INFLATION, MICROWAVE BACKGROUND ANISOTROPY, AND OPEN UNIVERSE MODELS

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1. Introduction

The inflationary scenario for the very early universe has proven very attractive, because it can simultaneously solve a number of cosmological puzzles, such as the homogeneity of the Universe on scales exceeding the particle horizon at early times, the flatness or entropy problem, and the origin of density fluctuations for large-scale structure [1]. In this scenario, the observed Universe (roughly, the present Hubble volume) represents part of a homogeneous inflated region embedded in an inhomogeneous space-time. On scales beyond the size of this homogeneous patch, the initially inhomogeneous distribution of energy-momentum that existed prior to inflation is preserved, the scale of the inhomogeneities merely being stretched by the expansion.

In its conventional form, inflation predicts a nearly scale-invariant spectrum of density perturbations produced by the inflaton field, and that the Universe is observationally indistinguishable from being spatially flat ($k = 0$). In the absence of a cosmological constant or exotic forms of matter, this implies that the present matter density parameter $\Omega_0 \equiv 8\pi G\rho_m(t_0)/3H_0^2$ is very close to unity. However, it is not clear that such an Einstein-de Sitter Universe jibes with astronomical observations. As is well known, dynamical estimates of mass-to-light ratios from galaxy rotation curves and cluster dynamics [2] typically indicate $\Omega_0 \simeq 0.1 - 0.2$. Similar conclusions have recently been reached from the consistency of the ROSAT observations of X-ray emission from the Coma cluster and Big Bang nucleosynthesis constraints on the baryon density Ω_B [3].

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Moreover, if $\Omega_0 = 1$ the age of the Universe is $t_0 = (2/3H_0) = 6.7 \times 10^9 h^{-1}$yrs (where the present Hubble parameter is $H_0 = 100h$ km/sec/Mpc). This is less than globular cluster age estimates of $t_{gc} \simeq 13 - 15 \times 10^9$ yr if $h \geq 0.5$, and a number of extragalactic distance indicators suggest $h \simeq 0.8$. A large age is also indicated by the colors of stellar populations of radio galaxies at high redshift, $z \simeq 4$ [4]. The presence of galaxies and perhaps even protoclusters at $z \geq 3.5$ is also easier to explain in a low-density Universe, where structures should have collapsed by $z \simeq \Omega_0^{-1} - 1$ [5]. On larger scales, the situation is still uncertain: several analyses of large-scale peculiar motions suggest higher values of Ω_0, consistent with unity [6], while other methods are consistent with low values of Ω_0 [7].

In sum, the current observational status of Ω_0 is at best inconclusive, with much of the data pointing to a low-density Universe. In the context of inflation, the simplest way to accomodate $\Omega_0 < 1$ is to incorporate a cosmological constant $\Lambda = 3H_0^2\Omega_\Lambda$, retaining spatial flatness by imposing $\Omega_0 + \Omega_\Lambda = 1$. However, initial studies of observed gravitational lens statistics indicate the bound $\Omega_\Lambda \lesssim 0.7$ [8], marginally disfavoring the spatially flat, low-density model.

The other logical possibility is an open, negatively curved universe, and various suggestions have been made to try to accommodate an open, low-Ω_0 Universe within inflation [9]. While the models differ in the mechanisms that drive inflation, their common feature is that the homogeneous patch that encompasses the presently observable Universe was inflated by just the right number of e-foldings to ensure that $1 - \Omega_0 \simeq 1$; generally, this implies that the present size L_0 of the inflated patch is comparable to the current Hubble distance, H_0^{-1}.

Points separated by distances larger than the scale of the inflated homogeneous patch have never been in causal contact, and one thus expects large density fluctuations, $(\delta\rho/\rho)_L \sim 1$, on scales $L \gtrsim L_0$. However, if the size of the homogeneous region is close to the present Hubble radius, such non-linear inhomogeneities on large scales will induce significant microwave background anisotropy via the Grischuk-Zel'dovich (GZ) effect [10]. In order of magnitude, the quadrupole anisotropy induced by superhorizon-size fluctuations of lengthscale L is $Q_L \simeq (\delta\rho/\rho)_L(LH_0)^{-2}$. The COBE DMR has measured a quadrupole anisotropy of $Q_{COBE} = (4.8 \pm 1.5) \times 10^{-6}$ from the first year of data and $Q_{COBE} = (2.2 \pm 1.1) \times 10^{-6}$ from the first two years of data [11]. Consequently, assuming order unity density fluctuations on scales $L \gtrsim L_0$, the size of the inflated patch must be significantly larger than the present Hubble radius, $L_0 > 500H_0^{-1}$ [12, 13]. However, the Grischuk-Zel'dovich analysis was performed for a spatially flat ($k = 0$) universe; to self-consistently exclude an open model, it must be extended to the case of negative curvature.

