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Inflationary Equilibrium in a Stochastic Economy with Independent Agents*

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Abstract

We argue that even when macroeconomic variables are constant, underlying microeconomic uncertainty and borrowing constraints generate inflation.

We study stochastic economies with fiat money, a central bank, one nondurable commodity, countably many time periods, and a continuum of agents. The aggregate amount of the commodity remains constant, but the endowments of individual agents fluctuate "independently" in a random fashion from period to period. Agents hold money and, prior to bidding in the commodity market each period, can either borrow from or deposit in a central bank at a fixed rate of interest. If the interest rate is strictly positive, then typically there will not exist an equilibrium with a stationary wealth distribution and a fixed price for the commodity. Consequently, we investigate stationary equilibria with inflation, in which aggregate wealth and prices rise deterministically and at the same rate. Such an equilibrium does exist under appropriate bounds on the interest rate set by the central bank and on the amount of borrowing by the agents.

*Much of this work originated in discussions at the Santa Fe Instiute and Yale University.

If there is no uncertainty, or if the stationary strategies of the agents select actions in the interior of their action sets in equilibrium, then the classical Fisher equation for the rate of inflation continues to hold and the real rate of interest is equal to the common discount rate of the agents. However, with genuine uncertainty in the endowments and with convex marginal utilities, no interior equilibrium can exist. The equilibrium inflation must then be higher than that predicted by the Fisher equation, and the equilibrium real rate of interest underestimates the discount rate of the agents.

Keywords: Inflation, Economic equilibrium and dynamics, Dynamic programming, Consumption

JEL Classification: D5, D8, E31, E58

1 Introduction

We seek to understand the behavior of prices and money in a simple infinite-horizon economy with a central bank and one nondurable commodity. Following Bewley (1986) we consider an economy in which a continuum of agents are subject to idiosyncratic, independent and identically distributed random shocks to their endowments. At the micro level the economy is in perpetual flux but, at the macro level, aggregate endowments remain constant across time and states. We prove the existence of a stationary equilibrium that also remains rock-steady at the macro level despite micro turmoil in individual consumption and saving. Stationary equilibrium means that markets clear, and prices and money grow at a deterministic rate τ, all the while maintaining the same distribution of real (inflation-corrected) wealth across agents. In each period some formerly rich agents may become poor, and vice versa, but the fraction of the population at every level of real wealth remains the same.

Bewley proved the existence of a stationary equilibrium in a more general economy than ours, allowing for example for multiple commodities[1]. But his model did not have a central bank that could change the stock of money over time, and therefore had no inflation in equilibrium. Inflation seems to complicate the existence problem. We are not aware of any other existence proof for stationary equilibrium with inflation in a Bewley-style model.

In this paper we study a model with a continuum of agents with a common discount rate β and common instantaneous utility function $u(\cdot)$, but

[1] Bewley also allowed for Markovian random endowments and for heterogeneous utility functions. All of these extensions could probably be accommodated in our setting as well.

with idiosyncratic shocks to their endowments that leave the aggregate endowment constant. For such a model, and without borrowing or lending, it is already known that there exists in great generality an equilibrium with a stationary distribution of nominal wealth and a constant commodity price; cf. Karatzas et al. (1994). Bewley (1986) showed that such a noninflationary stationary equilibrium also exists when there is borrowing and lending but at a zero rate of interest. We confirm this result in subsection 7.6.

We add to the model a central bank committed to borrowing or lending with every agent at a fixed nominal interest rate $\rho > 0$. With a central bank fixing a positive rate of interest, a noninflationary equilibrium rarely exists. (A necessary condition for existence is that the bank select an interest rate that "balances the books" so that all the lending comes from one agent to another and the aggregate money supply remains constant; see Karatzas et al. (1997) and Geanakoplos et al. (2000).)

We prove here the existence of stationary inflation-corrected equilibrium, under certain technical conditions and under a critical borrowing constraint (Theorem 7.1). More specifically, we assume that all agents have a strictly concave utility function $u(\cdot)$ whose derivative is bounded away from zero. Another important assumption is that agents can only borrow up to a fraction $\theta \in (0, \theta^*(\rho))$ of the discounted value of their current endowment, where the upper bound $\theta^*(\rho) \in (0,1)$ decreases as ρ increases. As long as $\theta \leq 1$, we have a model of secured lending, without any chance of default around equilibrium. We were not able to establish existence in general with $\theta = 1$, though we show that such an equilibrium does exist in the absence of microeconomic uncertainty (Example 6.1).

The existence of inflation-corrected equilibrium allows us to study the effect of micro uncertainty and the borrowing constraint on the rate of inflation and the real rate of interest. In a world of micro certainty, which we could obtain in our model by replacing each individual agent's random endowment with his mean endowment, the rate of inflation τ would necessarily satisfy the famous Fisher equation

$$\tau = \beta(1+\rho)\,,$$

provided $\theta = 1$. The Fisher equation also holds in our model, even with uncertainty, if the equilibrium is interior; that is, if agents are never consuming zero, and if they are never forced by the collateral constraint to borrow less than they would like (Theorem 5.1).

We prove, however, that if there is genuine micro uncertainty, and if the marginal utility function $u'(\cdot)$ is strictly convex, then all stationary equilibria have $\tau > \beta(1+\rho)$, irrespective of the bound θ on borrowing. Thus, with genuine randomness in the endowments and with $u'(\cdot)$ strictly convex, stationary equilibrium can only exist if a non-negligible fraction of the agents are up against their borrowing constraints (Theorem 5.2).

We prove that if $\theta \leq \theta^*(\rho)$, then indeed there is always an equilibrium in which $\tau > \beta(1+\rho)$, and the borrowing constraint is often binding, whether or not there is micro uncertainty and no matter what the sign of $u'''(\cdot)$ (Theorem 7.1). Micro uncertainty and borrowing constraints generate inflation.

We can also interpret our result in terms of the implied real rate of interest rather than in terms of the rate of inflation. Fisher defined the real rate of interest $\bar{\rho}$ by

$$1+\bar{\rho} \equiv \frac{1+\rho}{\tau}\,.$$

In our model with certainty and $\theta = 1$, the Fisher equation must hold; that is, the real rate of interest necessarily equals the reciprocal of the discount: $1/\beta = 1+\bar{\rho}$. Though historical consumption has been far from certain, historical real rates of interest are often mistakenly taken as indicators of the discounting in the population. Historically interest rates have exceeded inflation by about 3%, which has suggested to many economists that people regard next year as 3% less important than this year. But our theorem shows that when there are borrowing constraints, an historical real rate of interest of $\bar{\rho} = 3\%$ corresponds to discounting that might be much higher, say $\beta = 1/1.05$.

In an earlier paper on this subject (Karatzas et al. (2006)) we had a representative agent and random i.i.d. aggregate endowments. There, prices jumped around from period to period; but we showed that, in stationary equilibrium, the long-run rate of inflation was always uniquely defined. We also showed there that the long-run real rate of interest was typically smaller than the reciprocal of the representative agent's discount β.

A precise formulation of our model, and of equilibrium, is given in the next section. The notion of stationary equilibrium is defined in Section 3. Section 4 describes the optimization problem of the agent in our infinite-horizon economy. Section 5 is on interior equilibria and the Fisher equation. Section 6 presents two simple examples for which stationary interior equilibria exist, an example with a non-interior equilibrium with inflation rate $\tau > 1$, and also an example with no stationary equilibrium. Our existence proof is in section 7. The final section has a brief comparison of the representative agent model to the model of this paper.

2 The model

The model runs in discrete time units $n = 1, 2, \cdots$ and has a continuum of agents $\alpha \in I$ indexed by the unit interval $I = [0, 1]$. Each agent α seeks to maximize the expectation of his total discounted utility from consumption, namely

$$\mathbb{E}\left(\sum_{n=1}^{\infty} \beta^{n-1} u(c_n^\alpha)\right). \tag{2.1}$$

Here $\beta \in (0, 1)$ is a discount factor, c_n^α is his (possibly random) consumption in period n, and $u : [0, \infty) \to [0, \infty)$ is a concave, continuous, nondecreasing *utility function* with $u(0) = 0$. The utility function $u(\cdot)$ is assumed to be the same for all agents, but agent endowments are heterogeneous.

At time-period $n = 1$, each agent begins with a non-random amount $m_1^\alpha \in [0, \infty)$ of cash (fiat money, or nominal wealth); there is no dispensation of cash thereafter. The total amount of cash initially held by the agents is the constant

$$M_1 = \int_I m_1^\alpha \, d\alpha ,$$

which we assume to be finite and strictly positive.

At each time-period $n \geq 1$, every agent receives an endowment $Y_n^\alpha \geq 0$ of the (perishable) consumption good. The random variables Y_n^α, and all other random variables in this paper, are defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All the variables Y_n^α, $\alpha \in I$, $n \in \mathbb{N}$ are assumed to be nonnegative and to have a common distribution λ with

$$0 < Q := \int_{[0,\infty)} y \, \lambda(dy) < \infty .$$

For each agent α, the random variables $Y_1^\alpha, Y_2^\alpha, \cdots$ are assumed to be independent as well as identically distributed.

