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INFINITESIMAL VARIATIONS OF HODGE STRUCTURE (II): AN INFINITESIMAL INVARIANT OF HODGE CLASSES

Phillip Griffiths and Joe Harris

This is the second in the series of papers on infinitesimal variations of Hodge structure begun in [3]. There we introduced five invariants associated to an infinitesimal variation of Hodge structure and investigated the geometric interpretation of the first one. In this paper we shall study the geometry of the third construction, given in Section 1(c) of [3].

This invariant is associated to the pair (V, γ) consisting of an infinitesimal variation of Hodge structure $V=\{H_{\mathbf{Z}}, H^{p,q}, Q, T, \delta\}$ of even weight $n=2m$ and Hodge class $\gamma \in H_{\mathbf{Z}} \cap H^{m,m}$. If we think of V as a 1st order variation, with tangent space T, of the Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}, Q\}$, then the invariant is a linear subspace

$$H^{m+1,m-1}(-\gamma) \subset H^{m+1,m-1}$$

such that the quotient space $H^{m+1,m-1}/H^{m+1,m-1}(-\gamma)$ is naturally isomorphic to the *co*-normal space of the subspace

$$\left\{\begin{matrix}\xi \in \mathrm{T} \text{ such that } \gamma \text{ remains of type } (m,m) \text{ when the} \\ \text{Hodge structure moves infinitesimally in the direction } \xi\end{matrix}\right\} \subset T.$$

In case V arises from a family

$$\mathcal{X} \to S, \quad S = \operatorname{Spec} \mathbf{C}\left[\varepsilon^1, \dots, \varepsilon^r\right], \quad \varepsilon^i \varepsilon^j = 0$$

of polarized varieties and γ is the fundamental class of an algebraic cycle $\Gamma \subset X$ where X is the reduced fibre of $\mathcal{X} \to S$, then there is a geometrically defined subspace

$$H^{m+1,m-1}(-\Gamma) \subseteq H^{m+1,m-1}(-\gamma).$$

(*Note*: Actually, we shall study all the subspaces $H^{p,q}(-\gamma) \subset H^{p,q}$ where $p>q$, and for these there are corresponding geometrically defined subspaces $H^{p,q}(-\Gamma) \subseteq H^{p,q}(-\gamma)$. In particular, $H^{2m,0}(-\Gamma)$ is defined by the position of Γ relative to the canonical system $|K_X|$.) Our first main result is for the situation $\Gamma \subset X \subset \mathbb{P}^3$ where Γ is a smooth curve and X is a smooth surface containing Γ; the theorem computes the dimension of $H^{2,0}(-\gamma)/H^{2,0}(-\Gamma)$ as $h^1(N_{\Gamma/\mathbb{P}^3})$. A consequence is that when the degree of X is large (relative to the degree of Γ) and Γ has general moduli, then the equations of Γ are given purely in terms of the Hodge theoretic data (V, γ); it is this phenomenon that we should like to understand.

In a sense the simplest subvarieties (other than complete intersections) lying in a projective variety X are the linear subspaces Λ contained in X. Obviously it is of interest to be able to recognize which Hodge classes are fundamental classes of such linear spaces, and in Section 4(b) (cf. Theorem (4.b.2)) we give a first result of this kind. This result first appeared in [8], but here we give a different proof that extends to prove the following rather unexpected theorem (cf. (4.b.26)): "Let $\Gamma \subset X \subset \mathbb{P}^3$ be as above with fundamental class $\gamma \in H^2(X, \mathbb{Z})$. Suppose now that we are only given a Hodge class γ with the same numerical properties and that deg X is large relative to the numbers γ^2, $\gamma \cdot \omega$ $(\omega = c_1(\mathcal{O}_X(1)))$. Then γ is the fundamental class of an effective curve Γ on X." This result implies very subtle behaviour of the intersection of the "variable subspace" $H^{1,1}(X)$ with the "fixed lattice" $H^2(X, \mathbb{Z})$.

In Section 4(c) we discuss the infinitesimal variational aspects of the question of when a Hodge class on a smooth surface $X \subset \mathbb{P}^3$ remains effective under 1st order deformations. Some of this discussion is a special case of that in Section 4(a), but in the earlier section we choose to ignore crucial scheme-theoretic considerations in order to get to the essential point as quickly as possible. It is worthwhile pointing out that, although both Sections 4(b) and (c) are centered around the question of when a Hodge class on $X \subset \mathbb{P}^3$ is effective, the two discussions are complementary. Taken together they give a fairly reasonable picture of the scheme of pairs (X, Γ) where $\Gamma \subset X \subset \mathbb{P}^3$. However certain crucial questions remain open; for example: Let $R_{d,k} \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$ be the set of smooth surfaces $X \subset \mathbb{P}^3$ of degree d where the Picard number $\rho(X) \geqslant k + 1$. Then is codim $R_{d,1} \geqslant d - 3$ with equality holding only for the component of $R_{d,1}$ given by surfaces containing a line? (More generally, it is easy to check that the component of $R_{d,k}$ consisting of surfaces containing k coplanar lines has codimension $k(d-3) - \binom{k-2}{2}$ in $|\mathcal{O}_{\mathbb{P}^3}(d)|$. One may conjecture that this is the codimension of $R_{d,k}$.)

As an application of Sections 4(b) and (c), in Section 4(d) we show how to reconstruct the Fermat surface $F_d \subset \mathbb{P}^3$ $(d \geqslant 5)$ from its universal infinitesimal variation of Hodge structure. Actually, the proof gives a global Torelli theorem (in the form that a variety is uniquely determined by its universal infinitesimal variation of Hodge structure) for the family $\mathcal{F}_d$ of smooth surfaces $X \subset \mathbb{P}^3$ of degree d that contain $d-5$ sets of $d-3$ coplanar lines (it is well known that $F_d \in \mathcal{F}_d$). Although our Torelli theorem is to some extent now subsumed by the recent result of Donagi [5], our proof is of a geometric character (rather than algebraic, as in [5]) and illustrates very clearly our general premise that an infinitesimal variation of Hodge structure may at least to some extent serve as a surrogate for the theta divisor in the classical case of curves (in this regard, see the Schottky discussion in [5]). The argument also clearly points out the necessity of refining the data consisting of a Hodge structure alone in order to obtain geometric results.

In Section 4(e) we study the pair (V, γ) corresponding to a 2-plane Γ lying in a smooth fourfold $X \subset \mathbb{P}^5$. Although the global result from Section 4(b) is lacking (the Hodge conjecture is not known if $\deg X \geqslant 5$) we are able to establish the analogues of the infinitesimal results from Section 4(c); in particular, we obtain the equalities

$$\begin{cases} H^{3,1}(-\Gamma) = H^{3,1}(-\gamma) & (\deg X \geqslant 1) \\ H^{4,0}(-\Gamma) = H^{4,0}(-\gamma) & (\deg X \geqslant 7). \end{cases}$$

Partly the point here is to give computations involving Hodge classes and the "middle" Hodge groups $H^{p,q}$ ($p \neq 0$, $q \neq 0$).

The setup in both Sections 4(c) and (e) was first considered by Bloch in [2]. Part of our results may be interpreted as proving that certain subvarieties satisfy his condition of semi-regularity.

In Section 4(f), as an application of the infinitesimal Max Noether theorem in [3] and interpretation (4.a.3) of our infinitesimal invariant of Hodge classes, we may easily show that many smooth curves $C \subset S \subset \mathbb{P}^3$, where S is a smooth surface of degree $\geqslant 4$, have indecomposable normal bundles. Actually, the result is true for $\deg S \geqslant 2$ (cf. also Hulek [10]), but the remaining cases require a separate argument. An amusing offshoot is the formula

$$\begin{aligned} &\dim\big(H^{2,0}(-\gamma)/H^{2,0}(-C)\big) \\ &\quad = \dim\{\ker e : H^0(\mathcal{O}_C(d-4)) \to H^1(K_C(-d))\} \end{aligned}$$

where $e \in H^1(K_C(4-2d))$ is the extension class of $0 \to N_{C/S} \to N_{C/\mathbb{P}^3} \to N_{S/\mathbb{P}^3} \otimes \mathcal{O}_C \to 0$. Taking Section 4(a) and the main result of [6] into account, it follows that both sides are zero if C has general moduli.

In closing we should like to express our opinion that the geometry of the infinitesimal invariant of Hodge classes has turned out to be richer and more general than we originally thought (initially, it was designed to only study Hodge lines leading up to the Torelli for Fermat surfaces), and our study raises more questions then it answers.

We should also like to express particular gratitude to the referee, who did a marvelous job of deciphering and subsequently improving our original manuscript.

4. Infinitesimal invariants associated to Hodge classes

(a) The basic observation

An object of increasing interest in algebraic geometry is the global subvarieties of moduli spaces defined by considering all varieties of a certain type and having an additional specified geometric property. For example, the subvariety $\mathfrak{M}^r_{g,d} \subset \mathfrak{M}_g$ of the moduli space of curves, which consists of smooth curves having a linear series g^r_d, seems certain to play an important role in the global moduli theory of curves (cf. [9]). As a second example, we let U be a neighborhood (in the analytic topology) of a surface S in its global moduli space (assumed to exist). We may topologically identify all the surfaces $S' \in U$, and for a fixed class $\gamma \in H^2(S, \mathbb{Z})$ we define

$$U_\gamma = \{S' \in U : \gamma \in H^{1,1}(S')\}.$$

It is possible to show that U_γ is open in its Zariski-closure in the moduli space of S (cf. [12]).

In the first example the local structure of $\mathfrak{M}^r_{g,d}$ is one of the main concerns of Brill-Noether theory. In particular, when the Brill-Noether number

$$\rho = g - (r+1)(g-d+r) \leqslant 0$$

we have the "postulated dimension formula"

$$\dim \mathfrak{M}^r_{g,d} = 3g - 3 + \rho$$

which gives at least a first approximation to $\dim \mathfrak{M}^r_{g,d}$. Moreover the Zariski tangent space is given by

$$T(\mathfrak{M}^r_{g,d}) = (\text{image}\, \mu_1)^\perp$$

where

$$\begin{array}{l} \mu_0 : H^0(C, L) \otimes H^0(C, KL^{-1}) \to H^0(C, K) \\ \qquad\qquad\qquad \cup \\ \qquad\quad \mu_1 : \ker \mu_0 \longrightarrow H^0(C, K^2) \end{array}$$

are the usual maps of Brill-Noether theory [1]. The object of this section will be to analogously study the infinitesimal theory of the varieties U_γ and their generalizations.

We first recall from Section 1(c) of [3] the third invariant introduced there. Let $V = \{H_{\mathbb{Z}}, H^{p,q}, Q, T, \delta\}$ be an infinitesimal variation of (polarized) Hodge structure of even weight $2m$ and $\gamma \in H_{\mathbb{Z}}^{m,m}$ a Hodge class. Then we defined:

$$H^{m+k,m-k}(-\gamma)$$
$$= \{\psi \in H^{m+k,m-k} : Q(\delta^k(\xi)\psi, \gamma) = 0 \ \text{ for all } \xi \in T\}. \qquad (4.a.1)$$

As motivation we consider a family of polarized varieties $\{X_s\}_{s \in S}$ and suppose that V is the infinitesimal variation of Hodge structure corresponding to $X = X_{s_0}$ in this family. We denote by U a neighborhood of s_0 in S and as above we identify all the $H^*(X_s, \mathbb{Z})$ for $s \in U$. Given a Hodge class $\gamma \in H^{m,m}(X, \mathbb{Z})$ we define

$$U_\gamma = \{s \in U : \gamma \in H^{m,m}(X_s, \mathbb{Z})\}.$$

Using the Lefschetz decomposition it will suffice to consider only primitive cohomology, and then assuming that $\gamma \in H^{2m}_{\text{prim}}(X, \mathbb{Z})$ we have

$$U_\gamma = \{s \in U : Q(\psi, \gamma) = 0 \text{ for all } \psi \in F^{m+1}H^{2m}_{\text{prim}}(X_s)\}. \qquad (4.a.2)$$

The tangent space to U_γ at s_0 is obtained by differentiating the equations (4.a.2), and with the notations

$$\begin{array}{ccc} T & = & T_{s_0}(U) \\ \cup & & \cup \\ T_\gamma & = & T_{s_0}(U_\gamma) \end{array}$$

it is given by

$$T_\gamma = \{\xi \in T : Q(\delta(\xi)\psi, \gamma) = 0 \text{ for all } \psi \in H^{m+1,m-1}\}. \qquad (4.a.3)$$

Thus we have

$$0 \to H^{m+1,m-1}(-\gamma) \to H^{m+1,m-1} \to T_{s_0}(U_\gamma)^\perp \to 0,$$

and the number

$$h^{m+1,m-1} - h^{m+1,m-1}(-\gamma)$$

is *co*dimension of the Zariski tangent space to U_γ. (1) † For this reason we are led naturally to ask if there is an effective way of computing

† These numbers refer to notes at the end of the paper.

$h^{m+k,m-k}(-\gamma)$, at least in first approximation? It turns out that this question contains a surprising amount of geometry.

A key step is the following trivial observation: Suppose that $\{H_{\mathbf{Z}}, H^{p,q}, Q, T, \delta\}$ arises from an infinitesimal family

$$\begin{array}{l}\mathcal{X}\\ \downarrow\\ S = \mathrm{Spec}\left(\mathbf{C}\left[s^1,\dots,s^m\right]/m^2\right)\end{array}$$

of polarized varieties. If we denote the reduced fibre by X, then $H_{\mathbf{Z}} = H^{2m}(X, \mathbf{Z}) \cap H^{2m}_{\mathrm{prim}}(X)$ and the polarized Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}, Q\}$ is the usual one on primitive cohomology. The tangent space is given by $T = (m/m^2)^*$, and δ is induced by composing the cup-product and Kodaira-Spencer mappings (cf. Section 2(a) of [3]). Suppose also that γ is the primitive part of the fundamental class of a codimension-m algebraic cycle

$$\Gamma = \sum_i n_i Z_i$$

on X.[(2)] We define the *support* of Γ to be

$$\sigma(\Gamma) = \bigcup_i Z_i,$$

and denote by $I_{\sigma(\Gamma)}$ the ideal sheaf of $\sigma(\Gamma)$. We shall use the notation

$$H^{m-k}\left(X, \Omega_X^{m+k}(-\Gamma)\right)$$

to denote the image in $H^{m-k}_{\mathrm{prim}}(X, \Omega_X^{m+k}) = H^{m+k,m-k}$ of the composite map

$$H^{m-k}\left(X, \Omega_X^{m+k} \otimes I_{\sigma(\Gamma)}\right) \to H^{m-k}\left(X, \Omega_X^{m+k}\right) \to H^{m-k}_{\mathrm{prim}}\left(X, \Omega_X^{m+k}\right).$$

With this understood we have the

OBSERVATION:

$$\boxed{H^{m-k}\left(X, \Omega_X^{m+k}(-\Gamma)\right) \subseteq H^{m+k,m-k}(-\gamma).} \tag{4.a.4}$$

PROOF: Since for $\xi \in T$ and $k \geqslant 2$, the definition (4.a.1) immediately gives that

$$\xi \cdot H^{m+k,m-k}(-\gamma) \subseteq H^{m+k-1,m-k+1}(-\gamma),$$

it will suffice to verify (4.a.4) when $k=1$. Let $\psi \in H^{m-1}(X, \Omega_X^{m+1}(-\Gamma))$ and let $\xi \in T$ with

$$\rho(\xi) = \theta \in H^1(X, \Theta).$$

Then

$$\begin{aligned} Q(\delta(\xi)\psi, \gamma) &= \int_X \theta \cdot \psi \wedge \gamma \\ &= \sum n_i \int_{Z_i} \theta_\psi \\ &= 0. \end{aligned}$$

since ψ vanishes on the Z_i. Q.E.D.

When $n=2$ and $X=S$ is a smooth surface, (4.a.1) is the linear subsystem of the canonical system given by

$$H^{2,0}(-\gamma) = \{\psi \in H^{2,0} : Q(\delta(\xi)\psi, \gamma) = 0 \text{ for all } \xi \in T\}.$$

Also, $\Gamma = \Sigma n_i C_i$ is a (virtual) curve and (4.a.4) is (where we set $H^{2,0}(-\Gamma) = H^0(S, K_S(-\sigma(\Gamma))$

$$H^0(S, K_S(-\sigma(\Gamma))) \subseteq H^{2,0}(-\gamma) \tag{4.a.5}$$

where $\sigma(\Gamma) = \Sigma C_i$ is the support of Γ. We shall discuss the question: Under what circumstances can we expect equality in (4.a.5)? As will now be seen this has (to us) a suprising answer.

Let $C \subset \mathbb{P}^3$ be a smooth curve of degree m and genus g, and denote by $N \to C$ the normal bundle. We recall that C is said to be *non-special in the sense of Brill-Noether* if

$$H^1(C, N) = (0); \tag{4.a.6}$$

in particular, by the Gieseker-Petri Theorem this is true if C has general moduli (cf. [1]). We shall assume that C is a smooth point on the Chow variety Ξ of curves of degree m in $\mathbb{P}^3$; it is well known (cf. [11]) that this is true if (4.a.6) is satisfied. (*Note*: For the result (4.a.7) to be proved below it may not be necessary to make this assumption, but it simplifies the argument technically.)

We denote by $U \subset |\mathcal{O}_{\mathbb{P}^3}(d)|$ the Zariski open set of smooth surfaces S of degree d in $\mathbb{P}^3$, and

$$U(-C) \subset U$$

will denote the subset of U consisting of the smooth surfaces passing through C. We assume d is large enough that $U(-C) \neq \phi$, and for $S \in U(-C)$ we consider the infinitesimal variation of Hodge structure $\{H_{\mathbb{Z}}, H^{p,q}, Q, T, \delta\}$ corresponding to $H_{\mathbb{Z}} = H^2(S, \mathbb{Z}) \cap H^2_{\text{prim}}(S)$ and with tangent space $T = T_S(U)$ corresponding to *all* variations of S in $\mathbb{P}^3$. Finally, we denote by

$$\gamma \in H^2(S, \mathbb{Z}) \cap H^{1,1}_{\text{prim}}(S)$$

the primitive part of the fundamental class of C.

