A_a^h=0,$$

B is an infinitesimal affine transformation on (M, ∇),
C is a (1,1) tensor field on M satisfying

$$\nabla C=0, \quad T_{ja}^h C_i^a=T_{ji}^a C_a^h,$$

and F is a vector field on M.

We can obtain further conditions on A, B, C and F by considering terms involving y^h after putting $(ACB)=(hji)$ in (4.1) and so on. In theory, by carrying this process to the end, we should get the general form of an infinitesimal affine transformation on (TM, ∇^C). But in practice, this process becomes more and more involved and tends to be unmanagable. However, having succeeded in decomposing an infinitesimal affine transformation as the sum of vector fields, we now switch to the adapted frames.

5. Infinitesimal affine transformations on (TM, ∇^C)

Let $\tilde{X}$ be an infinitesimal affine transformation on (TM, ∇^C) and suppose that it is expressed uniquely in the form $\tilde{X}=A^*+B^C+\iota C+F^V$ as in Proposition 4.1. By (3.5) and (3.6), the frame components $\tilde{X}^\nu$ of $\tilde{X}$ are given by

$$(5.1)\qquad [\tilde{X}^\nu]=\begin{bmatrix} A_i^h y^i + B^h \\ y^i(\nabla_i B^h + C_i^h) + F^h \end{bmatrix}.$$

In what follows, we shall compute the frame components $\pounds_{\tilde{X}}\tilde{\Gamma}_{\lambda\mu}^\nu$ of $\pounds_{\tilde{X}}\nabla^C$ according to (cf. (2.2))

$$(5.2)\qquad \pounds_{\tilde{X}}\tilde{\Gamma}_{\lambda\mu}^\nu = \nabla_\lambda^C \nabla_\mu^* \tilde{X}^\nu + \tilde{R}_{\omega\lambda\mu}{}^\nu \tilde{X}^\omega.$$

By equating $\pounds_{\tilde{X}}\tilde{\Gamma}_{\lambda\mu}^\nu$ to zero, we then get the conditions on A, B, C, F for $\tilde{X}$ to be an infinitesimal affine transformation. We do our computations in stages.

First, the frame components $\nabla_\mu^* \tilde{X}^\nu$ of $\nabla^* \tilde{X}$ can be calculated by using (5.1), (3.13) and (3.8). The expressions thus obtained, after simplification by the conditions on A, B, C in Proposition 4.1, become

$$(5.3)\qquad \begin{aligned} &\nabla_i^* \tilde{X}^h = \hat{\nabla}_i B^h, \\ &\nabla_{\bar{i}}^* \tilde{X}^h = A_i^h, \\ &\nabla_i^* \tilde{X}^{\bar{h}} = y^a y^c R_{aib}{}^h A_c^b + y^c \nabla_c \hat{\nabla}_i B^h + \hat{\nabla}_i F^h, \\ &\nabla_{\bar{i}}^* \tilde{X}^{\bar{h}} = \hat{\nabla}_i B^h + C_i^h. \end{aligned}$$

We remark that to obtain the coefficient of y^c in $\nabla_i^* \tilde{X}^{\bar{h}}$, we have to use the Ricci identity (2.10) for the vector field B.

Next, the frame components $\nabla_\lambda^C \nabla_\mu^* \tilde{X}^\nu$ can be calculated by using (5.3) and (3.10). The expressions thus obtained, after simplification by the conditions on A, B, C in Proposition 4.1, become

$$(5.4)\qquad \nabla^C_j \nabla^*_i \tilde{X}^h = \nabla_j \hat{\nabla}_i B^h - y^k R_{kji}{}^a A^h_a,$$

$$\nabla^C_j \nabla^*_i \tilde{X}^h = \nabla^C_j \nabla^*_i \tilde{X}^h = \nabla^C_j \nabla^*_i \tilde{X}^h = 0,$$

$$\nabla^C_j \nabla^*_i \tilde{X}^{\bar{h}} = y^a y^c [\nabla_j (R_{aib}{}^h A^b_c) - T^l_{aj} R_{lib}{}^h A^b_c] + y^c [\nabla_j \nabla_c \hat{\nabla}_i B^h + R_{cjl}{}^h \hat{\nabla}_i B^l$$
$$- R_{cji}{}^l \hat{\nabla}_l B^h - R_{cji}{}^l C^h_l - T^l_{cj} \nabla_l \hat{\nabla}_i B^h] + \nabla_j \hat{\nabla}_i F^h,$$