If we were to assume that, for each $n \in \mathbb{N}$, the random commodity endowments Y_n^α were the same for all agents $\alpha \in I$, then we would have the representative agent model of Karatzas et al. (2006). The model of the present paper differs from the representative agent model, in that the random endowments vary from agent to agent and aggregate to a constant. Thus, we assume that the integrals

$$\int_I Y_n^\alpha(\omega) \, d\alpha = Q_n(\omega) = Q \tag{2.2}$$

are constant across time-periods $n \in \mathbb{N}$ and states $\omega \in \Omega$. We assume further that the random endowments $Y_n(\alpha, \omega) \equiv Y_n^\alpha(\omega)$ are jointly measurable in (α, ω), where the variable α ranges over the index set $I = [0,1]$ and the variable ω ranges over the probability space Ω. Furthermore, the distribution of $Y_n(\cdot, \omega)$ on I is the same for each fixed ω, as that of $Y_n(\alpha, \cdot)$ on Ω for each fixed α; this common distribution, namely λ, is assumed also to be constant in n. (A simple construction of random variables with these properties is in Feldman & Gilles (1985).) We often use Y to denote a generic random variable that has this distribution. A consequence of our assumptions about the random endowments is that

$$\mathbb{E}(Y) = \int_\Omega Y_n(\alpha, \omega)\, \mathbb{P}(d\omega) = \int_I Y_n(\alpha, \omega)\, d\alpha = Q\,.$$

2.1 Money and Commodity Markets

Two markets meet in every period. First, each agent can borrow cash from (or deposit cash into) a central bank at a fixed interest rate $\rho \geq 0$. Next, agents must sell all their endowment of the good for cash in a commodity market. At the same time, each agent α bids an amount $b_n^\alpha(\omega)$ of cash, to purchase goods for consumption from the market. At the end of the time-period, loans come due.

Suppose every agent and the central bank expect the price $p_n(\omega)$ for the commodity. The budget set of each agent $\alpha \in I$ is then defined recursively as follows: Let $m_n^\alpha(\omega)$ be the amount of cash or nominal wealth with which the agent enters period n, and recall that the initial amount of cash $m_1^\alpha(\omega) = m_1^\alpha$ has been fixed. In period n the agent can lend no more than the cash $m_n^\alpha(\omega)$ he has on hand; he is allowed to borrow no more than $\theta p_n(\omega) Y_n^\alpha(\omega)/(1+\rho)$. The agent will be able to pay back the loan with interest from the revenue derived from the sale of his endowment $Y_n^\alpha(\omega)$. The choice of the parameter $\theta = 1$ means the bank is willing to grant full credit, while the choice $\theta = 0$ means the bank will lend nothing at all; choices of $\theta \in (0,1)$ correspond to levels of partial credit.

After leaving the loan market the agent can bid at most the cash he has on hand, plus what he has borrowed, so $b_n^\alpha(\omega)$ is required to satisfy

$$0 \leq b_n^\alpha(\omega) \leq m_n^\alpha(\omega) + \frac{\theta p_n(\omega) Y_n^\alpha(\omega)}{1+\rho}\,.$$

The agent $\alpha \in I$ receives from the market his bid's worth of goods $c_n^\alpha(\omega) = b_n^\alpha(\omega)/p_n(\omega)$ and consumes these goods immediately, thereby receiving $u(c_n^\alpha(\omega))$ in utility. He then repays the bank in full and with interest, or is repaid himself with interest on his loan. Thus, at the beginning of the next period, the agent has a random endowment $Y^{\alpha}_{n+1}(\omega)$ of goods and

$$m^{\alpha}_{n+1}(\omega) = (1+\rho)\big(m^{\alpha}_{n}(\omega) - b^{\alpha}_{n}(\omega)\big) + p_n(\omega)Y^{\alpha}_{n}(\omega) \tag{2.3}$$

in cash. When the agent chooses his credit decision and his bid $b^{\alpha}_{n}(\omega)$ at time n, he is assumed to know his cash position $m^{\alpha}_{n}(\omega)$, his endowment $Y^{\alpha}_{n}(\omega)$, the price $p_n(\omega)$, as well as the bank interest rate ρ.

2.2 Equilibrium

We shall define formally a special kind of equilibrium in the next section. Roughly speaking, equilibrium is given by prices $p_n(\omega)$ and choices $b^{\alpha}_{n}(\omega)$ that maximize each agent's expected discounted utility over his budget set, in such a way that $b^{\alpha}_{n}(\omega)$ is measurable in α for every n, and such that every market clears. Denoting the total bid by $B_n(\omega) := \int b^{\alpha}_{n}(\omega)\, d\alpha$ and the total endowment of goods offered for sale by $Q_n(\omega) := \int Y^{\alpha}_{n}(\omega)\, d\alpha = Q$, commodity market-clearing means

$$p_n(\omega) = \frac{B_n(\omega)}{Q}\,.$$

Let $C_n(\omega) := \int_I c^{\alpha}_{n}(\omega)\, d\alpha$ be the total consumption in period n. When the commodity market clears, we have

$$C_n(\omega) = \frac{B_n(\omega)}{p_n(\omega)} = Q_n(\omega) = Q\,.$$

Note that the credit markets clear automatically, since the central bank stands ready to absorb any excess lending or borrowing.

Suppose we are in equilibrium, and denote the total amount of cash (or total nominal wealth) held by the agents in period n by

$$M_n(\omega) \;:=\; \int_I m^{\alpha}_{n}(\omega)\, d\alpha\,.$$

Integrating out α in the law of motion (2.3), we obtain

$$M_{n+1}(\omega) = (1+\rho)\big(M_n(\omega) - B_n(\omega)\big) + \frac{B_n(\omega)}{Q_n}\cdot Q_n = (1+\rho)M_n(\omega) - \rho B_n(\omega). \tag{2.4}$$

If $\rho = 0$, then $M_{n+1}(\omega) = M_n(\omega)$ and the money supply is the same in every period.

It is possible for the money supply to be conserved in equilibrium even for some $\rho > 0$; see Karatzas et al. (1997) and Geanakoplos et al. (2000). However, one expects that generically the money supply will not remain constant when $\rho > 0$: *genuine inflation must then be considered.*

3 Stationary Markovian equilibrium with inflation

In this section we define stationary equilibrium. As we saw in the last section, this will often require a non-zero inflation rate.

A *stationary Markovian equilibrium with inflation* (stationary equilibrium, or SE for short) is an equilibrium, as described briefly in the last section, that turns out to be completely deterministic in the aggregate level but possibly very random at the micro level. More precisely, it is an equilibrium in which the price level and the money stock grow deterministically at a constant inflation rate, and the distribution of real wealth stays the same each period. Formally, we require conditions (a), (b), and (c) below.

3.1 Definition of inflation-corrected SE

In period $n = 1$ the price level is p_1, and in every subsequent period it rises (or falls) at some constant rate $\tau > 0$:

$$(a) \qquad p_n(\omega) = \tau^{n-1} p_1 \quad \text{for all } n \geq 1\,,\ \omega \in \Omega\,.$$

Similarly, the aggregate money stock starts out at $M_1 = \int_I m_1^\alpha \, d\alpha$ and should rise (or fall) deterministically at the same inflation rate τ thereafter:

$$(b) \qquad M_n(\omega) = \tau^{n-1} M_1 \quad \text{for all } n \geq 1\,,\ \omega \in \Omega\,.$$

The purchasing power of one dollar of money, or a *real dollar,* is defined in any period or state by the reciprocal of the price level $1/p_n(\omega)$. The real money, or real wealth, with which an agent starts a period, is then given by $r_n^\alpha(\omega) = m_n^\alpha(\omega)/p_n(\omega)$ and his initial real wealth is

$$r_1^\alpha = m_1^\alpha / p_1.$$

The aggregate purchasing power, or *aggregate real wealth,* must then be a constant in SE, namely

$$R_n(\omega) \;:=\; \int_I r_n^\alpha(\omega)\, d\alpha \;=\; M_n(\omega)/p_n(\omega) \;=\; M_1/p_1 \;=\; R\,.$$

Let $\mu_n(\omega)$ be the distribution of real wealth in period n, defined by

$$\mu_n(\omega)(E) \;:=\; \mathcal{L}\left\{\, \alpha \in [0,1] \;:\; r_n^\alpha(\omega) \in E \,\right\},$$

where $\mathcal{L}$ is Lebesgue measure and E is a Borel subset of $[0, \infty)$ (we include ω in this definition to emphasize that, in general, the measures μ_n are random).

