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# INFINITESIMAL VARIATIONS OF THE RICCI TENSOR OF A SUBMANIFOLD

Dedicated to professor Tominosuke Ōtsuki on his sixtieth birthday

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## § 0. Introduction.

One of the present authors has recently studied infinitesimal variations of submanifolds of a Riemannian manifold, [5], [6], [7]. See also [1]. The method used is to displace the varied quantities back parallelly from the displaced point to the original point and to compare quantities obtained with the original quantities, [5], [7]. The variation is said to be *parallel* when the tangent space at a point of the submanifold and that at the corresponding point of the varied submanifold are parallel, [7], and the variation is said to be *normal* when the variation vector is normal to the submanifold, [7].

In the present paper we study normal parallel variations which preserve the Ricci tensor of a submanifold of a space of constant curvature and prove Theorem 3.8 using the following result of Sakamoto [4]. (See also [8])

THEOREM A ([4]). *Let $M^n$ be an n-dimensional connected complete submanifold with parallel second fundamental tensor immersed in an m-dimensional sphere $S^m(a)$ with radius $a>0$ $(1<n<m)$ and suppose that the normal bundle is locally trivial. Then $M^n$ is a small sphere, a great sphere or a Pythagorian product of a certain number of spheres.*

To prove Theorem 4.1 as a main result of the paper, we use the following theorem proved by Lawson [3] (See also [2]).

THEOREM B ([3]). *Let $M^{n+1}(c, R)$ be the simply connected space of constant curvature c, $S^{n+1}(R)$, $R^{n+1}$ or $D^{n+1}(R)$, depending on whether c is 1, 0 or $-1$ respectively. Suppose that $M^n$ is a submanifold of $M^{n+1}(c, R)$ over which the Ricci curvature is covariantly constant. Then, if $M^n$ is isometrically immersed into $M^{n+1}(c, R)$ with constant mean curvature, it must be an open submanifold of*

(i) $S^k(r)\times S^{n-k}(\sqrt{R^2-r^2})$ *for some r,* $R\geqq r\geqq 0$, *and* $k=0, \cdots, \frac{n}{2}$ *if* $c=1$.

(ii) $S^k(r)\times R^{n-k}$ *for some* $r\geqq 0$ *and* $k=0, \cdots, n$ *if* $c=0$.

(iii) $S^k(r)\times D^{n-k}(\sqrt{R^2+r^2})$ *for some* $r\geqq 0$ *and* $k=0, \cdots, n$, *or* $F^n$, *if* $c=-1$.

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## § 1. Structure equations of submanifolds.

Let $M^m$ be an $m$-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and denote by $g_{ji}$, $\Gamma_{ji}^h$, $\nabla_j$, $K_{kji}{}^h$ and $K_{ji}$ the metric tensor, the Christoffel symbols formed with $g_{ji}$, the operator of covariant differentiation with respect to $\Gamma_{ji}^h$, the curvature tensor and the Ricci tensor of $M^m$ respectively, where, here and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{\dot{1}, \dot{2}, \cdots, \dot{m}\}$.

Let $M^n$ be an $n$-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; y^a\}$ and denote by $g_{cb}$, $\Gamma_{cb}^a$, $\nabla_c$, $K_{dcb}{}^a$ and $K_{cb}$ the corresponding quantities of $M^n$ respectively, where, here and in the sequel, the indices $a, b, c, d, \cdots$ run over the range $\{1, 2, \cdots, n\}$.

We suppose that $M^n$ is isometrically immersed in $M^m$ by the immersion $i: M^n \to M^m$ and identify $i(M^n)$ with $M^n$ itself.

We represent the immersion by

$$x^h = x^h(y^a) \tag{1.1}$$

and put

$$B_b{}^h = \partial_b x^h, \quad (\partial_b = \partial/\partial y^b). \tag{1.2}$$

Then $B_b{}^h$ are $n$ linearly independent vectors of $M^m$ tangent to $M^n$. Since the immersion is isometric, we have

$$g_{cb} = B_{cb}^{ji} g_{ji}, \tag{1.3}$$

where $B_{cb}^{ji} = B_c{}^j B_b{}^i$.

We denote by $C_y{}^h$ $m-n$ mutually orthogonal unit normals to $M^n$, where, here and in the sequel, the indices $x, y, z$ run over the range $\{n+1, n+2, \cdots, m\}$. Then the metric tensor of the normal bundle of $M^n$ is given by

$$g_{zy} = C_z{}^j C_y{}^i g_{ji} \tag{1.4}$$

and has values $g_{zy} = \delta_{zy}$, $\delta_{zy}$ denoting the Kronecker delta.