$$\nabla^C_j \nabla^*_i \tilde{X}^{\bar{h}} = \nabla_j \hat{\nabla}_i B^h + y^k R_{kja}{}^h A^a_i,$$

$$\nabla^C_j \nabla^*_i \tilde{X}^{\bar{h}} = y^c (R_{jia}{}^h A^a_c + R_{cia}{}^h A^a_j) + \nabla_j \hat{\nabla}_i B^h,$$

$$\nabla^C_j \nabla^*_i \tilde{X}^{\bar{h}} = 0.$$

Finally, the frame components of $\pounds_{\tilde{X}} \tilde{\Gamma}^{\nu}_{\lambda\mu}$ can be calculated by using (5.2), (5.4), (3.11) and (5.1). The expressions thus obtained, after simplification by the conditions on A, B, C in Proposition 4.1, become

$$(5.5)\qquad \pounds_{\tilde{X}} \tilde{\Gamma}^h_{ji} = y^k (R_{aji}{}^h A^a_k - R_{kji}{}^a A^h_a),$$

$$\pounds_{\tilde{X}} \tilde{\Gamma}^h_{ji} = \pounds_{\tilde{X}} \tilde{\Gamma}^h_{ji} = \pounds_{\tilde{X}} \tilde{\Gamma}^h_{ji} = 0,$$

$$\pounds_{\tilde{X}} \tilde{\Gamma}^{\bar{h}}_{ji} = y^a y^c (\nabla_j R_{aib}{}^h A^b_c - T^l_{aj} R_{lib}{}^h A^b_c + \nabla_c R_{bji}{}^h A^b_a)$$
$$+ y^c (\pounds_B R_{cji}{}^h + R_{aji}{}^h C^a_c - R_{cji}{}^a C^h_a) + \pounds_F \Gamma^h_{ji},$$

(We remark that to obtain the coefficient of y^c in $\pounds_{\tilde{X}} \tilde{\Gamma}^{\bar{h}}_{ji}$, we have to use Bianchi's second identity (2.9))

$$\pounds_{\tilde{X}} \tilde{\Gamma}^{\bar{h}}_{ji} = y^k (R_{kja}{}^h A^a_i + R_{aji}{}^h A^a_k),$$

$$\pounds_{\tilde{X}} \tilde{\Gamma}^{\bar{h}}_{ji} = y^k (R_{jia}{}^h A^a_k + R_{kia}{}^h A^a_j + R_{aji}{}^h A^a_k),$$

$$\pounds_{\tilde{X}} \tilde{\Gamma}^{\bar{h}}_{ji} = 0.$$

Let us analyse the conditions we get when we equate (5.5) to zero. From $\pounds_{\tilde{X}} \tilde{\Gamma}^h_{ji} = \pounds_{\tilde{X}} \tilde{\Gamma}^{\bar{h}}_{ji} = 0$, we already get

$$R_{kji}{}^a A^h_a = R_{aji}{}^h A^a_k = -R_{kja}{}^h A^a_i.$$

Since $\nabla A = 0$, we get from the Ricci identity (2.11) that

$$R_{kja}{}^h A^a_i = R_{kji}{}^a A^h_a.$$

Hence, we have

(5.6) $$R_{kji}{}^{a}A_{a}^{h}=R_{aji}{}^{h}A_{k}^{a}=R_{kja}{}^{h}A_{i}^{a}=0.$$

With (5.6), what remains when (5.5) is equated to zero is

$$£_{\tilde{X}}\tilde{\Gamma}_{ji}^{\bar{h}}=y^{c}(£_{B}R_{cji}{}^{h}+R_{aji}{}^{h}C_{c}^{a}-R_{cji}{}^{a}C_{a}^{h})+£_{F}\Gamma_{ji}^{h}=0.$$

Since $£_{B}R_{cji}{}^{h}=0$ by the remark following (2.4), we are left with two more conditions:

(5.7) $$R_{aji}{}^{h}C_{k}^{a}=R_{kji}{}^{a}C_{a}^{h},\quad £_{F}\Gamma_{ji}^{h}=0.$$