The mean of $\mu_n(\omega)$ is just the aggregate real wealth in period n; that is,

$$\int_{[0,\infty)} r\,\mu_n(\omega)(dr) \;=\; \int_I r_n^\alpha(\omega)\,d\alpha \;=\; R_n(\omega) \;=\; R\,.$$

In stationary equilibrium we should have that μ_n remains the same in every period and in every state:

$$(c) \qquad \mu_n(\omega) \;=\; \mu \quad \text{for all } n \geq 1\,,\ \omega \in \Omega\,.$$

3.2 Macro equations in inflation-corrected SE

It is worthwhile noting five macro equations, namely (3.3)-(3.7) below, that must hold in stationary equilibrium. Dividing by $p_n(\omega)$ in the law of motion (2.3) and recalling that in stationary equilibrium $\tau = p_{n+1}(\omega)/p_n(\omega)$, we get

$$m_{n+1}^\alpha(\omega)/p_n(\omega) = (1+\rho)\big(m_n^\alpha(\omega) - b_n^\alpha(\omega)\big)/p_n(\omega) + Y_n^\alpha(\omega) \tag{3.1}$$

or equivalently

$$r_{n+1}^\alpha(\omega)\cdot\tau = (1+\rho)\big(r_n^\alpha(\omega) - c_n^\alpha(\omega)\big) + Y_n^\alpha(\omega). \tag{3.2}$$

Integrating the last equation over agents $\alpha \in I$, and noting that in stationary equilibrium the macro variables do not depend on ω, we get the macro law of motion

$$R_{n+1}\cdot\tau = (1+\rho)(R_n - C_n) + Q\,. \tag{3.3}$$

In stationary equilibrium we must also have

$$\mu_{n+1} = \mu_n = \mu\,, \tag{3.4}$$

and from this it follows

$$R_{n+1} = R_n = R\,. \tag{3.5}$$

We also need to clear the commodity markets; that is, we need

$$C_{n+1} = C_n = Q. \tag{3.6}$$

The macro law of motion (3.3), together with (3.5) and (3.6), guarantee that

$$\tau = 1 + \rho - \rho\cdot\frac{Q}{R}\ , \tag{3.7}$$

as can be seen by substitution.

Conversely, (3.3), (3.4), and (3.7) guarantee (3.5) and (3.6).

Remark: The conditions (a), (b), and (c) also follow from (3.3)-(3.7). Indeed, condition (c) is the same as (3.4). To obtain condition (a), we integrate with respect to α in (3.1) and use (3.3), to obtain

$$\begin{aligned} \frac{M_{n+1}}{p_n} &= (1+\rho)\left(\frac{M_n}{p_n} - \frac{B_n}{p_n}\right) + Q \\ &= (1+\rho)(R_n - C_n) + Q \\ &= \tau \cdot R_{n+1} = \tau \cdot \frac{M_{n+1}}{p_{n+1}} \,. \end{aligned}$$

Hence, $p_{n+1} = \tau \cdot p_n$ and (a) holds. We then have from (3.5) that

$$R = R_n = \frac{M_n}{p_n} = \frac{M_n}{\tau^{n-1} \cdot p_1}\,, \qquad \text{thus} \qquad M_n = \tau^{n-1}\, p_1 \cdot R = \tau^{n-1}\, M_1$$

and (b) also holds.

3.3 Finding stationary equilibrium

As long as the inflation and interest rates remain constant and the random endowments are independent and identically distributed from period to period, the optimal strategy for any agent is clearly to make a bid which determines a consumption that depends only on his current real wealth r and commodity endowment y.

Let $\mathfrak{c}(\cdot,\cdot)$ be a measurable function specifying the consumption $\mathfrak{c}(r,y)$ for an agent with real wealth r and goods y. The consumption function $\mathfrak{c}(\cdot,\cdot)$ is said to be *budget-feasible for the credit parameter* θ if, for all $r \geq 0$ and $y \geq 0$, we have

$$0 \leq \mathfrak{c}(r,y) \leq r + \frac{\theta y}{1+\rho}\,. \tag{3.8}$$

Such a function determines budget-feasible bids for all agents $\alpha \in I$ according to

$$b_n^\alpha(\omega) \;=\; p_n(\omega)\, \mathfrak{c}\big(r_n(\omega), Y_n^\alpha(\omega)\big)\,, \qquad \forall\;\; n \in N\,,\; \omega \in \Omega\,. \tag{3.9}$$

To find an SE, we can first guess a consumption function $\mathfrak{c}(\cdot,\cdot)$ and an initial distribution μ of real wealth. Let $R = R(\mu)$ be the mean of μ, and set $\tau = \tau(\mu)$ according to the equality (3.7). To see whether the pair $(\mu, \mathfrak{c})$ determines an SE we must check that, if every agent follows the consumption strategy $\mathfrak{c}(\cdot\,,\cdot)$, that is, bids according to the recipe in (3.9), and if the random variables $\mathcal{R}$ (real wealth) and Y (endowment) are independent with distributions μ and λ, respectively, then

$$\widetilde{\mathcal{R}} := \big[(1+\rho)\big(\mathcal{R} - \mathfrak{c}(\mathcal{R}, Y)\big) + Y\,\big] \,/\, \tau \ , \tag{3.10}$$

the new real wealth induced by the law of motion, has again distribution μ.

Indeed, the equalities (3.3) and (3.4) are immediate and (3.7) holds by construction. Conditions (a), (b), and (c) follow as was explained in the remark of the preceding section. Lastly, we must verify that the consumption function strategy $\mathfrak{c}(\cdot,\cdot)$ is optimal for the individual agents' problem. The next section discusses how.

4 Dynamic programming for a single agent

Suppose the economy is in stationary equilibrium with inflation rate τ. Then a single agent with real wealth r and goods y faces an infinite-horizon dynamic programming problem. In this section we collect several properties of this one-person problem for use in subsequent sections.

Recall that the utility function $u(\cdot)$ is assumed to be a concave, continuous, nondecreasing mapping from $[0,\infty)$ into itself. Let $V(r,y) \equiv V_{(\rho,\tau)}(r,y)$ be the agent's value function; that is, $V(r,y)$ is the supremum over all strategies of the agent's expected discounted total utility in (2.1).

If $\beta(1+\rho) > \tau$, then it can happen that $V(r,y) = \infty$. For example, if $u(x) = x$ and $r_1 = r > 0$, an agent can save all his money for the first n periods so that $r_n \geq [(1+\rho)/\tau]^{n-1}\, r$, then spend it all on consumption to obtain a discounted utility of at least $[\beta(1+\rho)/\tau]^{n-1}\, r$. This quantity can then be made arbitrarily large, by choosing n to be large.

If the utility function $u(\cdot)$ is bounded, or if $\beta(1+\rho) \leq \tau$, then it is not difficult to see that $V(r,y) < \infty$. (Notice that the concave, real-valued function $u(\cdot)$ is dominated by some affine function $\tilde{u}(x) = a + bx$ and so it suffices to check that $V(\cdot,\cdot)$ is finite for linear utilities.) In our search for stationary equilibria, we confine ourselves to the case $\beta(1+\rho) \leq \tau$. Thus, to avoid annoying technicalities, we assume that $V(\cdot,\cdot)$ is everywhere finite.

A plan π is called *optimal* if, for every initial position (r,y), the expected value under π of the total discounted utility is $V(r,y)$. If $\mathfrak{c}(\cdot,\cdot)$ is a feasible bid function as in (3.3), the *stationary plan* π corresponding to the use of $\mathfrak{c}(\cdot,\cdot)$ at every stage of play is written $\pi = \mathfrak{c}^\infty$.

Lemma 4.1. *(a) The value function $V(\cdot,\cdot)$ is concave, and satisfies the*

Bellman equation

$$V(r,y) = \sup_{0 \le c \le r + \frac{\theta y}{1+\rho}} \left[\psi_{(r,y)}(c\,; V) \right] \tag{4.1}$$

where

$$\psi_{(r,y)}(c\,; V) \;:=\; u(c) + \beta \cdot \mathbb{E}\, V\left(\frac{1+\rho}{\tau}(r-c) + \frac{y}{\tau}\,,\, Y \right). \tag{4.2}$$

(b) If the stationary plan $\pi = \mathfrak{c}^\infty$ is optimal, then $\mathfrak{c}(r,y) \in \arg\max\{\psi_{(r,y)}(\cdot\,; V)\}$ for all (r,y).

(c) If either $\tau > \beta(1+\rho)$ or $u(\cdot)$ is bounded, and $\mathfrak{c}(r,y) \in \arg\max\{\psi_{(r,y)}(\cdot\,; V)\}$ for all (r,y), then the plan $\pi = \mathfrak{c}^\infty$ is optimal.

Sketch of Proof: The proof that $V(\cdot,\cdot)$ is concave is similar to that given for Theorem 4.2 in Geanakoplos et al. (2000) (the main ideas go back at least to Bellman (1957)).

The Bellman equation holds in great generality; see, for example, section 9.4 of Bertsekas & Shreve (1978) – which also contains standard facts from dynamic programming that lead to assertion (b).

Part (c) follows from the characterization, originally due to Dubins and Savage (1965), of optimal strategies as being those that are both "thrifty" and "equalizing". In our context, "thriftiness" is equivalent to the condition that π selects actions which attain the supremum in the Bellman equation. On the other hand, every plan is equalizing if $\tau > \beta(1+\rho)$ holds, or if the utility function $u(\cdot)$ is bounded; see Rieder (1976) or Karatzas and Sudderth (2008) for a development of the Dubins-Savage characterization in the context of dynamic programming. □

For the rest of this section we impose the following additional requirements on the utility function.

