

Review

Inflationary Cosmology in Modified Gravity Theories

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Abstract: We review inflationary cosmology in modified gravity such as R^2 gravity with its extensions in order to generalize the Starobinsky inflation model. In particular, we explore inflation realized by three kinds of effects: modification of gravity, the quantum anomaly, and the R^2 term in loop quantum cosmology. It is explicitly demonstrated that in these inflationary models, the spectral index of scalar modes of the density perturbations and the tensor-to-scalar ratio can be consistent with the Planck results. Bounce cosmology in $F(R)$ gravity is also explained.

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1. Introduction

Inflation in the early universe has recently been studied much more extensively because of the BICEP2 experiment [1] in terms of the primordial gravitational waves, in addition to the Wilkinson Microwave anisotropy probe (WMAP) [2–6] and the Planck satellite [7,8] on the unisotropy of the cosmic microwave background (CMB) radiation. For a standard inflationary scenario like chaotic inflation [9], the existence of the inflaton field is assumed, whose potential contributes to inflation.

On the other hand, the accelerated expansion of the universe including inflation and the late-time acceleration, *i.e.*, dark energy problem, can be realized in modified gravity theories such as $F(R)$ gravity (for reviews on inflation, see, e.g., [10,11], whereas for dark energy and modified gravity, see, for example, References [12–20]). For instance, the trace-anomaly driven inflation such as the Starobinsky inflation [21,22] is well known.

Indeed, the WMAP and Planck data [2–8] support a kind of the trace-anomaly driven inflation with the R^2 term. Such a theory can be regarded as modified gravity because the R^2 term or its higher derivative term of the trace-anomaly term $\square R$, which leads to the long enough inflation and graceful exit from it [21], is the effective action of gravity, where \square is the covariant d’Alembertian for a scalar quantity. (In References [23–25], the features of inflation in non-local gravity including such a non-local term as $\square R$ has been analyzed in detail.)

In this paper, we review the main results in References [26–28]. The main purpose of this paper is to explain the recent developments on inflationary models to realize the Planck results in the so-called R^2 gravity (namely, the action consist of the Einstein-Hilbert term plus R^2 term) with further extensions, which can be regarded as a kind of $F(R)$ gravity. Particularly, we consider inflation (i) derived by modification terms of gravity [26]; (ii) through the quantum anomaly [27]; and (iii) in R^2 gravity in the framework of the so-called loop quantum cosmology (LQC) [29–33] to include quantum effects [28] (for reviews on LQC, see, for example, [34–40]). In addition, we state the recent progress of the bounce cosmology in $F(R)$ gravity by presenting the important consequences in References [41,42]. We use units of $k_B = c = \hbar = 1$ and express the gravitational constant $8\pi G_N$ by $\kappa^2 \equiv 8\pi/M_{\text{Pl}}^2$ with the Planck mass of $M_{\text{Pl}} = G_N^{-1/2} = 1.2 \times 10^{19}$ GeV.

The organization of the paper is the following. In Section 2, we study inflation induced by modification of gravity. In Section 3, we explore the trace-anomaly driven inflation in modified gravity. In Section 4, we investigate R^2 gravity in the context of LQC. Furthermore, in Section 5, we reconstruct $F(R)$ gravity to realize the cosmological bounce in LQC. Finally, conclusions are described in Section 6.

2. Inflation Induced by Modification of Gravity

In this section, we review inflation in modified gravity, particularly $F(R)$ gravity, based on Reference [26]. The deviation of $F(R)$ gravity from general relativity may be interpreted as a kind of quantum corrections in the early universe, or such a modification of gravity could be motivated by the so-called ultraviolet (UV) completion of quantum gravity. In fact, the Starobinsky inflation [21] can be regarded as inflation induced by the modification term of R^2 from general relativity. We here attempt to examine inflation by the other forms of modification of gravity.

2.1. Conformal Transformation

We first explain the conformal transformation from $F(R)$ gravity in the Jordan frame to the corresponding scalar field theory in the Einstein frame [19,43]. The action of $F(R)$ gravity is represented as $S = \int d^4x \sqrt{-g} [F(R)/(2\kappa^2)]$, where g is the determinant of the metric tensor $g_{\mu\nu}$. We use a conformal transformation $\hat{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu}$ with $\Omega^2 \equiv F_R$, where the hat denotes quantities in the Einstein frame, and the subscription of F_R denotes the derivative with respect to R as $F_R(R) \equiv dF(R)/dR$. Here, we introduce a scalar field $\varphi \equiv -\sqrt{3/2} (1/\kappa) \ln F_R$. Through the conformal transformation, the action in the Einstein frame reads [44,45]

$$S_E = \int d^4x \sqrt{-\hat{g}} \left(\frac{\hat{R}}{2\kappa^2} - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right) \quad (1)$$

$$V(\varphi) = \frac{F_R \hat{R} - F}{2\kappa^2 (F_R)^2} \quad (2)$$

with $F_R = \exp\left(-\sqrt{2/3}\kappa\varphi\right)$. This is the action for a canonical scalar field φ with its potential $V(\varphi)$. For the Starobinsky inflation model [21] with $F(R) = R + \alpha_S \kappa^2 R^2$, where α_S is a constant, we have $V(\varphi) = [1/(8\alpha_S \kappa^2)] \left(1 - \exp\left(-\sqrt{2/3}\kappa\varphi\right)\right)^2$.

2.2. Slow-Roll Inflation

We describe the procedures to deal with the so-called slow-roll inflation. We consider the action in Equation (1) and regard φ as the inflaton field. We suppose the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric $ds^2 = -dt^2 + a^2(t)d\vec{x}^2$. Here, $a(t)$ is the scale factor. The Hubble parameter is defined as $H \equiv \dot{a}/a$, where the dot denotes the time derivative. The gravitational field equations in this background read $3H^2/\kappa^2 = \dot{\varphi}^2/2 + V(\varphi)$ and $-(2\dot{H} + 3H^2)/\kappa^2 = \dot{\varphi}^2/2 - V(\varphi)$. Moreover, the equation of motion (EoM) for φ becomes $\ddot{\varphi} + 3H\dot{\varphi} + dV(\varphi)/d\varphi = 0$.

For the slow-roll regime, we impose the slow-roll approximations of $\dot{\varphi}^2/2 \ll V(\varphi)$ on the Friedmann equation and $|\ddot{\varphi}| \ll |3H\dot{\varphi}|$ on the EoM for φ , so that we can find $3H^2/\kappa^2 \approx V(\varphi) \approx \text{constant}$ and $3H\dot{\varphi} + dV(\varphi)/d\varphi \approx 0$. Furthermore, we define the slow-roll parameters $\epsilon \equiv -\dot{H}/H^2 = [1/(2\kappa^2)] [(dV(\varphi)/d\varphi)/V(\varphi)]^2 (\ll 1)$ and $\eta \equiv -\ddot{H}/(2H\dot{H}) = [1/\kappa^2] [(d^2V(\varphi)/d\varphi^2)/V(\varphi)] (\ll 1)$. During inflation, these parameters should be much smaller than unity. In addition, the number of e -folds is $N_e \equiv \ln(a_f/a_i) = \int_{t_i}^{t_f} H dt \approx \kappa^2 \int_{\varphi_i}^{\varphi_f} [V/(dV(\varphi)/d\varphi)] d\varphi$. Here, a_i (φ_i) and a_f (φ_f) are the values of the scale factor a (the scalar field φ) at the initial time t_i and end of time t_f of inflation, respectively. Moreover, in deriving the second approximate equality, we have used the second gravitational field equation with the slow-roll approximation. The amplitude of the power spectrum for the curvature perturbations is expressed as $\Delta_{\mathcal{R}}^2 = \kappa^2 H^2/(8\pi^2\epsilon) \approx \kappa^4 V/(24\pi^2\epsilon)$, where in the second approximate equality follows from the Friedmann equation operated the slow-roll approximation, and n_s and r are written as $n_s - 1 = -6\epsilon + 2\eta$ and $r = 16\epsilon$ [46,47]. The detailed explanations on the reconstruction of potential of inflationary models has been executed in Reference [48].

2.3. Reconstruction of $F(R)$ Gravity

There are two possible ways to reconstruct $F(R)$ gravity models to realize inflation. One is to start from the action in the Einstein frame. The other is to reconstruct the action in the Jordan frame. In this subsection, we consider the former way. We also explain the latter way in the next subsection.

The main purpose of our investigations is that we study the generalization of the Starobinsky inflation model. To execute it, we take an appropriate form of $V(\varphi)$ in the Jordan frame, which is an extended form from that in the Starobinsky inflation model. By taking the derivative of Equation (2) with respect to R , we find

$$RF_R = -\sqrt{6}\kappa \frac{d}{d\varphi} \left(V(\varphi) \exp \left(-2\sqrt{\frac{2}{3}}\kappa\varphi \right) \right) \quad (3)$$

Through this equation, *i.e.*, Equation (2), we reconstruct the form of $F(R)$ in the Jordan frame from the potential $V(\varphi)$ in the Einstein frame.

2.3.1. Extension of the Starobinsky Inflation Model

As the simplest model, we explore the following potential

$$V(\varphi) = c_1 + c_2 \exp \left(\sqrt{\frac{2}{3}}\kappa\varphi \right) + c_3 \exp \left(2\sqrt{\frac{2}{3}}\kappa\varphi \right) \quad (4)$$

where $c_1 (\neq 0)$, c_2 , and c_3 are constants. In this case, from Equation (3) we have $2c_1 F_R^2 + c_2 F_R - RF_R = 0$. By solving this equation with $F_R \neq 0$, we eventually find that the corresponding form of $F(R)$ can be expressed as

$$F(R) = R + \frac{R^2}{4c_1} + c_1 - c_3 \quad (5)$$

Here, we have set $-c_2/(2c_1) = 1$ in order to reproduce the Einstein-Hilbert term in $F(R)$ and used Equation (2) in determining the integration constant. If $c_1 = c_3$ (which leads to $c_3 = -c_2/2$), this model is equivalent to the Starobinsky inflation model with $V(\varphi) = c_1 \left(1 - \exp \left(-\sqrt{2/3}\kappa\varphi \right) \right)^2$, whereas for $c_1 > c_3$, this corresponds to an extended model of the Starobinsky inflation with a cosmological constant. As the possible origins of such a cosmological constant emerging at the large curvature regime could originate from the quantum effects, or a modification term of gravity removing the cosmological constant at the small curvature regime [49–52].