PROPOSITION: *There exists an integer $d(m, g)$ depending only on the degree m and genus g of $C \subset \mathbb{P}^3$ such that for $d \geqslant d(m, g)$ and $S \in U(-C)$*

$$h^0(S, K_S(-C)) = h^{2,0}(-\gamma) - h^1(C, N). \tag{4.a.7}$$

In particular, if C is non-special in the sense of Brill-Noether then

$$H^0(S, K_S(-C)) = H^{2,0}(-\gamma).$$

COROLLARY: *If C is non-special in the sense of Brill-Noether, then the equations of $C \subset S$ are given purely in terms of the fundamental class of C and infinitesimal variation of Hodge structure of S in $\mathbb{P}^3$.* (4.a.8)

PROOF OF COROLLARY: Recalling that γ is the primitive part of the fundamental class of C, C is the base locus of the linear subsystem $H^{2,0}(-\gamma) \subset H^0(S, K_S)$.

PROOF OF PROPOSITION: Shifting notation slightly, we now let Ξ be a smooth open neighborhood (Zariski or analytic; it doesn't matter) of C in the Chow variety of degree m curves in $\mathbb{P}^3$. Then by the Riemann-Roch theorem for vector bundles over C

$$\dim \Xi = h^0(C, N) = 4m - h^1(C, N). \tag{4.a.9}$$

We denote by

$$U_{\Xi} \subset U$$

the subvariety of smooth surfaces S that contain some curve $C' \in \Xi$ (thus set-theoretically

$$U_{\Xi} = \bigcup_{C' \in \Xi} U(-C')).$$

In giving the following argument we shall make a couple of dimension counts whose rigorous justification necessitates careful analysis of certain scheme structures. Since this analysis will be carried out in detail in Section 4(c) below in a special case (however, cf. remark (4.c.18)), but one where the techniques may be easily modified to apply in general, we shall not do this here.

We now focus attention on our curve $C \in \Xi$. From the adjunction formula

$$g = \tfrac{1}{2}\left(C^2 - c_1 \cdot C\right) + 1 \qquad \left(c_1 = -c_1(K_S)\right)$$

$$= \tfrac{1}{2}\left(C^2 + m(d-4)\right) + 1,$$

it follows that $C^2 < 0$ for large d. Thus

$$h^0\left(C, \mathcal{O}_C(C)\right) = 0$$

where $\mathcal{O}_C(C) = \mathcal{O}_S(C) \otimes \mathcal{O}_C$ is the normal bundle of C in S, and there are only a finite number of curves $C' \in \Xi$ that are contained in S. In particular, assuming that d is large enough to have $h^i(\mathcal{O}_{\mathbb{P}^3}(d) \otimes I_C) = 0$ for $i = 1, 2$, we may compute the codimension of U_Ξ near S as follows (this is one step that will be justified more rigorously in Section 4(c)):

$$\begin{aligned} \operatorname{codim} U_\Xi &= h^0\left(\mathcal{O}_{\mathbb{P}^3}(d)\right) - h^0\left(\mathcal{O}_{\mathbb{P}^3}(d) \otimes I_C\right) - \dim \Xi \\ &= h^0\left(\mathcal{O}_C(d)\right) - \dim \Xi \\ &= md - g + 1 - 4m + h^1(C, N) \end{aligned}$$

for $d \geqslant d(m, g)$, by the Riemann-Roch theorem for C and (4.a.9).

We now consider a neighborhood (in the analytic topology) $W \subset U$ of S and set

$$W(-C) = W \cap U(-C)$$

$$W_\Xi = W \cap U_\Xi.$$

By choosing W sufficiently small we may topologically identify all the surfaces $S' \in W$, and with the identification $H^2(S, \mathbb{Z}) = H^2(S', \mathbb{Z})$ we define

$$W_\gamma = \{S' \in W : \gamma \text{ is of type } (1, 1)\}.$$

This is an analytic subvariety and clearly

$$W_\Xi \subset W_\gamma.$$

LEMMA: *We have that*

$$W_{\Xi} = W_{\gamma}. \tag{4.a.10}$$

PROOF: If $L \to S$ is the line bundle corresponding to C, then according to the general variational theory of cohomology classes it will suffice to prove that

$$h^1(S, L) = 0. \tag{4.a.11}$$

Indeed, for $\{S_t\} \subset W_\gamma$ a variation of $S = S_0$, there will be a (unique since $\mathrm{Pic}^0(S_t) = (0)$) holomorphically varying family of line bundles $L_t \to S_t$ with $c_1(L_t) = \gamma$, and by obstruction theory (cf. [1]) the sufficient condition that the section $s_0 \in H^0(S_0, L_0)$ defining C be stable under small deformations is just (4.a.11).

By the Riemann-Roch theorem for surfaces,

$$\begin{aligned} h^1(S, L) &= \mathrm{X}(S, L) - h^0\big(S, K_S L^{-1}\big) - h^0(S, L) \\ &= \tfrac{1}{2}\big(C^2 + C \cdot c_1\big) + \mathrm{X}(\mathcal{O}_S) - h^0\big(S, K_S L^{-1}\big) - h^0(S, L) \\ &= \tfrac{1}{2}\big(C^2 - C \cdot c_1\big) + C \cdot c_1 + 1 + h^0\big(\mathcal{O}_{\mathbf{P}^3}(d-4) \otimes I_C\big) - 1 \\ &= \tfrac{1}{2}\big(C^2 - C \cdot c_1\big) + C \cdot c_1 + h^0\big(\mathcal{O}_C(d-4)\big) \\ &= g - 1 + C \cdot c_1 + m(d-4) - g + 1 \\ &= 0, \end{aligned}$$

where we have used the adjunction formula for $C \subset S$ and Riemann-Roch theorem for $h^0(\mathcal{O}_C(d-4))$ (assuming, of course, that d is large enough that $h^1(\mathcal{O}_C(d-4)) = 0$). Q.E.D. for the lemma.

Finally, we shall estimate codim W_γ Hodge-theoretically (again, the details for justifying this dimension count will be given in Section 4(c)): We have by (4.a.3)

$$\begin{aligned} \operatorname{codim} W_\gamma &= h^{2,0} - h^{2,0}(-\gamma) \qquad \big(h^{2,0} = h^{2,0}(S)\big) \\ &= h^0(S, K_S) - h^0\big(S, K_S(-C)\big) + \varepsilon \end{aligned} \tag{4.a.12}$$

where $\varepsilon = h^{2,0}(-\gamma) - h^{2,0}(-C) \geq 0$ by (4.a.5)

$$\begin{aligned} &= h^0\big(\mathcal{O}_{\mathbf{P}^3}(d-4)\big) - h^0\big(\mathcal{O}_{\mathbf{P}^3}(d-4) \otimes I_C\big) + \varepsilon \\ &= m(d-4) - g + 1 + \varepsilon \end{aligned}$$

by the Riemann-Roch theorem for C. Comparing this with the formula for codim $U_\Xi =$ codim W_Ξ and using lemma (4.a.10) gives that

$$\varepsilon = h^1(C, N)$$

in (4.a.12). By the definition of ε we conclude the proof of the Proposition. Q.E.D.

REMARK: Using lemma (4.a.10) let us agree to call the formula

$$\text{"codim } W_\Xi = h^0(S, K_S) - h^0(S, K_S(-\sigma(\Gamma)))\text{"}$$

the *näive Hodge dimension count.* This means: we count the number of *geometrically apparent* (at first glance) conditions that a variation of S should contain a variation of C; by (4.a.10) and (4.a.5) this number is $h^0(S, K_S) - h^0(S, K_S(-\sigma(\Gamma)))$. On the other hand, the naive dimension count for Ξ is the formula

$$\text{"dim } \Xi = h^0(C, N)\text{"};$$

for the reasons explained in [1] we shall refer to this as the *näive Brill-Noether dimension count.* Then the proposition says that the näive Hodge dimension count and näive Brill-Noether dimension count both fail by the same amount. This certainly suggests some interesting relation between Hodge theory and geometry.

(b) Lines on surfaces (i)

Let $S \subset \mathbb{P}^3$ be a smooth surface of degree d containing a line Λ with fundamental class $\lambda \in H^2(S, \mathbb{Z})$. Then λ satisfies the conditions

$$\begin{cases} \lambda^2 = 2 - d \\ \lambda \cdot \omega = 1 \text{ where } \omega = c_1(\mathcal{O}_S(1)) \\ \lambda \in H^{1,1}(S). \end{cases} \tag{4.b.1}$$

DEFINITION: A class $\lambda \in H^2(S, \mathbb{Z})$ satisfying the conditions (4.b.1) will be called a *Hodge line.*

In this section we will give a proof of the following result (cf. [8] for the original proof):

THEOREM: *A Hodge line is the fundamental class of a unique line* $\Lambda \subset S$. (4.b.2)

Given a Hodge line λ there is a unique holomorphic line bundle $L \to S$ with $c_1(L) = \lambda$, and one may try to use the Riemann-Roch theorem for $L \to S$. This gives

$$h^0(S, L) + h^0(S, K - L) \geqslant h^0(\mathcal{O}_{\mathbb{P}^3}(d-4)) - (d-4). \qquad (4.b.3)$$

To be able to use this we must know that

$$h^0(S, K - L) \leqslant h^0(\mathcal{O}_{\mathbb{P}^3}(d-4)) - (d-3),$$

and as will be seen below this is almost tantamount to assuming the theorem.[3] However, when $d = 5$, (4.b.3) gives

$$h^0(S, L) + h^0(S, K - L) \geqslant 3.$$

We want to show that

$$h^0(S, L) \neq 0,$$

and if this were false then

$$h^0(S, K - L) \geqslant 3.$$

Any effective divisor $D \in |K - L|$ satisfies

$$\begin{cases} \dim |D| = n \geqslant 2 \\ D^2 = (\omega - \lambda)^2 = 0 \\ D \cdot H = 4 \text{ where } H \text{ is a hyperplane.} \end{cases}$$

If any pencil from $|D|$ has a fixed component E, then we must have $\deg E = 1$, 2 or 3. If $\deg E = 2$ or 3 then S contains a pencil of rational curves, which implies that S is rational. If $\deg E = 1$ then S contains a pencil of elliptic or rational curves, and must then be an elliptic or rational surface. In all cases we obtain a contradiction to the fact that $|K_S|$ is very ample.

It follows that the linear system $|D|$ gives a *holomorphic* map

$$\varphi : S \to \Gamma \subset \mathbb{P}^n,$$

whose image $\varphi(S)$ cannot be a surface, and therefore must be a curve Γ. Since S is regular, Γ must be a rational curve non-degenerately embedded in $\mathbb{P}^n$ by a complete linear system. If $n \geqslant 2$ then $|D| = |\varphi^{-1}(\mathcal{O}_\Gamma(1))| = |\varphi^{-1}(\mathcal{O}_{\mathbb{P}^1}(n))|$ and the mapping φ is the composition

$$S \underset{\psi}{\rightarrow} \mathbb{P}^1 \underset{\eta}{\rightarrow} \mathbb{P}^n$$

where η is given by $|\mathcal{O}_{\mathbb{P}^1}(n)|$. Since $\deg D = 4$, $\deg \psi^{-1}(t) \leqslant 2$ for $t \in \mathbb{P}^1$ and S again contains a pencil of rational curves, which is a contradiction. This establishes the theorem when $d = 5$.

Turning to the general case, we choose a smooth curve

$$C \in |mH - L| \qquad m \gg 0,$$

where $H = \mathcal{O}_S(1)$ is the hyperplane bundle. We want to show that

For m sufficiently large there is a surface R of degree m not containing S and such that

$$C \subset S \cap R. \tag{4.b.4}$$

In this case

$$S \cdot R = C + \Lambda$$

where Λ is the desired line. Establishing (4.b.4) involves examining the postulation sequence of C, and this is what we shall do.[4] We first record the relevant data concerning the degree $d(C)$ and genus g of C:

$$\begin{cases} d(C) = md - 1 \\ K_C = \mathcal{O}_C((m+d-4)H - L) \\ g = \frac{1}{2}(m+d-4)(md-2), \end{cases} \tag{4.b.5}$$

where the last step follows from (4.b.1) and the adjunction formula.

We next choose a general hyperplane divisor $D = \mathbb{P}^2 \cdot C$ and consider the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & I_C(k-1) & \rightarrow & \mathcal{O}_{\mathbb{P}^3}(k-1) & \rightarrow & \mathcal{O}_C(k-1) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & I_C(k) & \rightarrow & \mathcal{O}_{\mathbb{P}^3}(k) & \rightarrow & \mathcal{O}_C(k) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & I_D(k) & \rightarrow & \mathcal{O}_{\mathbb{P}^2}(k) & \rightarrow & \mathcal{O}_D(k) & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $I_C \subset \mathcal{O}_{\mathbb{P}^3}$ is the ideal sheaf of C and $I_D \subset \mathcal{O}_{\mathbb{P}^2}$ is the ideal sheaf of D. A piece of the cohomology diagram is

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & H^0(I_C(k-1)) \arrow[r] \arrow[d] & H^0(\mathcal{O}_{\mathbb{P}^3}(k-1)) \arrow[r, "\rho_{k-1}"] \arrow[d] & H^0(\mathcal{O}_C(k-1)) \arrow[r] \arrow[d] & H^1(I_C(k-1)) \arrow[r] \arrow[d, "l_k"] & 0 \\
0 \arrow[r] & H^0(I_C(k)) \arrow[r] \arrow[d, "\tau_k"] & H^0(\mathcal{O}_{\mathbb{P}^3}(k)) \arrow[r, "\rho_k"] \arrow[d] & H^0(\mathcal{O}_C(k)) \arrow[r] \arrow[d] & H^1(I_C(k)) \arrow[r] & 0 \\
0 \arrow[r] & H^0(I_D(k)) \arrow[r] \arrow[d] & H^0(\mathcal{O}_{\mathbb{P}^2}(k)) \arrow[r, "\sigma_k"] \arrow[d] & H^0(\mathcal{O}_D(k)) \arrow[r] & \cdots & \\
 & H^1(I_C(k-1)) & 0 & & &
\end{tikzcd} \tag{4.b.6}$$

We use the notations

$$\begin{cases} V_k = \rho_k\big(H^0(\mathcal{O}_{\mathbb{P}^3}(k))\big) \subset H^0(\mathcal{O}_C(k)) & v_k = \dim V_k \\ W_k = \sigma_k\big(H^0(\mathcal{O}_{\mathbb{P}^2}(k))\big) \subset H^0(\mathcal{O}_D(k)) & w_k = \dim W_k, \end{cases} \tag{4.b.7}$$

so that $w_k = h^0(\mathcal{O}_{\mathbb{P}^2}(k)) - h^0(I_D(k))$ is the number of conditions imposed by D on $|\mathcal{O}_{\mathbb{P}^2}(k)|$. We thus have

$$\begin{cases} v_k - v_{k-1} = w_k + \dim \ker l_k, \quad k \geqslant 0, v_{-1} = 0; \\ \text{in particular } v_k - v_{k-1} \geqslant w_k, \\ \text{with equality holding for all } k \text{ if, and only if,} \\ C \text{ is projectively normal.}^{(5)} \end{cases} \tag{4.b.8}$$

We set

$$y_k = w_k - w_{k-1},$$

and note (as will be explained more fully below) that y_k is the dimension of a linear series cut out on a line $\mathbb{P}^1 \subset \mathbb{P}^2$.[(6)]

We now denote by Γ the intersection $S \cdot \mathbb{P}^2$; we may assume that Γ is a smooth plane curve of degree d and

$$C \cdot \Gamma = D$$

is a divisor of degree $md - 1$. We will study curves $\Phi_k \in |\mathcal{O}_{\mathbb{P}^2}(k)|$ that pass through D. We have

(i) if $0 < k < d$ then no curve Φ_k passes through D; and

(ii) if $d \leqslant k < m$ then any curve Φ_k through D must contain Γ.

Both of these follow from $D \subset \Gamma \cdot \Phi$ and $\deg D = md - 1 > kd$ if $k < m$.