Assumption 4.1. *The utility function $u(\cdot)$ is strictly concave and strictly increasing on $[0,\infty)$, differentiable on $(0,\infty)$ with $0 < u'_+(0) < \infty$.*

The function $\psi_{(r,y)}(\cdot) \equiv \psi_{(r,y)}(\cdot\,; V)$ of (4.2) is concave, since both $u(\cdot)$ and $V(\cdot,\cdot)$ are concave. Also, $\psi_{(r,y)}(\cdot)$ is strictly concave since $u(\cdot)$ is and therefore achieves its maximum at a unique point. We define a specific bid function $\mathfrak{c}(\cdot,\cdot) = \mathfrak{c}_{(\rho,\tau)}(\cdot,\cdot)$ by

$$\mathfrak{c}(r,y) = \arg\max\left\{ \psi_{(r,y)}(c) \;:\; 0 \le c \le r + \frac{\theta y}{1+\rho} \right\}. \tag{4.3}$$

Under Assumption 4.1 it follows from Lemma 4.1(b) that, if there is an optimal plan, then it must be the plan $\pi = \mathfrak{c}^\infty$. Furthermore, if either $\tau > \beta(1+\rho)$ or $u(\cdot)$ is bounded, then, by Lemma 4.1(b) and (c), $\pi = \mathfrak{c}^\infty$ is the unique optimal plan.

The next lemma and its proof are similar to results in Geanakoplos et al. (2000).

Lemma 4.2. *Under Assumption 4.1 we have:*

(a) The value function $V(r,y)$ is strictly increasing in each variable r, y.

If, in addition, the plan $\pi = \mathfrak{c}^\infty$ is optimal, then

(b) The value function $V(r,y)$ is strictly concave in each variable r and y.

(c) The functions $\mathfrak{c}(r,y)$ and $r - \mathfrak{c}(r,y)$ are strictly increasing in r, for each fixed y. Hence $\mathfrak{c}(r,y)$ is continuous in r, for each fixed y. Also, $\mathfrak{c}(r,y)$ is strictly increasing in y, for each fixed r.

(d) $\mathfrak{c}(r,y) \to \infty$ as $r \to \infty$ for fixed y.

(e) $\mathfrak{c}(r,y) > 0$ if $\max\{r,y\} > 0$ and $\tau > \beta(1+\rho)$.

Proof. Part (a) is clear from the fact that an agent with more cash or goods can spend more at the first stage and be in the same position at the next stage as an agent with less. For part (b) we will show the strict concavity of $V(r,y)$ in r. The proof of strict concavity in y is similar. Let us consider, then, two pairs (r,y) and $(\widetilde{r},y)$ with $r < \widetilde{r}$, and set $\bar{r} = (r+\widetilde{r})/2$. It suffices to show that $V(\bar{r},y) > (V(r,y)+V(\widetilde{r},y))/2$. Let $(r_1,y_1) = (r,y)$ and consider the sequence $\{(r_n,y_n)\}_{n\in\mathbb{N}}$ of successive positions of an agent who starts at (r_1,y_1) and follows the plan π. Also let $(\widetilde{r}_1,\widetilde{y}_1) = (\widetilde{r},\widetilde{y}) = (\widetilde{r},y)$ and consider the coupled sequence $\{(\widetilde{r}_n,\widetilde{y}_n)\}_{n\in\mathbb{N}}$ for an agent who starts at $(\widetilde{r},\widetilde{y})$, follows π, and receives the same income variables so that $\widetilde{y}_n = y_n$ for $n \geq 2$. Set $c_n = \mathfrak{c}(r_n,y_n)$ and $\widetilde{c}_n = \mathfrak{c}(\widetilde{r}_n,\widetilde{y}_n)$ for all n. Finally, consider a third agent who starts at $(\bar{r},\bar{y}) = (\bar{r},y))$, receives the same income variables $\bar{y}_n = y_n$, and, at every stage n, plays the action $\bar{c}_n = (c_n+\widetilde{c}_n)/2$. Let $(\bar{r}_n,\bar{y}_n) = (\bar{r}_n,y_n)$ be the successive positions of this third agent. It is easily verified that $\bar{r}_n = (r_n+\widetilde{r}_n)/2$ for all n and that the actions $\bar{c}_n$ are feasible at every stage. Hence,

$$V(\bar{r},y) \geq \mathbb{E}\left(\sum_{n=1}^{\infty}\beta^{n-1}u(\bar{c}_n)\right).$$

Since $u(\cdot)$ is concave, we have $u(\bar{c}_n) \geq \big(u(c_n)+u(\widetilde{c}_n)\big)/2$ for every $n \in \mathbb{N}$, and this inequality must be strict with positive probability for some n. This is because $u(\cdot)$ is strictly concave and $V(\widetilde{r}, y) > V(r, y)$ by part (a), which implies that $\widetilde{c}_n > c_n$ holds with positive probability for some n. Hence

$$\mathbb{E}\sum_{n=1}^{\infty} \beta^{n-1} u(\bar{c}_n) > \mathbb{E}\sum_{n=1}^{\infty} \beta^{n-1}\big((u(c_n) + u(\widetilde{c}_n))/2\big) = \big(V(r, y) + V(\widetilde{r}, y)\big)/2\,,$$

and the proof of (b) is now complete. For the proof of part (c), we first observe that if $u(\cdot)$ and $w(\cdot)$ are strictly concave functions defined on $[0, \infty)$, then it is an elementary exercise to show that

$$\text{argmax}_{0\le c\le r+\frac{\theta y}{1+\rho}} \big[u(c) + w(r - c)\big]$$

is strictly increasing in r. By part (b), the function

$$w(x) = \beta \cdot \mathbb{E}\, V\left(\frac{1+\rho}{\tau} \cdot x + \frac{y}{\tau}\,,\, Y\right)$$

is strictly concave. Since $\psi_{(r,y)}(c) = u(c) + w(r - c)$, the strict increase of $c(r, y)$ in r follows from our observation. A symmetric argument shows $r - c(r, y)$ is strictly increasing in r. We can also write $\psi_{(r,y)}(c) = u(c) + v(y - (1 + \rho)c)$, where

$$v(x) = \beta \cdot \mathbb{E}\, V\left(\frac{1}{\tau} \cdot x + \frac{1+\rho}{\tau} \cdot r\,,\, Y\right).$$

Thus, a very similar argument shows the strict increase of $c(r, y)$ in y. The proof of (d) is the same as the proof of Theorem 4.3 in Karatzas et al. (1994). The proof of (e) is given in detail for a similar problem in Geanakoplos et al. (2000) – see the proof there of Theorem 4.2. ☐

The bid c is an called *interior* at position (r, y), if

$$0 < c < r + \frac{\theta y}{1+\rho}\,.$$

The final result of this section establishes the Euler equation for interior actions.

Lemma 4.3. *Assume that $\pi = \mathfrak{c}^\infty$ is optimal. If $\mathfrak{c}(r, y) > 0$, then*

$$u'(\mathfrak{c}(r, y)) \ge \frac{\beta(1+\rho)}{\tau} \cdot \mathbb{E}[u'(\mathfrak{c}(\widetilde{r}, \widetilde{Y}))], \tag{4.4}$$

where

$$\widetilde{r} = \frac{1+\rho}{\tau}\big(r - \mathfrak{c}(r, y)\big) + \frac{y}{\tau} \tag{4.5}$$

and the random variable $\widetilde{Y}$ has distribution λ. If $\mathfrak{c}(r,y) < r + \theta y/(1+\rho)$, then the inequality opposite to (4.4) holds. Thus, if the action $\mathfrak{c}(r,y)$ is interior at (r,y), we have

$$u'(\mathfrak{c}(r,y)) = \frac{\beta(1+\rho)}{\tau} \cdot \mathbb{E}[u'(\mathfrak{c}(\widetilde{r},\widetilde{Y}))], \tag{4.6}$$

Proof. To prove the first assertion, let $c_1 = \mathfrak{c}(r,y)$ and $0 < \varepsilon < c_1$. Consider a plan $\widetilde{\pi}$ at (r,y) that bids $\widetilde{c}_1 = c_1 - \varepsilon$ in the first period and $\widetilde{c}_2 = c_2 + \frac{1+\rho}{\tau}\varepsilon$ in the second, where $c_2 = \mathfrak{c}(\widetilde{r},\widetilde{Y})$ is the bid of plan π in the second period. Thus, an agent using $\widetilde{\pi}$ is in the same position after two periods, as is an agent using π. Suppose that from the second period onward, $\widetilde{\pi}$ agrees with π. The return from $\widetilde{\pi}$ cannot exceed the return from the optimal plan π. Since the two plans agree after stage two, we have

$$u(c_1 - \varepsilon) + \beta \cdot \mathbb{E}\left[u\left(c_2 + \frac{1+\rho}{\tau}\varepsilon\right)\right] \le u(c_1) + \beta \cdot \mathbb{E}[u(c_2)]\,,$$

and inequality (4.4) follows easily from this. The proof of the reverse inequality when $\mathfrak{c}(r,y) < r + \theta y/(1+\rho)$ is similar. □

5 The Fisher equation

The most famous equation in monetary economic theory was proposed by Irving Fisher in 1931. It defines the "real" rate of interest as the nominal rate of interest, minus the rate of inflation (after taking logarithms). When the price level for the next period is certain, as it is in stationary equilibrium, the real rate of interest describes precisely the trade-off beteen consumption today and consumption tomorrow.