In the following, we set $c_3 = 0$ for simplicity and introduce a positive $\gamma_1 (>0)$ to express c_1 as $c_1 = \gamma_1/(4\kappa^2)$. Here, γ_1 has the mass dimension 2 and the dimensionless quantity $\gamma_1/M_{\text{Pl}} \ll 1$, so that in the higher-curvature regime, the correction to the Einstein gravity can appear. We explore the inflationary dynamics in this extended model with the potential $V(\varphi) = [\gamma_1/(4\kappa^2)] - [\gamma_1/(2\kappa^2)] \exp \left(\sqrt{2/3}\kappa\varphi \right)$. We consider the case that the inflaton slowly rolls from the initial value with its large negative amplitude down to the minimum of the potential as $V(\varphi = 0) = -\gamma_1/(4\kappa^2) (<0)$. In this case, from the gravitational field equations we find that the exponential inflation can be realized as $a(t) = a_i \exp(H_{\text{inf}}t)$ with the Hubble parameter $H_{\text{inf}} \equiv (1/2)\sqrt{\gamma_1/3}$ during inflation and a_i a constant. Furthermore, the solution of φ reads $\varphi = -\sqrt{3/2}\kappa \ln \left[(1/3)\sqrt{2\gamma_1/3}(t_i - t) \right]$. Around the beginning of inflation $t \simeq t_i$,

$|\varphi| \gg 1$ and the slow-roll parameters are $\epsilon = (4/3) \left[1 - \exp\left(-\sqrt{2/3}\kappa\varphi\right) \right]^{-2} \ll 1$ and $|\eta| = (4/3) \left| 1 - \exp\left(-\sqrt{2/3}\kappa\varphi\right) \right|^{-1} \ll 1$. These slow-roll parameters become of order of unity when φ approaches $\varphi_f \approx -0.17\sqrt{3/2}/\kappa$. For $|\varphi_i| \gg |\varphi_f|$, the number of e -folds is given by $N_e \approx (1/2) \sqrt{\gamma_1/6} t_i$. In addition, we find $t_f = t_i - 3\sqrt{3/(2\gamma_1)} \exp(0.17)$. For $N_e = 60$, we have $\varphi_i \approx 1.07M_{\text{Pl}}$. The slow-roll parameters are also represented as $\epsilon \approx 3/(4N_e^2)$ and $|\eta| \approx 1/N$. As a consequence, we obtain

$$\Delta_{\mathcal{R}}^2 \approx \frac{\kappa^2 \gamma_1 N_e^2}{72\pi^2}, \quad n_s - 1 \approx -\frac{2}{N_e}, \quad r \approx \frac{12}{N_e^2} \quad (6)$$

Here, we remark that $\Delta_{\mathcal{R}}^2 \approx \kappa^2 \gamma_1 N_e^2 / (72\pi^2) \ll \kappa^2 M_{\text{Pl}} N_e^2 / (72\pi^2)$ because $\gamma_1 \ll M_{\text{Pl}}$.

The observations obtained from the Planck satellite suggest $n_s = 0.9603 \pm 0.0073$ (68% CL) and $r < 0.11$ (95% CL) [7]. In this model, for $n_s < 1$ and $r < 0.11$, we see that $n_s > 1 - \sqrt{0.11/3} = 0.809$. Accordingly, for $N_e = 60$, we acquire $n_s = 0.967$ and $r = 3.00 \times 10^{-3}$. Thus, in this model, the spectral index n_s of the curvature perturbations and the tensor-to-scalar ratio r consistent with the Planck result can be realized. Various descriptions of inflationary models in terms of scalar field models [53] and perfect fluid as well as $F(R)$ gravity [54] have been examined. Moreover, the effects of quantum corrections on inflation have been explored in References [55–57]. We note that the BICEP2 experiment has recently detected the B -mode polarization of the cosmic microwave background (CMB) radiation with the tensor to scalar ratio $r = 0.20_{-0.05}^{+0.07}$ (68% CL) [1]. There have been proposed several discussions on the method to obtain this result regarding the subtraction of the foreground data, e.g., References [58–62]. A study to support the BICEP2 results has also been reported in Reference [63]. Very recently, the collaboration between BICEP2/Keck and Planck has released the result of $r < 0.12$ for the wave number $k = 0.05 \text{ Mpc}^{-1}$ of tensor mode of the density perturbations [64].

2.3.2. Power-Law Corrections to General Relativity

Next, we examine the following potential.

$$V(\varphi) = \frac{1}{2\kappa^2} \left\{ \left(\frac{1}{\beta q} \right)^{1/(q-1)} \left(1 - \exp\left(\sqrt{\frac{2}{3}}\kappa\varphi\right) \right)^{q/(q-1)} \exp\left[\left(\frac{q-2}{q-1}\right)\sqrt{\frac{2}{3}}\kappa\varphi\right] \left(\frac{q-1}{q}\right) + R_c \exp\left(\sqrt{\frac{2}{3}}\kappa\varphi\right) \left(\exp\left(\sqrt{\frac{2}{3}}\kappa\varphi\right) - 1 \right) - \Lambda_p \exp\left(2\sqrt{\frac{2}{3}}\kappa\varphi\right) \right\} \quad (7)$$

For this potential in the Einstein frame, a model in which a generic power-law correction term is added to the Einstein-Hilbert term is reconstructed as

$$F(R) = R + \beta (R + R_c)^q + \Lambda_p \quad (8)$$

Here, $\beta (>0)$ is a dimensionful positive constant, R_c and Λ_p are constant, and $q > 1$ ($q \neq 2$). Inflationary models in such a power-law type gravity with $q \lesssim 2$ has also been examined in Reference [65]. Through the same procedures as those executed for the previous case in Equations (4)

and (5), namely, by deriving the Hubble parameter at the inflationary stage, the scale factor, the solution of the inflaton φ , the slow-roll parameters, and the number of e -folds, if n is close to 2, we find

$$\begin{aligned} \Delta_{\mathcal{R}}^2 &\approx \frac{(q-1)^3}{16\pi^2 q (2-q)^2} \kappa^2 \exp \left[\frac{2}{3} N_e \frac{(q-2)^2}{(q-1)^2} \right] \left(\frac{1}{\beta q} \right)^{1/(q-1)}, \\ n_s - 1 &\approx -\frac{8(2-q)}{3(q-1)}, \quad r \approx \frac{16(2-q)^2}{3(q-1)^2} \end{aligned} \quad (9)$$

Indeed, for $q = 1.99$, we find $n_s = 0.962$ and $r = 1.08 \times 10^{-3}$. Therefore, this inflationary model can be compatible with the Planck analysis.

2.4. Reconstruction Method of $F(R)$ Gravity in the Jordan Frame

In this subsection, we reconstruct the form of $F(R)$ in the Jordan frame. We note that cosmology in the Einstein frame may differ from that in the Jordan frame due to their physical non-equivalence. Hence, it is more convenient to consider these theories in the Einstein and Jordan frames as different cosmological theories. Here, we discuss inflation in $F(R)$ gravity without its transformation to scalar-tensor theory. Eventually, the results may be different. Nevertheless, we demonstrate that $F(R)$ inflation is also consistent with the observations by the Planck satellite. From the other point of view, for the fairness, it should also be remarked that there are the debates on the issue of the equivalence between the (Jordan and Einstein) conformal frames in References [66–68]. Especially, the investigations in Reference [68] seems to support equivalence of conformal frames for inflationary scenarios. It is an issue of presentation rather than substance, but in the interest of fairness other points of view should be mentioned.

The reconstruction method of $F(R)$ gravity proposed in Reference [69] is as follows (for another reconstruction method of $F(R)$ gravity, see References [70–73]). We consider the action of $F(R)$ gravity with matter action S_{matter} as $S = \int d^4x \sqrt{-g} [F(R)/(2\kappa^2)] + S_{\text{matter}}$. We define the number of e -folds as $\bar{N} \equiv \ln(a_*/a)$ with a_* the scale factor at the fiducial time t_* . We define $\bar{G}(\bar{N}) \equiv H^2(\bar{N})$, so that R can be expressed as $R(\bar{N}) = 3 [(d\bar{G}(\bar{N})/d\bar{N}) + 4\bar{G}(\bar{N})]$. By solving this equation inversely, we get $\bar{N} = \bar{N}(R)$. In the flat FLRW space-time, the Friedmann equation can be represented as the second order differential equation of $F(R)$ with respect to R , given by

$$\begin{aligned} &-9\bar{G}(\bar{N}(R)) (4\bar{G}_{\bar{N}}(\bar{N}(R)) + G_{\bar{N}\bar{N}}(\bar{N}(R))) F_{RR}(R) \\ &+ 3 \left(\bar{G}(\bar{N}(R)) + \frac{1}{2} G_{\bar{N}}(\bar{N}(R)) \right) F_R(R) - \frac{F(R)}{2} + \kappa^2 \rho_{\text{matter}} = 0 \end{aligned} \quad (10)$$

where $F_{RR} \equiv d^2F(R)/dR^2$, $\bar{G}_{\bar{N}} \equiv d\bar{G}(\bar{N}(R))/d\bar{N}$, $G_{\bar{N}\bar{N}} \equiv d^2\bar{G}(\bar{N}(R))/d\bar{N}^2$, and ρ_{matter} is the energy density of matter.

As an example found in Reference [54], we study an exponential form $\bar{G}_{\bar{N}} = H^2(\bar{N}) = \bar{G}_1 e^{\tau\bar{N}} + \bar{G}_2$, where $G_1 (<0)$, $G_2 (>0)$, and $\tau (>0)$ are constants. For this expression, we have $e^{\tau\bar{N}} = (R - 12\bar{G}_2) / [3\bar{G}_1 (4 + \tau)]$. When the matter contribution is negligible, namely, $\rho_{\text{matter}} = 0$, the solution of Equation (10) is derived as

$$F(R) = Q_1 F(\omega_+, \omega_-, l; \vartheta) + Q_2 (12\bar{G}_2 - R)^{(1+1/\tau)} F\left(1 + \omega_- + \frac{1}{\tau}, 1 + \omega_+ + \frac{1}{\tau}, 2 + \frac{1}{\tau}; \vartheta\right) \quad (11)$$

Here, ω_{\pm} and ϑ are defined as

$$\omega_{\pm} \equiv \frac{-3\tau - 2 \pm \sqrt{\tau^2 - 20\tau + 4}}{4\tau}, \quad \vartheta \equiv \frac{12\bar{G}_2 - R}{3\bar{G}_2(4 + \tau)} \quad (12)$$

with $F(\varsigma_1, \varsigma_2, \varsigma_3; \vartheta)$ the hypergeometric function, where ς_i ($i = 1, \dots, 3$) are constants. If $(\bar{N}, G_1, G_2) = (50.0, -1.10, 10.0)$ and $(60.0, -1.20, 15.0)$, we obtain $(n_s, r) = (0.963, 6.89 \times 10^{-2})$ and $(0.965, 5.84 \times 10^{-2})$, respectively. (The running of the spectral index $\alpha_s \equiv dn_s/d \ln k$ is also estimated as $\alpha_s = -5.06 \times 10^{-5}$ and -4.51×10^{-5} for $(\bar{N}, G_1, G_2) = (50.0, -1.10, 10.0)$ and $(60.0, -1.20, 15.0)$, respectively.) Therefore, this model can yield the values of n_s and r indicated by the Planck analysis.