If we define

$$n = \left\{ \begin{array}{l} \text{least integer such that there exists a} \\ \text{curve } \Phi_n \text{ not containing } \Gamma \text{ and passing through } D \end{array} \right\}$$

then $n \geqslant m$ and from (i) and (ii) we have

$$\begin{aligned} w_k &= \binom{k+2}{2} & & k < d \\ w_k &= \binom{k+2}{2} - \binom{k-d+2}{2} & & d \leqslant k < n. \end{aligned}$$

This gives

$$\left\{ \begin{array}{ll} y_k = k+1 & 0 \leqslant k \leqslant d-1 \\ y_k = d & d \leqslant k \leqslant n-1. \end{array} \right. \tag{4.b.9}$$

A crucial step in the proof is provided by the

LEMMA: *For $k \geqslant n$, either $y_k < y_{k-1}$ or $y_{k-1} = 0$.* (4.b.10)

PROOF: Let $\mathbb{P}^1 \subset \mathbb{P}^2$ be a general line and set

$$U_k = \operatorname{Im}\{H^0(I_D(k)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(k))\}$$

$$u_k = \dim U_k.$$

Since $H^0(I_D(k)) \supset \Gamma \cdot H^0(\mathcal{O}_{\mathbb{P}^2}(k-d)) + \Phi_n \cdot H^0(\mathcal{O}_{\mathbb{P}^2}(k-n))$, and since we may assume that $\mathbb{P}^1$ misses the intersection $\Phi \cap \Gamma$, it follows from $k \geqslant n$ that $U_k \subset H^0(\mathcal{O}_{\mathbb{P}^1}(k))$ is a base point free linear subsystem, and that the image of

$$U_{k-1} \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(k))$$

contains U_k. On the other hand

$$w_k = \binom{k+2}{2} - h^0(I_D(k));$$

$$\begin{aligned} y_k &= w_k - w_{k-1} \\ &= (k+1) - \big(h^0(I_D(k)) - h^0(I_D(k-1))\big); \end{aligned}$$

$$y_{k-1} - y_k = w_k - w_{k-1} - 1.$$

Lemma (4.b.10) then follows from the

LEMMA(Gieseker): *Let* $U_j \subset H^0(\mathcal{O}_{\mathbb{P}^1}(j))$ $(j = k-1, k)$ *be base point free linear systems such that*

$$U_{k-1} \otimes H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big) \to U_k. \tag{4.b.11}$$

If $\dim U_j = u_j$, *then either*

$$u_k > u_{k-1} + 1$$

or else $u_{k-1} = k$ *(i.e.,* $U_{k-1} = H^0(\mathcal{O}_{\mathbb{P}^1}(k-1))$*).*

PROOF: Set $U = U_{k-1} \subset H^0(\mathcal{O}_{\mathbb{P}^1}(k-1))$ and $u = \dim U$. Then by Koszul's complex we have

$$0 \to E \to U \otimes \mathcal{O}_{\mathbb{P}^1}(1) \to \mathcal{O}_{\mathbb{P}^1}(k) \to 0,$$

where E is a vector bundle of rank $u-1$ over $\mathbb{P}^1$. Since by assumption we have

$$0 \to H^0(E) \to U \otimes H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)\big) \to U_k$$

it follows that

$$u_k \geqslant 2u_{k-1} - h^0(E). \tag{4.b.12}$$

To estimate $h^0(E)$ we use Grothendieck's decomposition to write

$$E = \bigoplus_{i=1}^{u-1} \mathcal{O}(l_i).$$

Since $c_1(E) = u - k$ the conditions on the l_i are

$$\begin{cases} l_i \leqslant 1 & \big(\text{since } \mathcal{O}(l_i) \subset U \otimes \mathcal{O}_{\mathbb{P}^1}(1)\big) \\ \sum l_i = u - k \end{cases}. \tag{4.b.13}$$

We let

$$\alpha = \#\{i : l_i = 1\}$$

$$\beta = \#\{i : l_i = 0\}$$

$$\gamma = \#\{i : l_i \leqslant -1\}.$$

Then $h^0(E) = 2\alpha + \beta$, and by (4.b.12) and (4.b.13)

$$u_k - u_{k-1} \geqslant u - h^0(E)$$

$$= u - (2\alpha + \beta), \tag{4.b.14}$$

$$\alpha - \gamma = \deg E = u - k; \quad \text{and}$$

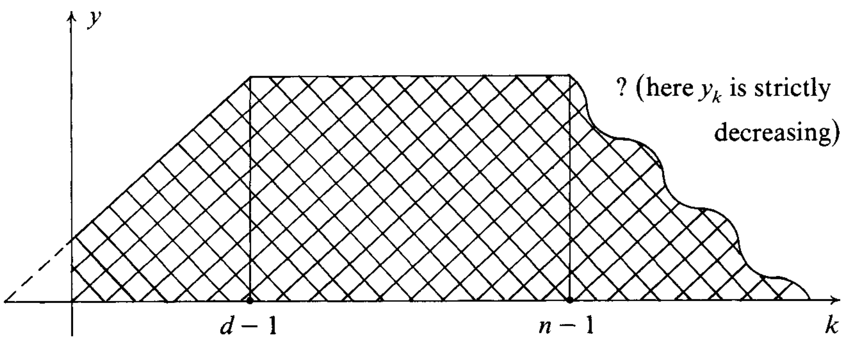
$$\alpha + \beta + \gamma = \operatorname{rank} E = u - 1.$$

Adding the last two and substituting into (4.b.14) gives

$$u_k - u_{k-1} \geqslant k + 1 - u,$$

from which Gieseker's lemma is an immediate consequence. Q.E.D.

Using (4.b.9) and (4.b.10) we may picture the graph of y_k as a function of k as follows:



From

$$w_k = md - 1 \qquad k \gg 0$$

$$y_k = w_k - w_{k-1}$$

$$w_0 = 1 \text{ and } w_{-1} = 0$$

it follows by telescoping that

$$\sum_{i=0}^{k} y_i = md - 1, \qquad k \gg 0. \tag{4.b.15}$$

On the other hand, by the cohomology diagram (4.b.6)

$$\begin{aligned} h^0(\mathcal{O}_C(k)) &\geqslant \sum_{i=0}^{k} w_i \\ &= \sum_{i=0}^{k}\left(\sum_{j=0}^{i} y_i\right) \\ &= \sum_{i=0}^{k}(k-i+1)y_i. \end{aligned} \tag{4.b.16}$$

By the Riemann-Roch theorem for C, for $k \gg 0$

$$g = k \deg D - h^0(\mathcal{O}_C(k)) + 1$$

$$\leqslant k\left(\sum_{i=0}^{k} y_i\right) - \sum_{i=0}^{k}(k-i+1)y_i + 1;$$

$$g \leqslant \sum_{i=0}^{\infty}(i-1)y_i + 1, \tag{4.b.17}$$

which by (4.b.5) gives

$$\tfrac{1}{2}(m+d-4)(md-2) \leqslant \sum_{i=0}^{\infty}(i-1)y_i + 1. \tag{4.b.18}$$

At this stage, using (4.b.9), (4.b.10), and (4.b.15) our result will be a consequence of the following combinatorial

LEMMA: *Let* $\{y_k\}$, $k = 0, 1, 2, \ldots,$ *be a sequence of non-negative integers* satisfying

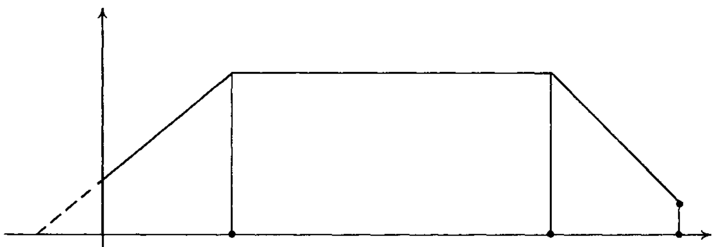
$$\begin{cases} y_k = k+1 & 0 \leqslant k \leqslant d-1 \\ y_k = d & d \leqslant k \leqslant n-1,\ n \geqslant m \\ y_k < y_{k-1} \text{ or } y_{k-1} = 0 & \text{for } n \leqslant k \\ \sum_{i=0}^{\infty} y_i = md - 1 & (= \text{shaded area in above figure}). \end{cases}$$

Then

$$\sum_{i=0}^{\infty} (i-1)y_i + 1 \leqslant \tfrac{1}{2}(m+d-4)(md-2)$$

with equality holding if, and only if, $n = m$ *and* $y_{m-1+j} = d - j$ *for* $j \leqslant d-2$, $y_{m+d-2} = 0$.

In other words, the sum in (4.b.17) is maximized, subject to the constraints (4.b.9), (4.b.10), and (4.b.15), by the figure



We will not give a proof of this lemma, which may be found in [12].

Using it and comparing with (4.b.18), it follows that $n = m$ and that (4.b.17) and (4.b.16) are equalities. From the definitions (4.b.7) and cohomology diagrams (4.b.6) for all k, it follows that

$$\begin{cases} n = m, \text{ and} \\ \rho_k \text{ is surjective for all } k. \end{cases}$$

Thus $C \subset \mathbb{P}^3$ is projectively normal, and consequently τ_k in (4.b.6) is surjective for all k. Therefore there exists a surface $R \in |I_C(m)|$ which cuts out the curve $\Phi \in |I_D(m)|$ in $\mathbb{P}^2$, and this is the surface required in (4.b.4). Q.E.D.

We ask now how far this argument can be pushed. One extension is immediate: we can replace the "line" in (4.b.2) by a "plane curve" of degree $\varepsilon \leqslant d$. Specifically, the class λ of a plane curve of degree ε on the surface S will satisfy

$$\begin{cases} \lambda^2 = \varepsilon(\varepsilon - d + 1) \\ \lambda \cdot \omega = \varepsilon \end{cases} \tag{4.b.19}$$

and we claim that in fact

THEOREM: *Any class* $\lambda \in H^{1,1}(S) \cap H^2(S, \mathbb{Z})$ *satisfying* (4.b.19) *is the fundamental class of a plane curve of degree* ε *contained in* S, *unique if* $\varepsilon \leqslant d-2$. (4.b.20)

PROOF: The proof follows exactly the same lines as that of (4.b.2). We let L, C and D be as before, but instead of (4.b.5) we have

$$\begin{cases} d(C) = md - \varepsilon \\ g = \pi(m, d, \varepsilon) = \frac{1}{2}(md(m+d-4) - \varepsilon(2m+2d-\varepsilon-5)+2) \end{cases} \tag{4.b.21}$$

As before, we analyze the sequences v_k, w_k and y_k for C and D, and again find that the sequence y_k must be extremal; specifically, we have the

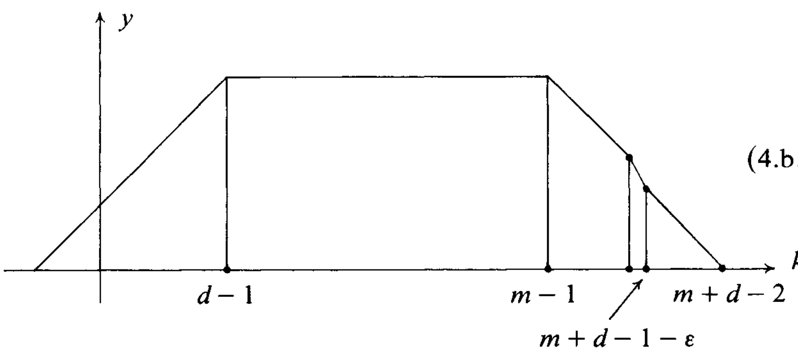
LEMMA: *Let $\{y_k\}$ be a sequence satisfying the first three conditions of* (4.b.19), *but with $\sum y_i = md - \varepsilon$. Then*

$$\sum (i-1)y_i + 1 \leqslant \pi(m, d, \varepsilon) \tag{4.b.22}$$

with equality holding if, and only if $n = m$ and

$$y_{m-1+j} = \begin{cases} d-j & \text{for} \quad j \leqslant d-1-\varepsilon \\ d-j-1 & \text{for} \quad d-\varepsilon \leqslant j \leqslant d-1 \\ 0 & \text{for} \quad j \geqslant d-1 \end{cases}$$

In other words, the sum (4.b.17) is maximized by the figure



(4.b.23)

Thus, as in the case of Hodge lines, we see that C must be projectively normal, and that D lies on a curve of degree m in $\mathbb{P}^2$ not containing Γ, so that C lies on a surface R of degree m not containing S. Writing

$$R \cdot S = C + E$$

we see that E is a plane curve of degree ε and class λ. Q.E.D.
Note that this argument establishes as well the

COROLLARY: If λ is an integral class of type (1, 1) on S, and $\lambda \cdot \omega = \varepsilon \leqslant d$, *then*

$$\lambda^2 \leqslant \varepsilon(\varepsilon - d + 1). \tag{4.b.24}$$

PROOF: If $\lambda^2 > \varepsilon(\varepsilon - d + 1)$, then for the curve C appearing in the proof of the theorem we have $\pi(C) > \pi(m, d, \varepsilon)$. Tracing through the proof of that result, this latter inequality violates the first statement in lemma (4.b.22). Q.E.D.
More generally, we have the

COROLLARY: *If* λ *is an integral class of type* $(1, 1)$ *on* S, *writing*

$$\lambda\omega = ld + \varepsilon \tag{4.b.25}$$

with $0 \leqslant \varepsilon \leqslant d$ *then*

$$\lambda^2 \leqslant l^2 d + \varepsilon(2l + \varepsilon - d + 1).$$

PROOF: This is just corollary (4.b.24) applied to the class $\lambda - l\omega$.

The next question to ask, clearly, is what about classes λ of "degree" $\lambda \cdot \omega = \varepsilon$ with $\lambda^2 < \varepsilon(\varepsilon - d + 1)$ – or, equivalently, with virtual genus

$$\pi(\lambda) = \frac{\lambda^2 + (d-4)\lambda - \omega}{2} + 1 < \frac{(\varepsilon - 1)(\varepsilon - 2)}{2}.$$

(Note that $\pi(\lambda)$ may be negative, as in the case of a pair of skew lines.) To answer this, let δ represents the *defect* $\binom{\varepsilon - 1}{2} - \pi(\lambda)$ of the virtual genus of λ; equivalently, write

$$\lambda^2 = \varepsilon(\varepsilon - d + 1) - 2\delta.$$

Clearly, theorem (4.b.20) cannot be true as stated for classes λ with defect $\delta > 0$. Surprisingly, however, it is true if the degree d of S is large enough compared to ε and δ. Precisely, we have the

THEOREM: *Let* S *be a smooth surface of degree* d, λ *an integral class of type* $(1, 1)$ *on* S *with*

$$\lambda \cdot \omega = \varepsilon$$

$$\lambda^2 = \varepsilon(\varepsilon - d + 1) - 2\delta.$$

(i.e., $\pi(\lambda) = \binom{\varepsilon - 1}{2} - \delta$). *Assume* $d > \varepsilon + \delta + 2$. *Then* λ *is the class of an effective divisor on* S. (4.b.26)

PROOF: The proof follows, at first, the same lines as those above: If we let L be the line bundle on S with Chern class λ, $m \gg 0$, and C an irreducible smooth curve in the linear system $|mH - L|$, then we find that

$$d(C) = md - \varepsilon$$

$$g = \pi(m, d, \varepsilon) - \delta$$

where $\pi(m, d, \varepsilon)$ is given by (4.b.21). We let D, Γ, w_k, and y_k be as before. Here, of course, y_k need not be the sequence specified in (4.b.23), inasmuch as we know only that

$$\sum (i-1) y_i + 1 \geqslant \pi(m, d, \varepsilon) - \delta.$$

There is, however, one thing we can say, using our assumption that d is large, and that is that

$$n = m$$

or, equivalently, that

$$y_m = d - 1,$$

or, equivalently, that D *lies on a curve of degree* m *not containing* Γ. To see this, note that if we add 1 to y_m in the sequence represented by (4.b.23), we have to subtract 1 from some y_{m+j}, and the first j for which we can do this without violating the condition that $\{y_k\}$ be strictly decreasing (cf. (4.b.10)) in this range is $j = m + d - 2 - \varepsilon$. From this it follows that if $\{y_i\}$ is a sequence satisfying the conditions of (4.b.22) and such that $y_m = d$,

$$\sum (i-1) y_i \leqslant \pi(m, d, \varepsilon) - d + 2 + \varepsilon.$$

Since our present curve C has genus $g = \pi(m, d, \varepsilon) - \delta > \pi(m, d, \varepsilon) - d + 2 + \varepsilon$, we conclude that $y_m = d - 1$ and hence that D lies on a curve of degree m not containing Γ.

We now encounter the second difficulty in applying the previous argument: since C need not be projectively normal, the fact that D lies on a curve of degree m not containing Γ does not insure (as it did before) that C lies on a surface R of degree m not containing S. Indeed, from the sequence

$$0 \to \mathcal{I}_{C,\mathbb{P}^3}(k-1) \to I_{C,\mathbb{P}^3}(k) \to I_{D,\mathbb{P}^2}(k) \to 0$$

we see that the space of curves of degree k in $\mathbb{P}^2$ containing D, modulo those which are restrictions of surfaces of degree k containing C, has dimension exactly

$$\dim \ker l_k,$$

where l_k is the map in diagram (4.b.6). What we can do, however, is to bound this discrepancy: since by (4.b.8) and (4.b.16), we have

$$g \leqslant \sum (i-1) y_i - \sum_i \dim \ker l_k$$

we have in our present circumstances

$$\sum_k \dim \ker l_k \leqslant \delta.$$

From this we may deduce the

LEMMA: *For some* ν, $0 \leqslant \nu \leqslant \delta$, *the linear series* $|\mathcal{I}_C(m+\nu)|$ *in* $\mathbb{P}^3$ *cuts out the complete series* $|\mathcal{I}_D(m+\nu)|$ *in* $\mathbb{P}^2$. (4.b.27)

Now, let ν be as in Lemma (4.b.27). The series $|\mathcal{I}_C(m+\nu)|$ of surfaces of degree $m+\nu$ through C then cuts out on S the complete linear series $|E| = |(m+\nu)H - C| = |\nu H + L|$. We now make the

CLAIM: *The base locus of the linear series* $|E|$ is a curve $E_0 \subset S$ *with fundamental class* λ. (4.b.28)

To see this, consider the series cut out on Γ by $|E|$. To begin with, since the divisor D lies on a curve of degree m not containing Γ, we can write on Γ

$$mH \sim D + F$$

for some effective divisor F of degree ε; in fact, we have $\mathcal{O}_\Gamma(F) = L \otimes \mathcal{O}_\Gamma$. The linear series $|E|$ on S thus cuts out, on the general hyperplane section Γ of S, the complete linear series

$$\begin{aligned} |E\|_\Gamma &= |(m+\nu)H - D| \\ &= |\nu H + F| \end{aligned}$$

What does this linear system on Γ look like? The key point here is the

LEMMA: *Let* Γ *be a plane curve of degree* d, F *an effective divisor of degree* ε *on* Γ. *Then for any* ν *such that* $d-1 > \nu + \varepsilon$, *the linear series* $|\nu H + F|$ *on* Γ *has* F *as fixed divisor*; *i.e., it consists of the linear series* $|\nu H|$ *plus* F. (4.b.29)

PROOF OF LEMMA. Write $F = p_1 + \ldots + p_\varepsilon$, let $L_i \subset \mathbb{P}^2$ be a general line through p_i, and write

$$L_i \cdot \Gamma = p_i + q_{i,1} + \ldots + q_{i,d-1}$$

Then the complete linear series $|\nu H + F|$ will be cut out by curves of degree $\nu + \varepsilon$ passing through the $\varepsilon(d-1)$ points $\{q_{ij}\}$. But by Bezout's theorem any curve of degree $\nu + \varepsilon < d-1$ passing through $q_{i,1}, \ldots, q_{i,d-1}$ must contain L_i, and the lemma follows. Q.E.D. for (4.b.29).