It is well known that $\Gamma_{cb}^a$ and $\Gamma_{ji}^h$ are related by

$$\Gamma_{cb}^a = (\partial_c B_b{}^h + \Gamma_{ji}^h B_{cb}^{ji}) B^a{}_h, \tag{1.5}$$

where $B^a{}_h = B_b{}^i g^{ba} g_{ih}$, $g^{ba}$ being contravariant components of the metric tensor $g_{cb}$ of $M^n$ and the components $\Gamma_{cy}^x$ of the connection induced in the normal bundle are given by

$$\Gamma_{cy}^x = (\partial_c C_y{}^h + \Gamma_{ji}^h B_c{}^j C_y{}^i) C^x{}_h, \tag{1.6}$$

where $C^x{}_h = C_y{}^i g^{yx} g_{ih}$, $g^{yx}$ being contravariant components of the metric tensor $g_{yx}$ of the normal bundle.

If we denote by $\nabla_c B_b{}^h$ and $\nabla_c C_y{}^h$ the van der Waerden-Bortolotti covariant derivatives of $B_b{}^h$ and $C_y{}^h$ along $M^n$ respectively, that is, if we put

(1.7) $$\nabla_c B_b{}^h = \partial_c B_b{}^h + \Gamma^h_{ji} B^{ji}_{cb} - \Gamma^a_{cb} B_a{}^h$$

and

(1.8) $$\nabla_c C_y{}^h = \partial_c C_y{}^h + \Gamma^h_{ji} B_c{}^j C_y{}^i - \Gamma^x_{cy} C_x{}^h,$$

then we can write equations of Gauss and those of Weingarten in the form

(1.9) $$\nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h$$

and

(1.10) $$\nabla_c C_y{}^h = -h_c{}^a{}_y B_a{}^h$$

respectively, where $h_{cb}{}^x$ are the second fundamental tensors of $M^n$ with respect to the normals $C_x{}^h$ and $h_c{}^a{}_x = h_{cbx} g^{ba} = h_{cb}{}^y g^{ba} g_{yx}$.

Equations of Gauss, Codazzi and Ricci are respectively

(1.11) $$K_{dcb}{}^a = K_{kji}{}^h B^{kjia}_{dcbh} + h_d{}^a{}_x h_{cb}{}^x - h_c{}^a{}_x h_{db}{}^x,$$

(1.12) $$0 = K_{kji}{}^h B^{kji}_{dcb} C^x{}_h - (\nabla_d h_{cb}{}^x - \nabla_c h_{db}{}^x)$$

and

(1.13) $$K_{dcy}{}^x = K_{kji}{}^h B^{kj}_{dc} C_y{}^i C^x{}_h + (h_{de}{}^x h_c{}^e{}_y - h_{ce}{}^x h_d{}^e{}_y),$$

where

(1.14) $$K_{dcy}{}^x = \partial_d \Gamma^x_{cy} - \partial_c \Gamma^x_{dy} + \Gamma^x_{dz} \Gamma^z_{cy} - \Gamma^x_{cz} \Gamma^z_{dy}$$

and

$$B^{kjia}_{dcbh} = B_d{}^k B_c{}^j B_b{}^i B^a{}_h, \quad B^{kji}_{dcb} = B_d{}^k B_c{}^j B_b{}^i, \quad C^x{}_h = C_y{}^i g^{yx} g_{ih},$$

$K_{dcy}{}^x$ being the curvature tensor of the connection induced in the normal bundle.

## § 2. Infinitesimal variations of submanifolds. [7]

We now consider an infinitesimal variation of $M^n$ of $M^m$ given by

(2.1) $$\bar{x}^h = x^h + \xi^h(y)\varepsilon,$$

where $g_{ji}\xi^j\xi^i > 0$ and $\varepsilon$ is an infinitesimal. We then have

(2.2) $$\bar{B}_b{}^h = B_b{}^h + (\partial_b \xi^h)\varepsilon,$$

where $\bar{B}_b{}^h = \partial_b \bar{x}^h$ are $n$ linearly independent vectors tangent to the varied submanifold at the varied point $(\bar{x}^h)$.

If we displace $\bar{B}_b{}^h$ back parallelly from the point $(\bar{x}^h)$ to $(x^h)$, then we obtain

$$\tilde{B}_b{}^h = \bar{B}_b{}^h + \Gamma^h_{ji}(x+\xi\varepsilon)\xi^j \bar{B}_b{}^i \varepsilon,$$

that is,

$$\tilde{B}_b{}^h = B_b{}^h + (\nabla_b \xi^h)\varepsilon, \tag{2.3}$$

neglecting the terms of order higher than one with respect to $\varepsilon$, where

$$\nabla_b \xi^h = \partial_b \xi^h + \Gamma^h_{ji} B_b{}^j \xi^i. \tag{2.4}$$

In the sequel we always neglect terms of order higher than one with respect to the infinitesimal $\varepsilon$.

Thus putting

$$\delta B_b{}^h = \tilde{B}_b{}^h - B_b{}^h, \tag{2.5}$$

we have

$$\delta B_b{}^h = (\nabla_b \xi^h)\varepsilon. \tag{2.6}$$

If we put

$$\xi^h = \xi^a B_a{}^h + \xi^x C_x{}^h, \tag{2.7}$$

we have

$$\nabla_b \xi^h = (\nabla_b \xi^a - h_b{}^a{}_x \xi^x) B_a{}^h + (\nabla_b \xi^x + h_{ba}{}^x \xi^a) C_x{}^h. \tag{2.8}$$

When $\xi^x = 0$, that is, when the variation vector $\xi^h$ is tangent to the submanifold we say that the variation is *tangential* and when $\xi^a = 0$, that is, when the variation vector $\xi^h$ is normal to the submanifold we say that the variation is *normal*.