In a world of consumption without uncertainty, one can infer the discount rate of the agents from the real rate of interest. If there is no uncertainty in the endowments, then in stationary equilibrium our model must have

$$1/\beta \;=\; (1+\rho)/\tau\,.$$

We shall see, however, that when there is uncertainty in consumption, Fisher's real rate of interest typically underestimates how much agents discount the future.

We shall assume in this section that the economy is in stationary equilibrium with relative wealth distribution μ and optimal bid function $\mathfrak{c}(\cdot,\cdot)$. We continue to impose Assumption 4.1.

Definition 5.1. *We say that the SE* $(\mu, \mathfrak{c})$ *is* interior*, if for all* (r, y) *in a set of full* $\mu \otimes \lambda$ *measure, we have*

$$0 < \mathfrak{c}(r, y) < r + \frac{\theta y}{1+\rho}\,.$$

If an SE is interior, then the Fisher equation must hold.

Theorem 5.1. *In an interior SE, we have* $\tau = \beta(1+\rho)$*.*

Proof. By interiority, the Euler equation (4.6) holds for almost every (r, y) with respect to the product measure $\nu = \mu \otimes \lambda$. By stationarity, if the random vector $(\mathcal{R}, Y)$ has distribution ν, and if $\widetilde{Y}$ is an independent random variable with distribution λ, then in the notation of (3.10) the vector $(\widetilde{\mathcal{R}}, \widetilde{Y})$ has also distribution ν. But then the Euler equation (4.6) gives a.s.

$$u'(\mathcal{R}, Y) \;=\; \frac{\beta(1+\rho)}{\tau} \cdot \mathbb{E}\big[\, u'\big(\mathfrak{c}(\widetilde{\mathcal{R}}, \widetilde{Y})\big) \,|\, \mathcal{R}, Y \,\big]\,;$$

taking expectations on both sides, recalling that the random vectors $(\mathcal{R}, Y)$ and $(\widetilde{\mathcal{R}}, \widetilde{Y})$ have common distribution ν, and canceling the common integral on both sides of the resulting equality, we obtain $1 = \beta(1+\rho)/\tau$. □

We develop in the next section two simple examples of interior SE's, for which the Fisher equation holds. However, these two examples do not involve uncertainty coupled with risk-aversion; they give a completely misleading picture.

When the marginal utility function $u'(\cdot)$ is strictly convex, and there is uncertainty in the endowments, there is no interior SE and $\tau > \beta(1+\rho)$. Then *the Fisher equation fails in many situations of interest that include the exponential utility* function $u(x) = 1 - e^{-x}$, $x \geq 0$.

Theorem 5.2. *Suppose that* $u'(\cdot)$ *is strictly convex and the endowment random variable* Y *is not constant. Then, in any SE, we have* $\tau > \beta(1+\rho)$*. In particular, there cannot exist an interior SE.*

Proof. The final assertion is immediate from the first, in conjunction with Theorem 5.1. To prove the first assertion, assume by way of contradiction that $(\mu, \mathfrak{c})$ is an SE with $\tau \leq \beta(1+\rho)$. At any position (r, y) such that $\mathfrak{c}(r, y) > 0$, Lemma 4.3 gives

$$u'\left(\mathfrak{c}(r, y)\right) \geq \frac{\beta(1+\rho)}{\tau} \cdot \mathbb{E}\left[u'\left(\mathfrak{c}\big(\widetilde{r}, \widetilde{Y}\big)\right)\right] \geq \mathbb{E}\left[u'\left(\mathfrak{c}\big(\widetilde{r}, \widetilde{Y}\big)\right)\right]. \tag{5.1}$$

Even if $\mathfrak{c}(r,y)=0$, it is clear from the concavity of $u(\cdot)$ that the first term in (5.1) is at least as large as the last term, if we set $u'(0)$ equal to the derivative from the right at zero. By assumption, the distribution λ of $\widetilde{Y}$ is not a point-mass and $u'(\cdot)$ is strictly convex. Furthermore, by Lemma 4.2(c), the mapping $y \mapsto \mathfrak{c}(r,y)$ is strictly increasing, so the distribution of $\mathfrak{c}(\widetilde{r},\widetilde{Y})$ is nontrivial and Jensen's inequality gives

$$\mathbb{E}\left[u'\left(\mathfrak{c}(\widetilde{r},\widetilde{Y})\right)\right] > u'\left(\mathbb{E}\left[\mathfrak{c}(\widetilde{r},\widetilde{Y})\right]\right). \tag{5.2}$$

Since $u'(\cdot)$ is strictly decreasing, we deduce from (5.1) and (5.2) that $\mathfrak{c}(r,y) < \mathbb{E}[\mathfrak{c}(\widetilde{r},\widetilde{Y})]$ holds for all (r,y); in particular, we obtain the a.s. inequality

$$\mathfrak{c}(\mathcal{R},Y) < \mathbb{E}\left[\,\mathfrak{c}(\widetilde{\mathcal{R}},\widetilde{Y})\,\right]$$

in the notation developed for the proof of Theorem 5.1. But this is impossible, since the random vectors $\mathfrak{c}(\mathcal{R},Y)$ and $\mathfrak{c}\big(\widetilde{\mathcal{R}},\widetilde{Y}\big)$ have the same distribution in stationary equilibrium. □

6 Three simple examples, and a counterexample

The first example is the case where there is no randomness in the economy.

Example 6.1. *Assume that the random variable Y is identically equal to a constant $y>0$, that the interest rate ρ is strictly positive, and that $\theta=1$. Suppose that the utility function $u(\cdot)$ satisfies Assumption 4.1. We shall find a class of stationary equilibria.*

From the last section we guess that, without uncertainty, there will be an interior equilibrium, and that the Fisher equation will hold (Theorem 5.1), so we conjecture

$$\tau = \beta(1+\rho).$$

We know from (3.7) that in stationary equilibrium the aggregate real money supply must be

$$R = \frac{\rho}{1+\rho-\tau}\cdot Q = \frac{\rho}{(1-\beta)(1+\rho)}\cdot Q\,. \tag{6.1}$$

Let μ be an arbitrary real wealth distribution with mean $\int_{[0,\infty]} r\,\mu(dr) = R$. Since $\beta(1+\rho)/\tau = 1$ and Y is the constant y, the Euler equation (4.6) takes the form

$$u'\left(\mathfrak{c}(r,y)\right) = u'\left(\mathfrak{c}(\tilde{r},y)\right),$$

which holds when consumption $c = \mathfrak{c}(r, y) = \mathfrak{c}(\tilde{r}, y)$ remains constant. Now an agent's consumption will be constant if his real wealth remains constant, hence we need

$$r = \widetilde{r} = \frac{1+\rho}{\tau}(r-c) + \frac{y}{\tau},$$

or equivalently

$$\mathfrak{c}(r, y) \equiv c = [(1+\rho-\tau)r + y]/(1+\rho) = (1-\beta)r + \frac{y}{1+\rho}. \qquad (6.2)$$

This stationary bidding strategy is clearly interior, and satisfies the Euler equation by construction. Likewise, the usual transversality condition clearly holds, since $r_n = r$ for all n and so $\beta^n r_n u'(\mathfrak{c}(r_n, y_n)) = \beta^n r u'(\mathfrak{c}(r, y)) \to 0$ as $n \to \infty$. Hence, the bid function $\mathfrak{c}(\cdot, \cdot)$ of (6.2) determines an optimal strategy (see Stokey & Lucas (1989)).

Since every individual maintains the same real wealth r, the distribution μ of wealth is stationary. It follows that the pair $\big(\mu, \mathfrak{c}(\cdot, \cdot)\big)$ is an SE. □

For the next example we take the agents to be risk-neutral.

Example 6.2. *Assume that $\rho > 0$, $\theta \in [0, 1]$, and take $u(x) = x$ for all x. For this utility function the Bellman equation of (4.1) is relatively straightforward to solve, and a stationary equilibrium easy to compute. This time we allow for random Y. We showed in Theorem 5.2 that there cannot exist an interior stationary equilibrium for $u(\cdot)$ strictly concave. Here, however, $u(\cdot)$ is linear, so we look for an interior SE anyway.*

If there is an interior SE, then by Theorem 5.1 the Fisher equation will hold and we can set

$$\tau = \beta(1+\rho).$$

As before, the aggregate wealth will have to be given by (6.1).