Applying the lemma in our present circumstances (which we may do, since by hypothesis $d > \delta + \varepsilon + 2 \geqslant \nu + \varepsilon + 2$), we see that the linear system $|E|$ on S cuts out on Γ a linear system with *fixed divisor* exactly F. The fixed divisor E_0 of $|E|$ is thus a curve on S whose restriction to Γ is F. Since $\mathcal{O}_\Gamma(F) = L \otimes \mathcal{O}_\Gamma$, then, we conclude that for a general hyperplane section Γ of S,

$$\mathcal{O}_S(E_0) \otimes \mathcal{O}_\Gamma = L \otimes \mathcal{O}_\Gamma,$$

and hence that

$$\mathcal{O}_S(E_0) = L.$$

This establishes our claim (4.b.28) and thereby our theorem (4.b.26).

REMARK: The variational form of Theorem (4.b.26) will be discussed below (cf. remark (4.c.18)).

As an example we may take

$$C = L_1 + \ldots + L_\varepsilon$$

where the $L_i \subset \mathbb{P}^3$ are non-intersecting lines. Then for S a smooth surface passing through the L_i and with λ denoting the fundamental class of C, the virtual genus

$$\pi(\lambda) = -\varepsilon + 1.$$

Our theorem then implies the interesting fact:

If S is a smooth surface of degree

$$d > \tfrac{1}{2}(\varepsilon^2 + \varepsilon + 4) \tag{4.b.30}$$

that contains a Hodge class λ with

$$\lambda \cdot \omega = \varepsilon$$

$$\pi(\lambda) = -\varepsilon + 1,$$

then necessarily

$$\lambda = \lambda_1 + \ldots + \lambda_\varepsilon$$

where the λ_i are Hodge lines.

PROOF: By Theorem (4.b.26), λ is the fundamental class of a curve C.

Assuming that all components of C are reduced, it is clear from $\pi(\lambda) = -\varepsilon + 1$ that C must have $\geqslant \varepsilon$ irreducible components, while from $\lambda \cdot \omega = \varepsilon$ it follows that C must have exactly ε components L_i each of degree one. Thus L_i is a line, and these lines must be pairwise disjoint. Now take λ_i to be the fundamental class of L_i.

It remains to justify our assumption that all components of C have multiplicity one. Suppose first that

$$C = 2\Lambda + C' \tag{4.b.31}$$

where Λ is a line and C' is an irreducible curve of degree $\varepsilon - 2$. Using the formula

$$\pi(D + E) = \pi(D) + \pi(E) + (D \cdot E) - 1 \tag{4.b.32}$$

for divisors D, E on S, we have

$$\pi(C') \leqslant \frac{(\varepsilon - 3)(\varepsilon - 4)}{2}$$

(since the genus of $C' \subset \mathbb{P}^3$ is maximized when C' is a smooth plane curve);

$$\Rightarrow 1 - \varepsilon \leqslant 1 - d + \frac{(\varepsilon - 3)(\varepsilon - 4)}{2} + 2(\varepsilon - 2) - 1$$

$$\Rightarrow d \leqslant \frac{\varepsilon(\varepsilon - 1)}{2} + 1$$

But our assumption on d rules this possibility out.

Now suppose that C is any effective divisor having at least one non-reduced component. Then an easy computation using (4.b.32) shows that $\pi(C)$ is maximized when C is of the form (4.b.31). From this it now follows that our C must have all components reduced. Q.E.D.

This result shows that the variable Hodge decomposition

$$H = H^{2,0}(S) \oplus H^{1,1}(S) \oplus H^{0,2}(S)$$

intersects the fixed lattice $H_{\mathbb{Z}} \subset H$ in an extremely subtle manner.

(c) Lines on surfaces (ii)

Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $d \geqslant 3$ and denote by $\{H_{\mathbb{Z}}, H^{p,q}, Q, T, \delta\}$ the infinitesimal variation of Hodge structure on $H_{\mathbb{Z}} = H^2(S, \mathbb{Z}) \cap H^2_{\text{prim}}(S)$ whose tangent space $T = H^0(S, \mathcal{O}_S(d))$ corresponds to *all* infinitesimal deformations of S in $\mathbb{P}^3$. Let $\Lambda \subset S$ be a line with fundamental class λ. In this section we shall prove the following (cf. (4.a.5))

THEOREM: *With the above notations we have*

$$H^0(S, K_S(-\Lambda)) = H^{2,0}(-\lambda). \tag{4.c.1}$$

Since this result is infinitesimal its proof will require a scheme-theoretic study of both Hodge-theoretic and projective conditions that a surface contain a line (the latter is certainly well-known to experts). For this we denote by

$$U_d \subset \mathbb{P}^{\binom{d+2}{3}-1}$$

$$W \subset U_d$$

respectively the family of smooth surfaces $S \in |\mathcal{O}_{\mathbb{P}^3}(d)|$, and those S that contain a line.

Projective study. We begin by proving that

$$\operatorname{codim} W = d - 3. \tag{4.c.2}$$

For this we denote by $W_\Lambda \subset U_d$ the subvariety of surfaces that contain a fixed line Λ. The exact cohomology sequence of

$$0 \to \mathcal{O}_{\mathbb{P}^3}(d) \otimes I_\Lambda \to \mathcal{O}_{\mathbb{P}^3}(d) \to \mathcal{O}_\Lambda(d) \to 0$$

gives

$$0 \to H^0(\mathcal{O}_{\mathbb{P}^3}(d) \otimes I_\Lambda) \to H^0(\mathcal{O}_{\mathbb{P}^3}(d)) \to H^0(\mathcal{O}_\Lambda(d)) \to 0.$$

Since $h^0(\mathcal{O}_\Lambda(d)) = d + 1$ this implies that

$$\operatorname{codim} W_\Lambda = d + 1. \tag{4.c.3}$$

Finally, since the Grassmannian $G = \mathbb{G}(1, 3)$ of lines in $\mathbb{P}^3$ has dimension 4, (4.c.2) follows from (4.c.3).

We will reestablish (4.c.2) and at the same time give a natural desingularization of W. For this we consider the incidence correspondence

$$I \subset U_d \times G$$

defined by

$$I = \{(S, \Lambda) : \Lambda \subset S\}.$$

Denoting by π_1 and π_2 the respective projections, we begin by noting that for $d \geqslant 3$

$$\pi_1^{-1}(S) = \{\text{set of lines } \Lambda \subset S\}$$

is finite. (*Proof*: Denoting the hyperplane class by ω, the class λ of any line satisfies

$$\lambda^2 = 2 - d, \lambda\omega = 1. \tag{4.c.4}$$

From the Hodge index theorem it follows that the number of solutions $\lambda \in H^2(S, \mathbb{Z})$ to the equation (4.c.4) is finite. Since $h^0(\mathcal{O}_S(\Lambda)) = 1$ our claim follows.) The proof of (4.c.3) gives that

$$\begin{aligned}\pi_2^{-1}(\Lambda) &= \{\text{set of surfaces } S \supset \Lambda\}\\ &= W_\Lambda\end{aligned}$$

has codimension $d+1$ in $U_d \times G$. Putting these two observations together yields

$$\begin{aligned}\operatorname{codim} W &= \operatorname{codim} \pi_1(I)\\ &= \operatorname{codim} I - \dim G.\\ &= d - 3.\end{aligned}$$

For our infinitesimal purposes we need the

PROPOSITION: *I is smooth* (*including reduced*). (4.c.5)

PROOF: We begin by identifying tangent spaces. Assuming that $S \in U_d$ is given by

$$F_0(x) = F_0(x_0, x_1, x_2, x_3) = 0,$$

tangent vectors $F_1 \in T_S(U_d)$ may be considered as variations $F_0(x) + tF_1(x)$ (i.e., $T_S(U_d) \cong \operatorname{Sym}^d\mathbb{C}^4/\mathbb{C} \cdot F_0$). Next, we also think of G as 2-planes in $\mathbb{C}^4$ and use the standard isomorphism

$$T_\Lambda(G) \cong \operatorname{Hom}(\Lambda, \mathbb{C}^4/\Lambda).$$

Explicitly, if Λ is given parametrically by

$$x : \mathbb{P}^1 \to \mathbb{P}^3$$

where

$$x(s_0, s_1) = [s_0, s_1, 0, 0],$$

then any arc $\{\Lambda_t\}$ in G with $\Lambda_0 = \Lambda$ and with tangent vector ϕ corresponding to the 2×2 matrix $\|\phi_{i,2+j}\|$ $(i, j = 0, 1)$ is given parametrically by

$$x(t,s) = \left[s_0 + t\lambda_0(s), s_1 + t\lambda_1(s), t(s_0\phi_{02} + s_1\phi_{12}),\right.$$
$$\left. t(s_0\phi_{03} + s_1\phi_{13})\right] + (\dots)$$

where $(\dots)$ are higher order terms in t and λ_0, λ_1 are linear functions of s. The condition that

$$\zeta = (F_1, \phi) \in T_{(S,\Lambda)}(U_d \times G)$$

be tangent to I is

$$F_t(x(t,s)) \equiv 0 \bmod t^2. \tag{4.c.6}$$

Setting

$$\phi(x(s))_\alpha = \left.\frac{\partial x_\alpha(t,s)}{\partial t}\right|_{t=0}$$

(4.c.6) is

$$F_1(x(s)) + \sum_{\alpha=2}^{3} \frac{\partial F_0}{\partial x_\alpha}(x(s))\phi(x(s))_\alpha = 0. ^{(8)} \tag{4.c.7}$$

Equivalently, we consider the map

$$\psi : T_{(S,\Lambda)}(U_d \times G) \to H^0(\mathcal{O}_\Lambda(d))$$

given by sending (F_1, φ) to the polynomial on the left-hand side of (4.c.7). It is clear that ψ is surjective and the sequence

$$0 \to T_{(S,\Lambda)}(I) \to T_{(S,\Lambda)}(U_d \times G) \to H^0(\mathcal{O}_\Lambda(d)) \to 0$$

is exact (for *any* $S \in |\mathcal{O}_{\mathbb{P}^3}(d)|$). This proves that the Zariski tangent space to I is everywhere of codimension $d + 1$, which establishes (4.c.5). Q.E.D.
We will next show that

PROPOSITION: *The map $\pi_1 : I \to U_d$ is everywhere of maximal rank. Thus W is the union of smooth branches each of codimension $d - 3$.* (4.c.8)

PROOF: The differential

$$(\pi_1)_* : T_{(S,\Lambda)}(I) \to T_S(U_d)$$

is given by

$$(F_1, \phi) \to F_1 \tag{4.c.9}$$

where (F_1, ϕ) satisfies (4.c.7). The kernel of (4.c.9) is given by fields of tangent vectors to $\mathbb{P}^3$

$$\xi = \sum_{\alpha=0}^{3} \phi(x(s))_\alpha \partial/\partial x_\alpha$$

defined along Λ and satisfying

$$\xi \cdot F = \langle dF, \xi \rangle = 0$$

along Λ. Thus ξ is tangent to S along Λ and gives a section $\xi \in H^0(\mathcal{O}_\Lambda(N_{\Lambda/S}))$ of the normal bundle to Λ in S. Since $N_{\Lambda/S} \cong \mathcal{O}_\Lambda(2-d)$ we have $h^0(\mathcal{O}_\Lambda(N_{\Lambda/S})) = 0$ for $d \geq 3$. Q.E.D.

REMARK: As indicated in Section 4(a), the same result holds if we replace Λ by any smooth curve $C \subset \mathbb{P}^3$ with $h^1(N_{C/\mathbb{P}^3}) = 0$, G by an irreducible Zariski neighborhood Ξ in the Chow variety corresponding to C, and W by smooth surfaces of sufficiently high degree passing through some $C' \in \Xi$. Moreover, if we assume that Ξ is smooth (scheme-theoretically), then we may drop the condition $h^1(N_{C/\mathbb{P}^3}) = 0$.

Hodge theoretic study. Given a smooth surface $S \in U_d$ containing a line Λ, we denote by V a neighborhood (in the analytic topology) of S in U_d and by V_Λ the subvariety of surfaces $S' \in V$ that contain a line Λ' close to Λ. More precisely, V_Λ is the intersection of V with the branch of $W = \pi_1(I)$ determined by Λ. By (4.c.8), $V_\Lambda \subset V$ *is a smooth submanifold of codimension* $d-3$.

We may assume that the $S' \in V$ are diffeomorphic to S by a diffeomorphism preserving the hyperplane class, and we denote by $\lambda' \in H_2(S', \mathbb{Z})$ the class corresponding to the fundamental class λ of Λ. The condition that λ' be of type (1, 1) defines the following subscheme $V_\lambda \subset V$:

$$V_\lambda = \{S' \in V : \langle \psi, \lambda' \rangle = 0 \quad \text{for all} \quad \psi \in H^{2,0}(S')\}. \tag{4.c.10}$$

By Theorem (4.b.2) we have

$$\text{support } V_\lambda = V_\Lambda.$$

PROPOSITION: *The schemes V_λ and V_Λ coincide.* (4.c.11)

Assuming the proposition we shall complete the proof of Theorem (4.c.1), and then we shall give the proof of (4.c.11).

Proof of Theorem (4.c.1): We first note the

LEMMA: *The codimension of $H^0(S, K_S(-\Lambda))$ in $H^0(S, K_S)$ is $d-3$.* (4.c.12)

PROOF: This follows from

$$\begin{cases} H^0(S, K_S) \cong H^0(\mathcal{O}_{\mathbb{P}^3}(d-4)) \\ H^0(S, K_S(-\Lambda)) \cong H^0(\mathcal{O}_{\mathbb{P}^3}(d-4) \otimes I_\Lambda) \end{cases}$$

and the exact cohomology sequence of

$$0 \to \mathcal{O}_{\mathbb{P}^3}(d-4) \otimes I_\Lambda \to \mathcal{O}_{\mathbb{P}^3}(d-4) \to \mathcal{O}_\Lambda(d-4) \to 0.$$

Q.E.D. for Lemma.

Recall that $\{H_{\mathbb{Z}}, H^{p,q}, Q, T, \delta\}$ denotes the infinitesimal variation of Hodge structure on $H_{\mathbb{Z}} = H^2(S, \mathbb{Z}) \cap H^2_{\text{prim}}(S)$ with tangent space

$$T = T_S(U_d) \cong H^0(\mathcal{O}_S(d)).$$

The Zariski tangent space to the scheme V_λ given by (4.c.10) is (cf. (4.a.3)

$$T_S(V_\lambda) = \{\xi \in T : \langle \delta(\xi)\psi, \lambda \rangle = 0 \qquad \text{for all} \quad \psi \in H^{2,0}\}.$$

From the definition (4.a.1) it follows that $T_S(V_\lambda)$ *has codimension* $h^{2,0} - h^{2,0}(-\lambda)$ *in* $T_S(V)$ (i.e., of the $h^{2,0}$ equations that define $T_S(V_\lambda)$, exactly $h^{2,0} - h^{2,0}(-\lambda)$ are independent). We then have

$$d-3 = h^0(S, K_S) - h^0(S, K_S(-\Lambda))$$

by (4.c.12)

$$\leqslant h^{2,0} - h^{2,0}(-\lambda) \tag{4.c.13}$$

by (4.a.5)

$$= \operatorname{codim} T_S(V_\lambda)$$

$$= d-3$$

by Proposition (4.c.11). It follows that (4.c.13) must be an equality, i.e.

$$h^0(S, K_S(-\Lambda)) = h^{2,0}(-\lambda),$$

which proves the theorem.

Proof of Proposition (4.c.11): With the notation $N_{X/Y}$ for the normal bundle to X in Y we have exact sequences

$$\begin{cases} \text{(a)} & 0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(\Lambda) \longrightarrow N_{\Lambda/S} \longrightarrow 0 \\ \text{(b)} & 0 \longrightarrow N_{\Lambda/S} \longrightarrow N_{\Lambda/\mathbb{P}} \longrightarrow N_{S/\mathbb{P}} \otimes \mathcal{O}_\Lambda \longrightarrow 0 \qquad (\mathbb{P} = \mathbb{P}^3) \\ \text{(c)} & 0 \longrightarrow N_{S/\mathbb{P}}(-\Lambda) \longrightarrow N_{S/\mathbb{P}} \longrightarrow N_{S/\mathbb{P}} \otimes \mathcal{O}_\Lambda \longrightarrow 0 \end{cases}$$

A piece of the exact cohomology diagram is

$$\begin{tikzcd}
 & & {} & & \\
{} & H^2(\mathcal{O}_S) \arrow[l] & H^1(N_{\Lambda/S}) \arrow[l, "\delta"'] \arrow[u] & H^1(\mathcal{O}_S(\Lambda)) \arrow[l] & \ldots \arrow[l] \\
{} \arrow[r] & H^0(N_{S/\mathbb{P}}) \arrow[r, "\beta"] \arrow[u] & H^0(N_{S/\mathbb{P}} \otimes \mathcal{O}_\Lambda) \arrow[r] \arrow[u, "\gamma"'] & \ldots & \\
 & & H^0(N_{\Lambda/\mathbb{P}}) \arrow[u, "\alpha"'] & & \\
 & & {} \arrow[u] & &
\end{tikzcd} \tag{4.c.15}$$

(the top row is the exact cohomology sequence of (a), the vertical row that of (b), and the bottom row that of (c)). We note the interpretations: [9]

$\beta^{-1}(\text{image } \alpha) =$ infinitesimal deformations of S in $\mathbb{P}^3$ such that there is an infinitesimal deformation of $\Lambda \subset \mathbb{P}^3$ that remains in S.