From (2.5), (2.6) and (2.8), we have

$$\tilde{B}_b{}^h = [\delta_b^a + (\nabla_b \xi^a - h_b{}^a{}_x \xi^x)\varepsilon] B_a{}^h + (\nabla_b \xi^x + h_{ba}{}^x \xi^a) C_x{}^h \varepsilon. \tag{2.9}$$

When the tangent space at a point $(x^h)$ of the submanifold and that at the corresponding point $(\bar{x}^h)$ of the varied submanifold are parallel, we say that the variation is *parallel*. [7].

From (2.9), we have

PROPOSITION 2.1 [7]. *In order for a normal variation of a submanifold to be parallel, it is necessary and sufficient that*

$$\nabla_b \xi^x = 0, \tag{2.10}$$

*that is, the variation vector* $\xi^x C_x{}^h$ *is parallel in the normal bundle.*

When the submanifold is a hypersurface, a normal variation is given by $\bar{x}^h = x^h + \lambda C^h \varepsilon$, $C^h$ being the unique unit normal to the hypersurface and $\lambda$ a function. In this case (2.10) reduces to $\nabla_b \lambda = 0$ and we have

PROPOSITION 2.2 [7]. *In order for a normal variation of a hypersurface to be parallel, it is necessary and sufficient that the normal variation displaces each point of the hypersurface the same distance.*

Denoting by $\bar{C}_y{}^h$ $m-n$ mutually orthogonal unit normals to the varied submanifold and by $\tilde{C}_y{}^h$ the vectors obtained from $\bar{C}_y{}^h$ by parallel displacement of $\bar{C}_y{}^h$ from the point $(\bar{x}^h)$ to $(x^h)$, we have

$$\tilde{C}_y{}^h = \bar{C}_y{}^h + \Gamma^h_{ji}(x + \xi\varepsilon)\xi^j \bar{C}_y{}^i \varepsilon. \tag{2.11}$$

We put

$$\delta C_y{}^h = \tilde{C}_y{}^h - C_y{}^h \tag{2.12}$$

and assume that $\delta C_y{}^h$ is of the form

$$\delta C_y{}^h = (\eta_y{}^a B_a{}^h + \eta_y{}^x C_x{}^h)\varepsilon. \tag{2.13}$$

Then (2.11), (2.12) and (2.13) give

$$\bar{C}_y{}^h = C_y{}^h - \Gamma^h_{ji}\xi^j C_y{}^i \varepsilon + (\eta_y{}^a B_a{}^h + \eta_y{}^x C_x{}^h)\varepsilon. \tag{2.14}$$

Applying the operator $\delta$ to $B_b{}^j C_y{}^i g_{ji} = 0$ and using (2.6), (2.8), (2.13) and $\delta g_{ji} = 0$, we find

$$(\nabla_b \xi_y + h_{bay}\xi^a) + \eta_{yb} = 0,$$

where $\xi_y = \xi^z g_{zy}$ and $\eta_{yb} = \eta_y{}^c g_{cb}$, or, putting $\nabla^a = g^{ba}\nabla_b$,

$$\eta_y{}^a = -(\nabla^a \xi_y + h_b{}^a{}_y \xi^b). \tag{2.15}$$

Applying the operator $\delta$ to $C_z{}^j C_y{}^i g_{ji} = \delta_{zy}$ and using (2.13) and $\delta g_{ji} = 0$, we find

$$\eta_{yx} + \eta_{xy} = 0, \tag{2.16}$$

where $\eta_{yx} = \eta_y{}^z g_{zx}$.

From (2. 12) and (2. 13), we have

$$\tilde{C}_y{}^h=[\eta_y{}^a B_a{}^h+(\delta_y^x+\eta_y^x)C_x{}^h]\varepsilon. \tag{2.17}$$

## § 3. Variations of the curvature tensor.

In this section we compute infinitesimal variations of the Christoffel symbols, the second fundamental tensors and curvature tensor of the submanifold.