3. Trace-Anomaly Driven Inflation in Modified Gravity

In this section, we review inflation by the quantum anomaly in the framework of $F(R)$ gravity by following Reference [27]. The effect of the trace anomaly on inflation in $F(T)$ gravity with T the torsion scalar in teleparallelism has also been studied in Reference [74] (the explanations of teleparallelism exist, e.g., in References [15,19]).

3.1. Quantum Anomaly

It is known that the quantum anomaly appears via the procedure of the renormalization. For four-dimensional space-time, the trace of the energy momentum tensor $T_{\mu\nu}^{(QA)}$ originating from the quantum anomaly becomes [75–79]

$$\langle T_{\mu}^{(QA)\mu} \rangle = \alpha_1 \left(\mathcal{W} + \frac{2}{3} \square R \right) - \alpha_2 \mathcal{G} + \alpha_3 \square R \quad (13)$$

$$\mathcal{W} \equiv \mathcal{C}^{\mu\nu\rho\sigma} \mathcal{C}_{\mu\nu\rho\sigma} = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 2R^{\mu\nu} R_{\mu\nu} + \frac{1}{3} R^2 \quad (14)$$

$$\mathcal{G} \equiv R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2 \quad (15)$$

Here, the brackets $\langle \rangle$ denotes the vacuum expectation value. Moreover, $R_{\mu\nu\rho\sigma}$ is the Riemann tensor, $R_{\mu\nu}$ is the Ricci tensor, R is the scalar curvature, $\mathcal{C}_{\mu\nu\rho\sigma}$ is the Weyl tensor, to whose square \mathcal{W} corresponds, \mathcal{G} is the Gauss-Bonnet invariant, and $\square = g^{\mu\nu} \nabla_{\mu} \nabla_{\nu}$ with ∇_{μ} the covariant derivative associated with the metric tensor $g_{\mu\nu}$ is the covariant d'Alembertian. In addition, the coefficients are defined as $\alpha_1 \equiv (N_S + 6N_F + 12N_V) / (1920\pi^2)$, $\alpha_2 \equiv (N_S + 11N_F + 62N_V) / (5760\pi^2)$, $\alpha_3 \equiv -N_V / (96\pi^2)$ with the number of real scalar fields N_S , that of the Dirac (fermion) fields N_F , and that of vector fields N_V , where we have neglected the contributions from gravitons and higher-derivative conformal scalars.

We set the values of α_1 and α_2 positive, but the qualitative consequences do not depend on α_1 , α_2 , and α_3 . For example, in the $\mathcal{N} = 4$ SU(N) super Yang-Mills theory, we have $\alpha_1 = \alpha_2 = \bar{N}^2 / (64\pi^2)$ (>0) and $\alpha_3 = -\bar{N}^2 / (96\pi^2)$, where we have used $N_S = 6\bar{N}^2$, $N_F = 2\bar{N}^2$, and $N_V = \bar{N}^2$ with $\bar{N} \gg 1$. Here, $(2/3)\alpha_1 + \alpha_2 = 0$. However, if the action has an additional R^2 term as [80] $[\alpha_4 \bar{N}^2 / (192\pi^2)] \int d^4x \sqrt{-g} R^2$, where $\alpha_4 (>0)$ is a positive constant, we find $(2/3)\alpha_1 + \alpha_2 = -\alpha_4 \bar{N}^2 / (16\pi^2)$. In the classical

level, the vacuum expectation value of $\langle T_{\mu\nu}^{(\text{QA})} \rangle$ in Equation (13) can be regarded as a contribution of matter in the right-hand side as $R_{\mu\nu} - (1/2)g_{\mu\nu}R = \kappa^2 \langle T_{\mu\nu}^{(\text{QA})} \rangle$. Its trace reads $R = \alpha_1 [\mathcal{W} + (2/3)\square R] - \alpha_2 \mathcal{G} + \alpha_3 \square R + [\alpha_4 \bar{N}^2 \kappa^2 / (16\pi^2)] \square R$. Accordingly, for the Yang-Mills theory in the curved space-time, the R^2 term plays a role of correction of the higher curvature to the Einstein gravity or it contributes to the energy-momentum tensor as matter.

3.2. $F(R)$ Gravity with the Quantum Anomaly

The action describing $F(R)$ gravity is given by

$$S = \int d^4x \sqrt{-g} \frac{F(R)}{2\kappa^2} + S^{(\text{QA})} \quad (16)$$

$$F(R) \equiv R + 2\kappa^2 \left(\frac{\alpha_4 \bar{N}^2}{192\pi^2} \right) R^2 + f(R) \quad (17)$$

where $S^{(\text{QA})}$ is the action of the quantum anomaly and $f(R)$ is a function of R . For the original Starobinsky inflation [21], $f(R) = 0$. From the action in Equation (16), the gravitational field equation reads

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \kappa^2 \langle T_{\mu\nu}^{(\text{QA})} \rangle + \kappa^2 \left(\frac{\alpha_4 \bar{N}^2}{48\pi^2} \right) \left(-RR_{\mu\nu} + \frac{1}{4}R^2 g_{\mu\nu} + \nabla_\mu \nabla_\nu R - g_{\mu\nu} \square R^2 \right) \\ &\quad - f_R(R) \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \right) + \frac{1}{2}g_{\mu\nu} (f(R) - Rf_R(R)) \\ &\quad + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square) f_R(R) \end{aligned} \quad (18)$$

where $f_R(R) \equiv df(R)/dR$. The trace of this equation is expressed as $R = -\kappa^2 \{ \alpha_1 \mathcal{W} - \alpha_2 \mathcal{G} - [\alpha_4 \bar{N}^2 / (16\pi^2)] \square R \} - 2f(R) + Rf_R(R) + 3\square f_R(R)$.

In the FLRW background, the gravitational field equations are represented as

$$\frac{3}{\kappa^2} H^2 = \rho_{\text{eff}} \quad - \frac{1}{\kappa^2} (2\dot{H} + 3H^2) = P_{\text{eff}} \quad (19)$$

Here, ρ_{eff} and P_{eff} are the effective energy density and pressure and they obey the equation of the conservation law $\dot{\rho}_{\text{eff}} + 3H(\rho_{\text{eff}} + P_{\text{eff}}) = 0$. Their expressions are given by

$$\rho_{\text{eff}} \equiv \rho^{(\text{QA})} + \frac{1}{2\kappa^2} \left(Rf_R(R) - f(R) - 6H^2 f_R(R) - 6H\dot{f}_R(R) \right) \quad (20)$$

$$P_{\text{eff}} \equiv P^{(\text{QA})} + \frac{1}{2\kappa^2} \left[-Rf_R(R) + f(R) + (4\dot{H} + 6H^2)f_R(R) + 4H\dot{f}_R(R) + 2\ddot{f}_R(R) \right] \quad (21)$$

with the contributions from the quantum anomaly to the effective energy density and pressure

$$\rho^{(\text{QA})} \equiv \frac{\bar{\rho}}{a^4} + 6\alpha_2 H^4 - \left(\frac{\alpha_4 \bar{N}^2}{16\pi^2} \right) \left(18H^2 \dot{H} + 6\ddot{H}H - 3\dot{H}^2 \right) \quad (22)$$

$$P^{(\text{QA})} \equiv \frac{\bar{\rho}}{3a^4} - \alpha_2 \left(6H^4 + 8H^2 \dot{H} \right) + \left(\frac{\alpha_4 \bar{N}^2}{16\pi^2} \right) \left(9\dot{H}^2 + 12H\ddot{H} + 2\ddot{H} + 18H^2 \dot{H} \right) \quad (23)$$

where $\bar{\rho}$ is an integration constant, and in deriving these expressions, we have used the conservation equation for ρ_{eff} and P_{eff} . The $\bar{\rho}$ term corresponds to the energy density of radiation of the quantum

state [80]. In what follows, we take $\bar{\rho} = 0$ because at the inflationary stage around the Planck scale, the contribution of radiation can be neglected in comparison with that of the quantum anomaly as well as that of deviation of modified gravity from general relativity.

3.3. De Sitter Solutions by the Trace Anomaly

We consider the case that $f(R)$ is given by an exponential form [49,50,81]

$$f(R) = -2\Lambda_c \left[1 - \exp\left(-\frac{R}{R_c}\right) \right] \quad (24)$$

where $\Lambda_c (>0)$ and $R_c (>0)$ are positive constants. For $R/R_c \ll 1$, *i.e.*, in the late-time (e.g., present) universe, $f(R)$ approaches zero, and therefore our model becomes equivalent to R^2 gravity with the quantum anomaly. While for $R/R_c \gg 1$, namely, in the early universe such as the inflationary stage, the term of Λ_c plays a role of the cosmological constant. When we expand the exponential term as $\exp(-R/R_c) = 1 - R/R_c + O((R/R_c)^2)$, these terms make the de Sitter solution realized by the trace anomaly unstable. This property can lead to the graceful exit from inflation.

We derive the de Sitter solution at the inflationary stage. In the limit $R/R_c \gg 1$, from Equation (24) we get $f(R) \approx -2\Lambda_c$. For this limit, the Friedmann (first) equation in Equation (19) becomes $(3/\kappa^2) H^2 \approx 6\alpha_2 H^4 - 3\alpha_4 \bar{N}^2 / (16\pi^2) (6H^2 \dot{H} + 2\ddot{H}H - \dot{H}^2) + (\Lambda_c/\kappa^2)$. Solving this equation, we acquire the de Sitter solution

$$H_{\text{de Sitter}} = \sqrt{\frac{1}{4\alpha_2\kappa^2} \left[1 \pm \sqrt{1 - \frac{8\Lambda_c\alpha_2\kappa^2}{3}} \right]} \quad (25)$$

where we impose the condition $\Lambda_c < 3/(8\alpha_2\kappa^2)$ with $\alpha_2 > 0$ so that the solution can be real.