It follows that

$\ker(\gamma \circ \beta) =$ infinitesimal deformations of $S \subset \mathbb{P}^3$ under which Λ is stable (i.e., Λ moves with S).

We note finally the interpretation

$\ker(\delta \circ \gamma \circ \beta) =$ infinitesimal deformations of $S \subset \mathbb{P}^3$ under which λ remains of type (1, 1) (the verification of this requires a computation-c.f. [2]).

Combining the interpretations it follows that the proposition is equivalent to:

$$\ker(\delta \circ \gamma \circ \beta) = \ker(\gamma \circ \beta). \tag{4.c.16}$$

We will prove this by showing that δ is injective (this implies semi-regularity in Bloch's sense).

The dual of the top row of (4.c.15) is the top row of

$$\begin{array}{ccccc} \cdots \longrightarrow H^0(S, K_S) & \xrightarrow{\delta^*} & H^0(N^*_{\Lambda/S} \otimes \Omega^1_\Lambda) & \to H^1(\mathcal{O}_S(\Lambda))^* \to \cdots \\ \wr\| & & \wr\| & \\ H^0(\mathcal{O}_{\mathbb{P}^3}(d-4)) & \xrightarrow{r} & H^0(\mathcal{O}_\Lambda(d-4)) & \quad (r = \text{restriction}), \end{array}$$

(4.c.17)

where we have used that

$$N_{\Lambda/S} \cong \mathcal{O}_\Lambda(2-d).$$

The commutativity of (4.c.17) is again a straightforward verification, and Proposition (4.c.11) follows from the obvious surjectivity of the bottom row in (4.c.17). Q.E.D.

Remark: We may view Proposition (4.c.11) as an infinitesimal analogue of Theorem (4.b.2). However, the proof gives also an infinitesimal analogue of Theorem (4.b.26), as follows: Let $\Lambda \subset \mathbb{P} = \mathbb{P}^3$ be a smooth curve and $S \in |\mathcal{O}_{\mathbb{P}}(d)|$ a smooth surface containing O. Define V_λ and V_Λ as before (using the Chow variety to replace the Grassmannian, as was done in 4.a)). Then: (4.c.18)

If $h^1(I_\Lambda(d-4)) = h^1(I_\Lambda(d)) = 0$, *then proposition* (4.c.11) *is still true.* (4.c.19)

Proof: In the diagram (4.c.15),

$$h^1(I_\Lambda(d)) = 0 \Rightarrow \beta \text{ is surjective.}$$

Moreover, as in (4.c.17) the dual of the top row in (4.c.15) is

$$\cdots \to H^0(\mathcal{O}_{\mathbb{P}}(d-4)) \xrightarrow{\delta^*} H^0(\mathcal{O}_\Lambda(d-4)) \to H^1(\mathcal{O}_S(\Lambda))^* \to \cdots$$

where δ^* is the usual restriction map. Thus, from the exact cohomology sequence of $0 \to I_\Lambda(d-4) \to \mathcal{O}_{\mathbb{P}}(d-4) \to \mathcal{O}_\Lambda(d-4) \to 0$,

$$h^1(I_\Lambda(d-4)) = 0 \Rightarrow \delta^* \text{ is surjective.}$$

$$\Rightarrow \delta \text{ is injective.}$$

Consequently, (4.c.16) holds and the argument is just as before. Q.E.D.

(d) Infinitesimal variation of Hodge structure for Fermat surfaces

We denote by F_d the *Fermat surface*

$$x_0^d + x_1^d + x_2^d + x_3^d = 0 \tag{4.d.1}$$

of degree d in $\mathbb{P}^3$. These surfaces have been the object of considerable recent study (cf. [13], [14], and [15]). Given any smooth surface $S \in |\mathcal{O}_{\mathbb{P}^3}(d)|$ we denote by

$$V(S) = \{H_{\mathbb{Z}}, H^{p,q}, Q, \omega, T, \delta\}$$

the infinitesimal variation of Hodge structure with (i) $H_{\mathbb{Z}} = H^2(S, \mathbb{Z})$; (ii) $H^{p,q} = H^{p,q}(S)$; (iii) $Q =$ cup-product form on $H_{\mathbb{Z}}$; (iv) $\omega = c_1(\mathcal{O}_S(1))$ is the polarizing form; (v) $T \cong H^0(\mathcal{O}_{\mathbb{P}^3}(d))/H^0(\Theta_{\mathbb{P}^3})$ is the tangent space to $|\mathcal{O}_{\mathbb{P}^3}(d)|/PGL$ at S; and (vi) δ is the usual differential of the variation of Hodge structure. We note that $H^2_{\text{prim}}(S, \mathbb{Z}) = \{\psi \in H_{\mathbb{Z}} : Q(\psi, \omega) = 0\}$, and that from $V(S)$ we may construct the usual polarized Hodge structure $\varphi(S)$ associated to $H^2_{\text{prim}}(S, \mathbb{Z})$. We also note that $\delta(\xi)\omega = 0$ for all $\xi \in T$, and that

$$\delta : T \to \operatorname{Hom}\left(H^{2,0}, H^{1,1}_{\text{prim}}\right)$$

is injective. [10] Thus $V(S)$ gives a point $\varphi(S) \in D$, the classifying space for polarized Hodge structures, together with a subspace $T \subset T_{\varphi(S)}(D)$ (more precisely, $V(S)$ gives a point in $\Gamma\backslash D$, and we choose a lift $\varphi(S) \in D$ of this point). In this section we will prove the following

THEOREM: (i) *If $S \in |\mathcal{O}_{\mathbb{P}^3}(d)|$ is a smooth surface of degree $d \geqslant 5$ with*

$$V(S) \cong V(F_d), \tag{4.d.2}$$

then S is projectively equivalent to F_d. (ii) *The automorphisms of $V(F_d)$ are exactly the automorphisms of $\mathbb{P}^3$ that leave F_d invariant.*

The idea is, of course, to use the lines in S. The proof will show quite clearly that just giving the Hodge structure does not seem sufficient to reconstruct the Fermat surface, and we also feel that there are automorphisms of the polarized Hodge structure on $H^2(F_d, \mathbb{Z})$ that are not induced by automorphisms of $\mathbb{P}^3$.

We begin by describing the configuration of lines on F_d.

DEFINITION: A *star* is given by set of d coplanar lines in $\mathbb{P}^3$ all passing through a common point.

Briefly, a star consists of d members of a pencil of lines in $\mathbb{P}^3$. In suitable homogeneous coordinates $[y_0, y_1, y_2, y_3]$ the plane of the star is $y_3 = 0$, the common point is $y_1 = y_2 = y_3 = 0$, and the lines are

$$y_2 = A_\mu y_1 \qquad \mu = 1, \dots, d. \tag{4.d.3}$$

We will say that the star is *special* if we can take

$$A_\mu = \zeta^\mu \qquad \zeta = e^{2\pi\sqrt{-1}/d}. \tag{4.d.4}$$

Intrinsically, the lines through $[1, 0, 0, 0]$ in the plane $y_2 = 0$ form a $\mathbb{P}^1$, and these d points in $\mathbb{P}^1$ should be the orbit of a single point under a cyclic automorphism. It follows that, if $d \geqslant 4$, not every star is special.

It is well known that there are $6d$ special stars of lines on F_d, and that every line belongs to exactly two stars. For example, setting $\xi = e^{\pi\sqrt{-1}/d}$ and fixing an integer μ with $1 \leqslant \mu \leqslant d$, the lines given parametrically by

$$[t_0, t_1] \to [t_0, \xi^\nu t_0, t_1, \xi^\mu t_1]$$

form a star lying in the plane

$$\xi^\mu y_2 - y_3 = 0$$

and passing through the point $[0, 0, 1, \xi^\mu]$. The other stars are obtained by applying the automorphisms of F_d to this one (cf. [13], [15]).

Thus there are $3d^2$ lines on F_d, and two lines meet if, and only if, they belong to the same star. We will denote these lines by $L_{\mu,i}$ where $\mu = 1, \dots, d$ indexes the lines in a star and $i = 1, \dots, 6d$ indexes the stars. With this labelling every line appears twice. We will also denote by $\lambda_{\mu,i} \in H^2(F_d, \mathbb{Z})$ the respective classes of these lines.

Now let $S \subset \mathbb{P}^3$ be a smooth surface with the same polarized Hodge structure as F_d. Thus there is an isomorphism

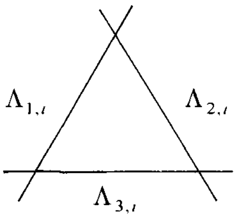
$$\eta : H^2(S, \mathbb{Z}) \xrightarrow{\sim} H^2(F_d, \mathbb{Z}) \tag{4.d.5}$$

that preserves (i) the polarizing class $\omega = c_1(\mathcal{O}(1))$; (ii) the intersection form Q; and (iii) the Hodge decomposition. Using η we will identify the two cohomology groups, and will denote by $\{H_{\mathbb{Z}}, H^{p,q}, Q, \omega\}$ the Hodge structure on $H_{\mathbb{Z}} = H^2(F_d, \mathbb{Z})$.

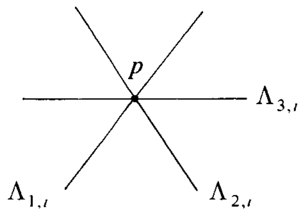
By Theorem (4.b.2) there are lines $\Lambda_{\mu,i} \subset S$ whose fundamental classes are $\lambda_{\mu,i}$. We can even say that the lines

$$\Lambda_{1,i}, \dots, \Lambda_{d,i}, \qquad i \text{ fixed},$$

are coplanar. (*Proof*: First, using the intersection form any two of these lines must intersect in a point. Choose three from among them; say $\Lambda_{1,l}$, $\Lambda_{2,l}$, $\Lambda_{3,l}$. Then these are the possibilities:



(*i*)



(*ii*)

In case (i) it is clear that all three lines lie in a plane. This is also true in case (ii), since they must lie in the tangent plane $T_p(S)$.) In summary, using only the Hodge structure we may say that S contains $3d^2$ lines that fall in $6d$ sets of d coplanar lines, and that every line belong to exactly two of these sets. *But it is not at all clear that each set forms a star, much less a special star.* For this we must use the infinitesimal variation of Hodge structure.

We now assume that $d \geqslant 5$ and that (4.d.5) is induced by an isomorphism of the infinitesimal variations of Hodge structure corresponding to S and F_d. (Equivalently, the period map

$$\varphi : U_d \to \Gamma \backslash D$$

should satisfy the conditions

$$\varphi(S) = \varphi(F_d)$$

$$T_{\varphi(S)}(\varphi(U_d)) = T_{\varphi(F_d)}(\varphi(U_d)).)$$

We may thus identify the spaces (cf. (4.a.1)) $H^{2,0}(-\lambda_{\mu,l})$ for S and F_d. Denote by $\Lambda_{\mu,l}$, $L_{\mu,l}$ the respective lines on S, F_d with fundamental classes $\lambda_{\mu,l}$. Then by Theorem (4.c.1) the isomorphism (4.c.5) induces

$$\eta : H^0\left(S, \mathcal{O}_S(d-4) \otimes I_{\Lambda_{\mu,l}}\right) \xrightarrow{\sim} H^0\left(F_d, \mathcal{O}_{F_d}(d-4) \otimes L_{L_{\mu,l}}\right) {}^{(11)} \tag{4.d.6}$$

(Note: we do not know yet that (4.d.6) is induced by a *linear* automorphism of $\mathbb{P}^3$ – in a certain sense this is the whole point). Using (4.d.6) we will now show how to distinguish between the possibilities (i) and (ii) on S. Although not logically necessary for the proof this will show clearly the use of the infinitesimal variation of Hodge structure. We claim that the possibilities (i) and (ii) correspond respectively to

$$H^{2,0}(-\lambda_{1,t}) + H^{2,0}(-\lambda_{2,t}) + H^{2,0}(-\lambda_{3,t}) = H^{2,0}$$

$$H^{2,0}(-\lambda_{1,t}) + H^{2,0}(-\lambda_{2,t}) + H^{2,0}(-\lambda_{3,t}) \neq H^{2,0}.$$

Proof: In case (i) we may assume that the lines all lie in the plane $y_3 = 0$ and are given by $y_0 = 0$, $y_1 = 0$, $y_2 = 0$ respectively. It is then clear that the union of their ideals consists of all homogeneous forms of positive degree.

In case (ii), all the forms in

$$H^{2,0}(-\lambda_{1,t}) + H^{2,0}(-\lambda_{2,t}) + H^{2,0}(-\lambda_{3,t}) \subset H^0(S, \mathcal{O}_S(d-4))$$

vanish at the common point p. Q.E.D. for the claim.

At this juncture we now know that the $6d$ sets of coplanar lines on S fall into stars, and presumably using the automorphisms of the infinitesimal variation of Hodge structure we could even show that these are special stars. Instead, we will give a direct argument for (i) in Theorem (4.d.2).

For this we consider the intersection

$$\bigcap_{\mu=1}^{d} H^{2,0}(-\lambda_{\mu,t}) = H^{2,0}(-\pi_t) \subset H^0(S, \mathcal{O}_S(d-4)),$$

where the equality is a definition. Since the lines $\Lambda_{\mu,t}$ are coplanar, $H^{2,0}(-\pi_t)$ consists of the forms in $H^0(S, \mathcal{O}_S(d-4)) = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-4))$ that vanish on a plane $\mathbb{P}_t^2 \subset \mathbb{P}^3$. (This is because any form $P \in H^{2,0}(-\pi_t)$ will either vanish on $\mathbb{P}_t^2$ or will cut out a curve Γ of degree $d-4$. The latter is impossible since Γ must contain the d lines $\Lambda_{\mu,t}$.)

We next consider the intersection of $d-5$ of the $H^{2,0}(-\pi_t)$; say

$$\bigcap_{i=1}^{d-5} H^{2,0}(-\pi_t). \tag{4.d.7}$$

The elements in this intersection are all of the form

$$P(x) = L(x) L_1(x) \dots L_{d-5}(x)$$

where $L_t(x)$ defines $\mathbb{P}_t^2$ and $L(x)$ is an arbitrary linear form. In other words, if $V = H^0(S, \mathcal{O}_S(1))^*$ is the four-dimensional vector space such that $S = \mathbb{P}V$, then the intersection (4.d.7) is naturally ismorphic to V^*.

DEFINITION: Given any class $\lambda_{\nu,j}$ that does not correspond to a line in one of the planes $\mathbb{P}_1^2, \ldots, \mathbb{P}_{d-5}^2$, we define

$$\Lambda_{\nu,j}^{\perp} = H^{2,0}(-\lambda_{\nu,j}) \cap \left(\bigcap_{i=1}^{d-5} H^{2,0}(-\pi_i) \right). \tag{4.d.8}$$

By what we have said above, $\Lambda_{\nu,j}^{\perp} \subset V^*$ consists of the linear forms on $\mathbb{P}V$ that vanish on $\Lambda_{\nu,j}$.

Now the isomorphism (4.d.5) induces an isomorphism between the corresponding spaces (4.d.7). Using this isomorphism we have that both S and F_d are embedded in the same $\mathbb{P}^3 = \mathbb{P}V$ and that $\Lambda_{\nu,j} = L_{\nu,j}$. Since there are at most $d(d-5)$ lines in the planes $\mathbb{P}_1^2, \ldots, \mathbb{P}_{d-5}^2$, there are at least

$$3d^2 - (d^2 - 5d) = 2d^2 + 5d \geqslant d^2 + 1$$

lines $\Lambda_{\nu,j} = L_{\nu,j}$ in the intersection $S \cap F_d$. By Bezout's theorem it follows that $S = F_d$ and we are done for part (i) of the theorem.

The proof of part (ii) is similar. Given an automorphism of only the Hodge structure, by Theorem (4.b.2) it induces an automorphism $L_{\mu,i} \to L_{\sigma(\mu),\sigma(i)}$ of the configuration of lines in F_d. Since σ preserves incidence it takes stars into stars, but it is by no means evident that σ is induced by a linear automorphism of $\mathbb{P}^3$.

However, suppose that we have an automorphism of the infinitesimal variation of Hodge structure inducing the permutation $L_{\mu,i} \to L_{\sigma(\mu),\sigma(i)}$ of the configuration of lines. Then there is an induced *linear* transformation

$$\bigcap_{i=1}^{d-5} H^{2,0}(-\pi_i) \to \bigcap_{i=1}^{d-5} H^{2,0}(-\pi_i)$$

of the vector space V^*, where $F_d \subset \mathbb{P}V$. Denote by

$$A : V \to V$$

the contragredient transformation. Then (cf. (4.d.8))

$$A\left(L_{\nu,j}^{\perp}\right) = A\left(L_{\sigma(\nu),\sigma(j)}^{\perp}\right)$$

for all lines $L_{\nu,j}$ not in one of the planes $\mathbb{P}_1^2, \ldots, \mathbb{P}_{d-5}^2$. As in the proof of (i) it follows that $A(F_d) = F_d$. Q.E.D.