Suppose that $v^h$ is a vector field of $M^m$ defined intrinsically along the submanifold $M^n$. When we displace the submanifold $M^n$ by $\bar{x}^h=x^h+\xi^h(y)\varepsilon$ in the direction $\xi^h$, we obtain a vector field $\bar{v}^h$ which is defined also intrinsically along the varied submanifold. If we displace $\bar{v}^h$ back parallelly from the point $(\bar{x}^h)$ to $(x^h)$, we obtain

$$\tilde{v}^h=\bar{v}^h+\Gamma_{ji}^h(x+\xi\varepsilon)\xi^j\bar{v}^i\varepsilon$$

and hence putting $\delta v^h=\tilde{v}^h-v^h$, we find

$$\delta v^h=\bar{v}^h-v^h+\Gamma_{ji}^h\xi^j v^i\varepsilon. \tag{3.1}$$

Similarly we have

$$\delta\nabla_c v^h=\bar{\nabla}_c\bar{v}^h-\nabla_c v^h+\Gamma_{ji}^h\xi^j\nabla_c v^i\varepsilon,$$

that is,

$$\begin{aligned}\delta\nabla_c v^h=&\nabla_c\bar{v}^h-\nabla_c v^h+(\partial_k\Gamma_{ji}^h+\Gamma_{kt}^h\Gamma_{ji}^t)\xi^k B_c{}^j v^i\varepsilon\\&+\Gamma_{ji}^h[(\partial_c\xi^j)v^i+\xi^j(\partial_c v^i)]\varepsilon.\end{aligned} \tag{3.2}$$

On the other hand, from (3. 1) we have

$$\begin{aligned}\nabla_c\delta v^h=&\nabla_c\bar{v}^h-\nabla_c v^h+(\partial_j\Gamma_{ki}^h+\Gamma_{jt}^h\Gamma_{ki}^t)\xi^k B_c{}^j v^i\varepsilon\\&+\Gamma_{ji}^h[(\partial_c\xi^j)v^i+\xi^j(\partial_c v^i)]\varepsilon.\end{aligned} \tag{3.3}$$

Thus forming (3. 2)—(3. 3), we find

$$\delta\nabla_c v^h-\nabla_c\delta v^h=K_{kji}{}^h\xi^k B_c{}^j v^i\varepsilon. \tag{3.4}$$

For a tensor field carrying three kinds of indices, say, $T_{by}{}^h$, we have

$$\delta\nabla_c T_{by}{}^h-\nabla_c\delta T_{by}{}^h=K_{kji}{}^h\xi^k B_c{}^j T_{by}{}^i\varepsilon-(\delta\Gamma_{cb}^a)T_{ay}{}^h-(\delta\Gamma_{cy}^x)T_{bx}{}^h, \tag{3.5}$$

where $\delta\Gamma_{cb}^a$ and $\delta\Gamma_{cy}^x$ are variations of $\Gamma_{cb}^a$ and $\Gamma_{cy}^x$ respectively.

Applying formula (3.5) to $B_b{}^h$, we find

$$\delta\nabla_c B_b{}^h - \nabla_c \delta B_b{}^h = K_{kji}{}^h \xi^k B_c{}^j B_b{}^i \varepsilon - (\delta\Gamma_{cb}^a) B_a{}^h,$$

or using (1.9) and (2.6)

$$\delta(h_{cb}{}^x C_x{}^h) = (\nabla_c \nabla_b \xi^h + K_{kji}{}^h \xi^k B_c{}^j B_b{}^i)\varepsilon - (\delta\Gamma_{cb}^a) B_a{}^h,$$

from which, using (2.13),

$$(\delta h_{cb}{}^x) C_x{}^h + h_{cb}{}^x (\eta_x{}^a B_a{}^h + \eta_x{}^y C_y{}^h)\varepsilon$$
$$= (\nabla_c \nabla_b \xi^h + K_{kji}{}^h \xi^k B_c{}^j B_b{}^i)\varepsilon - (\delta\Gamma_{cb}^a) B_a{}^h.$$

Thus we have

$$\delta\Gamma_{cb}^a = (\nabla_c \nabla_b \xi^h + K_{kji}{}^h \xi^k B_c{}^j B_b{}^i) B^a{}_h \varepsilon - h_{cb}{}^y \eta_y{}^a \varepsilon \tag{3.6}$$

and

$$\delta h_{cb}{}^x = -h_{cb}{}^y \eta_y{}^x \varepsilon + (\nabla_c \nabla_b \xi^h + K_{kji}{}^h \xi^k B_c{}^j B_b{}^i) C^x{}_h \varepsilon,$$

from which, using (1.12) and (2.8),

$$\begin{aligned}\delta h_{cb}{}^x = & [\xi^d \nabla_d h_{cb}{}^x + h_{eb}{}^x (\nabla_c \xi^e) + h_{ce}{}^x (\nabla_b \xi^e) - h_{cb}{}^y \eta_y{}^x]\varepsilon \\ & + [\nabla_c \nabla_b \xi^x + K_{kji}{}^h C_y{}^k B_{cb}^{ji} C^x{}_h \xi^y - h_{ce}{}^x h_b{}^e{}_y \xi^y]\varepsilon.\end{aligned} \tag{3.7}$$