We examine the instability condition of the de Sitter solution. If the de Sitter solution describes inflation, it should be unstable because inflation has to end. We represent the perturbation as $H = H_{\text{de Sitter}} + \delta H(t)$, where $|\delta H(t)| \ll 1$. By combining it with the Friedmann equation, we obtain $\delta\ddot{H}(t) + 3H_{\text{de Sitter}}\delta\dot{H}(t) = -[\alpha_4\bar{N}^2/(16\pi^2)] [(1/\kappa^2) - 4\alpha_2 H_{\text{de Sitter}}^2] \delta H(t)$, where we have neglected the terms proportional to $\exp(-R/R_c)$ in Equation (24) because the stability of the solution is only related to Λ_c . The solution for $\delta H(t)$ is written as $\delta H(t) = \bar{H} \exp(\lambda_{\pm}t)$, where \bar{H} is a constant and $\lambda_{\pm} \equiv (-3H_{\text{de Sitter}} \pm \sqrt{D})/2$ (the subscriptions \pm of λ_{\pm} correspond to the signs of \pm in the r.h.s.). The de Sitter solutions are unstable (and adopted to describe the inflation) only if the value of λ_+ is a real and positive number, namely,

$$D \equiv 9H_{\text{de Sitter}}^2 - \frac{64\pi^2}{\alpha_4\bar{N}^2} J > 0, \quad J \equiv \frac{1}{\kappa^2} - 4\alpha_2 H_{\text{de Sitter}}^2 > 0 \quad (26)$$

where $\alpha_2 > 0$ and $\alpha_4 > 0$ have been used.

3.4. Trace-Anomaly Driven Inflation

We investigate the observable quantities of the spectral index n_s of the power spectrum for the the scalar mode of the density perturbations and the tensor-to-scalar ratio r in the trace-anomaly driven

inflation in exponential gravity, namely, inflation is described by the de Sitter solutions in Equation (25). In the slow-roll inflation, for the exponential form of $f(R)$ in Equation (24), we have

$$\epsilon \approx \frac{u^2 e^{-u} \Lambda_c \alpha_2 \kappa^2 (u+2)}{N_e^2 (1 - 8\Lambda_c \alpha_2 \kappa^2 / 3)} \ll 1, \quad |\eta| \approx \left| -\frac{u}{2N_e} \right| \ll 1 \quad (27)$$

where $1 \ll u \leq (1 + \sqrt{1-y}) / (1 - \sqrt{1-y})$ with $y \equiv 8\Lambda_c \alpha_2 \kappa^2 / 3$. As a result, we acquire

$$\Delta_{\mathcal{R}}^2 = \frac{1}{32\pi^2 \alpha_2 \epsilon} \left(1 + \sqrt{1 - \frac{8\Lambda_c \alpha_2 \kappa^2}{3}} \right) \quad (28)$$

$$n_s - 1 = -\frac{u}{N_e} - \frac{6u^2 e^{-u} \Lambda_c \alpha_2 \kappa^2 (u+2)}{N_e^2 (1 - 8\Lambda_c \alpha_2 \kappa^2 / 3)} \quad (29)$$

$$r = \frac{16u^2 e^{-u} \Lambda_c \alpha_2 \kappa^2 (u+2)}{N_e^2 (1 - 8\Lambda_c \alpha_2 \kappa^2 / 3)} \quad (30)$$

For $u = 3$, $\Lambda_c \alpha_2 \kappa^2 = 0.125$, and $N_e = 76$, we acquire $n_s = 0.960$ and $r = 1.20 \times 10^{-3}$. Consequently, the trace-anomaly driven inflation in exponential gravity can explain the Planck results.

4. R^2 Gravity in Loop Quantum Cosmology (LQC)

In this section, we review R^2 gravity and its cosmological dynamics in LQC with the holonomy corrections along the investigations in Reference [28]. (For LQC in teleparallelism, finite-time future singularities [82,83] have been examined in Reference [84], and bouncing behaviors have also been studied in References [85,86].)

4.1. $F(R)$ Gravity in LQC

We explain $F(R)$ gravity in the framework of LQC [87–89]. We consider the Einstein frame as in usual LQC only for the FLRW background with its spatially flatness [90]. In this case, the relation $\{\hat{\beta}_{\text{LQC}}, \hat{V}_{\text{volume}}\} = \gamma_{\text{BI}}/2$ is satisfied [91]. Here, $\{\}$ denotes the Poisson bracket in terms of classical variables $\hat{\beta}_{\text{LQC}} \equiv \gamma_{\text{BI}} \hat{H}$ with γ_{BI} the Barbero-Immirzi parameter and the volume $\hat{V}_{\text{volume}} \equiv \hat{a}^3$, where $\hat{a} = \sqrt{F_R a}$. Moreover, $\hat{\beta}_{\text{LQC}}$ and \hat{V}_{volume} are canonically conjugated quantities with each other, and these variables are the unique combination for a loop quantization [92]. We note that the hat shows the quantities in the Einstein frame.

It is necessary to take the Hilbert space, in which the quantum states are described by (almost) periodic functions, so that the property of the discrete space can be included. For this purpose, we use the Hamiltonian with the general holonomy corrections [35,93]. Namely, we introduce $\lambda_{\text{LQC}} \equiv \sqrt{(\sqrt{3}/2) \gamma_{\text{BI}}}$ and replace $\hat{\beta}_{\text{LQC}}$ with $\sin(\lambda_{\text{LQC}} \hat{\beta}_{\text{LQC}}) / \lambda_{\text{LQC}}$ in the Hamiltonian $\hat{\mathcal{H}} = -3 \left(\hat{\beta}_{\text{LQC}}^2 / \gamma_{\text{BI}}^2 \right) \hat{V}_{\text{volume}} + \hat{V}_{\text{volume}} [(1/2) (d\hat{\varphi}/d\hat{t})^2 + V(\hat{\varphi})] = 0$ with $d\hat{t} = \sqrt{F_R} dt$ by retaining $\{\hat{\beta}_{\text{LQC}}, \hat{V}_{\text{volume}}\} = \gamma_{\text{BI}}/2$ [94–96]. As a result, we acquire the novel Hamiltonian $\hat{\mathcal{H}}_{\text{LQC}}$. With the Hamiltonian equation $d\hat{V}_{\text{volume}}/d\hat{t} = \{\hat{V}_{\text{volume}}, \hat{\mathcal{H}}_{\text{LQC}}\}$ and the Hamiltonian constraint $\hat{\mathcal{H}}_{\text{LQC}} = 0$, the Friedmann equation with the holonomy corrections reads [91]

$$\hat{H}^2 = \frac{1}{3} \hat{\rho} \left(1 - \frac{\hat{\rho}}{\hat{\rho}_{\text{critical}}} \right) \quad (31)$$

Here, $\hat{\rho}$ is the energy density of matter, and $\hat{\rho}_{\text{critical}} \equiv 3/(\lambda_{\text{LQC}}\gamma_{\text{BI}})^2$ is the critical energy density.

4.2. R^2 Gravity in LQC

For R^2 gravity, whose action is given by $S = \int d^4x \sqrt{-g} [F(R)/(2\kappa^2)]$ with $F(R) = R + \alpha_S \kappa^2 R^2$, there appears curvature singularities in the early universe. In what follows, when we consider R^2 gravity, we analyze this action. On the other hand, for R^2 gravity in the context of LQC, it is possible that no singularity happens. We show this point below. It follows from Equation (31) that $0 \leq \hat{\rho} \leq \hat{\rho}_{\text{critical}}$ and $-\sqrt{\hat{\rho}_{\text{critical}}/12} \leq \hat{H} \leq \sqrt{\hat{\rho}_{\text{critical}}/12}$. In addition, for R^2 gravity, we have $V(\hat{\varphi}) = [1/(8\alpha_S \kappa^2)] \left(1 - \exp\left(-\sqrt{2/3}\kappa\hat{\varphi}\right)\right)^2$ in the Einstein frame. Therefore, we find $0 \leq (d\hat{\varphi}/d\hat{t})^2 \leq 2\hat{\rho}_{\text{critical}}$ and $0 \leq V(\hat{\rho}) \leq \hat{\rho}_{\text{critical}}$. For LQC, the Raychaudhuri equation becomes $d\hat{H}/d\hat{t} = -(1/2)(d\hat{\varphi}/d\hat{t})^2(1 - 2\hat{\rho}/\hat{\rho}_{\text{critical}})$. With this equation, we find $|d\hat{H}/d\hat{t}| \leq (1/2)(d\hat{\varphi}/d\hat{t})^2 \leq \hat{\rho}_{\text{critical}}$, and thus, it follows from this relation that $|\hat{R}| \leq 7\hat{\rho}_{\text{critical}}$.

Furthermore, the potential $V(\hat{\varphi})$ in the Einstein frame described above obeys $dV(\hat{\varphi})/d\hat{\varphi} = (1/F_R)\sqrt{V(\hat{\varphi})/(3\alpha_S \kappa^2)}$. Substituting this equation into the relation between R and \hat{R} as $R = F_R \left[\hat{R} + (d\hat{\varphi}/d\hat{t})^2 + \sqrt{6}(dV(\hat{\varphi})/d\hat{\varphi})\right]$ and using the relation $1 - 2\alpha_S \kappa^2 \left[\hat{R} + (d\hat{\varphi}/d\hat{t})^2\right] \geq 1 - 18\alpha_S \kappa^2 \hat{\rho}_{\text{critical}}$, we acquire

$$|R| \leq \frac{1}{1 - 18\alpha_S \kappa^2 \hat{\rho}_{\text{critical}}} \left(18\hat{\rho}_{\text{critical}} + \sqrt{\frac{2\hat{\rho}_{\text{critical}}}{\alpha_S \kappa^2}}\right) \quad (32)$$

For $\alpha_S \kappa^2 < 1/(18\hat{\rho}_{\text{critical}})$, the absolute value of R is bounded as in Equation (32). In addition, the relation between H and \hat{H} is represented as $H = \sqrt{F_R} \left[\hat{H} - (1/\sqrt{6})(d\hat{\varphi}/d\hat{t})\right]$. Accordingly, $|H|$ is bounded. As a result, $|\dot{H}| = (1/6)|R - 12H^2|$ does not diverge. Thus, there does not appear any singularity in R^2 gravity for LQC.