Hence, in the presence of Proposition 4.1, conditions (5.5) when equated to zero is equivalent to (5.6) and (5.7). Thus, the conditions in Proposition 4.1 together with (5.6) and (5.7) determine the most general infinitesimal affine transformation on (TM, ∇^{C}). We state this as our main result in

Theorem 5.1. *Let ∇^{C} be the complete lift of an affine connection ∇ on M to TM. The most general infinitesimal affine transformation $\tilde{X}$ on (TM, ∇^{C}) can be expressed uniquely in the form*

(5.8) $$\tilde{X}=A^{*}+B^{C}+\iota C+F^{V},$$

where A is a (1,1) *tensor field on M satisfying*

$$\nabla A=0,\quad T_{ja}^{h}A_{i}^{a}=T_{ji}^{a}A_{a}^{h}=0,$$

$$R_{kji}{}^{a}A_{a}^{h}=R_{aji}{}^{h}A_{k}^{a}=R_{kja}{}^{h}A_{i}^{a}=0,$$

C is a (1,1) *tensor field on M satisfying*

$$\nabla C=0,\quad T_{ja}^{h}C_{i}^{a}=T_{ji}^{a}C_{a}^{h},$$

$$R_{kji}{}^{a}C_{a}^{h}=R_{aji}{}^{h}C_{k}^{a}=R_{kja}{}^{h}C_{i}^{a},$$

and B, F are infinitesimal affine transformations on (M, ∇).

Let us consider some consequences of our theorem.

1. By putting $B=0$, $C=0$, $F=0$, we see that for a (1,1) tensor A on M satisfying the conditions stated in the theorem, A^{*} is an infinitesimal affine transformation. Similarly, for B, C and F satisfying the conditions stated in the theorem, B^{C}, ιC and F^{V} are infinitesimal affine transformations. The cases B^{C} and F^{V} have appeared in [3, Proposition 7.6].

2. A transformation on TM is said to be fibre-preserving if it sends each fibre of TM into a fibre. An infinitesimal transformation on TM is said to be *fibre-preserving* if it generates a local 1-parameter group of fibre-preserving transformations. The infinitesimal transformation $\tilde{X}$ in the theorem is fibre-preserving iff $A=0$. Thus, by putting $A=0$, we get a characterization of fibre-preserving infinitesimal affine transformation on TM (cf. [4, Lemma 3.6]). Similarly, by putting $A=0$, $B=0$ in the theorem, we get a characterization of vertical infinitesimal affine transformation on TM (cf. [4, Lemma 3.1]).

3. With our theorem, we can easily deduce

Proposition 5.2. *Every infinitesimal affine transformation on* (TM, ∇^C) *is fibre-preserving iff the only* $(1,1)$ *tensor field* A *on* (M, ∇) *satisfying the conditions stated in Theorem* 5.1 *is the zero tensor.*

We note that the proof of the necessity part of Proposition 5.2 requires the fact that if A satisfies the conditions stated in Theorem 5.1, then A^* is an infinitesimal affine transformation. This fact is a consequence of our theorem (consequence 1 above), and does not follow from any of the discussions in [4]. From Proposition 5.2, we see that what Yano and Kobayashi [4] did was precisely to characterize the infinitesimal affine transformations that are fibre-preserving (cf. their Remark 3).

6. Infinitesimal isometries on (TM, g^C)

Let g be a pseudo-Riemannian metric on M and g^C its complete lift to TM. In this last section, the affine connection ∇ is taken to be the Riemannian connection associated with g $\left(\text{so that } \Gamma^h_{ji}=\left\{ {h \atop ji} \right\}\right)$. It is well-known that ∇^C is just the Riemannian connection associated with g^C. Tanno [1] has determined the general form of infinitesimal isometries on (TM, g^C). Here, as an application of our results and further illustration of our method, we present an alternative proof of his theorem.