Substituting (2.8) and (2.15) into (3.6) and using equations (1.11) of Gauss and (1.12) of Codazzi, we get

$$\begin{aligned}\delta\Gamma_{cb}^a = & (\nabla_c \nabla_b \xi^a + K_{dcb}{}^a \xi^d)\varepsilon \\ & - [\nabla_c (h_{bex}\xi^x) + \nabla_b (h_{cex}\xi^x) - \nabla_e (h_{cbx}\xi^x)] g^{ea}\varepsilon,\end{aligned}$$

or, equivalently,

$$\delta\Gamma_{cb}^a = [\mathcal{L}\Gamma_{cb}^a - \nabla_c (h_b{}^a{}_x \xi^x) - \nabla_b (h_c{}^a{}_x \xi^x) + \nabla^a (h_{cbx}\xi^x)]\varepsilon, \tag{3.8}$$

where $\mathcal{L}\Gamma_{cb}^a$ denotes the Lie derivative of $\Gamma_{cb}^a$ with respect to $\xi^a$ [6], that is,

$$\mathcal{L}\Gamma_{cb}^a = \nabla_c \nabla_b \xi^a + K_{dcb}{}^a \xi^d.$$

For the varied submanifold, the curvature tensor of the submanifold can be written as

$$\bar{K}_{dcb}{}^a = \partial_d \bar{\Gamma}_{cb}^a - \partial_c \bar{\Gamma}_{db}^a + \bar{\Gamma}_{de}^a \bar{\Gamma}_{cb}^e - \bar{\Gamma}_{ce}^a \bar{\Gamma}_{db}^e. \tag{3.9}$$

Thus denoting by $K_{dcb}{}^a + \delta K_{dcb}{}^a$ the curvature tensor and by $\Gamma_{cb}^a + \delta\Gamma_{cb}^a$ the Christoffel symbols of the varied submanifold, we have

$$K_{dcb}{}^{a}+\delta K_{dcb}{}^{a}=\partial_{d}(\Gamma^{a}_{cb}+\delta\Gamma^{a}_{cb})-\partial_{c}(\Gamma^{a}_{db}+\delta\Gamma^{a}_{db})$$
$$+(\Gamma^{a}_{de}+\delta\Gamma^{a}_{de})(\Gamma^{e}_{cb}+\delta\Gamma^{e}_{cb})-(\Gamma^{a}_{ce}+\delta\Gamma^{a}_{ce})(\Gamma^{e}_{db}+\delta\Gamma^{e}_{db}),$$

from which

$$\delta K_{dcb}{}^{a}=\nabla_{d}(\delta\Gamma^{a}_{cb})-\nabla_{c}(\delta\Gamma^{a}_{db}).$$

Substituting (3.8) into this and using (1.14), we find by a straightforward computation

$$(3.10)\qquad \delta K_{dcb}{}^{a}=[\mathcal{L}K_{dcb}{}^{a}-\nabla_{d}\nabla_{c}(h_{b}{}^{a}{}_{x}\xi^{x})-\nabla_{d}\nabla_{b}(h_{c}{}^{a}{}_{x}\xi^{x})+\nabla_{d}\nabla^{a}(h_{cbx}\xi^{x})$$
$$+\nabla_{c}\nabla_{d}(h_{b}{}^{a}{}_{x}\xi^{x})+\nabla_{c}\nabla_{b}(h_{d}{}^{a}{}_{x}\xi^{x})-\nabla_{c}\nabla^{a}(h_{dbx}\xi^{x})]\varepsilon,$$

where [6]

$$(3.11)\qquad \mathcal{L}K_{dcb}{}^{a}=\nabla_{d}\mathcal{L}\Gamma^{a}_{cb}-\nabla_{c}\mathcal{L}\Gamma^{a}_{db},$$

from which, using the Ricci identity,

$$(3.12)\qquad \delta K_{dcb}{}^{a}=[\mathcal{L}K_{dcb}{}^{a}-K_{dce}{}^{a}h_{b}{}^{e}{}_{x}\xi^{x}+K_{dcb}{}^{e}h_{e}{}^{a}{}_{x}\xi^{x}-\nabla_{d}\nabla_{b}(h_{c}{}^{a}{}_{x}\xi^{x})$$
$$+\nabla_{d}\nabla^{a}(h_{cbx}\xi^{x})+\nabla_{c}\nabla_{b}(h_{d}{}^{a}{}_{x}\xi^{x})-\nabla_{c}\nabla^{a}(h_{dbx}\xi^{x})]\varepsilon,$$

which implies that

$$(3.13)\qquad \delta K_{cb}=[\mathcal{L}K_{cb}-K_{ce}h_{b}{}^{e}{}_{x}\xi^{x}+K_{dcba}h^{da}{}_{x}\xi^{x}$$
$$-\nabla^{a}\nabla_{b}(h_{cax}\xi^{x})+\nabla^{a}\nabla_{a}(h_{cbx}\xi^{x})$$
$$+\nabla_{c}\nabla_{b}(h_{e}{}^{e}{}_{x}\xi^{x})-\nabla_{c}\nabla^{a}(h_{bax}\xi^{x})]\varepsilon.$$