(e) Planes contained in hypersurfaces in $\mathbb{P}^5$

When we try to generalize Theorems (4.b.2) and (4.c.1) about lines contained in smooth surfaces in $\mathbb{P}^3$ to projective planes contained in smooth hypersurfaces $X \subset \mathbb{P}^5$, three difficulties arise: (i) since the Hodge conjecture is not known for algebraic 2-cycles on a fourfold, we cannot say that all of $H^4(X, \mathbb{Z}) \cap H^{2,2}(X) = H^{2,2}(X, \mathbb{Z})$ consists of algebraic cycles (modulo torsion questions); (ii) even if we are given a class $\lambda \in H^{2,2}(X, \mathbb{Z})$ represented by an algebraic cycle, there are not yet methods such as used in the proof of Theorem (4.b.2) to determine if λ contains an effective class (obviously (i) and (ii) are related); and (iii) in contrast to $H^0(S, K_S(-\Lambda)) = \{$space of holomorphic 2-forms on $S \subset \mathbb{P}^3$ vanishing on a line $\Lambda \subset S\}$, we do not have an obvious geometric interpretation of $H^1(X, \Omega_X^3 \otimes I_\Lambda)$, where Λ is a 2-plane lying in $X \subset \mathbb{P}^5$.[(12)] On the other hand the fact that, among all surfaces in $\mathbb{P}^5$, planes impose the least number of conditions on hypersurfaces leads us to expect a positive answer to the following

QUESTION: Given a smooth hypersurface $X \subset \mathbb{P}^5$ of degree d with $\omega = c_1(\mathcal{O}_X(1))$, and given a *Hodge plane*; that is, a class $\lambda \in H^{2,2}(X, \mathbb{Z})$ satisfying (4.e.1)

$$\begin{cases} \lambda^2 = 3 - d \\ \lambda \cdot \omega^2 = 1, \end{cases}$$

does there exist a unique projective plane $\Lambda \subset X$ whose fundamental class is λ?

In this section we shall give an affirmative answer to the variational form of (4.e.1), and in so doing shall in this case overcome the difficulty iii) above. Specifically, we shall prove the

THEOREM: *Let X be a smooth hypersurface of degree d in $\mathbb{P}^5$ and $\Lambda \subset X$ be a projective plane. Then equality holds in* (4.*a*.4); *i.e.*,

$$H^1(X, \Omega_X^3(-\Lambda)) = H^{3,1}(-\lambda). \tag{4.e.2}$$

As was the case for lines contained in surfaces, the proof of this theorem will give an affirmative answer to the variational form of (4.e.1) (cf. (4.c.11)).

Let $X \in |\mathcal{O}_{\mathbb{P}^5}(d)|$ be a smooth hypersurface and U a small neighborhood of X in $|\mathcal{O}_{\mathbb{P}^3}(d)|$. We may topologically identify all $X' \in U$ with X, and using this we will make the identification

$$H^4(X', \mathbb{Z}) \cong H^4(X, \mathbb{Z}) \qquad (X' \in U). \tag{4.e.3}$$

Of course (4.e.3) will preserve the hyperplane class and cup-product and therefore induces an isomorphism on primitive cohomology. Given a Hodge class

$$\lambda \in H^{2,2}(X, \mathbb{Z})$$

we let $\lambda' \in H^4(X', \mathbb{Z})$ be the class corresponding under (4.e.3) and define, in the obvious natural way, a subscheme $U_\lambda \subset U$ whose support is

$$\operatorname{supp} U_\lambda = \{X' \in U : \lambda' \text{ is of type } (2,2)\}$$
$$= \{X' \in U : Q(\psi, \lambda') = 0 \text{ for all } \psi \in F^3H^4(X')\}.$$

The Zariski tangent space to U_λ at X is given by (cf. (4.a.3))

$$T_X(U_\lambda) = \{\xi \in T_X(U) : Q(\delta(\xi)\psi, \lambda) = 0 \text{ for all } \psi \in H^{3,1}(X)\} \tag{4.e.4}$$

where

$$\delta : T_X(U) \to \operatorname{Hom}(H^{3,1}(X), H^{2,2}(X))$$

is the differential of the variation of Hodge structure.

Now suppose X contains a 2-plane Λ with fundamental class λ. We may define a subscheme

$$U_\Lambda \subset U$$

whose support is

$$\operatorname{supp} U_\Lambda = \{X' \in U : X' \text{ contains a 2-plane } \Lambda' \text{ close to } \Lambda\}.$$

To justify this we need to give a discussion parallel to that for lines in surfaces in section 4(c), but since this is completely analogous to the previous case we will not give the details.

As in the proof of Theorem (4.c.1), a major step in the proof of (4.e.2) is given by the

PROPOSITION: *The schemes U_λ, U_Λ are smooth, and in fact*

$$U_\lambda = U_\Lambda. \tag{4.e.5}$$

PROOF: In general we suppose that we have three compact, complex manifolds

$$\Lambda \subset X \subset P$$

and we set

$$N_\Lambda = \text{normal bundle of } \Lambda \text{ in } P$$

$$N_X = \text{normal bundle of } X \text{ in } P$$

$$N_{\Lambda/X} = \text{normal bundle of } \Lambda \subset X.$$

Then we have the pair of exact sequences

$$\begin{cases} 0 \to N_{\Lambda/X} \longrightarrow N_\Lambda \to N_X \otimes \mathcal{O}_\Lambda \to 0 \\ 0 \to N_X \otimes I_\Lambda \to N_X \to N_X \otimes \mathcal{O}_\Lambda \to 0. \end{cases} \tag{4.e.6}$$

A piece of resulting cohomology diagram is (where δ will be defined below)

$$\begin{tikzcd} & {} \\ H^{m+1}(X, \Omega_X^{m-1}) & H^1(\Lambda, N_{\Lambda/X}) \arrow[l, "\delta"'] \arrow[u] \\ H^0(X, N_X) \arrow[r, "\beta"] & H^0(\Lambda, N_X \otimes \mathcal{O}_\lambda) \arrow[u, "\gamma"'] \\ & H^0(\Lambda, N_\Lambda) \arrow[u, "\alpha"'] \end{tikzcd} \tag{4.e.7}$$

As before we have the interpretation:

ker $\gamma \circ \beta$ = infinitesimal deformations of $X \subset P$ under which Λ is stable (P is fixed). (4.e.8)

We will define δ as the dual of a natural mapping

$$H^{m-1}\left(X, \Omega_X^{m+1}\right) \to H^{m-1}\left(\Lambda, \Omega_X^m \otimes N_{\Lambda/X}^*\right) \tag{4.e.9}$$

where we assume that

$$\begin{cases} \dim X = 2m \\ \dim \Lambda = m. \end{cases}$$

The mapping (4.e.9) is derived from the 1st two cohomology sequences of the diagram

$$\begin{cases} 0 \to \Omega_X^{m+1} \otimes I_\Lambda \to \Omega_X^{m+1} \to \Omega_X^{m+1} \otimes \mathcal{O}_\Lambda \longrightarrow 0 \\ 0 \to \mathcal{F}_1 \longrightarrow \Omega_X^{m+1} \otimes \mathcal{O}_\Lambda \to \Omega_\Lambda^m \otimes N_{\Lambda/X}^* \longrightarrow 0 \\ 0 \to \mathcal{F}_2 \longrightarrow \mathcal{F}_1 \longrightarrow \Omega_\Lambda^{m-1} \otimes \Lambda^2 N_{\Lambda/X}^* \to 0 \\ \qquad \vdots \end{cases}$$

Again, as before we have the interpretation (cf. Bloch [2]):

ker $\delta \circ \gamma \circ \beta$ = infinitesimal deformations of $X \subset P$ under which the fundamental class of Λ remains of type (m, m). (4.e.10)

Comparing (4.e.8) and (4.e.10), Proposition (4.e.5) is equivalent to

$$\ker \delta \circ \gamma \circ \beta = \ker \gamma \circ \beta \tag{4.e.11}$$

Note: The referee points out that β and γ are surjective under the hypothesis

$$\begin{cases} H^0(N_X) \to H^0(N_X \otimes \mathcal{O}_\Lambda) \to 0 \\ H^1(N_\Lambda) = 0; \end{cases} \tag{$*$}$$

moreover, $(*)$ is satisfied for $\Lambda \subset X^4 \subset \mathbb{P}^5$. Thus $\ker(\gamma\beta) = \ker(\delta\gamma\beta)$ *if and only if* δ is injective; i.e., $\Lambda \in X$ is *semi-regular* in the sense of Bloch (loc. cit.) So our proof of (4.e.5) reduces to the following residue-theoretic computation establishing the semi-regularity of $\Lambda \cong \mathbb{P}^2$ in a smooth hypersurface $X \subset \mathbb{P}^5$.

To prove (4.e.11) we introduce coordinates $[x, y] = [x_0, x_1, x_2, y_0, y_1, y_2]$ in P^5 so that Λ is given by $y_0 = y_1 = y_2 = 0$ and X by $F(x, y) = 0$. For any form $P(x, y)$ we denote by $P(x, 0)$ the restriction of $P(x, y)$ to Λ. The first sequence in (4.e.6) is (using $N_X \otimes \mathcal{O}_\Lambda \cong \mathcal{O}_\Lambda(d)$ and $N_\Lambda \cong \oplus^3 \mathcal{O}_\Lambda(1)$)

$$0 \to N_{\Lambda/X} \to \bigoplus^3 \mathcal{O}_\Lambda(1) \xrightarrow{\alpha} \mathcal{O}_\Lambda(d) \to 0$$

where, for

$$L(x) = \bigoplus_{j=0}^{2} L_j(x) \in \bigoplus^3 H^0(\mathcal{O}_\Lambda(1)),$$

$$\alpha(L(x)) = \sum_{j=0}^{2} \frac{\partial F}{\partial y_j}(x, 0) L_j(x). \tag{4.e.12}$$

By the discussion in Section 3(b) of [3] we have (using the notations there)

$$\begin{cases} \bigoplus^3 H^0(\Lambda, \mathcal{O}_\Lambda(1)) \xrightarrow{\alpha} H^0(\Lambda, \mathcal{O}_\Lambda(d)) \to H^1(\Lambda, N_{\Lambda/X}) \\ \text{Res}\colon S_{2d-6}/J_{F,2d-6} \xrightarrow{\sim} H^1(X, \Omega_X^3) \qquad (\textit{Notation}\colon \text{Res}\, P = \omega_P) \end{cases}$$

$$\tag{4.e.13}$$

The main computational step in the proof of (4.e.5) is provided by the following lemma, whose proof is entirely analogous to the main computation in [4] and will therefore be omitted.

LEMMA: *Taking $m = 2$ in (4.e.7) and making the identification $H^0(\Lambda, N_X \otimes \mathcal{O}_\Lambda) = H^0(\Lambda, \mathcal{O}_\Lambda(d))$, for $P(x, y) \in S_{2d-6}$ and $Q(x) \in H^0(\Lambda, \mathcal{O}_\Lambda(d))$,*

$$\langle \delta\gamma Q, \omega_P \rangle = \operatorname{Res}\left\{ \frac{P(x,0)Q(x)\mathrm{d}x_0 \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2}{\dfrac{\partial F}{\partial y_0}(x,0)\dfrac{\partial F}{\partial y_1}(x,0)\dfrac{\partial F}{\partial y_2}(x,0)} \right\} \tag{4.e.14}$$

Here the left-hand side is the duality pairing

$$H^3(X, \Omega_X^1) \otimes H^1(X, \Omega_X^3) \to \mathbb{C},$$

and the right-hand side is the Grothendieck residue symbol.

It follows from (4.e.14) and the local duality theorem (cf. Chapter V of [7]) that

$$\delta\gamma Q = 0 \Leftrightarrow Q \in \left\{ \frac{\partial F}{\partial y_0}(x,0), \frac{\partial F}{\partial y_1}(x,0), \frac{\partial F}{\partial y_2}(x,0) \right\}.$$

By (4.e.12) this is exactly the condition that $Q \in$ image α, which proves (4.e.11) and therefore also Proposition (4.e.5).

We may now complete the proof of Theorem (4.e.2). By the proof of the proposition (cf. (4.e.10)) we have

$$H^{3,1}(X, -\gamma) = \operatorname{image}(\delta \circ \gamma \circ \beta)^{\perp}.$$

By Lemma (4.e.14) and the local duality theorem,

$$\begin{aligned} &\omega_P \in H^{3,1}(-\gamma) \\ &\quad \Leftrightarrow P(x,0) \in \left\{ \frac{\partial F}{\partial y_0}(x,0), \frac{\partial F}{\partial y_1}(x,0), \frac{\partial F}{\partial y_2}(x,0) \right\}. \end{aligned}$$

This is clearly equivalent to

$$P(x, y) \in J_{F, 2d-6} + H^0(\mathcal{O}_{\mathbb{P}^5}(2d-6) \otimes I_\Lambda).$$

To complete the proof of the theorem we need only observe that the proof of Theorem (3.b.7) in [3] gives that the residue mapping induces

$$\operatorname{Res}: H^0(\mathcal{O}_{\mathbb{P}^5}(2d-6) \otimes I_\Lambda) \to H^1(X, \Omega_X^3 \otimes I_\Lambda). \qquad \text{Q.E.D.}$$

The method of proof also gives the

PROPOSITION: *With the notations of Theorem (4.e.2),*

$$H^0(X, K_X \otimes I_\Lambda) = H^{4,0}(-\lambda). \tag{4.e.15}$$

PROOF: If $P \in S_{d-6}$ has residue

$$\omega_P \in H^{4,0}(-\lambda),$$

then by the proof of Theorem (4.e.2) we have

$$P(x,0)Q(x,0) \in \left\{ \frac{\partial F}{\partial y_0}(x,0), \frac{\partial F}{\partial y_1}(x,0), \frac{\partial F}{\partial y_2}(x,0) \right\}$$

for all $Q(x, y) \in S_d$. By the local duality theorem and fact that deg $P = d - 6 < d - 1$ we conclude that $P(x, 0) \equiv 0$. Thus $P \in H^0(\mathcal{O}_{\mathbb{P}^5}(d-6) \otimes I_\Lambda)$ and $\omega_P \in H^0(X, K_X \otimes I_\Lambda)$. Q.E.D.

COROLLARY: *If $d \geqslant 7$, then the plane Λ is the base locus of the linear system $H^{4,0}(-\lambda) \subset H^0(X, K_X)$.* (4.e.16)

Once again, the equations of the algebraic cycle Λ (assumed to exist) are given by purely Hodge-theoretic data.

Put differently, for $d \geqslant 7$ the base locus of $\mathbb{P}H^{4,0}(-\lambda) \subset \mathbb{P}H^0(X, K_X)$ gives the *unique candidate* for the equations of Λ. In particular, if Λ exists it is unique. The trouble with using $H^{4,0}(-\lambda)$ to show the existence of Λ is the *arithmetic* properties of the Hodge class λ must somehow be used. In Part III of this series of papers we will give a general way of constructing a candidate for equations of a cycle that does use these arithmetic properties (to construct a global normal function). Of course, we have no idea how viable this candidate is.

Note: Suppose that $X \subset \mathbb{P}^5$ is a smooth hypersurface of degree d containing a pair Λ_1, Λ_2 of skew 2-planes and set

$$\Lambda = \Lambda_1 + \Lambda_2. \tag{4.e.17}$$

Then the fundamental class of Λ is

$$\lambda = \lambda_1 + \lambda_2$$

where $\lambda_j \in H^{2,2}(X, \mathbb{Z})$ is the fundamental class of Λ_j. We note that if $\omega = c_1(\mathcal{O}_X(1))$, then λ_1, λ_2, ω^2 are linearly independent. We claim that:

Theorem (4.e.2) remains valid when Λ is given by (4.e.17). (4.e.18)

PROOF: We may define

$$U_\Lambda = \{X' \in U \colon X' \text{ contains a pair of}$$
$$\text{2-planes } \Lambda_1', \Lambda_2' \text{ close to } \Lambda_1, \Lambda_2\}$$

and U_λ as before, and then the first step is to verify (4.e.5). Choose coordinates $[x_0, x_1, x_2, y_0, y_1, y_2]$ so that

$$\begin{cases} \Lambda_1 = \{[x, 0] : x \in \mathbb{P}^2\} \\ \Lambda_2 = \{[0, y] : y \in \mathbb{P}^2\}. \end{cases}$$

Then the 1st sequence in (4.e.6) is

$$0 \to N_{\Lambda/X} \to \left(\overset{3}{\oplus}\, \mathcal{O}_{\Lambda_1}(1)\right) \hat{\oplus} \left(\overset{3}{\oplus}\, \mathcal{O}_{\Lambda_2}(1)\right) \overset{\alpha}{\to} \mathcal{O}_{\Lambda_1}(d) \hat{\oplus} \mathcal{O}_{\Lambda_1}(d) \to 0$$

where, for $\mathcal{F}_i$ a sheaf on Λ_i, we denote by $\mathcal{F}_1 \hat{\oplus} \mathcal{F}_2$ the corresponding sheaf on Λ (thus, $H^0(\mathcal{F}_1 \hat{\oplus} \mathcal{F}_2) \cong \mathrm{H}^0(\mathcal{F}_1) \oplus H^0(\mathcal{F}_2)$). To describe α we remark that for

$$(L_1(x), L_2(y)) = \left(\overset{3}{\oplus}\, L_{1,j}(x)\right) \oplus \left(\overset{3}{\oplus}\, L_{2,j}(y)\right)$$

in $H^0((\overset{3}{\oplus}\, \mathcal{O}_{\Lambda_1}(1)) \hat{\oplus} (\overset{3}{\oplus}\, \mathcal{O}_{\Lambda_2}(1)))$,

$$\alpha(L_1(x), L_2(y)) = \left(\sum_j \frac{\partial F}{\partial y_j}(x, 0) L_{1,j}(x)\right)$$
$$\oplus \left(\sum_j \frac{\partial F}{\partial x_j}(0, y) L_{2,j}(y)\right)$$

in $H^0(\mathcal{O}_{\Lambda_1}(d) \hat{\oplus} \mathcal{O}_{\Lambda_2}(d))$. The analogue of (4.e.14) is (this is the main step in the proof):

$$\langle \delta\gamma Q, \omega_p \rangle = \mathrm{Res}\left\{ \frac{P(x, 0) Q_1(x) \mathrm{d}x}{F_0(x, 0) F_1(x, 0) F_2(x, 0)} \right\}$$
$$+ \mathrm{Res}\left\{ \frac{P(0, y) Q_2(y) \mathrm{d}y}{F_0(0, y) F_1(0, y) F_2(0, y)} \right\} \tag{4.e.19}$$

where $Q = Q_1(x) \oplus Q_2(y) \in H^0(\mathcal{O}_{\Lambda_1}(d) \hat{\oplus} \mathcal{O}_{\Lambda_2}(d))$, $F_j(x, 0) =$

$\partial F/\partial y_j(x, 0)$ and $F_j(0, y) = \partial F/\partial x_j(0, y)$, and $dx = dx_0 \wedge dx_1 \wedge dx_2$, $dy = dy_0 \wedge dy_1 \wedge dy_2$. It follows that

$$\delta\gamma Q = 0 \Leftrightarrow \begin{cases} Q_1(x) \in \{F_0(x,0), F_1(x,0), F_2(x,0)\}, \text{ and} \\ Q_2(y) \in \{F_0(0,y), F_1(0,y), F_2(0,y) \end{cases}$$

$$\Rightarrow Q = \alpha(L_1(x), L_2(y))$$

To complete the proof of (4.e.18) we have again from (4.e.19) that

$$\omega_p \in H^{3,1}(-\lambda) \Leftrightarrow \begin{cases} P(x,0) \in \{F_0(x,0), F_1(x,0), F_2(x,0)\} \\ P(0,y) \in \{F_0(0,y), F_1(0,y), F_2(0,y)\} \end{cases}$$

$$\Leftrightarrow P(x,y) \in J_{F,2d-6} + H^0(\mathcal{O}_{\mathbb{P}^5}(2d-6) \otimes I_\Lambda);$$

the remainder of the argument is just as before.