This talk summarizes an investigation of the Grischuk-Zel'dovich effect in an open universe, done in collaboration with Alexander Kashlinsky and Igor Tkachev [14]. We found that the constraint on L_0 generally becomes even tighter when $\Omega_0 < 1$. If the Universe began from inhomogeneous initial conditions, the comoving size of the quasi-homogeneous patch that encompasses our observable universe must extend to at least 500 – 2000 times the present Hubble radius. Thus, the required large size of the inflated patch is very improbable in low-Ω inflationary models.

To relate the size of the inflated patch to the local value of Ω_0 we write the Friedmann equation as $-K = [1 - \Omega(t)]H^2(t)a^2(t)$, where $a(t)$ is the global expansion factor and $K = +1, -1$, or 0 is the spatial curvature constant. Note that the global topology of the inflationary Universe could be quite complex, with *e.g.,* locally Friedmann universes of both positive and negative spatial curvature connected by wormhole throats. We will focus on the open, negatively curved ($K = -1$) model, since it is the open model that attracts attention as an alternative to the flat Universe on observational grounds. Thus, we can relate the present scale L_0 of the homogeneous patch to its size L_s at the start of inflation, $(1 - \Omega_0)H_0^2L_0^2 = (H_s^2L_s^2)(1 - \Omega_s)$ (where subscript 's' denotes quantitites at the onset of inflation). By the onset of inflation, we expect that causal microphysical processes could have smoothed out initial inhomogeneities only on scales up to the Hubble radius, so that $H_sL_s \sim 1$. This is also a sufficient condition for spatial gradients to be subdominant compared to the vacuum energy density driving accelerated expansion [15]. Inflation was proposed in part to allow $1 - \Omega_s \sim 1$ as an initial condition, but in any case $1 - \Omega_s \leq 1$. Consequently, we expect the present size of the homogeneous patch to satisfy $L_0^2 \lesssim H_0^{-2}/(1 - \Omega_0) \equiv R_{\rm curv}^2$, *i.e.,* the present size of the inflated patch is at most comparable to the present curvature radius $R_{\rm curv}$. If $1 - \Omega_0 \ll 1$, the Universe is nearly spatially flat, and the present curvature radius is much larger than the Hubble radius. On the other hand, if $1-\Omega \simeq 1$, then $R_{\rm curv} \sim H_0^{-1}$, implying $L_0 \lesssim H_0^{-1}$, and in particular $L_0 \ll 500H_0^{-1}$. This simple argument shows that the GZ effect is only naturally suppressed in the limit $\Omega_0 \to 1$, and that the required large size of the homogeneous domain of our observable Universe implied by the microwave background measurements is difficult to produce in $\Omega_0 \ll 1$ inflationary models. However, as noted above, the effect of spatial curvature on the GZ anisotropy can be significant, and this calculation should be done self-consistently in an open universe.

Microwave background anisotropies in an open universe have been studied by a number of authors [16]. We write the background metric of the open universe in the form $ds^2 = a^2(\eta)[d\eta^2 - d\chi^2 - \sinh^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)]$, where $\eta = \int dt/a(t)$ is conformal time, and χ is the comoving radial distance in units of the curvature scale (*i.e.,* the physical distance $\chi_{\rm phys} =$

$R_{\rm curv}\chi$). For the matter-dominated universe, the scale factor is given by $a(\eta) = a_m(\cosh\eta - 1)$, where a_m is a constant and $\eta = 0$ corresponds to the initial singularity. At a given conformal time, the density parameter is given by $\Omega(\eta) = 2(\cosh\eta - 1)/\sinh^2\eta$. Thus, at early times, $\eta \ll 1$, the universe is effectively flat, $\Omega(\eta \ll 1) \simeq 1$, and at late times, $\eta \gtrsim 1$, it is curvature-dominated.