4.3. Loop Quantum R^2 Gravity in the Einstein Frame

We further analyze the dynamics of R^2 gravity (*i.e.*, $F(R) = R + \alpha_S \kappa^2 R^2$) in the Einstein frame, where equations become simpler than those in the Jordan frame. The equation of motion for $\hat{\varphi}$ is expressed by $d^2\hat{\varphi}/d\hat{t}^2 + 3\hat{H}(d\hat{\varphi}/d\hat{t}) + dV(\hat{\varphi})/d\hat{\varphi} = 0$ with $V(\hat{\varphi}) = [1/(8\alpha_S \kappa^2)] \left(1 - \exp\left(-\sqrt{2/3}\kappa\hat{\varphi}\right)\right)^2$. We introduce a variable $\hat{\Psi} \equiv \exp\left(\sqrt{2/3}\kappa\hat{\varphi}\right)$. From the equation of motion for $\hat{\varphi}$, we obtain

$$\frac{d^2\hat{\Psi}}{d\hat{t}^2}\hat{\Psi} - \left(\frac{d\hat{\Psi}}{d\hat{t}}\right)^2 + 3\hat{H}\frac{d\hat{\Psi}}{d\hat{t}}\hat{\Psi} + \frac{1}{6\alpha_S \kappa^2}(\hat{\Psi} - 1) = 0 \quad (33)$$

Since this equation is invariant under the transformation $(\hat{H}, \hat{t}) \rightarrow (-\hat{H}, -\hat{t})$, the solution orbit draws the symmetric trajectories of the expansion and contraction phases on the $(\hat{\Psi}, d\hat{\Psi}/d\hat{t})$ plane in terms of the $d\hat{\Psi}/d\hat{t} = 0$ axis. That is, if there is a trajectory $(\hat{\Psi}(\hat{t}), d\hat{\Psi}(\hat{t})/d\hat{t})$ for the contraction phase ($\hat{H} < 0$), we have a trajectory $(\hat{\Psi}(-\hat{t}), -d\hat{\Psi}(-\hat{t})/d\hat{t})$ for the expansion phase ($\hat{H} > 0$).

In addition, the energy density is expressed as

$$\hat{\rho} = \frac{3}{4\hat{\Psi}^2} \left[\left(\frac{d\hat{\Psi}}{d\hat{t}} \right)^2 + \frac{1}{6\alpha_S \kappa^2} (\hat{\Psi} - 1)^2 \right] \quad (34)$$

This implies that $\hat{H} = 0$ at $(\hat{\Psi}(\hat{t}), d\hat{\Psi}(\hat{t})/d\hat{t}) = (1, 0)$. For $\hat{\rho} = \hat{\rho}_{\text{critical}}$, we find

$$\frac{\left(d\hat{\Psi}/d\hat{t} \right)^2}{4\hat{\rho}_{\text{critical}}/(3A)} + \frac{\left(\hat{\Psi} - 1/A \right)^2}{8\alpha_S \kappa^2 \hat{\rho}_{\text{critical}}/A^2} = 1 \quad (35)$$

with $A \equiv 1 - 8\alpha_S \kappa^2 \hat{\rho}_{\text{critical}}$. This depicts an ellipse for $A > 0$, a parabola for $A = 0$, and a hyperbola for $A < 0$. There exists only the critical point at $(\hat{\Psi}(\hat{t}), d\hat{\Psi}(\hat{t})/d\hat{t}) = (1, 0)$, where $\hat{\rho} = 0$. All of the trajectories start from this point and come back to it. Therefore, it corresponds to both the beginning and end points of the universe.

As a consequence, thanks to the holonomy corrections, in the Einstein frame, the bounces can occur when $\hat{\rho} = \hat{\rho}_{\text{critical}}$. The universe evolves from the contraction phase ($\hat{H} < 0$). Its trajectory oscillates near the critical point $(\hat{\Psi}(\hat{t}), d\hat{\Psi}(\hat{t})/d\hat{t}) = (1, 0)$ and the oscillatory amplitude becomes large. Eventually, the trajectory approaches the line $\hat{\rho} = \hat{\rho}_{\text{critical}}$ and the bounce happens. After that, the expansion phase ($\hat{H} > 0$) begins and the trajectory goes back to the critical point $(\hat{\Psi}(\hat{t}), d\hat{\Psi}(\hat{t})/d\hat{t}) = (1, 0)$ with its oscillating behavior.

Next, we explore the slow-roll inflation. With the slow-roll approximations $(d\hat{\varphi}/d\hat{t})^2 \ll V(\hat{\varphi})$ and $|d^2\hat{\varphi}/d\hat{t}^2| \ll |3\hat{H}(d\hat{\varphi}/d\hat{t})|$, the Friedmann equation with the holonomy corrections and equation of motion for $\hat{\varphi}$ are written as $\hat{H}^2 = (1/3)V(\hat{\varphi})(1 - V(\hat{\varphi})/\hat{\rho}_{\text{critical}})$ and $3\hat{H}(d\hat{\varphi}/d\hat{t}) + dV(\hat{\varphi})/d\hat{\varphi} = 0$, respectively. The number of e -folds during inflation is derived by $\hat{N}_e \equiv \int_{\hat{t}_i}^{\hat{t}_f} \hat{H} d\hat{t} = \int_{\hat{\varphi}_i}^{\hat{\varphi}_f} \hat{H}/(d\hat{\varphi}/d\hat{t}) d\hat{\varphi} \approx \int_{\hat{\varphi}_e}^{\hat{\varphi}_i} [V(\hat{\varphi})/(dV(\hat{\varphi})/d\hat{\varphi})] (1 - V(\hat{\varphi})/\hat{\rho}_{\text{critical}}) d\hat{\varphi}$. For $V(\hat{\varphi}) = [1/(8\alpha_S \kappa^2)] \left(1 - \exp\left(-\sqrt{2/3}\kappa\hat{\varphi}\right) \right)^2$, we find $\hat{N}_e \approx (3/4) \exp\left(\sqrt{2/3}\kappa\hat{\varphi}_i\right)$. Moreover, for $\hat{N}_e \gg 1$, the slow-roll parameters are described as

$$\hat{\epsilon} \approx \frac{3}{4\hat{N}_e^2} \frac{[1 - 1/(4\alpha_S \kappa^2 \hat{\rho}_{\text{critical}})]}{[1 - 1/(8\alpha_S \kappa^2 \hat{\rho}_{\text{critical}})]^2}, \quad \hat{\eta} \approx -\frac{1}{\hat{N}_e} \frac{1}{[1 - 1/(8\alpha_S \kappa^2 \hat{\rho}_{\text{critical}})]} \quad (36)$$

By using these expressions, the spectral index of scalar mode of the density perturbations $\hat{n}_s - 1 = -6\hat{\eta} + 2\hat{\epsilon}$ and the tensor-to-scalar ratio $\hat{r} = 16\hat{\epsilon}$ become

$$\hat{n}_s - 1 \approx -\frac{2}{\hat{N}_e} \frac{1}{[1 - 1/(8\alpha_S \kappa^2 \hat{\rho}_{\text{critical}})]} \quad (37)$$

$$\hat{r} \approx \frac{12}{\hat{N}_e^2} \frac{[1 - 1/(4\alpha_S \kappa^2 \hat{\rho}_{\text{critical}})]}{[1 - 1/(8\alpha_S \kappa^2 \hat{\rho}_{\text{critical}})]^2} \quad (38)$$

In the limit $\hat{\rho}_{\text{critical}} \rightarrow \infty$, these expressions of \hat{n}_s and \hat{r} become equivalent to those for pure R^2 gravity without the holonomy corrections, *i.e.*, the original Starobinsky inflation model. For instance, if $\hat{N}_e = 68.0$ and $8\alpha_S \kappa^2 \hat{\rho}_{\text{critical}} = 8.50$, we acquire $\hat{n}_s = 0.967$ and $\hat{r} = 2.55 \times 10^{-3}$. Thus, in an R^2 gravity model with the holonomy corrections in the context of LQC, the spectral index of scalar mode of the density perturbations and the tensor-to-scalar ratio can be compatible with the Planck data.

We mention that in the Starobinsky inflation model, for $\hat{N}_e = 60.0$ (68.0), we have $\hat{n}_s = 0.967$ (0.971) and $\hat{r} = 3.33 \times 10^{-3}$ (2.60×10^{-3}).

4.4. Loop Quantum R^2 Gravity in the Jordan Frame

Furthermore, in the Jordan frame, we analyze the cosmological behaviors in R^2 gravity for LQC. For the comparison with the consequences in the Einstein frame, at the point on the $(\hat{\Psi}, d\hat{\Psi}/d\hat{t})$ plane, the bounce occurs, *i.e.*, H becomes zero. From $H = \sqrt{F_R} \left[\hat{H} - (1/\sqrt{6}) (d\hat{\phi}/d\hat{t}) \right]$, if $H = 0$, we get $\hat{H}^2 = (1/4) \left(d\hat{\Psi}/d\hat{t} \right)^2 / \hat{\Psi}^2$. This equation leads to

$$\frac{\left(d\hat{\Psi}/d\hat{t} \right)^2}{\hat{\rho}_{\text{critical}} / (12B_{\pm})} + \frac{\left(\hat{\Psi} - C_{\pm}/B_{\pm} \right)^2}{2\alpha_S \kappa^2 \hat{\rho}_{\text{critical}} / B_{\pm}^2} = 1 \quad (39)$$

with $B_{\pm} \equiv 1 \pm \sqrt{8\alpha_S \kappa^2 \hat{\rho}_{\text{critical}}}$ and $C_{\pm} \equiv 1 \pm \sqrt{2\alpha_S \kappa^2 \hat{\rho}_{\text{critical}}}$, where the subscription \pm in B_{\pm} and C_{\pm} corresponds to the sign \pm . The case of $+$ sign is for $0 < \hat{\Psi} < 1$. In this case, this curve draws an ellipse for $B_+ > 0$, a parabola for $B_+ = 0$, and an hyperbola for $B_+ < 0$. On the other hand, the case of $-$ sign is for $\hat{\Psi} > 1$. In this case, it shows an ellipse. If the trajectory intersects this curve in the Einstein frame, the bounce happens in the Jordan frame. At the bouncing point, the relation $\hat{H} = (1/2) \left(d\hat{\Psi}/d\hat{t} \right) / \hat{\Psi}$ has to be met.