COROLLARY: *Under any infinitesimal deformation of* $X \subset \mathbb{P}^5$ *such that* $\lambda = \lambda_1 + \lambda_2$ *remains a Hodge class, both* λ_1 *and* λ_2 *also remain Hodge classes.* (4.e.20)

This again illustrates how the variable Hodge decomposition meets the integral lattice in a very subtle manner.

(f)

As an application of the infinitesimal Max Noether theorem and scheme-theoretic interpretation (4.a.3) of our infinitesimal invariant $H^{m+k,m-k}(-\gamma)$, we will give a criterion for when the normal bundle of a smooth curve $C \subset \mathbb{P}^3 = P$ is indecomposable. Suppose, in fact, that we have

$$C \subset S \subset P$$

where S is a smooth surface, and consider the normal bundle sequence

$$0 \to N_{C/S} \to N_{C/P} \xrightarrow{\alpha} N_{S/P} \otimes \mathcal{O}_C \to 0. \tag{4.f.1}$$

Note that

$$N_{C/S} \cong \Omega^1_C(-n+4) \tag{4.f.2}$$

where $\deg S = n$. We will prove the

THEOREM: *The sequence* (4.f.1) *splits if, and only if,* $C = S \cap R$ *is a complete intersection.* (4.f.3)

COROLLARY: *If C has degree d and genus g, and if*

$$g \geq d(n-2)+1, \tag{4.f.4}$$

then the normal bundle $N_{C/P}$ is indecomposable.

PROOF OF COROLLARY: We first note that if N', N'' are any two distinct line sub-bundles of a rank 2 vector bundle $N \to C$, then the obvious map

$$N' \oplus N'' \to N$$

is surjective on a general fibre, and hence

$$c_1(N) \geq c_1(N') + c_1(N'')$$

with equality holding if, and only if,

$$N = N' \oplus N''.$$

It follows that if $M \subset N$ is a line sub-bundle with $c_1(M) \geq \frac{1}{2}c_1(N)$, then N is decomposable if, and only if, the sequence

$$0 \to M \to N \to N/M \to 0 \tag{4.f.5}$$

splits. Indeed, if $N = N' \oplus N''$ with say $c_1(N') \geq c_1(N'')$, then $c_1(N') \geq c_1(N/M)$ and either the map $N' \to N/M$ is an isomorphism or else it is zero, in which case $N' \to M$ is an isomorphism. In either alternative, (4.f.5) splits.

Applying this observation to (4.f.1) and using (4.f.2), we obtain the corollary. Q.E.D.

EXAMPLE: If $n \geq 2$ and $\Lambda \subset S$ is a line, then for sufficiently large m we may find surfaces $R \in |\mathcal{O}_{\mathbb{P}^3}(m)|$ such that (4.f.6)

$$S \cap R = C + \Lambda$$

where C is smooth. From (4.b.5) it follows that the normal bundle to $C \subset \mathbb{P}^3$ is indecomposable.

EXAMPLE: In particular, when $n = 2$ it follows easily that *any* irrational curve, other than a complete intersection, on a quadric has an indecomposable normal bundle. (4.f.7)

PROOF OF THEOREM (4.f.3): We first treat the case

$$n \geqslant 4,$$

as this is an immediate consequence of (4.c.20) and the infinitesimal Max Noether theorem (3.a.16) in [3]. To see this, we look at the maps

$$\begin{array}{ccc} N_{C/P} & \xrightarrow{\alpha} & N_{S/P} \otimes \mathcal{O}_C \\ & & \uparrow \beta \\ & & N_{S/\mathbb{P}} \end{array}$$

where α is an in (4.f.1) and β is the obvious restriction. By the general principle (4.c.20)[9], a first order deformation $\tilde{S}$ of S corresponding to a section $\tau \in H^0(N_{S/\mathbb{P}})$ will contain the first order deformation $\tilde{C}$ of C corresponding to $\sigma \in H^0(N_{C/\mathbb{P}})$ if and only if $\beta(\tau) = \alpha(\sigma)$. But at the same time, the infinitesimal Max Noether theorem asserts that, in case $n \geqslant 4$ and C is not a complete intersection on S, there exist first-order deformations of S that contain no first-order deformations of C. We conclude, then, that *the map α is not surjective on global sections*, and hence that the sequence (4.f.1) does not split; this proves the theorem in case $n \geqslant 4$ (cf. remark (4.f.13) below).

The remaining cases $n = 2$, 3 of the theorem are naturally more delicate, inasmuch as a deformation of a smooth quadric or cubic surface will preserve the Picard number (and indeed, the map α *will* be surjective on global sections in these cases).

The key idea here will be to look at the base locus E of a first-order deformation of the surface S, and at the divisor classes cut out on E by a first-order deformation of the curve $C \subset S$. We will show: (i) if C is not a complete intersection with S, then this divisor class must indeed vary as we deform C; and (ii) if the sequence (4.f.1) splits, we can make this divisor class constant.

To make this precise in case $n = 2$, let S be a quadric in $\mathbb{P}^3$, given by the polynomial Q_0. Let $T = \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2)$, and let the subscheme

$$\tilde{S} \subset \mathbb{P}^3 \times T$$

be the first-order deformation of S given by the equation

$$Q_0 + \varepsilon Q_1 = 0$$

for some quadric Q_1. We may assume that the quadrics Q_0 and Q_1 meet transversely, so that the curve $E \subset \mathbb{P}^3$ given by $Q_0 = Q_1 = 0$ is a smooth, irreducible elliptic quartic curve; note that if $\pi : \mathbb{P}^3 \times T \to \mathbb{P}^3$ is the projection, then

$$\tilde{E} =: E \times T = \pi^{-1}(E) \subset \tilde{S}.$$

Now, let M_1 and $M_2 \subset S$ be lines of the two rulings of S, and let $\tilde{L}$ be a line bundle on $\tilde{S}$ whose restriction to S is $\mathcal{O}_S(M_1)$. The restriction $\tilde{L} \otimes \mathcal{O}_{\tilde{E}}$ of $\tilde{L}$ to $\tilde{E} = E \times T$ is then a first-order deformation of the line bundle $\mathcal{O}_E(M_1)$, and so gives, via the induced map $T \xrightarrow{\phi} \operatorname{Pic} E$, a tangent vector

$$v = \phi_* \frac{\partial}{\partial \varepsilon}$$

to Pic E. The first half of our argument is now expressed in the

LEMMA: $v \neq 0$ (4.f.9)

PROOF: We prove this by globalizing. Let $\{Q_\lambda\}_{\lambda \in \mathbb{P}^1}$ be the pencil of quadrics generated by Q_0 and Q_1, and let F be the curve consisting of pairs (λ, M) where $\lambda \in \mathbb{P}^1$ and M is the divisor class of a line on Q_λ; let

$$\bar{\phi}: F \to \operatorname{Pic}^2(E)$$

be the map given by

$$\bar{\phi}(\lambda, M) = M|_E.$$

Since F is an elliptic curve – it is the double cover of $\mathbb{P}^1$ branched at the four values of λ corresponding to the singular quadrics in the pencil † – and non-constant, we conclude that $\bar{\phi}$ *is an isogeny*. In fact, $\bar{\phi}$ is an isomorphism, but that is irrelevant; for our present purposes what is important is the fact that *the differential of* $\bar{\phi}$ *is never zero*; since ϕ is the restriction of $\bar{\phi}$ to $T \subset F$, this establishes the lemma.

Lemma (4.f.9) asserts that as we vary the quadric Q_0 (to first order) in the pencil $\overline{Q_0Q_1}$, the divisor class cut on $E = Q_0 \cap Q_1$ by a line on Q_0 (and hence by any linear combination $aM_1 + bM_2$ with $a \neq b$) *must* vary. We claim now that if the sequence (4.f.1) splits for a curve $C \subset S$ linearly equivalent to $aM_1 + bM_2$, $a \neq b$, then we obtain a contradiction.

To see this, suppose that C is such a curve, and that (4.f.1) splits; let Q_0, Q_1 and $\tilde{E} \subset \tilde{S} \subset \mathbb{P}^3 \times T$ be as before. Let $\tau \in H^0(S, N_{S/\mathbb{P}^3})$ be the section corresponding to the deformation $\tilde{S}$ of S. (i.e. a section with zero-divisor $E \subset S$). Then since (4.f.1) splits, there exists a section $\sigma \in H^0(C, N_{C/\mathbb{P}^3})$ whose image in $N_{S/\mathbb{P}^3} \otimes \mathcal{O}_C$ is the restriction of τ to C; and moreover *we may choose* σ *to vanish whenever* $\tau|_C$ *does*; that is, along the intersection $C \cap E = Q_1 \cap C$. Now let $\tilde{C} \subset \mathbb{P}^3 \times T$ be the first-order

† In fact, a pencil of quadrics in $\mathbb{P}^3$ will have exactly four singular elements if and only if its base locus is smooth (as was assumed in the present case).

deformation of C corresponding to σ; by the general principle (4.c.21), we have

$$\pi^{-1}(C \cap E) \subset \tilde{C} \subset \tilde{S}; \tag{4.f.10}$$

note, moreover, that since $\tilde{S}$ is smooth over T, $\tilde{C} \subset \tilde{S}$ is a Cartier divisor. But now the divisor $\tilde{C}$ restricted to $\tilde{E}$ is flat of degree $2 \cdot \deg C$ over T, as is $\pi^{-1}(C \cap E)$; since $\tilde{C} \cap \tilde{E} \supset \pi^{-1}(C \cap E)$, then, it follows from (4.f.10) that

$$\tilde{C} \cap \tilde{E} = \pi^{-1}(C \cap E)$$

Simply put, the divisor cut on E by C does not vary as we vary C. Thus, in particular, the line bundle

$$\mathcal{O}_{\tilde{S}}(C) \otimes \mathcal{O}_{\tilde{E}} = \pi^*(\mathcal{O}_S(C) \otimes \mathcal{O}_E); \text{ i.e.,}$$

the line bundle $\mathcal{O}_{\tilde{S}}(\tilde{C})$ restricted to $\tilde{E}$ is *constant* on $\tilde{E} = E \times T$. If $a \neq b$, however, this contradicts Lemma (4.f.9), and so we may conclude that (4.f.1) does not split.

It remains to treat the case $n = 3$. Our argument here will follow exactly the lines of the case $n = 2$. Indeed, it is clear that the latter half of our argument goes over word for word if Q_0 and Q_1 are cubics rather than quadrics. What we have to analyze, then in case $n = 3$ is how the various divisor classes on a varying cubic surface behave under restriction to the infinitesimal base locus of that variation. Specifically, suppose S is the cubic surface given by the cubic polynomial Q_0, Q_1 a second cubic polynomial cutting a smooth curve E on S, and

$$\tilde{S} = (Q_0 + \varepsilon Q_1) \subset \mathbb{P}^3 \times T$$

the corresponding first-order deformation. Let $L_1, \ldots, L_6 \subset S$ be skew lines (and thus generators, together with the hyperplane class, of Pic S) and let $\tilde{L}_1, \ldots, \tilde{L}_6 \subset \tilde{S}$ be the corresponding deformations of $L_1, \ldots, L_6$. Then if we set, as before, $\tilde{E} = \pi^{-1}(E) \subset \tilde{S}$, the restrictions $\mathcal{O}_{\tilde{S}}(\tilde{L}_i) \otimes \mathcal{O}_{\tilde{E}}$ define first order deformations of the line bundles $\mathcal{O}_S(L_i) \otimes \mathcal{O}_E$ and, correspondingly tangent vectors $v_1, \ldots, v_6$ to Pic(E). In these terms, it is not hard to see that the remaining case $n = 3$ of (4.f.3) will follow, as in the case $n = 2$, from the

LEMMA: *The vectors $v_1, \ldots, v_6$ are linearly independent.* (4.f.11)

PROOF: Let us first check that the vectors v_i are non-zero. This is not hard; since $\tilde{L}_i \neq \pi^{-1}L_i$ (Q_i does not vanish identically on any line $L_i \subset S$), the first order deformation $\tilde{L}_i|_{\tilde{E}}$ of the divisor $L_i|_E$ represents a non-zero tangent vector $\tilde{v}_i$ to the symmetric product E_3 of E. But since C is not trigonal (any three points of E impose independent conditions on the canonical series $|K_E|$, which is cut on E by quadrics in $\mathbb{P}^3$), the natural map $E_3 \overset{u}{\to} \mathrm{Pic}_3(E)$ is an embedding; in particular the differential u_* is everywhere injective and hence $v_i = u_* \tilde{v}_i \neq 0$.

This said, we now want to identify the line in the tangent space $T(\mathrm{Pic}^3(E))$ generated by each v_i or equivalently, via the identifications

$$T(\mathrm{Pic}^3(E)) = H^0(E, \Omega_E^1)^*,$$

$$H^0(E, \Omega_E^1) = H^0(E, \mathcal{O}_E(2)) = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)),$$

the codimension 1 linear system $\Gamma_i \subset |\mathcal{O}_{\mathbb{P}^3}(2)|$ of quadrics which annihilate the tangent vector v_i. To see what Γ_i is, note that if S is the locus of the homogeneous polynomial $F(X)$, then of the 4-dimensional vector space of polars $\{\partial F/\partial X_\alpha\}$ of F, a 2-dimensional subspace vanishes identically on the line L_i. The remaining ones cut on L_i a pencil of what we may call the *polar divisors*; these are just the divisors $p + q$ of degree 2 on L_i such that

$$T_p(S) = T_q(S),$$

or, equivalently, such that there is a plane $H \subset \mathbb{P}^3$ through L_i meeting S in L_i plus a conic curve C with

$$C \cdot L_i = p + q.$$

This said, we claim that *the linear system Γ_i consists exactly of those quadrics which cut a polar divisor on L_i.*

To see this, note first that since the tangent vector v_i is a linear combination of tangent vectors to $E \subset J(E))$ at the points p_i of $L_i \in E$, its annihilator in $H^0(E, \Omega_E^1)$ certainly contains the l-forms on E vanishing at all three points p_i – that is, the quadrics in $\mathbb{P}^3$ containing the line L_i.

The second point needed to establish the claim is more subtle. We observe that since the divisors $D_i = L_i \cdot E$ and $D_{i,\varepsilon} = L_{i,\varepsilon} \cdot E \cdot$ are both colinear – that is, $h^0(E, \mathcal{O}(F - D_i)) = h^0(E, \mathcal{O}(F - D_{i,\varepsilon})) = 2$ where F is the hyperplane divisor on $E \subset \mathbb{P}^3$ – *the tangent vector v_i lies in the tangent space to the subscheme $W_6^1 \subset J(E)$ at the point $F - D_i$*. This tangent space is in turn identified as the annihilator in $H^0(E, \Omega_E^1)^*$ of the image of the tensor product map (cf. [1])

$$H^0(E, \mathcal{O}(F - D_i)) \otimes H^0(E, \Omega_E^1(D_i - F)) \to H^0(E, \Omega_E^1).$$

Dually, then it follows that Γ_i *contains any quadric whose intersection with E contains a divisor of the pencil $|F - D_i|$ on E*. But now any polar divisor

$p+q$ on L_i is, as noted, cut on L_i by a conic $C \subset S$ residual to L_i in the intersection of S with a plane in $\mathbb{P}^3$; and for any such conic C we can certainly find a quadric $Q \subset \mathbb{P}^3$ containing C but not L_i. Since C cuts on E a divisor of the pencil $|F - D_i|$, then, $Q \in \Gamma_i$; and since $Q \supset C$ but $Q \not\supset L$, Q must cut on L_i exactly the divisor $p+q$. Thus Γ_i cuts out on L_i the polar pencil; since it includes as well all quadrics containing L_i our claim is proved.