To describe the propagation of waves in curved space, we expand them in terms of eigenfunctions of the Helmholtz equation $(\nabla^2 + k^2 + 1)f(\chi, \theta, \phi) = 0$, where ∇^2 is the Laplace operator on the three-surface of constant negative curvature. The solutions are of the form $X_l(k;\chi)Y_l^m(\theta,\phi)$, where Y_l^m are the spherical harmonics and the radial eigenfunctions are given by

$$X_l(k;\chi) = (-1)^{l+1}\frac{(k^2+1)^{l/2}}{N_l^{-1}(k)}\sinh^l\chi\frac{d^{l+1}(\cos k\chi)}{d(\cosh\chi)^{l+1}} \, . \tag{1}$$

Here $N_l^k = k^2(k^2+1)...(k^2+l^2)$. The normalization is chosen such that in the limit $\Omega \longrightarrow 1$ the radial eigenfunctions become spherical Bessel functions. For a perturbation of comoving wavelength λ, the comoving wavenumber $k = 2\pi/\lambda = k_{\rm phys}R_{\rm curv}$. Using this relation and the Friedmann equation, the comoving wavenumber corresponding to the size of the inflated patch is $k_0 \simeq R_{\rm curv}/L_0 = 1/L_sH_s\sqrt{1-\Omega_s} \gtrsim 1$.

Similarly, the microwave background temperature can be expanded in spherical harmonics on the sky, $\delta T/T = \sum a_{lm}Y_{lm}(\theta,\phi)$, and the multipole moments of the anisotropy are then given by the Sachs-Wolfe relation

$$\langle |a_l|^2\rangle = \frac{2}{\pi}\int |\Phi_k(\eta=0)|^2|\tilde{\theta}_l(k)|^2\frac{N_l(k)}{(k^2+1)^l}dk \quad , \tag{2}$$

where

$$\tilde{\theta}_l(k) = \frac{F(\eta_{ls})}{3}X_k^l(\eta_0 - \eta_{ls}) + 2\int_{ls}^{\eta_0}\frac{dF}{d\eta}X_k^l(\eta_0-\eta)d\eta \; , \tag{3}$$

η_{ls} denotes the epoch of last scattering, and the gravitational potential fluctuation satisfies $\Phi_k(\eta) = \Phi_k(\eta = 0)F(\eta)$, with (ignoring the decaying mode)[17]

$$F(\eta) = 5\frac{\sinh^2\eta - 3\eta\sinh\eta + 4\cosh\eta - 4}{(\cosh\eta - 1)^3} \, . \tag{4}$$

Note that $F(\eta) = 1$ for $\Omega_0 = 1$; in an open universe, $F(\eta) \simeq 1$ for $\eta \lesssim 1$ and decays as $1/a(\eta)$ for $\eta \gtrsim 1$. Eqs. (1) - (4) allow one to estimate the anisotropy due to superhorizon-size perturbations, with wavelengths $\lambda \gg H_0^{-1}$. The potential Φ is a gauge-invariant measure of the spatial curvature perturbation, related to the density fluctuation by the relativistic curved-space analogue of the Poisson equation [18]. For perturbations on scales larger than the Hubble radius, $k\eta \lesssim 1$, it satisfies $\Phi_k \simeq -\delta_k/2 \simeq$ constant, where δ is a gauge-invariant measure of the density perturbation amplitude, equal to the density fluctuation in the longitudinal (conformal Newtonian) gauge [17]. For such long wavelengths, the dominant anisotropy is generally the quadrupole $l = 2$ (for some values of Ω_0, the quadrupole is accidentally suppressed, and the main contribution would be the $l = 3$ octupole moment, as we discuss below). The quadrupole anisotropy due to such superhorizon-scale modes is thus

$$\langle |a_2|^2 \rangle \simeq \frac{1}{2\pi} \int_0^{k_0} \frac{k^2+4}{k^2+1} |\tilde{\theta}_2(k)|^2 \langle |\delta_k|^2 \rangle k^2 dk \ . \tag{5}$$

For superhorizon-size modes, the quadrupole mode contribution $|\tilde{\theta}_2(k)|^2$ quadrupole depends on Ω_0: for example, for $\Omega_0 = 0.1$, we find $|\tilde{\theta}_2(k)|^2 \simeq 0.1$ for these modes, while for $\Omega_0 = 0.7$, $|\tilde{\theta}_2(k)|^2 \simeq 0.02$. For $\Omega_0 = 0.4$, the mode contribution is strongly suppressed, due to a near cancellation of the line-of-sight contribution (the second term on the RHS of eq. (3)) with the last scattering term (see below). We emphasize that for modes outside the scale of the homogeneous patch, $k \leq k_0$, the pre-inflation perturbation amplitude δ_k is preserved and expected to be of order unity. To study the implications of this result, we consider two limits: Ω_0 close to unity ($1 - \Omega_0 \ll 1$) and low-density models with $1 - \Omega_0 \sim 1$.