To begin with, we recall that the components of g^C are given by

$$(6.1)\qquad \begin{bmatrix} y^k\partial_k g_{ji} & g_{ji} \\ g_{ji} & 0 \end{bmatrix}.$$

It follows from this that the frame components of g^C are

$$(6.2)\qquad [G_{\lambda\mu}]=\begin{bmatrix} 0 & g_{ji} \\ g_{ji} & 0 \end{bmatrix}.$$

Now, let $\tilde{X}$ be an infinitesimal isometry on (TM, g^C). $\tilde{X}$ is then an infinitesimal affine transformation on (TM, ∇^C) and according to Proposition 4.1, it can be expressed uniquely in the form

$$\tilde{X} = A^* + B^C + \iota C + F^V,$$

where A, B, C, F satisfy the conditions listed in Proposition 4.1 (with $T_{ji}^h = 0$). In addition A satisfies the condition

$$R_{kja}{}^h A_i^a = 0 \tag{6.3}$$

by Theorem 5.1. What we now do is to compute the frame components $\pounds_{\tilde{X}} G_{\lambda\mu}$ of $\pounds_{\tilde{X}} g^C$ according to (cf. (2.6))

$$\pounds_{\tilde{X}} G_{\lambda\mu} = G_{\nu\lambda} \nabla_\mu^C \tilde{X}^\nu + G_{\nu\mu} \nabla_\lambda^C \tilde{X}^\nu \tag{6.4}$$

and then equate the expressions thus obtained to zero.

Since $T_{ji}^h = 0$, we have $\nabla = \hat{\nabla}$ and $\nabla^C = \nabla^*$. The frame components $\nabla_\mu^C \tilde{X}^\nu$ can thus be obtained from (5.3). On account of (6.3), they become

$$\begin{aligned} \nabla_i^C \tilde{X}^h &= \nabla_i B^h, \\ \nabla_{\bar{i}}^C \tilde{X}^h &= A_i^h, \\ \nabla_i^C \tilde{X}^{\bar{h}} &= y^c \nabla_c \nabla_i B^h + \nabla_i F^h, \\ \nabla_{\bar{i}}^C \tilde{X}^{\bar{h}} &= \nabla_i B^h + C_i^h. \end{aligned} \tag{6.5}$$

By (6.4), (6.2) and (6.5), the frame components $\pounds_{\tilde{X}} G_{\lambda\mu}$ are

$$\begin{aligned} \pounds_{\tilde{X}} G_{ji} &= y^c \nabla_c \pounds_B g_{ji} + \pounds_F g_{ji}, \\ \pounds_{\tilde{X}} G_{j\bar{i}} &= \pounds_{\tilde{X}} G_{\bar{i}j} = \pounds_B g_{ji} + g_{jh} C_i^h, \\ \pounds_{\tilde{X}} G_{\bar{j}\bar{i}} &= g_{jh} A_i^h + g_{ih} A_j^h. \end{aligned} \tag{6.6}$$

By (2.7), we can easily show that $\nabla_c \pounds_B g_{ji} = 0$ iff $\pounds_B \left\{ {h \atop ji} \right\} = 0$. Hence, the result of equating (6.6) to zero is

$$\begin{aligned} &\pounds_F g_{ji} = 0, \\ &C_i^h = -g^{hj} \pounds_B g_{ji}, \\ &g_{jh} A_i^h + g_{ih} A_j^h = 0. \end{aligned} \tag{6.7}$$

Our discussion can now be summarized in the following theorem, first obtained by Tanno [1, Theorem A]:

Theorem 6.1. *Let g^C be the complete lift of a pseudo-Riemannian metric g on M to TM. The most general infinitesimal isometry $\tilde{X}$ on (TM, g^C) can be expressed uniquely in the form*

$$\tilde{X} = A^* + B^C + \iota C + F^V,$$

where A is a (1,1) tensor field on M satisfying

$$\nabla A = 0, \quad R_{kja}{}^h A_i^a = 0, \quad g_{jh} A_i^h + g_{ih} A_j^h = 0,$$

B is an infinitesimal affine transformation on (M, g),
C is the (1,1) tensor field $C_i^h = -g^{hj} £_B g_{ji}$ on M,
and F is an infinitesimal isometry on (M, g).

Let B be an infinitesimal affine transformation on (M, g) and consider the (1,1) tensor field C: $C_i^h = -g^{hj} £_B g_{ji}$. By Theorem 6.1, $B^C + \iota C$ is an infinitesimal isometry on TM and so ιC is an infinitesimal affine transformation. It follows from Theorem 5.1 that $C_i^h = -g^{hj} £_B g_{ji}$ satisfies (5.7), i.e.,

$$R_{aji}{}^h g^{ab} £_B g_{bk} = R_{kji}{}^a g^{hb} £_B g_{ba}. \tag{6.8}$$

It would be interesting to see a direct proof of (6.8).

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