Thus we have

PROPOSITION 3.1. *An infinitesimal variation of a submanifold gives the variation* (3.12) *to the curvature tensor and consequently it preserves the curvature tensor if and only if*

$$(3.14)\qquad \mathcal{L}K_{dcb}{}^{a}=K_{dce}{}^{a}h_{b}{}^{e}{}_{x}\xi^{x}-K_{dcb}{}^{e}h_{e}{}^{a}{}_{x}\xi^{x}$$
$$+\nabla_{d}\nabla_{b}(h_{c}{}^{a}{}_{x}\xi^{x})-\nabla_{d}\nabla^{a}(h_{cbx}\xi^{x})-\nabla_{c}\nabla_{b}(h_{d}{}^{a}{}_{x}\xi^{x})$$
$$+\nabla_{c}\nabla^{a}(h_{dbx}\xi^{x}).$$

PROPOSITION 3.2. *An infinitesimal variation of a submanifold gives the variation* (3.13) *to the Ricci tensor and consequently it preserves the Ricci tensor if and only if*

(3.15)
$$\mathcal{L}K_{cb}=K_{ce}h_b{}^e{}_x\xi^x-K_{dcba}h^{da}{}_x\xi^x$$
$$+\nabla^a\nabla_b(h_{cax}\xi^x)-\nabla^a\nabla_a(h_{cbx}\xi^x)$$
$$-\nabla_c\nabla_b(h_e{}^e{}_x\xi^x)+\nabla_c\nabla^a(h_{bax}\xi^x)].$$

COROLLARY 3.3. *For an infinitesimal normal variation of a submanifold, we have*

(3.16)
$$\delta K_{cb}=[-K_{ce}h_b{}^e{}_x\xi^x+K_{dcba}h^{da}{}_x\xi^x$$
$$-\nabla^a\nabla_b(h_{cax}\xi^x)+\nabla^a\nabla_a(h_{cbx}\xi^x)$$
$$+\nabla_c\nabla_b(h_e{}^e{}_x\xi^x)-\nabla_c\nabla^a(h_{bax}\xi^x)]\varepsilon$$

*and consequently a normal variation of a submanifold preserves the Ricci tensor if and only if*

(3.17)
$$-K_{ce}h_b{}^e{}_x\xi^x+K_{dcba}h^{da}{}_x\xi^x-\nabla^a\nabla_b(h_{cax}\xi^x)$$
$$+\nabla^a\nabla_a(h_{cbx}\xi^x)+\nabla_c\nabla_b(h_e{}^e{}_x\xi^x)-\nabla_c\nabla^a(h_{bax}\xi^x)=0.$$

From Proposition 2.1 and Corollary 3.3, we have immediately

COROLLARY 3.4. *An infinitesimal normal parallel variation of a submanifold preserves the Ricci tensor if and only if*

(3.18)
$$[K_{dcba}h^{da}{}_x-K_{ce}h_b{}^e{}_x-\nabla^a\nabla_b h_{cax}+\nabla^a\nabla_a h_{cbx}$$
$$+\nabla_c\nabla_b(h_e{}^e{}_x)-\nabla_c\nabla^a h_{bax}]\xi^x=0.$$

We now prepare a lemma for later use.

LEMMA 3.5. *If a submanifold $M^n$ of a Riemannian manifold $M^m$ admits $m-n$ linearly independent infinitesimal normal parallel variations, then the connection induced in the normal bundle is of zero curvature.*

*Proof.* By Proposition 2.1, a normal parallel variation satisfies $\nabla_b\xi^x=0$, from which

$$0=\nabla_d\nabla_c\xi^x-\nabla_c\nabla_d\xi^x=K_{dcy}{}^x\xi^y.$$

Thus if $M^n$ admits $m-n$ linearly independent infinitesimal normal parallel variations, then we have $K_{dcy}{}^x=0$, which proves the lemma.

We now suppose that the ambient manifold $M^m$ is a space of constant curvature $c$. Then we have from (1.11), (1.12) and (1.13),

(3.19)
$$K_{dcb}{}^a=c(\delta_d^a g_{cb}-\delta_c^a g_{db})+h_d{}^a{}_y h_{cb}{}^y-h_c{}^a{}_y h_{db}{}^y,$$

(3.20) $$\nabla_d h_{cb}{}^x - \nabla_c h_{db}{}^x = 0$$

and

(3.21) $$K_{dcy}{}^x = h_{de}{}^x h_c{}^e{}_y - h_{ce}{}^x h_d{}^e{}_y$$

respectively.

From (3.18), (3.19) and (3.20) we have

(3.22) $$[h_{cay} h_{db}{}^y h^{da}{}_x - h_c{}^d{}_y h_{de}{}^y h_b{}^e{}_x + h_e{}^e{}_y h_{cd}{}^y h_b{}^d{}_x + nch_{cbx} - h_{dey} h^{de}{}_x h_{cb}{}^y - ch_e{}^e{}_x g_{cb}]\xi^x = 0.$$

We now prove the following

LEMMA 3.6. *Let $M^n$ be a minimal submanifold of a space $M^m$ of constant curvature c. If the submanifold $M^n$ admits $m-n$ linearly independent infinitesimal normal parallel variations preserving the Ricci tensor of $M^n$, then the length of the second fundamental tensor is constant.*