Ω_0 **close to 1**: Using the relation $\cosh\eta - 1 = 2(1-\Omega)/\Omega$, the limit $\Omega_0 \to 1$ corresponds to taking $\eta_0^2 \simeq 4(1-\Omega_0) \to 0$. Taking this limit in Eq. (1) while keeping $k_{\rm phys}$ fixed, we find $X_2(k;\eta \to 0) \simeq (1+k^2)\eta^2/15 \simeq 4\,(1+k^2)(1-\Omega_0)/15$. In this limit, the line-of-sight integral in Eq. (3) becomes $\int (dF/d\eta) X_k^2 d\eta \simeq -(1+k^2)\eta_0^4/630$, which can be neglected compared to the last scattering term. As a result, the quadrupole arising from modes with $k\eta_0 \lesssim 1$ can be expressed as

$$\langle |a_2|^2 \rangle \simeq \frac{8}{225\pi} \frac{(1-\Omega)^2}{9} \int_0^{k_0} dk\ k^2(k^2+1)(k^2+4)\langle |\delta_k|^2 \rangle \ . \tag{6}$$

The usual flat-space result can be recovered from Eq. (6) by taking the limit $k \gg 1$ and keeping $k_{\rm phys}$ fixed in the relation $k_{\rm phys} = kH_0\sqrt{1-\Omega_0}$. Eq. (6) can be used to constrain Ω_0 with any given pre-inflation power spectrum $\langle |\delta_k|^2 \rangle$ on scales $k \leq k_0$. A plausible assumption is that $\langle |\delta_k|^2 \rangle \sim k^n$ with $n \geq 0$, *i.e.*, random Poisson fluctuations (or less). For example, such a spectrum would arise if one imagines that prior to inflation the universe consisted of uncorrelated, quasi-homogeneous regions of size k_0^{-1}. However, quantitatively the result does not depend strongly upon the shape of the power spectrum. With the assumption of no fine tuning prior to inflation, *i.e.*, $\langle |\delta(k_0)|^2 \rangle \simeq 1$, and since in inflationary models $k_0 \gtrsim 1$, the

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COBE measurement of the quadrupole moment translates eq. (6) into the constraint

$$\Omega_0 > 1 - a_2(\mathrm{COBE}) \simeq 1 - 10^{-6} \quad . \tag{7}$$

Thus, *if* an epoch of inflationary expansion was responsible for the homogeneity of our observable Universe, the density parameter Ω_0 cannot differ from 1 by more than one part in $Q^{-1} \sim 10^6$.

Low Ω_0: We now consider the case of low Ω_0 and estimate the scale out to which the Universe must be homogeneous in light of the COBE results, independent of considerations of inflation, namely we allow $k_0 \ll 1$. If Ω_0 is not very close to 0.4 or 1, $\tilde{\theta}_2(k)$ is nearly independent of k for small k, and we can set $\tilde{\theta}_2(k) \simeq \tilde{\theta}_2(0)$ to good approximation for $k < 1$. (This is very different from the spatially flat model, where $|\tilde{\theta}_2(k)| = j_2(2k)$ and goes to zero as k^2 at small k). The zero-mode contribution $|\tilde{\theta}_2(0)|$ as a function of Ω_0 is shown in Fig. 2. In this case the quadrupole becomes:

$$\langle |a_2|^2 \rangle \simeq |\tilde{\theta}_2(0)|^2 \frac{1}{2\pi} \int_0^{k_0} \frac{k^2+4}{k^2+1} \langle |\delta_k|^2 \rangle k^2 dk \quad . \tag{8}$$

Again conservatively assuming an initial spectrum that falls at least as white noise ($n \geq 0$), eq. (8) yields a lower bound on the scale $k_0^{-1} \sim L_0 H_0 \sqrt{1-\Omega_0}$ over which the Universe must be homogeneous if $\Omega_0 \lesssim 1$,

$$k_0^{-1} > \left(\frac{|\tilde{\theta}_2(0)|}{a_{2,COBE}} \right)^{2/3} \simeq 10^4 |\tilde{\theta}_2(0)|^{2/3} \quad . \tag{9}$$

Eq. (9) implies that the Universe must to be homogeneous over scales $k_0^{-1} \gtrsim 2000$ for $\Omega_0 \lesssim 0.1$ and over scales $k_0^{-1} \gtrsim 500$ for $\Omega \simeq 0.5 - 0.8$. In inflation models these bounds on $k_0 \ll 1$ require superhorizon-sized correlations prior to inflation. Note that for a given constant value of $|\delta_k|^2$ the quadrupole anisotropy for $k_0 \ll 1$ scales as $\sim k_0^{7/2}$ for $\Omega = 1$ and only as $\sim k_0^{3/2}$ for $\Omega \ll 1$.