In the Einstein frame, the universe first contracts and finally expands at the critical point $(\hat{\Psi}(\hat{t}), d\hat{\Psi}(\hat{t})/d\hat{t}) = (1, 0)$. With the relation between H and \hat{H} and its time derivative

$$H = \sqrt{\hat{\Psi}} \left(\hat{H} - \frac{1}{2} \frac{d\hat{\Psi}/d\hat{t}}{\hat{\Psi}} \right) \quad (40)$$

$$\dot{H} = \frac{1}{2} \frac{d\hat{\Psi}}{d\hat{t}} \left(\hat{H} - \frac{1}{2} \frac{d\hat{\Psi}/d\hat{t}}{\hat{\Psi}} \right) + \hat{\Psi} \left[\frac{d\hat{H}}{d\hat{t}} - \frac{1}{2} \frac{d}{d\hat{t}} \left(\frac{d\hat{\Psi}/d\hat{t}}{\hat{\Psi}} \right) \right] \quad (41)$$

it is seen that in the Jordan frame, the universe begins and ends at the point $(H, \dot{H}) = (0, 0)$. It should be emphasized that the holonomy corrections yield the bounce in the Jordan frame, and hence, if they are absent, a singularity appears at the early stage of the universe.

5. Bouncing Cosmology in $F(R)$ Gravity

In this section, we review the cosmological bounce from $F(R)$ gravity. Especially, we present the consequences found in References [41,42]. The bouncing behaviors in various modified gravity theories have also been investigated in References [97–102]. We show that it is possible to reconstruct an $F(R)$ gravity theory in which the matter bounce can happen in the framework of LQC.

For the Friedmann equation with the holonomy corrections Equation (31), the energy density of matter can be represented as $\rho = \bar{\rho}_m / [(3/4)t^2 + 1]$ with $\bar{\rho}_m$ a constant. In this case, the scale factor and the Hubble parameter read [86,99]

$$a(t) = \left(\frac{3}{4} \bar{\rho}_m t^2 + 1 \right)^{1/3}, \quad H(t) = \frac{(1/2) \bar{\rho}_m t}{(3/4) \bar{\rho}_m t^2 + 1} \quad (42)$$

For these expressions, the solution of Equation (10) yields the form of $F(R)$ to realize the matter bounce described by a and H in Equation (42) for LQC. We solve Equation (42), and consequently acquire

$$F(R) = \mathcal{I}_1 R + \mathcal{I}_2 R^{1/2} \quad (43)$$

where \mathcal{I}_1 and \mathcal{I}_2 are constants. We can set $\mathcal{I}_1 = 1$, so that the Einstein-Hilbert term should be included.

In Reference [42], the reconstruction of various modified gravity theories including $F(R)$, $F(\mathcal{G})$, and $F(T)$ gravity theories has been performed, where $F(\mathcal{G})$ is an arbitrary function of the Gauss-Bonnet invariant \mathcal{G} , to describe the two-times bouncing phenomena, called super-bounce [103,104], and the ekpyrotic scenario [105] in the context of LQC.

6. Conclusions

In the present paper, we have reviewed inflationary models in modified gravity theories such as $F(R)$ gravity including R^2 gravity with extended terms so that we can generalize the Starobinsky inflation in R^2 gravity and derive its important properties to be useful clues to obtain the information on physics in the early universe.

First, we have studied inflationary cosmology by modification terms of gravity, especially, inflation in $F(R)$ gravity, by following Reference [26]. The Starobinsky inflation in R^2 gravity is considered to be the seminal and significant idea of inflationary models in modified gravity theories. We have made the coformal transformation from the Jordan frame (namely, $F(R)$ gravity) to the Einstein frame (*i.e.*, general gravity plus the scalar field theory), and given slow-roll dynamics of inflation in the Einstein frame. In addition, we have reconstructed $F(R)$ gravity models, which are an extended version of the Starobinsky inflation model in R^2 gravity and general relativity with power-law correction terms.

Second, we have explored the trace-anomaly driven inflation in $F(R)$ gravity along the discussions in Reference [27]. We have first explained the quantum anomaly appearing through the process of the renormalization in four-dimensional space-time. We have further discussed $F(R)$ gravity with the quantum anomaly and the de Sitter solutions for inflation due to the trace anomaly.

Third, based on Reference [28], we have examined inflation in R^2 gravity and the cosmological evolutions for LQC with the holonomy corrections. We have analyzed R^2 gravity for LQC in both the Einstein and Jordan frames. We have found that in the Jordan frame, owing to the holonomy corrections, the bounce can happen, and accordingly the cosmic singularities can be removed, although such singularities appear in ordinary R^2 gravity.

In these three inflationary models, we have shown that the spectral index of scalar modes of the density perturbations and the tensor-to-scalar ratio can be compatible with the Planck analysis.

Furthermore, we have presented the recent developments of the bounce cosmology in $F(R)$ gravity obtained in References [41,42]. It has been performed that an $F(R)$ gravity theory can be reconstructed, where the matter bounce occurs in the context of LQC, thanks to the reconstruction method of $F(R)$ gravity. Recently, the reconstruction of $F(R)$, $F(\mathcal{G})$, and $F(T)$ gravity theories have also been executed, in which the super-bounce (*i.e.*, two-times bounce behaviors) and the ekpyrotic scenario for LQC can be realized.

In this work, we have concentrated on the accelerating universe from $F(R)$ gravity. Note, however, that it is possible to extend this study for more complicated versions of effective gravity, which comes from quantum gravity. Particularly, it has recently been demonstrated that successful inflation consistent with the Planck data may emerge from multiplicatively-renormalizable higher derivative quantum gravity in Reference [106].

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Author Contributions

All authors contributed equally to this work.

Conflicts of Interest

The authors declare no conflict of interest.

References

1. Ade, P.A.R.; Aikin, R.W.; Barkats, D.; Benton S.J.; Bischoff, C.A.; Bock, J.J.; Brevik, J.A.; Buder, I.; Bullock, E.; Dowell, C.D.; *et al.* Detection of B-Mode Polarization at Degree Angular Scales by BICEP2. *Phys. Rev. Lett.* **2014**, *112*, doi:10.1103/PhysRevLett.112.241101.
2. Spergel, D.N.; Verde, L.; Peiris, H.V.; Komatsu, E.; Nolta, M.R.; Bennett, C.L.; Halpern, M.; Hinshaw, G.; Jarosik, N.; Kogut, A.; *et al.* First Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Determination of Cosmological Parameters. *Astrophys. J. Suppl.* **2003**, *148*, 175–194.
3. Spergel, D.N.; Bean, R.; Dore, O.; Nolta, M.R.; Bennett, C.L.; Hinshaw, G.; Jarosik, N.; Komatsu, E.; Page, L.; Peiris, H.V.; *et al.* Wilkinson Microwave Anisotropy Probe (WMAP) three year results: Implications for cosmology. *Astrophys. J. Suppl.* **2007**, *170*, 377–408.
4. Komatsu, E.; Dunkley, J.; Nolta, M.R.; Bennett, C.L.; Gold, B.; Hinshaw, G.; Jarosik, N.; Larson, D.; Limon, M.; Page, L.; *et al.* Five-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation. *Astrophys. J. Suppl.* **2009**, *180*, 330–376.
5. Komatsu, E.; Smith, K.M.; Dunkley, J.; Bennett, C.L.; Gold, B.; Hinshaw, G.; Jarosik, N.; Larson, D.; Nolta, M.R.; Page, L.; *et al.* Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation. *Astrophys. J. Suppl.* **2011**, *192*, doi:10.1088/0067-0049/192/2/18.
6. Hinshaw, G.; Larson, D.; Komatsu, E.; Spergel, D.N.; Bennett, C.L.; Dunkley, J.; Nolta, M.R.; Halpern, M.; Hill, R.S.; Odegard, N.; *et al.* Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results. *Astrophys. J. Suppl.* **2013**, *208*, doi:10.1088/0067-0049/208/2/19.

7. Ade, P.A.R.; Aghanim, N.; Armitage-Caplan, C.; Arnaud, M.; Ashdown, M.; Atrio-Barandela, F.; Aumont, J.; Baccigalupi, C.; Banday, A.J.; Barreiro, R.B.; *et al.* Planck 2013 results. XVI. Cosmological parameters. *Astron. Astrophys.* **2014**, *571*, doi:10.1051/0004-6361/201321591
8. Ade, P.A.R.; Aghanim, N.; Armitage-Caplan, C.; Arnaud, M.; Ashdown, M.; Atrio-Barandela, F.; Aumont, J.; Baccigalupi, C.; Banday, A.J.; Barreiro, R.B.; *et al.* Planck 2013 results. XXII. Constraints on inflation. *Astron. Astrophys.* **2014**, *571*, doi:10.1051/0004-6361/201321569
9. Linde, A.D. Chaotic Inflation. *Phys. Lett. B* **1983**, *129*, 177–181.
10. Linde, A. Inflationary Cosmology after Planck 2013. 2014, arXiv:1402.0526, arXiv.org e-Print archive. Available online: <http://arxiv.org/abs/1402.0526> (accessed on 9 March 2015).
11. Gorbunov, D.S.; Rubakov, V.A. *Introduction to the Theory of the Early Universe: Cosmological Perturbations and Inflationary Theory*; World Scientific: Hackensack, NJ, USA, 2011.
12. Nojiri, S.; Odintsov, S.D. Unified cosmic history in modified gravity: From $F(R)$ theory to Lorentz non-invariant models. *Phys. Rep.* **2011**, *505*, 59–144.
13. Nojiri, S.; Odintsov, S.D. Introduction to modified gravity and gravitational alternative for dark energy. *ECONF* **2007**, *4*, 115–145.
14. Bamba, K.; Nojiri, S.; Odintsov, S.D. Modified gravity: Walk through accelerating cosmology. 2013, arXiv:1302.4831, arXiv.org e-Print archive. Available online: <http://arxiv.org/abs/1302.4831> (accessed on 9 March 2015).
15. Bamba, K.; Odintsov, S.D. Universe acceleration in modified gravities: $F(R)$ and $F(T)$ cases. 2014, arXiv:1402.7114, arXiv.org e-Print archive. Available online: <http://arxiv.org/abs/1402.7114> (accessed on 9 March 2015).
16. Capozziello, S.; Faraoni, V. *Beyond Einstein Gravity*; Springer: London, UK, 2010.
17. Capozziello, S.; de Laurentis, M. Extended Theories of Gravity. *Phys. Rep.* **2011**, *509*, 167–321.
18. De la Cruz-Dombriz, A.; Sáez-Gómez, D. Black holes, cosmological solutions, future singularities, and their thermodynamical properties in modified gravity theories. *Entropy* **2012**, *14*, 1717–1770.
19. Bamba, K.; Capozziello, S.; Nojiri, S.; Odintsov, S.D. Dark energy cosmology: The equivalent description via different theoretical models and cosmography tests. *Astrophys. Space Sci.* **2012**, *342*, 155–228.
20. Joyce, A.; Jain, B.; Khoury, J.; Trodden, M. Beyond the Cosmological Standard Model. 2014, arXiv:1407.0059, arXiv.org e-Print archive. Available online: <http://arxiv.org/abs/1407.0059> (accessed on 9 March 2015).
21. Starobinsky, A.A. A New Type of Isotropic Cosmological Models Without Singularity. *Phys. Lett. B* **1980**, *91*, 99–102.
22. Vilenkin, A. Classical and Quantum Cosmology of the Starobinsky Inflationary Model. *Phys. Rev. D* **1985**, *32*, 2511–2521.
23. Nojiri, S.; Odintsov, S.D.; Sasaki, M.; Zhang, Y.L. Screening of cosmological constant in non-local gravity. *Phys. Lett. B* **2011**, *696*, 278–282.
24. Bamba, K.; Nojiri, S.; Odintsov, S.D.; Sasaki, M. Screening of cosmological constant for De Sitter Universe in non-local gravity, phantom-divide crossing and finite-time future singularities. *Gen. Rel. Grav.* **2012**, *44*, 1321–1356.