*If, moreover, $c \leqq 0$, then $M^n$ is totally geodesic.*

*Proof.* First of all, by Lemma 3.5, we have $K_{dcy}{}^x = 0$ and consequently by (3.21)

$$h_{de}{}^x h_c{}^e{}_y - h_{ce}{}^x h_d{}^e{}_y = 0.$$

Thus, $M^n$ being minimal, we have from (3.22)

(3.23) $$nch_{cby} = \alpha_{yx} h_{cb}{}^x,$$

where we have put

(3.24) $$\alpha_{yx} = h_{dey} h^{de}{}_x.$$

Applying $\nabla_d$ to (3.23) and taking skew-symmetric part with respect to $d$ and $c$, we find

(3.25) $$(\nabla_d \alpha_{yx}) h_{cb}{}^x - (\nabla_c \alpha_{yx}) h_{db}{}^x = 0$$

because of (3.20), from which, $M^n$ being minimal,

(3.26) $$(\nabla_d \alpha_{yx}) h_c{}^{dx} = 0.$$

If we transvect $h^{cby}$ to (3.25) and make use of (3.24) and (3.26), then we have

$$(\nabla_d \alpha_{yx}) \alpha^{yx} = \frac{1}{2} \nabla_d (\alpha_{yx} \alpha^{yx}) = 0,$$

from which we see that $\alpha_{yx} \alpha^{yx}$ is constant.

Now, from (3.24), we find

$$\alpha_{yx}\alpha^{yx}=h_{dey}h^{de}{}_{x}\alpha^{yx},$$

from which, using (3.23)

$$\alpha_{yx}\alpha^{yx}=nch_{dey}h^{dey}=nc\alpha_{y}{}^{y}. \tag{3.27}$$

Thus $\alpha_y{}^y$ is also constant. The last assertion follows immediately from (3.24) and (3.27). This completes the proof of the lemma.

Finally we prepare the following lemma.

LEMMA 3.7. *Let $M^n$ be a minimal submanifold of a space $M^m$ of constant curvature c. If the submanifold $M^n$ admits $m-n$ linearly independent infinitesimal normal parallel variations preserving the Ricci tensor of $M^n$, then the second fundamental tensor is parallel.*

*Proof.* We compute the Laplacian $\Delta F$ of the function $F=h_{cb}{}^{x}h^{cb}{}_{x}$, which is globally defined in $M^n$, where $\Delta=g^{cb}\nabla_c\nabla_b$. We then have

$$\frac{1}{2}\Delta F=g^{ed}(\nabla_e\nabla_d h_{cb}{}^{x})h^{cb}{}_{x}+(\nabla_c h_{ba}{}^{x})(\nabla^c h^{ba}{}_{x}).$$

By using the Ricci identity and equations (3.20) of Codazzi, we can easily find

$$\frac{1}{2}\Delta F=K_c{}^a h_{ba}{}^{x}h^{cb}{}_{x}-K_{ecba}h^{ea}{}_{x}h^{cbx}+(\nabla_c h_{ba}{}^{x})(\nabla^c h^{ba}{}_{x})$$

with the help of Lemma 3.5 and $g^{cb}h_{cb}{}^{x}=0$, where $K_c{}^a$ is defined to be $K_c{}^a=K_{cb}g^{ba}$ and, as we can see from (3.19), is given by

$$K_c{}^a=c(n-1)\delta_c^a-h_c{}^e{}_x h_e{}^{ax} \tag{3.28}$$

under our assumptions. If we substitute (3.19) and (3.28) into the expression above of $\frac{1}{2}\Delta F$, then we have

$$\frac{1}{2}\Delta F=nch_{ba}{}^{x}h^{ba}{}_{x}-\alpha_{yx}\alpha^{yx}+(\nabla_c h_{ba}{}^{x})(\nabla^c h^{ba}{}_{x}),$$

from which, taking account of Lemma 3.6 and (3.27),

$$\nabla_c h_{ba}{}^{x}=0,$$

which proves the lemma.

Combining Theorem A, Lemmas 3.5, 3.6 and 3.7, we have

THEOREM 3.7. *Let $M^n$ be a simply connected and complete minimal submanifold of a space $M^m$ of constant curvature c. If $M^n$ admits $m-n$ linearly independent infinitesimal normal parallel variations preserving the Ricci tensor of $M^n$, then $M^n$ is totally geodesic if $c\leqq 0$, $M^n$ is $S^n(r)$ or $S^p(r_1)\times S^{n-p}(r_2)$ if $c>0$, where $S^n(r)$ denotes an n-sphere of radius $r>0$.*

## § 4. Variations of hypersurfaces preserving the Ricci tensor.

In this section, we consider a normal parallel variation $\bar{x}^h=x^h+\lambda C^h\varepsilon$ of a hypersurface $M^n$, where $\lambda$ is a positive function and $C^h$ the unit normal to $M^n$. In this case (2.10) reduces to $\nabla_b\lambda=0$ and (3.13) to

$$(4.1)\qquad \delta K_{cb}=[\mathcal{L}K_{cb}-\lambda K_{ce}h_b{}^e+\lambda K_{dcba}h^{da}-\nabla^a\nabla_b(\lambda h_{ca})$$
$$+\nabla^a\nabla_a(\lambda h_{cb})+\nabla_c\nabla_b(\lambda h_e{}^e)-\nabla_c\nabla^a(\lambda h_{ba})]\varepsilon.$$