Having now identified the linear system Γ_i as those quadrics which cut polar divisors on L_i, to show that the original vectors v_i are independent we have to show that

$$\dim \bigcap_i \Gamma_i = 9 - 6 = 3;$$

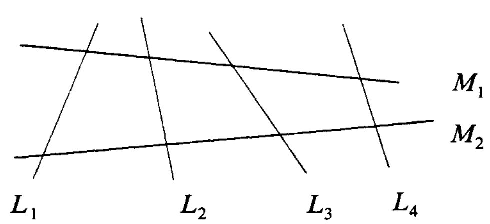
and since we can spot right away a 3-dimensional (sub)series of Γ_i – just the system of polar quadrics of S – this is equivalent to the final

LEMMA: *Any quadric which cuts a polar divisor on each of the lines $L_1, \ldots, L_6$ is in fact a polar quadric of S.*

Now, this statement as it stands appears difficult; we know of no way to establish it directly. We can, however, prove it easily by going back a step or two and transposing it. The point is that since

(i) any linear relation $\Sigma a_i L_i = 0$ in the classes of the lines L_i in $\operatorname{Pic}(S) \cong H^2(S, \mathbb{Z})$ modulo the hyperplane class implies the same relation $\Sigma a_i v_i = 0$ among the vectors v_i; and since

(ii) the classes $L_1, \ldots, L_6$ generate $H^2(S, \mathbb{Z})$ modulo the hyperplane class, we see that *we can replace the lines $L_1, \ldots, L_6$ in Lemma 2 with the six lines on S independent in $H^2(S, \mathbb{Z})$ modulo the hyperplane class.* In particular, we may replace L_5 and L_6 with the two lines M_1, M_2 on S meeting each of L_1, L_2, L_3 and L_4 (cf. [7], p. 486) forming a configuration



(in fact, the classes $L_1, \ldots, L_4$, M_1, M_2 and H form a unimodular basis for $H^2(S, \mathbb{Z})$) and now the Lemma is easy: if there were in fact five independent quadrics in $\mathbb{P}^3$, each cutting a polar divisor on each of the lines L_1, L_2, L_3, L_4, M_1, and M_2, then there would be a non-zero quadric Q containing M_1 and M_2 and cutting a polar divisor on each of $L_1, \ldots, L_4$. But since M_1 and M_2 are skew, the divisor $p_i + q_i$, $p_i = M_1 \cdot L_i$,

$q_t = M_2 \cdot L_t$ on L_t is *not* polar; since Q contains p_t and q_t it follows that Q must contain L_1, L_2, L_3 and L_4. This finally is impossible: a quadric $Q \subset \mathbb{P}^3$ cannot contain four skew lines of intersection with a cubic. Q.E.D.

REMARK: While Corollary (4.f.4) applies to a large range of curves in $\mathbb{P}^3$ – for example, if S is any smooth surface and C any curve on S not a complete intersection, for all m sufficiently large the linear system $|\mathcal{O}_S(C)(m)|$ on S consists of generically smooth curves satisfying the requisite inequality

$$g \geqslant d(n-2)+1, \tag{4.f.12}$$

it does not apply to a general embedding in $\mathbb{P}^3$ of degree d of a *general* curve of genus $g \geqslant 7$: for such a curve, estimates on $h^0(C, \mathcal{O}_C(n))$ and the inequality $g \geqslant d(n-2)+1$ would combine to say that $h^1(C, \mathcal{O}_C(n))$, and hence $h^1(C, N_{C/\mathbb{P}^3})$, were non-zero; but we know the latter vanishes (cf. [6]). Thus the question remains of whether the normal bundle of such a curve is, in general, indecomposable.

REMARK: By comparing Proposition (4.a.7) and the extension (4.c.19) of Proposition (4.c.11) we are not only able to say that the sequence (4.f.1) does not split but, in a rather curious way, are able to measure how *non*-split it is. To explain this, let $C \subset P = \mathbb{P}^3$ be a smooth non-complete intersection curve of genus g and degree m, and let $S \subset P$ be a smooth surface of degree $d \geqslant d(g, m)$ containing C. Using the fact that (4.f.13)

$$\det N_{C/P} = K_C(4)$$

the exact sequence (4.f.1) is

$$0 \to K_C(4-d) \to N_{C/P} \to \mathcal{O}_C(d) \to 0, \tag{4.f.14}$$

and we let

$$e \in H^1(C, K_C(4-2d))$$

be the extension class. Finally we consider the natural pairing

$$H^0(\mathcal{O}_C(d-4)) \otimes H^1(K_C(4-2d)) \to H^1(K_C(-d)),$$

and denote by

$$e: H^0(\mathcal{O}_C(d-4)) \to H^1(K_C(-d))$$

the natural mapping

$$H^0(\mathcal{O}_C(d-4)) \otimes \{e\} \to H^1(K_C(-d)).$$

Then we have the

PROPOSITION: (4.f.15)

$$\dim(\ker e) = h^1(N_{C/P})$$

COROLLARY: $e \neq 0$ *for* d *sufficiently large.* (4.f.16)

PROOF OF COROLLARY: By the Riemann-Roch for C

$$h^0(\mathcal{O}_C(d-4)) = m(d-4) - g + 1 > h^1(N_{C/P})$$

for sufficiently large d. By the proposition we must then have $e \neq 0$. Q.E.D.

PROOF OF PROPOSITION: Let $\gamma \in H^{1,1}(S) \cap H^2(S, \mathbb{Z})$ be the fundamental class of C. By (4.a.7)

$$h^{2,0}(-\gamma) = h^0(K_S(-C)) + h^1(N_{C/P}) \tag{4.f.17}$$

On the other hand, we consider the diagram (4.c.15) (with C replacing Λ). If $F(x) = 0$ is the equation of S, then there is a natural identification

$$H^0(N_{S/P}) \cong H^0(\mathcal{O}_S(d)) \cong H^0(\mathcal{O}_P(d))/\mathbb{C}F,$$

and the diagram gives (with $\tilde{\beta}$ denoting restriction)

$$\begin{array}{ccc} H^2(\mathcal{O}_S) & \xleftarrow{\delta} & H^1(K_C(4-d)) \\ & & \uparrow{\scriptstyle\tilde{\gamma}} \\ H^0(\mathcal{O}_P(d)) & \xrightarrow{\tilde{\beta}} & H^0(\mathcal{O}_C(d)) \end{array}$$

Set $S_k = H^0(\mathcal{O}_P(k))$ and think of elements $Q \in S_d$ as giving the tangent to the deformation $\{F + tQ = 0\}$ of S. Using the duality isomorphism

$$H^2(\mathcal{O}_S)^* \cong S_{d-4}$$

we have

$$\left\{\begin{array}{l}\gamma \text{ remains of type } (1,1) \text{ along the} \\ \text{infinitesimal deformation given by } Q \in S_d\end{array}\right\}$$
$$\Updownarrow$$
$$\langle \delta\tilde{\gamma}\tilde{\beta}Q, P\rangle = 0 \text{ for all } P \in S_{d-4},$$

On the other hand, any $G \in S_k$ gives by restriction an element $g \in H^0(\mathcal{O}_C(k))$ and an easy computation gives

$$\langle \delta\tilde{\gamma}\tilde{\beta}Q, P\rangle = \langle PQ, e\rangle \tag{4.f.18}$$

where $e \in H^1(K_C(4-2d)) \cong H^0(\mathcal{O}_C(2d-4))^*$ is the extension class of (4.f.14). (In fact, the cohomology map $H^0(\mathcal{O}(d)) \to H^1(K_C(4-d))$ is well-known to be given by cup-product with e, and (4.f.18) is an obvious consequence of this.) From (4.f.18) we have

$$\left\{\begin{array}{l}\gamma \text{ remains of type } (1,1) \text{ along the} \\ \text{infinitesimal deformation given by } Q \in S_d\end{array}\right\} \tag{4.f.19}$$
$$\Updownarrow$$
$$\langle Q, Pe\rangle = 0 \text{ for all } P \in S_{d-4}.$$

From (4.f.19) we have

$$\left\{\begin{array}{l}\sharp \text{ conditions that} \\ \gamma \text{ remains of type } (1,1)\end{array}\right\} = h^0(\mathcal{O}_C(d-4)) - \dim(\ker e)$$

$$= h^0(K_S) - h^0(K_S(-C)) - \dim(\ker e)$$

But by (4.f.17) the left hand side is also equal to

$$h^{2,0} - h^{2,0}(-\gamma) = h^{2,0} - h^0(K_S(-C)) - h^1(N_{C/P}),$$

and the proposition follows from these two equations. Q.E.D.

Remark: Theorem (4.f.3) and Proposition (4.f.15) may to some extent be generalized to an arbitrary smooth $\Lambda^m \subset \mathbb{P}^{2m+1}$, as follows: For simplicity we consider the case of a smooth surface

$$\Lambda \subset P = \mathbb{P}^5$$

and let $X \subset P$ be a smooth hypersurface of degree d containing Λ. The analogue of (4.f.1) is

$$0 \to N_{\Lambda/X} \to N_{\Lambda/P} \to \mathcal{O}_\Lambda(d) \to 0, \tag{4.f.21}$$

and using the infinitesimal Max Noether theorem (3.a.16) we have that:

If Λ *is not homologous to a complete intersection, then the sequence* (4.f.21) *does not split for* $d \geqslant 3$. (4.f.22)

PROOF: If it does split, then $\gamma = 0$ in (4.e.7). This then implies that the fundamental class of Λ remains of type (2, 2) under *all* infinitesimal deformations of X, which contradicts (3.a.16) in [3]. Q.E.D.

As before we may prove a stronger result for d assumed to be sufficiently large (this automatically implies that Λ is not homologous to a complete intersection). To explain this we let

$$e \in H^1\big(N_{\Lambda/X}(-d)\big)$$

be the extension class of (4.f.21). Observe that

$$\Lambda^2 N_{\Lambda/X} = \det N_{\Lambda/X} \cong K_\Lambda(6-d), \tag{4.f.23}$$

and let

$$\Lambda^2 e \in H^2\big(K_\Lambda(6-3d)\big)$$

be the image of $e \otimes e$ under the natural map

$$H^1\big(N_{\Lambda/X}(-d)\big) \otimes H^1\big(N_{\Lambda/X}(-d)\big) \to H^2\big(\Lambda^2 N_{\Lambda/X}(-2d)\big)$$

PROPOSITION: $\Lambda^2 e \neq 0$. *In particular, the sequence (4.f.21) does not split.*

PROOF: We set $S_k = H^0(\mathcal{O}_P(k))$ and denote the restriction mapping

$$S_k \to H^0\big(\mathcal{O}_\Lambda(k)\big)$$

by

$$P \to p.$$

If X has equation $F(x) = 0$ ($F \in S_d$), then each $P \in S_d$ may be considered as tangent to the family $\{F + tP = 0\}$ of hypersurfaces, and we denote by

$$H^{p,q}(X) \xrightarrow{\xi_P} H^{p-1,q+1}(X)$$

the corresponding infinitesimal change in Hodge structure (ξ_P is cup-product with the Kodaira-Spencer class of P). The diagram (4.e.7) gives, for each $P \in S_d$,

$$\begin{array}{ccccc} H^4(\mathcal{O}_X) & \xleftarrow{\xi_P} & H^3(\Omega^1_X) & \xleftarrow{\delta} & H^1(N_{\Lambda/X}) \\ & & & & \uparrow{\scriptstyle\gamma} \\ & & S_d & \xrightarrow{\tilde{\beta}} & H^0(\mathcal{O}_\Lambda(d)) \end{array}$$

where $\tilde{\beta}(Q) = q$. Given

$$R \in S_{d-6} \cong H^0(K_X) \cong H^4(\mathcal{O}_X)^*$$

we have the following analogue of (4.f.18):

$$\langle \xi_P \delta \gamma \tilde{\beta} Q, R \rangle = \langle pqr, \Lambda^2 e \rangle \tag{4.f.25}$$

(*Note*: This makes sense since $pqr \in H^0(\mathcal{O}_\Lambda(3d-6))$ and $\Lambda^2 e \in H^2(K_\Lambda(6-3d)) \cong H^0(\mathcal{O}_\Lambda(3d-6))^*$.) The proof of (4.f.25) is based on the remark that $\gamma: H^0(\mathcal{O}_\Lambda(d)) \to H^1(N_{\Lambda/X})$ is given by cup-product with e; further details will be omitted.)

Since

$$\langle pqr, \Lambda^2 e \rangle = \langle r, pq\Lambda^2 e \rangle$$

we may conclude that

$$\begin{gathered} \xi_P \xi_Q \lambda = 0 \text{ in } H^{4,0}(X) \\ \Updownarrow \\ pq\Lambda^2 e = 0 \text{ in } H^2(K_\Lambda(6-d)) \end{gathered}$$

where $\lambda \in H^4(X)$ is the fundamental class of Λ. On the other hand, if $d \geqslant 6$ and Λ is not a complete intersection, then the infinitesimal Max Noether theorem (3.a.16) implies that

$$\xi_P \xi_Q \lambda \neq 0 \text{ for some } P, Q \in S_d.$$

This proves the proposition. Q.E.D.

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Notes

(1) Put slightly differently, we may identify

$$H^{m+1,m-1}/H^{m+1,m-1}(-\gamma)$$

with the conormal space to $T(U_\gamma)$ in $T(U)$.

(2) This means that, in the Lefschetz decomposition

$$H^{2m}(X) = H^{2m}_{\text{prim}}(X) \oplus LH^{2m-2}(X),$$

γ is the projection to $H^{2m}_{\text{prim}}(X)$ of the fundamental class of Γ.

(3) In a certain sense, the Riemann-Roch theorem

$$\chi(S,L) = \chi(S,\mathcal{O}_S) + \tfrac{1}{2}(L^2 - KL)$$

for surfaces is just a reflection of the Riemann-Roch theorem for the curves on S. What (4.b.2) seems to require is the deeper understanding of special linear series on curves lying on S.

We remark that the theorem is easy for $d = 3$ (cf. Chapter IV of [6]), and follows immediately from (4.b.3) when $d = 4$. Thus we may restrict attention to the case $d \geqslant 5$.

(4) This is equivalent to examining the sequence $h^0(\mathcal{O}_{\mathbf{P}^3}(k) \otimes I_C)$, which is clearly what is relevant to the proof of (4.b.4).

(5) We recall that C is *projectively normal* if

$$V_k = H^0(\mathcal{O}_C(k)), \qquad k \geqslant 0.$$

By (4.b.6) this is equivalent to

$$H^0(I_C(k)) \xrightarrow{\tau_k} H^0(I_D(k)) \to 0$$

being surjective for all $k \geqslant 0$.

(6) The proof consists in examining the restrictions of the linear series $|H^0(\mathcal{O}_{\mathbb{P}^3}(k) \otimes I_C|$ first to planes and then to lines.

(7) Put differently, the failure of projective normality or of the postulation sequence for C would force the genus of C to be less than that of a smooth curve that is residual to a line in a complete intersection $S \cap R$. But from (4.b.1) we know what the genus of C must be, so that all inequalities must be equalities.

(8) We note that

$$\frac{\partial F_0}{\partial x_0}(x(s)) = \frac{\partial F_0}{\partial x_1}(x(s)) \equiv 0,$$

and that the vector

$$\frac{\partial x}{\partial t}(t, s)|_{t=0}$$

may be intrinsically interpreted as a section in $H^0(\Lambda, N_{\Lambda/\mathbb{P}^3})$ where $N_{\Lambda/\mathbb{P}^3} \cong \mathcal{O}_\Lambda(1) \oplus \mathcal{O}_\Lambda(1)$ is the normal bundle to Λ in $\mathbb{P}^3$. With this interpretation

$$\varphi = \sum_{\alpha=0}^{1} s_\alpha \varphi_{\alpha, 2+\beta} \partial/\partial x_\beta \in H^0(\Lambda, N_{\Lambda/\mathbb{P}^2}).$$

(9) These interpretations together with the analogous ones to be used in 4(e) and 4(f) are consequences and variants of the following:

General Principle: *Let $X \subset Y \subset Z$ be smooth varieties, and in the diagram*

$$\begin{tikzcd} N_{X/Z} \arrow[r, "\alpha"] & N_{Y/Z} \otimes \mathcal{O}_X \\ & N_{Y/Z} \arrow[u, "\beta"'] \end{tikzcd}$$

let α and β be the natural quotient and restriction maps. Let $\sigma \in H^0(N_{X/Z})$ and $\tau \in H^0(N_{Y/Z})$, and let $\tilde{X}$, $\tilde{Y} \subset Z \times \operatorname{Spec} \mathbb{C}[\varepsilon]/(\varepsilon^2)$ be the corresponding first order deformations of X, Y in Z (cf. Kodaira [11]). Then $\tilde{X} \subset \tilde{Y}$ if, and only if, $\alpha(\sigma) = \beta(\tau)$ (4.c.20)

Corollary: *Keeping the above notations, let $W \subset Y$ be the scheme of zeroes of τ and $\pi: Z \times \mathbb{C}[\varepsilon]/(\varepsilon^2) \to Z$ the projection. Then*

$$\pi^{-1}(X) \subset \tilde{Y} \Leftrightarrow X \subset W. \tag{4.c.21}$$

Proof: This follows from (4.c.20) in the case $\sigma \equiv 0$.

(10) Using the notations of § III b) of [4] (with S corresponding to X), we may identify T with $S_d/J_{F,d}$ and $H^{2,0}$ with S_{d-4}, $H^{1,1}$ with $S_{2d-4}/J_{F,2d-4}$. When this is done the differential

$$T \otimes H^{2,0} \to H^{1,1}$$

becomes ordinary multiplication

$$(S_d/J_{F,d}) \otimes S_{d-4} \to S_{2d-4}/J_{F,2d-4},$$

which by Macaulay's theorem is injective in each factor.

(11) Briefly, an isomorphism between the Hodge structure for S and F_d induces an isomorphism

$$H^0(S, K_S) \xrightarrow{\sim} H^0(F_d, K_{F_d}). \qquad (*)$$

Moreover, each of S, F_d will contain configurations of lines $\Lambda_{\mu,\iota}$, $L_{\mu,\iota}$ whose fundamental classes correspond under (4.c.5). However, more geometric information relating these is required, and this is provided by using the infinitesimal variations of Hodge structure to prove that $(*)$ induces

$$H^0(K_S(-\Lambda_{\mu,\iota})) \xrightarrow{\sim} H^0(K_{F_d}(-L_{\mu,\iota})),$$

which is just (4.d.6).

(12) In fact, (iii) is also related to (i) and (ii) in that a basic difficulty in constructing cycles is the lack of a suitable geometric interpretation of the middle Hodge groups $H^{p,q}$ for $p, q > 0$.

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