25. Zhang, Y.L.; Sasaki, M. Screening of cosmological constant in non-local cosmology. *Int. J. Mod. Phys. D* **2012**, *21*, doi:10.1142/S021827181250006X.
26. Sebastiani, L.; Cognola, G.; Myrzakulov, R.; Odintsov, S.D.; Zerbini, S. Nearly Starobinsky inflation from modified gravity. *Phys. Rev. D* **2014**, *89*, doi:10.1103/PhysRevD.89.023518
27. Bamba, K.; Myrzakulov, R.; Odintsov, S.D.; Sebastiani, L. Trace-anomaly driven inflation in modified gravity and the BICEP2 result. *Phys. Rev. D* **2014**, *90*, doi:10.1103/PhysRevD.90.043505.
28. Amorós, J.; de Haro, J.; Odintsov, S.D. $R + \alpha R^2$ Loop Quantum Cosmology. *Phys. Rev. D* **2014**, *89*, doi:10.1103/PhysRevD.89.104010.
29. Bojowald, M. Absence of singularity in loop quantum cosmology. *Phys. Rev. Lett.* **2001**, *86*, 5227–5230.
30. Bojowald, M. Inflation from quantum geometry. *Phys. Rev. Lett.* **2002**, *89*, doi:10.1103/PhysRevLett.89.261301.
31. Ashtekar, A. An Introduction to Loop Quantum Gravity Through Cosmology. *Nuovo Cim. B* **2007**, *122*, 135–155.
32. Ashtekar, A.; Bojowald, M. Quantum geometry and the Schwarzschild singularity. *Class. Quant. Grav.* **2006**, *23*, 391–411.
33. Ashtekar, A.; Corichi, A.; Singh, P. Robustness of key features of loop quantum cosmology. *Phys. Rev. D* **2008**, *77*, doi:10.1103/PhysRevD.77.024046.
34. Thiemann, T. Lectures on loop quantum gravity. *Lect. Notes Phys.* **2003**, *631*, 41–135.
35. Ashtekar, A.; Bojowald, M.; Lewandowski, J. Mathematical structure of loop quantum cosmology. *Adv. Theor. Math. Phys.* **2003**, *7*, 233–268.
36. Bojowald, M. Loop quantum cosmology. *Living Rev. Relativ.* **2005**, *8*, doi:10.12942/lrr-2005-11.
37. Thiemann, T. *Modern Canonical Quantum General Relativity*; Cambridge University Press: Cambridge, UK, 2007.
38. Bojowald, M. Loop quantum cosmology. *Living Rev. Relativ.* **2008**, *11*, doi:10.12942/lrr-2008-4.
39. Ashtekar, A.; Singh, P. Loop Quantum Cosmology: A Status Report. *Class. Quant. Grav.* **2011**, *28*, doi:10.1088/0264-9381/28/21/213001.
40. Bojowald, M. Quantum Cosmology: Effective Theory. *Class. Quant. Grav.* **2012**, *29*, doi:10.1088/0264-9381/29/21/213001.
41. Odintsov, S.D.; Oikonomou, V.K. Matter Bounce Loop Quantum Cosmology from $F(R)$ Gravity. *Phys. Rev. D* **2014**, *90*, doi:10.1103/PhysRevD.90.124083.
42. Odintsov, S.D.; Oikonomou, V.K.; Saridakis, E.N. Superbounce and Loop Quantum Ekpyrotic Cosmologies from Modified Gravity: $F(R)$, $F(G)$ and $F(T)$ Theories. 2015, arXiv:1501.06591, arXiv.org e-Print archive. Available online: <http://arxiv.org/abs/1501.06591> (accessed on 9 March 2015).
43. Bamba, K.; Geng, C.Q.; Nojiri, S.; Odintsov, S.D. Crossing of the phantom divide in modified gravity. *Phys. Rev. D* **2009**, *79*, doi:10.1103/PhysRevD.79.083014.
44. Maeda, K.I. Towards the Einstein-Hilbert Action via Conformal Transformation. *Phys. Rev. D* **1989**, *39*, 3159–3162.

45. Fujii, Y.; Maeda, K.I. *The Scalar-Tensor Theory of Gravitation*; Cambridge University Press: Cambridge, UK, 2003.
46. Mukhanov, V.F.; Chibisov, G.V. Quantum Fluctuation and Nonsingular Universe. *JETP Lett.* **1981**, *33*, 532–535. (In Russian)
47. Liddle, A.R.; Lyth, D.H. *Cosmological Inflation and Large-Scale Structure*; Cambridge University Press: Cambridge, UK, 2000.
48. Lidsey, J.E.; Liddle, A.R.; Kolb, E.W.; Copeland, E.J.; Barreiro, T.; Abney, M. Reconstructing the inflation potential: An overview. *Rev. Mod. Phys.* **1997**, *69*, 373–410.
49. Cognola, G.; Elizalde, E.; Nojiri, S.; Odintsov, S.D.; Sebastiani, L.; Zerbini, S. A Class of viable modified $f(R)$ gravities describing inflation and the onset of accelerated expansion. *Phys. Rev. D* **2008**, *77*, doi:10.1103/PhysRevD.77.046009.
50. Linder, E.V. Exponential Gravity. *Phys. Rev. D* **2009**, *80*, doi:10.1103/PhysRevD.80.123528.
51. Elizalde, E.; Nojiri, S.; Odintsov, S.D.; Sebastiani, L.; Zerbini, S. Non-singular exponential gravity: A simple theory for early- and late-time accelerated expansion. *Phys. Rev. D* **2011**, *83*, doi:10.1103/PhysRevD.83.086006.
52. Oikonomou, V.K. An Exponential $F(R)$ Dark Energy Model. *Gen. Rel. Grav.* **2013**, *45*, 2467–2481.
53. Bamba, K.; Nojiri, S.; Odintsov, S.D. Reconstruction of scalar field theories realizing inflation consistent with the Planck and BICEP2 results. *Phys. Lett. B* **2014**, *737*, 374–378.
54. Bamba, K.; Nojiri, S.; Odintsov, S.D.; Sáez-Gómez, D. Inflationary universe from perfect fluid and $F(R)$ gravity and its comparison with observational data. *Phys. Rev. D* **2014**, *90*, doi:10.1103/PhysRevD.90.124061.
55. Bamba, K.; Cognola, G.; Odintsov, S.D.; Zerbini, S. One-loop Modified Gravity in de Sitter Universe, Quantum Corrected Inflation, and its Confrontation with the Planck Result. *Phys. Rev. D* **2014**, *90*, doi:10.1103/PhysRevD.90.023525.
56. Cognola, G.; Elizalde, E.; Zerbini, S. Functional Determinant of the Massive Laplace Operator and the Multiplicative Anomaly. 2014, arXiv:1408.1766, arXiv.org e-Print archive. Available online: <http://arxiv.org/abs/1408.1766> (accessed on 9 March 2015).
57. Rinaldi, M.; Cognola, G.; Vanzo, L.; Zerbini, S. Inflation in scale-invariant theories of gravity. 2015, arXiv:1410.0631, arXiv.org e-Print archive. Available online: <http://arxiv.org/abs/1410.0631> (accessed on 9 March 2015).
58. Ade, P.A.R.; Aghanim, N.; Alina, D.; Alves, M.I.R.; Armitage-Caplan, C.; Arnaud, M.; Arzoumanian, D.; Ashdown, M.; Atrio-Barandela, F.; Aumont, J.; *et al.* Planck intermediate results. XIX. An overview of the polarized thermal emission from Galactic dust. 2014, arXiv:1405.0871, arXiv.org e-Print archive. Available online: <http://arxiv.org/abs/1405.0871> (accessed on 9 March 2015).
59. Ade, P.A.R.; Alves, M.I.R.; Aniano, G.; Armitage-Caplan, C.; Arnaud, M.; Atrio-Barandela, F.; Aumont, J.; Baccigalupi, C.; Banday, A.J.; Barreiro, R.B.; *et al.* Planck intermediate results. XXII. Frequency dependence of thermal emission from Galactic dust in intensity and polarization. 2014, arXiv:1405.0874, arXiv.org e-Print archive. Available online: <http://arxiv.org/abs/1405.0874> (accessed on 9 March 2015).