$\Omega_0 \simeq 0.4$: The quadrupole due to long wavelength modes is suppressed not only at $\Omega_0 \to 1$ but also accidentally for $\Omega_0 \sim 0.4$, due to cancellation between the last scattering term and the line-of-sight integral. (The positive last scattering term dominates at $\Omega_0 \to 1$, while the negative line-of-sight term dominates at $\Omega_0 \to 0$.) As Ω_0 is varied over a small interval around 0.4, the wavenumber where the two terms cancel varies over the interval $(0, \eta_0^{-1})$. While interesting, this suppression cannot make inflation and low-Ω_0 compatible, for in this case the contribution to the octupole ($l = 3$) mode will be dominant and lead to similarly severe constraints on L_0.

We have arrived at two results of significance for inflation and open universe models. (1) Inflation can produce a homogeneous patch encompassing the observable Universe (the present Hubble volume) and be consistent with the microwave background observations only if the present density parameter Ω_0 differs from unity by no more than 1 part in $Q_{COBE}^{-1} \sim 10^6$. (2) On the other hand, if Ω_0 is significantly below 1, the Universe must be homogeneous on scales $k_0^{-1} > (500 - 2000)$. If this is the case, inflation does not by itself solve the horizon problem. Indeed, if we assume that the distribution of quasi-homogeneous regions satisfies Poisson statistics, the probability of finding one such region per volume k_0^{-3} in curvature units is $P \simeq k_0^{-3} \times \exp(-k_0^{-3})$, which is negligibly small for the k_0^{-1} values above. If it turns out that the universe is open, $\Omega_0 < 1$, this implies that our Hubble volume occupies a very special place in the space of initial conditions, which is precisely the condition inflation was meant to alleviate.

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References

1. For a review see K. A. Olive, Phys. Rep. C **190**, 307 (1990).
2. See, *e.g.,* S. Kent and J. Gunn, Astron. J. **87**, 945 (1982).
3. S. D. M. White *et al.*, Nature **366**, 429 (1993).
4. S. Lilly, Ap. J **333**, 161 (1988); K. C. Chambers and S. Charlot, Ap.J. **348**, L1 (1990); K. C. Chambers, *et al.*, Ap. J. **363**, 21 (1990).
5. J. Uson *et al.*, Phys. Rev. Lett. **67**, 3328 (1992); M. Giavalisco *et al.*, Ap. J. **425**, L5 (1994); A. Kashlinsky, Ap. J. **406**, L1 (1993).
6. A. Dekel, Ann. Rev. Astr. Astrophys., in press (1994).
7. A. Kashlinsky, Ap. J. **386**, L37 (1992), and in "*Evolution of the Universe and its observational quest*", ed. K. Sato, in press (1994); R. Scaramella *et al.*, Ap. J. **422**, 1 (1994).
8. M. Fukugita and E. L. Turner, MNRAS **253**, 99 (1991); D. Maoz and H. W. Rix, Ap. J. **416**, 425 (1993); C. Kochanek, Ap. J. **419**, 12 (1993).
9. J. R. Gott, Nature **295**, 304 (1982); G. F. R. Ellis, Class. Quantum Grav. **5**, 891 (1988); P. Steinhardt, Nature **345**, 47 (1989); D. H. Lyth and E. D. Stewart, Phys. Lett. **B252**, 336 (1990); B. Ratra and P. J. E. Peebles, preprints PUPT-1444, PUPT-1445 (1994); M. Bucher, A. S. Goldhaber, and N. Turok, preprint PUPT-94-1507 (1994).
10. L. Grischuk and Ya. B. Zeldovich, Sov. Astron. **22**, 125 (1978).
11. G. Smoot *et al* Ap. J. **396**, L1 (1992); C. Bennett *et al.* Ap. J. (1994), submitted.
12. M. Turner, Phys. Rev. D **44**, 3737 (1991).
13. L. Grischuk, Phys. Rev. D **45**, 4717 (1992).
14. A. Kashlinsky, I. Tkachev, and J. Frieman, Phys. Rev. Lett. **73**, 1582 (1994).
15. D.S. Goldwirth and T. Piran, Phys. Rev. Lett. **64**, 2852 (1990).
16. See, *e.g.,* M. Wilson, Ap. J. **253**, L53 (1983); L. F. Abbott and R. K. Schaefer, Ap. J. **308**, 546 (1986); J. Traschen and D. Eardley, Phys. Rev. D **34**, 1665 (1986), and the last two references in [9]; M. Kamionkowski and D. Spergel, preprint IASSNS-HEP-93/73 (1993).
17. V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rep. C **215**, 203 (1992).
18. J. Bardeen, Phys. Rev. D **22**, 1882 (1980).