Let μ be an arbitrary real wealth distribution with $\int_{[0,\infty)} r\, \mu(dr) = R$. Agents could very well pursue exactly the same strategy as in the last example, bidding for consumption

$$\mathfrak{c}(r, y) = (1-\beta)r + \frac{y}{1+\rho},$$

just as in (6.2), when their real wealth is $r \geq 0$ and their commodity endowment is $y \geq 0$. With such a bidding strategy, the relative wealth of the agent does not change from one period to the next, no matter what his endowment; thus, aggregate wealth R remains constant. It follows that the the commodity market also clears.

All that remains is to check that this bidding strategy is optimal. But the Euler equation is trivially satisfied (provided that consumption is feasible), since the marginal utility of consumption is constant and $\beta(1+\rho)/\tau = 1$. As before, transversality is obviously satisfied, so this strategy will indeed be optimal if it is always budget-feasible.

Feasibility requires

$$\mathfrak{c}(r,y) = (1-\beta)r + \frac{y}{1+\rho} \le r + \frac{\theta y}{1+\rho}\,,$$

or equivalently

$$(1-\theta)y \;\le\; \beta(1+\rho)r\,,$$

to hold on a set of full $\mu \otimes \lambda-$measure. This is the case rather obviously when $\theta = 1$, as well as when

$$(1-\theta)\,Q \;\le\; \beta(1+\rho)R$$

and the distributions λ and μ are sufficiently concentrated around their respective means Q and R. With the aid of (6.1), this last condition can be written in the form $(1-\beta)(1-\theta) \;<\; \rho\beta$. □

Suppose that no borrowing is permitted in the economy. In other words, the credit parameter is set at $\theta = 0$. Typically, some agents will save and receive interest from the bank. Thus the money supply and prices will increase, so that the inflation parameter τ will be greater than 1. The next example illustrates this phenomenon.

Example 6.3. *Let $\theta = 0$ and assume that the endowment variable Y takes on the values 0 and 1 with probability 1/2 each. Assume that the utility $u(\cdot)$ is stictly concave, so that in particular $u(1) < u(1/2) + \beta u(1/2)$ when the discount factor β is sufficiently close to 1.*

Suppose, by way of contradiction, that there is an equilibrium in which agents do not save. Thus, since agents cannot borrow, the optimal consumption must be $\mathfrak{c}(r,y) = r$. Now with probability 1/2, agents reach the position $(1,0)$ starting from any position $(r,1)$. But an agent at $(1,0)$ can consume 1/2 and save 1/2 the first period and then consume $(1+\rho)/2$ at the next position $((1+\rho)/2, Y)$. The agent thus obtains $u(1/2) + \beta u((1+\rho)/2)$ in the first two periods while the agent who follows $\mathfrak{c}$ gets only $u(1)$. Since both agents are in the same position at the beginning of period three, it is better to spend 1/2 the first period. We have reached a contradiction. □

Our final example shows that an SE need not exist when the utility function saturates.

Example 6.4. *Assume that*

$$u(x) = \begin{cases} x, & 0 \le x \le 1, \\ 1, & x > 1. \end{cases}$$

Let the random variable Y equal 0 with probability $\gamma \in (0,1)$, and equal 4 with probability $1-\gamma$. Set the interest rate ρ and the credit parameter θ equal to 1, and let the discount factor $\beta \in (0,1)$ be arbitrary. We shall show by contradiction, that an SE cannot exist.

To get a contradiction, assume that there is an SE with bid function $\mathfrak{c}(\cdot,\cdot)$, and relative wealth distribution μ. We shall reach a contradiction after a few steps.

Step 1: $\tau < 2$. This is because $\tau = 1 + \rho - \rho \cdot Q/R = 2 - Q/R$. Let

$$1 + \rho' := \frac{1+\rho}{\tau} = \frac{2}{\tau} > 1.$$

Step 2: For all (r, y), $\mathfrak{c}(r, y) \le 1$. It is never optimal for an agent to bid more than 1 because, if he does so, he gains nothing in immediate utility and has less money at the next stage.

Step 3: Let k be any positive number. Then from any initial real wealth r_1, an agent using the bid function $\mathfrak{c}(\cdot,\cdot)$, will reach real wealth positions in $[k, \infty)$ with positive probability. Indeed, on the event $\{Y_1 = Y_2 = \cdots = Y_n = 4\}$ (which has probability $(1-\gamma)^n > 0$), we have almost surely

$$r_n \ge (1+\rho')^{n-1} \ge k \,, \qquad \forall \;\; n \ge 1 + \frac{\log k}{\log (1+\rho')}. \tag{6.3}$$

We shall prove (6.3) by induction. By Step 2, $c_1 = \mathfrak{c}(r_1, 4) \le 1$. So

$$r_2 \;=\; (1+\rho')(r_1 - c_1) + \frac{4}{\tau} \;\ge\; (1+\rho')(-1) + 2(1+\rho') \;=\; (1+\rho')\,.$$

Now assume (6.3) holds for n and $y_n = 4$. Then $c_n = \mathfrak{c}(r_n, 4) \le 1$ and

$$r_{n+1} = (1+\rho')(r_n - c_n) + \frac{4}{\tau} \ge (1+\rho')\big((1+\rho')^{n-1} - 1\big) + 2(1+\rho')\,.$$

Hence $r_{n+1} \ge (1+\rho')^n$.

Step 4: Let $k^* = 2(1+\rho')/\rho'$. If $r_n \ge k^*$, then it follows from Step 2 that

$$r_{n+1} \ge (1+\rho')(r_n - 1) \ge r_n + \rho' \cdot k^* - (1+\rho') = r_n + (1+\rho')$$

holds almost surely on $\{r_n \ge k^*\}$.

Steps 3 and 4 imply that the Markov chain $\{r_n\}$ of an agent's relative wealth positions diverges to infinity, with positive probability; thus, the chain is transient and cannot have a stationary distribution. This contradicts the invariance of μ.

7 Existence of Stationary Equilibrium

Most of this section is devoted to the proof of a general existence theorem. In the final subsection we present a simpler result for the special case when the interest rate ρ is zero.

For the proof of the first theorem we shall impose the following additional assumptions:

Assumption 7.1. *(a) The distribution of the random endowments is bounded from above by some $y^* \in (0, \infty)$; equivalently, $\lambda([0, y^*]) = 1$.*

(b) $\inf_{x \in [0,\infty)} u'(x) > 0$.

(c) The interest rate ρ satisfies

$$\rho < (1 - \beta)(1 - \theta) .$$

Condition (c) is equivalent to the inequality $\theta < \theta^*(\rho) := 1 - \big(\rho/(1 - \beta)\big)$.

Theorem 7.1. *Under Assumptions 4.1 and 7.1, there exists a stationary Markov equilibrium (SE) with inflation rate $\tau > \beta(1 + \rho)$.*

Assumptions 4.1 and 7.1 will be in force until the end of subsection 7.4.

Theorem 7.1 asserts, in particular, that a small enough interest rate will induce, in equilibrium, a rate of inflation τ *higher* than that predicted by the Fisher equation. By Theorem 5.1, such an equilibrium cannot be interior.

It seems likely that some of the conditions in Assumption 7.1 could be relaxed. However, our proof of Theorem 7.1 will use them all.

The proof will be given in a number of steps. The idea is familiar. We shall define an appropriate mapping from the set of real wealth distributions into itself, and argue that the mapping has a fixed point that corresponds to an SE.

7.1 The mapping Ψ

Let Δ^+ be the set of probability measures μ defined on the Borel subsets of $[0, \infty)$ that have a finite, positive mean:

$$0 < R \equiv R(\mu) = \int_{[0,\infty)} r\mu(dr) < \infty .$$

We regard Δ^+ as the set of possible real wealth distributions. For $\mu \in \Delta^+$, let

$$\tau \equiv \tau(\mu) = 1 + \rho - \rho \cdot \frac{Q}{R(\mu)} \,.$$

In SE, this quantity τ will be the rate of inflation. Note that τ can never exceed $1 + \rho$.

Now let $\Delta^* := \{\, \mu \in \Delta^+ \;:\; \tau(\mu) > \beta(1+\rho) \,\}$. Select a probability measure $\mu \in \Delta^*$, and let $\mathfrak{c}(r,y) = \mathfrak{c}_{(\rho,\tau)}(r,y) = \mathfrak{c}_\tau(r,y)$ be the optimal bid of an agent at (r,y) as defined in (4.3); in particular, $\mathfrak{c}(r,y)$ satisfies (3.8). Next, let the random vector $(\mathcal{R}, Y)$ have distribution $\mu \otimes \lambda$, and define

$$\widetilde{\mathcal{R}} \equiv \widetilde{\mathcal{R}}\big(\mathcal{R}, Y\big) := \frac{1+\rho}{\tau}\big(\mathcal{R} - \mathfrak{c}_\tau(\mathcal{R}, Y)\big) + \frac{Y}{\tau} \tag{7.1}$$

by analogy with (3.10) (equivalently, $\widetilde{\mathcal{R}}$ is the quantity of (4.5) corresponding to the next real wealth position of an agent who begins at $(\mathcal{R}, Y)$ and bids according to the rule $\mathfrak{c}_\tau(\cdot\,,\cdot)\big)$). We now define the mapping

$$\Psi(\mu) \;=\; \widetilde{\mu}$$

by taking the probability measure $\widetilde{\mu}$ to be the distribution of the random variable $\widetilde{\mathcal{R}}$ in (7.1). Let

$$\widetilde{\tau} \equiv \tau(\widetilde{\mu}) = 1 + \rho - \frac{\rho\, Q}{R(\widetilde{\mu})} \,,$$

where $\widetilde{R} \equiv R(\widetilde{\mu}) = \int_{[0,\infty)} r\, \widetilde{\mu}(dr)$ is the mean of the probability measure $\widetilde{\mu}$.