In the sequel we suppose that the normal parallel variation of a hypersurface with constant mean curvature of a space of constant curvature preserves the Ricci tensor. Then we have from (3.19), (3.20) and (3.22)

$$(4.2)\qquad (h_e{}^e)h_{cd}h_b{}^d+(cn-h_{ed}h^{ed})h_{cb}-ch_e{}^e g_{cb}=0.$$

Since the mean curvature $h_e{}^e$ is constant, we have only to consider two cases $h_e{}^e=0$ and $h_e{}^e\neq 0$.

In the first case, we have from (4.2),

$$(4.3)\qquad h_{ed}h^{ed}=nc \quad\text{or}\quad h_{cb}=0.$$

In the second case we have

$$(4.4)\qquad h_{ce}h_b{}^e=kh_{cb}+cg_{cb},$$

where we have put

$$(4.5)\qquad k=\frac{1}{h_e{}^e}(h_{de}h^{de}-nc).$$

Differentiating (4.4) covariantly along $M^n$, we find

$$(4.6)\qquad (\nabla_d h_{ce})h_b{}^e+h_{ce}\nabla_d h_b{}^e=(\nabla_d k)h_{cb}+k\nabla_d h_{cb},$$

from which, taking skew-symmetric part with respect to $d$ and $c$ and using the fact that $\nabla_d h_{cb}-\nabla_c h_{db}=0$, we have

$$(4.7)\qquad h_{ce}\nabla_d h_b{}^e-h_{de}\nabla_c h_b{}^e=(\nabla_d k)h_{cb}-(\nabla_c k)h_{db}.$$

Interchanging indices $d$ and $b$ in (4.7), we get

$$h_{ce}\nabla_b h_d{}^e - h_{be}\nabla_c h_d{}^e = (\nabla_b k) h_{cd} - (\nabla_c k) h_{bd}. \tag{4.8}$$

Adding (4.6) and (4.8) and using $\nabla_d h_{ce} - \nabla_c h_{de} = 0$, we find

$$2h_{ce}\nabla_d h_b{}^e = k\nabla_d h_{cb} + (\nabla_d k) h_{cb} + (\nabla_b k) h_{cd} - (\nabla_c k) h_{db}. \tag{4.9}$$

If we transvect $g^{db}$ to this and use the fact that $h_e{}^e$ is constant, then we have

$$h_c{}^e \nabla_e k = \frac{1}{2} h_e{}^e \nabla_c k. \tag{4.10}$$

Moreover, transvecting (4.9) with $h_a{}^c$ and taking account of (4.4) and (4.10), we find

$$kh_a{}^e\nabla_d h_{be} + 2c\nabla_d h_{ba} = (kh_{ba} + cg_{ba})\nabla_d k + (kh_{da} + cg_{da})\nabla_b k - \frac{1}{2} h_e{}^e (\nabla_a k) h_{db}, \tag{4.11}$$

from which, transvecting $g^{db}$ and using (4.10)

$$\left[kh_e{}^e + 2c - \frac{1}{2}(h_e{}^e)^2\right]\nabla_a k = 0,$$

from which, $h_e{}^e$ being a constant, we have $k$=constant on $M^n$. Thus (4.9) and (4.11) imply that

$$(k^2 + 4c)\nabla_d h_{cb} = 0. \tag{4.12}$$

Thus, if $k^2+4c \neq 0$, we have $\nabla_d h_{cb}=0$. If $k^2+4c=0$, then we see from (4.4) that

$$\left(h_{cb} - \frac{1}{2}kg_{cb}\right)\left(h^{cb} - \frac{1}{2}kg^{cb}\right) = 0$$

and consequently $h_{cb} = \frac{1}{2}kg_{cb}$ which implies that $\nabla_d h_{cb}=0$. Therefore in any case we have

$$\nabla_d h_{cb} = 0, \tag{4.13}$$

from which, using the equations of Gauss, we see that the Ricci tensor is covariantly constant. Thus we conclude that

(i) If $h_e{}^e=0$, then $h_{ed}h^{ed}=nc$ or $h_{cb}=0$,

(ii) If $h_e{}^e \neq 0$, then $h_{ce}h_b{}^e = kh_{cb} + cg_{cb}$, $k$=constant and $\nabla_d h_{cb}=0$.

Therefore by Theorem A (See also Chern, do Carmo and Kobayashi [2]) we have

THEOREM 4.1. *Let $M^n$ be a complete hypersurface with constant mean curvature of a unit sphere. If an infinitesimal normal parallel variation $\bar{x}^h = x^h + \lambda C^h \varepsilon$, $\lambda > 0$, preserves the Ricci tensor of $M^n$, then $M^n$ is a sphere $S^n$ or $S^r \times S^{n-r}$.*

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