60. Adam, R.; Ade, P.A.R.; Aghanim, N.; Arnaud, M.; Aumont, J.; Baccigalupi, C.; Banday, A.J.; Barreiro, R.B.; Bartlett, J.G.; Bartolo, N.; *et al.* Planck intermediate results. XXX. The angular power spectrum of polarized dust emission at intermediate and high Galactic latitudes. 2014, arXiv:1409.5738, arXiv.org e-Print archive. Available online: <http://arxiv.org/abs/1409.5738> (accessed on 9 March 2015).
61. Mortonson, M.J.; Seljak, U. A joint analysis of Planck and BICEP2 B modes including dust polarization uncertainty. *JCAP* **2014**, *1410*, doi:10.1088/1475-7516/2014/10/035.
62. Kamionkowski M.; Kovetz, E.D. Statistical diagnostics to identify Galactic foregrounds in B-mode maps. *Phys. Rev. Lett.* **2014**, *113*, doi:10.1103/PhysRevLett.113.191303.
63. Colley W.N.; Gott, J.R. Genus Topology and Cross-Correlation of BICEP2 and Planck 353 GHz B-Modes: Further Evidence Favoring Gravity Wave Detection. *Mon. Not. Roy. Astron. Soc.* **2015**, *447*, 2034–2045.
64. Ade, P.A.R.; Aghanim, N.; Ahmed, Z.; Aikin, R.W.; Alexander, K.D.; Arnaud, M.; Aumont, J.; Baccigalupi, C.; Banday, A.J.; Barkats, D.; *et al.* A Joint Analysis of BICEP2/Keck Array and Planck Data. 2015, arXiv:1502.00612, arXiv.org e-Print archive. Available online: <http://arxiv.org/abs/1502.00612> (accessed on 9 March 2015).
65. Motohashi, H. Consistency relation for R^p inflation. 2014, arXiv:1411.2972, arXiv.org e-Print archive. Available online: <http://arxiv.org/abs/1411.2972> (accessed on 9 March 2015).
66. Prokopec, T.; Weenink, J. Uniqueness of the gauge invariant action for cosmological perturbations. *JCAP* **2012**, *1212*, doi:10.1088/1475-7516/2012/12/031.
67. George, D.P.; Mooij, S.; Postma, M. Quantum corrections in Higgs inflation: The real scalar case. *JCAP* **2014**, *1402*, doi:10.1088/1475-7516/2014/02/024.
68. Kaiser, D.I. Frame independent calculation of spectral indices from inflation. 2014, arXiv:astro-ph/9507048, arXiv.org e-Print archive. Available online: <http://arxiv.org/abs/astro-ph/9507048> (accessed on 9 March 2015).
69. Nojiri, S.; Odintsov, S.D.; Sáez-Gómez, D. Cosmological reconstruction of realistic modified $F(R)$ gravities. *Phys. Lett. B* **2009**, *681*, 74–80.
70. Nojiri, S.; Odintsov, S.D. Modified $f(R)$ gravity consistent with realistic cosmology: From matter dominated epoch to dark energy universe. *Phys. Rev. D* **2006**, *74*, doi:10.1103/PhysRevD.74.086005.
71. Nojiri, S.; Odintsov, S.D. Modified gravity and its reconstruction from the universe expansion history. *J. Phys. Conf. Ser.* **2007**, *66*, doi:10.1088/1742-6596/66/1/012005.
72. Nojiri, S.; Odintsov, S.D.; Sáez-Gómez, D. Cyclic, ekpyrotic and little rip universe in modified gravity. *J. Phys. Conf. Ser.* **2011**, *1458*, 207–221.
73. De la Cruz-Dombriz, A.; Dobado, A. $f(R)$ gravity without cosmological constant. *Phys. Rev. D* **2006**, *74*, doi:10.1103/PhysRevD.74.087501.
74. Bamba, K.; Nojiri, S.; Odintsov, S.D. Trace-anomaly driven inflation in $f(T)$ gravity and in minimal massive bigravity. *Phys. Lett. B* **2014**, *731*, 257–264.
75. Deser, S.; Duff, M.J.; Isham, C.J. Nonlocal Conformal Anomalies. *Nucl. Phys. B* **1976**, *111*, 45–55.
76. Duff, M.J. Observations on Conformal Anomalies. *Nucl. Phys. B* **1977**, *125*, 334–348.

77. Birell, N.D.; Davies, P.C.W. *Quantum Fields in Curved Space*; Cambridge University Press: Cambridge, UK, 1982.
78. Duff, M.J. Twenty years of the Weyl anomaly. *Class. Quant. Grav.* **1994**, *11*, 1387–1404.
79. Nojiri, S.; Odintsov, S.D. Conformal anomaly for dilaton coupled theories from AdS/CFT correspondence. *Phys. Lett. B* **1998**, *444*, 92–97.
80. Hawking, S.W.; Hertog, T.; Reall, H.S. Trace anomaly driven inflation. *Phys. Rev. D* **2001**, *63*, doi:10.1103/PhysRevD.63.083504.
81. Bamba, K.; Geng, C.Q.; Lee, C.C. Cosmological evolution in exponential gravity. *JCAP* **2010**, *1008*, doi:10.1088/1475-7516/2010/08/021.
82. Nojiri, S.; Odintsov, S.D.; Tsujikawa, S. Properties of singularities in (phantom) dark energy universe. *Phys. Rev. D* **2005**, *71*, doi:10.1103/PhysRevD.71.063004.
83. Bamba, K.; Nojiri, S.; Odintsov, S.D. The future of the universe in modified gravitational theories: Approaching a finite-time future singularity *JCAP* **2008**, *10*, doi:10.1088/1475-7516/2008/10/045.
84. Bamba, K.; de Haro, J.; Odintsov, S.D. Future Singularities and Teleparallelism in Loop Quantum Cosmology. *JCAP* **2013**, *1302*, doi:10.1088/1475-7516/2013/02/008.
85. Amorós, J.; de Haro, J.; Odintsov, S.D. Bouncing loop quantum cosmology from $F(T)$ gravity. *Phys. Rev. D* **2013**, *87*, doi:10.1103/PhysRevD.87.104037.
86. Haro, J.; Amorós, J. Viability of the matter bounce scenario in $F(T)$ gravity and Loop Quantum Cosmology for general potentials. *JCAP* **2014**, *1412*, doi:10.1088/1475-7516/2014/12/031.
87. Zhang, X.; Ma, Y. Extension of loop quantum gravity to $f(R)$ theories. *Phys. Rev. Lett.* **2011**, *106*, doi:10.1103/PhysRevLett.106.171301.
88. Zhang, X.; Ma, Y. Loop quantum $f(R)$ theories. *Phys. Rev. D* **2011**, *84*, doi:10.1103/PhysRevD.84.064040.
89. Zhang, X.D.; Ma, Y.G. Loop quantum modified gravity and its cosmological application. *Front. Phys. China* **2013**, *8*, 80–93.
90. Gupt, B.; Singh, P. Contrasting features of anisotropic loop quantum cosmologies: The Role of spatial curvature. *Phys. Rev. D* **2012**, *85*, doi:10.1103/PhysRevD.85.044011.
91. Singh, P. Are loop quantum cosmos never singular? *Class. Quant. Grav.* **2009**, *26*, doi:10.1088/0264-9381/26/12/125005.
92. Corichi, A.; Singh, P. Is loop quantization in cosmology unique? *Phys. Rev. D* **2008**, *78*, doi:10.1103/PhysRevD.78.024034.
93. Ashtekar, A.; Pawłowski, T.; Singh, P. Quantum Nature of the Big Bang: Improved dynamics. *Phys. Rev. D* **2006**, *74*, doi:10.1103/PhysRevD.74.084003.
94. Haro, J.; Elizalde, E. Loop cosmology: Regularization vs. quantization. *Europhys. Lett.* **2010**, *89*, doi:10.1209/0295-5075/89/69001.
95. Dzierzak, P.; Malkiewicz, P.; Piechocki, W. Turning big bang into big bounce. 1. Classical dynamics. *Phys. Rev. D* **2009**, *80*, doi:10.1103/PhysRevD.80.104001.
96. Bojowald, M. Consistent Loop Quantum Cosmology. *Class. Quant. Grav.* **2009**, *26*, doi:10.1088/0264-9381/26/7/075020.

97. De Haro, J.; Amorós, J. Viability of the Matter Bounce Scenario. 2014, arXiv:1411.7611, arXiv.org e-Print archive. Available online: <http://arxiv.org/abs/1411.7611> (accessed on 9 March 2015).
98. Elizalde, E.; Haro, J.; Odintsov, S.D. Quasi-matter domination parameters in bouncing cosmologies. 2015, arXiv:1411.3475, arXiv.org e-Print archive. Available online: <http://arxiv.org/abs/1411.3475> (accessed on 9 March 2015).
99. De Haro, J.; Amorós, J. Viability of the matter bounce scenario in Loop Quantum Cosmology from BICEP2 last data. *JCAP* **2014**, *1408*, doi:10.1088/1475-7516/2014/08/025.
100. Bamba, K.; Makarenko, A.N.; Myagky, A.N.; Odintsov, S.D. Bounce universe from string-inspired Gauss-Bonnet gravity. 2014, arXiv:1411.3852, arXiv.org e-Print archive. Available online: <http://arxiv.org/abs/1411.3852> (accessed on 9 March 2015).
101. Bamba, K.; Makarenko, A.N.; Myagky, A.N.; Odintsov, S.D. Bouncing cosmology in modified Gauss-Bonnet gravity. *Phys. Lett. B* **2014**, *732*, 349–355.
102. Bamba, K.; Makarenko, A.N.; Myagky, A.N.; Nojiri, S.; Odintsov, S.D. Bounce cosmology from $F(R)$ gravity and $F(R)$ bigravity. *JCAP* **2014**, *1401*, doi:10.1088/1475-7516/2014/01/008.
103. Koehn, M.; Lehnert, J.L.; Ovrut, B.A. Cosmological super-bounce. *Phys. Rev. D* **2014**, *90*, doi:10.1103/PhysRevD.90.025005.
104. Cai, Y.F.; Easson, D.A.; Brandenberger, R. Towards a Nonsingular Bouncing Cosmology. *JCAP* **2012**, *1208*, doi:10.1088/1475-7516/2012/08/020.
105. Khoury, J.; Ovrut, B.A.; Steinhardt, P.J.; Turok, N. The Ekpyrotic universe: Colliding branes and the origin of the hot big bang. *Phys. Rev. D* **2001**, *64*, doi:10.1103/PhysRevD.64.123522.
106. Myrzakulov, R.; Odintsov, S.D.; Sebastiani, L. Inflationary universe from higher-derivative quantum gravity. 2015, arXiv:1412.1073, arXiv.org e-Print archive. Available online: <http://arxiv.org/abs/1412.1073> (accessed on 9 March 2015).

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