Suppose that μ is a fixed point of this mapping Ψ. Then $\widetilde{\mu} = \mu$, and consequently $\widetilde{R} = R$ as well as $\widetilde{\tau} = \tau$. Thus the distribution of real wealth in the economy remains constant when the initial real wealth distribution is μ and the agents use the optimal bid function $\mathfrak{c}_\tau(\cdot,\cdot)$. It then follows from the discussion in section 3.3 that $\big(\mu, \mathfrak{c}_\tau(\cdot,\cdot)\big)$ is an SE.

To establish that Ψ has a fixed point, it suffices to find a nonempty compact convex set $K \subset \Delta^*$ such that $\Psi(K) \subseteq K$ and ψ is continuous on K. (This is the Brouwer-Schauder-Tychonoff fixed point theorem; see Aliprantis & Border (1999).) We now set out to find such a set K.

7.2 Bounding R and τ

Define

$$R_* := \frac{1-\theta}{1+\rho} \cdot Q \qquad \text{and} \qquad \tau_* := \frac{(1+\rho)(1-\rho-\theta)}{1-\theta} \,.$$

We continue to use the notation of the previous section.

Lemma 7.1. *Let $\mu \in \Delta^*$, and suppose that the random vector $(\mathcal{R}, Y)$ has distribution $\mu \otimes \lambda$. Then the quantities $\widetilde{R} = R(\widetilde{\mu})$ and $\widetilde{\tau} = \tau(\widetilde{\mu})$ satisfy*

$$\infty > \widetilde{R} \geq R_*,$$

and

$$1+\rho \geq \widetilde{\tau} \geq \tau_* > \beta(1+\rho) > 0.$$

Proof. Integration with respect to $\mu \otimes \lambda$ in (7.1) gives

$$\widetilde{R} \leq \frac{1+\rho}{\tau} \int r\, \mu(dr) + \frac{1}{\tau} \int y\, \lambda(dy) = \frac{1+\rho}{\tau} \cdot R + \frac{1}{\tau} \cdot Q < \infty.$$

To prove the second inequality of the first line of the lemma, first consider the quantity

$$\widetilde{C} := \int_{[0,\infty)} \int_{[0,\infty)} \mathfrak{c}_\tau(r,y)\, \mu(dr)\lambda(dy),$$

which is the total consumption when all agents use $\mathfrak{c}_\tau(\cdot,\cdot)$. By (3.8), we have

$$\widetilde{C} \leq \int_{[0,\infty)} \int_{[0,\infty)} \left(r + \frac{\theta y}{1+\rho} \right) \mu(dr)\lambda(dy) = R + \frac{\theta Q}{1+\rho}.$$

Thus

$$\begin{aligned} \widetilde{R} &= \int_{[0,\infty)} \int_{[0,\infty)} \frac{1+\rho}{\tau} \Big(\big(r - \mathfrak{c}_\tau(r,y)\big) + y \Big)\, \mu(dr)\lambda(dy) \\ &= \frac{1+\rho}{\tau} \big(R - \widetilde{C}\big) + \frac{Q}{\tau} \geq \frac{1+\rho}{\tau} \cdot \frac{(-\theta Q)}{1+\rho} + \frac{Q}{\tau} = \frac{1-\theta}{\tau} \cdot Q \\ &\geq \frac{1-\theta}{1+\rho} \cdot Q = R_*, \end{aligned}$$

which establishes the second inequality. In the second row of the lemma, the first inequality is obvious; The second inequality follows from $\widetilde{R} \geq R_*$ and

$$\widetilde{\tau} = 1 + \rho - \rho \cdot \frac{Q}{\widetilde{R}} \geq 1 + \rho - \rho \cdot \frac{Q}{R_*} = \tau_*.$$

The third inequality in the second row of the lemma amounts to $(1-\beta)(1-\theta) > \rho$, which holds by Assumption 7.1(c). The final inequality is obvious. □

7.3 Bounding the wealth distribution

Define the random variable $\widetilde{\mathcal{R}}$ as in (7.1), and let $\tau \geq \tau_*$. Then by Lemma 4.2(c) and Assumption 7.1(a), we have

$$\widetilde{\mathcal{R}} \;\leq\; \frac{1}{\tau_*}\left[(1+\rho)\big(\mathcal{R}-\mathfrak{c}_\tau(\mathcal{R},Y)\big)+Y\right] \;\leq\; \frac{1}{\tau_*}\left[(1+\rho)\big(\mathcal{R}-\mathfrak{c}_\tau(\mathcal{R},0)\big)+y^*\right].$$

Define

$$\eta^* := \sup\{\,r-\mathfrak{c}_\tau(r,0) \,:\, r\geq 0,\ \tau_*\leq\tau\leq 1+\rho\,\}, \quad J := \left[0\,,\frac{(1+\rho)\eta^*+y^*}{\tau_*}\right].$$

Lemma 7.2. *The constant η^* is finite and, for every $\mu\in\Delta^*$ such that $\tau=\tau(\mu)\geq\tau_*$, the measure $\widetilde{\mu}$ is supported by the compact interval J.*

Proof. To bound $r-\mathfrak{c}_\tau(r,0)$ we can assume without loss of generality that $r>0$ and, by Lemma 4.2(e), that the inequalities in (3.8) are strict. In the following calculation we set $\xi=\inf_{x\geq 0}u'(x)$ and $y=0$. Thus the quantity of (4.5) becomes $\widetilde{r}=\frac{1+\rho}{\tau}\big(r-\mathfrak{c}_\tau(r,0)\big)$, and by Lemma 4.3 and Lemma 4.2(c) we have

$$\begin{aligned}\xi\leq u'(\mathfrak{c}_\tau(r,0)) &= \frac{\beta(1+\rho)}{\tau}\cdot\mathbb{E}\left[u'\left(\mathfrak{c}_\tau(\widetilde{r},\widetilde{Y})\right)\right]\\ &\leq \frac{\beta(1+\rho)}{\tau_*}\,u'\big(\mathfrak{c}_\tau(\widetilde{r},0)\big),\end{aligned} \tag{7.2}$$

where the random variable $\widetilde{Y}$ has distribution λ. Now $u'(c)\downarrow\xi$ as $c\to\infty$ and, by Lemma 7.1, $\tau_*>\beta(1+\rho)$, so for all c sufficiently large we have

$$u'(c) < \frac{\tau_*\,\xi}{\beta(1+\rho)}\,.$$

From Lemma 4.2(d), we obtain $\mathfrak{c}_\tau(r,0)\to\infty$ as $r\to\infty$. Hence

$$\eta(\tau) := \sup\left\{\,r\geq 0\,:\,u'\big(\mathfrak{c}_\tau(r,0)\big)\geq\frac{\tau_*\,\xi}{\beta(1+\rho)}\,\right\}<\infty$$

for all $\tau\in[\tau_*\,,1+\rho]$, and (7.2) gives

$$\eta(\tau)\geq\widetilde{r}=\frac{1+\rho}{\tau}\cdot\big(r-\mathfrak{c}_\tau(r,0)\big)\geq r-\mathfrak{c}_\tau(r,0)$$

for τ in this range. Thus

$$\eta^* \leq \sup_{\tau_*\leq\tau\leq 1+\rho}\eta(\tau).$$

Finally, as in Proposition 3.4 of Karatzas et al. (1997), the function $\tau \mapsto \mathfrak{c}_\tau(r,0)$ is continuous for fixed r. This fact, together with the continuity and monotonicity of $\mathfrak{c}_\tau(\cdot,0)$, can be used to check that $\eta(\cdot)$ is upper-semicontinuous. Hence,

$$\sup_{\tau_* \le \tau \le 1+\rho} \eta(\tau) < \infty\,,$$

and the interval J is compact. The assertion that $\widetilde{\mu}$ is supported by J follows from the calculation preceding the lemma. □

7.4 Completion of the proof of Theorem 7.1

Let $K := \{\, \mu \in \Delta^* \,:\, \tau(\mu) \ge \tau_*,\, \mu(J) = 1,\, R(\mu) \ge R_* \,\}$. This set is clearly compact and convex, since $\tau(\mu) = 1 + \rho - \rho Q / R(\mu)$ is a concave function of μ. Also, by Lemmas 7.1 and 7.2, we have $\Psi(K) \subseteq K$. The continuity of Ψ on K follows from Theorem 3.5 in Langen (1981). By the Brouwer-Schauder-Tychonoff Theorem, ψ has a fixed point μ. It follows that $\big(\mu, \mathfrak{c}_\tau(\cdot,\cdot)\big)$ is an SE.

7.5 An Open Question

With exponential utility function $u(x) = 1 - e^{-x}$, $x \ge 0$ and endowment random variable Y that is not a.s. equal to a constant, does an SE exist, at least for small enough values of $\rho > 0$? This case is not covered by Theorem 7.1; but if an SE exists, it cannot be interior and we must have $\tau > \beta(1+\rho)$ (Theorem 5.2).

7.6 The special case $\rho = 0$

If the interest rate ρ is zero, then Assumption 7.1 is not needed to prove the existence of a stationary equilibrium. We can take the credit parameter θ to be 1, thus allowing full credit to the agents.

Theorem 7.2. *Suppose that $\rho = 0$, $\theta = 1$, and that the utility function $u(\cdot)$ is strictly concave and satisfies Assumption 4.1. Suppose also that the endowment variable Y has a finite second moment. Then there is a stationary equilibrium in which the price and wealth distribution remain constant.*

The proof of Theorem 7.2 uses a similar result for a different model that was studied in Karatzas et al. (1994). We begin with a description of this model, which we call "Model 2".

As in the present paper, there is in Model 2 a continuum of agents $\alpha \in I$, one nondurable commodity, fiat money, and countably-many time periods $n = 1, 2, \ldots$. At the beginning of each period n, every agent α holds cash $S_n^\alpha(\omega) \geq 0$, but does not hold goods. There is no bank or loan market, so each agent α bids an amount $b_n^\alpha(\omega) \in [0, S_n^\alpha(\omega)]$ of cash in order to purchase goods for consumption. *After* bidding, agent α receives a random endowment $Y_n^\alpha(\omega)$ of the commodity, which is then sold in a market. The price for goods is formed as

$$p_n(\omega) \;=\; \frac{B_n(\omega)}{Q_n(\omega)}\,,$$

where

$$B_n(\omega) \;=\; \int_I b_n^\alpha(\omega)\,d\alpha\,, \qquad Q_n(\omega) \;=\; \int_I Y_n^\alpha(\omega)\,d\alpha \;=\; Q$$

are the aggregates of the bids and endowments, respectively. The agent α then receives the quantity $x_n^\alpha(\omega) = b_n^\alpha(\omega)/p_n(\omega)$ of the commodity, gets $u(x_n^\alpha)$ in utility, and begins the next period with cash

$$S_{n+1}^\alpha(\omega) \;=\; S_n^\alpha(\omega) - b_n^\alpha(\omega) + p_n(\omega)Y_n^\alpha(\omega)\,.$$

The agent seeks to maximize the expected total discounted utility

$$\mathbb{E}\left(\sum_{n=1}^{\infty} \beta^{n-1} u\big(x_n^\alpha\big)\right).$$

We shall assume that the utility function $u(\cdot)$ satisfies the hypotheses of Theorem 7.2, and that the random variables Y_n^α satisfy the assumptions made in Section 2 above.

The result that follows corresponds to Theorem 7.3 in Karatzas et al. (1994).

Lemma 7.3. *There is an equilibrium for Model 2 with a constant price $p \in (0, \infty)$ and a constant wealth distribution ν defined on the Borel subsets of $[0, \infty)$, and in which every agent $\alpha \in I$ bids according to a stationary plan $\mathfrak{c}^\infty$, namely, $b_n^\alpha(\omega) = \mathfrak{c}\big(S_n^\alpha(\omega)\big)$ for all n, ω and α.*

Suppose now $S \sim \nu$ and $Y \sim \lambda$, where "$\sim$" means "is distributed as". (If both X and Ξ are random variables, then $X \sim \Xi$ means that they have the same distribution.) Assume also that S and Y are independent. Then, by Theorem 7.3 (loc. cit.) we have

$$S - \mathfrak{c}(S) + pY \sim \nu\,.$$

Denote by μ be the distribution of $S - \mathfrak{c}(S)$ and define

$$\mathfrak{a}(m, y) := \mathfrak{c}(m + py) \qquad \text{for} \quad m \geq 0, \quad y \geq 0\,.$$

We now claim that *the price* p*, wealth distribution* μ*, and stationary plan* $\mathfrak{a}^\infty$*, form a stationary equilibrium for the original model.* To verify the claim, first observe that, if $M \sim \mu$, $Y \sim \lambda$, and M and Y are independent, then

$$M + pY - \mathfrak{a}(m, y) = M + pY - \mathfrak{c}(M + pY) \sim S - \mathfrak{c}(S) \sim \mu\,.$$

Thus the distribution of wealth is preserved. Consequently, the price

$$p \;=\; \frac{\int_I \int_I \mathfrak{a}(m, y)\, \mu(dm) \lambda(dy)}{Q} \;=\; \frac{\int_I \mathfrak{c}(s)\, \nu(ds)}{Q}$$

also remains constant. It remains to be shown that the plan $\mathfrak{a}^\infty$ is optimal for a given agent, when all other agents follow it.

Let $W(\cdot)$ be the optimal reward function for an agent playing in the equilibrium of Lemma 7.3. Then $W(\cdot)$ satisfies the Bellman equation

$$W(s) = \sup_{0 \leq b \leq s} \left[u\Big(\frac{b}{p}\Big) + \beta \cdot \mathbb{E} W\big(s - b + pY\big) \right]\,.$$

The Bellman equation for an agent in the original model is

$$V(m, y) = \sup_{0 \leq b \leq m + py} \left[u\Big(\frac{b}{p}\Big) + \beta \cdot \mathbb{E} V\big(m + py - b, Y\big) \right]\,.$$

It is easy to see that

$$V(m, y) = V(m', y') \qquad \text{whenever} \;\; m + py = m' + py'\,.$$

Indeed, it is not difficult to show that $V(m, y) = W(m + py)$. (One method is to verify the corresponding equality for $n-$day optimal returns using backward induction, and then pass to the limit as $n \to \infty$.) It is also straightforward to check that the expected total reward to an agent in Model 2 who plays $\mathfrak{c}^\infty$ from the initial position $s = m + py$, is the same as that of an agent in the original model who plays $\mathfrak{a}^\infty$ starting from (m, y). Since $\mathfrak{c}^\infty$ is optimal for Model 2, it follows that $\mathfrak{a}^\infty$ is optimal for the original model.

This completes the proof of the claim and also of Theorem 7.2.

8 Comments on representative and independent agents

The repesentative agent model of Karatzas et al (2006) and the independent agents model of this paper are at two extremes. The representative agent model is far easier to analyze than the independent agents model. Both call for a modification of the Fisher equation. In the representative agent model the rate of inflation is a random variable $\mathcal{T}(Y)$, where the function $\mathcal{T}(\cdot)$ is given by the closed form expression

$$\mathcal{T}(y) \;=\; \beta(1+\rho)\cdot\frac{\mathbb{E}[\,Y\,u'(Y)\,]}{(1-\beta)\cdot y\,u'(y)+\beta\cdot\mathbb{E}[\,Y\,u'(Y)\,]}\;.$$

The following *harmonic Fisher equation* follows readily:

$$\mathbb{E}\left(\frac{1}{\mathcal{T}(Y)}\right)=\frac{1}{\beta(1+\rho)}\,.$$

Consequently, the expectation $\mathbb{E}(\mathcal{T}(Y))$ exceeds $\beta(1+\rho)$, as does the long run rate of inflation.

It is not clear how the random variable $\mathcal{T}(Y)$ (inflation rate in the representative agent model) compares with τ (inflation rate in the model with independent agents) – a quantity for which we have no analytic expression. We do note, however, that under the conditions of Theorem 7.1, or with $u'(\cdot)$ strictly convex and λ a non-degenerate distribution as in Theorem 5.2, the latter dominates the harmonic mean of the former, to wit:

$$\mathbb{E}\left(\frac{1}{\mathcal{T}(Y)}\right)\;>\;\frac{1}{\tau}\,.$$

In this sense, the inflationary pressure is even greater for the independent-agent model than for the repesentative-agent model.

References

Aliprantis, C.D. & K. Border (1999) *Infinite-Dimensional Analysis: A Hitchhiker's Guide*. Berlin: Springer.

Bellman, R.E. (1957) *Dynamic Programming*. Princeton: Princeton University Press.

Bertsekas, D.P. & S.E. Shreve (1978) *Stochastic Optimal Control: The Discrete-Time Case*. New York: Academic Press.

Bewley, T. (1986) “Stationary Monetary Equilibrium with a Continuum of Independently Fluctuating Consumers." In Werner Hildenbrand and Andreu Mas-Colell, eds., *Contributions to Mathematical Economics in Honor of Gerard Debreu*, North-Holland, 79-102

Dubins, L.E. & L.J. Savage (1965) *How to Gamble If You Must: Inequalities for Stochastic Processes.* New York: McGraw-Hill.

Feldman, M. & C. Gilles (1985) “An expository note on individual risk without aggregate uncertainty”. *Journal of Economic Theory* **35**, 26–32.

Geanakoplos, J., I. Karatzas, M. Shubik & W. Sudderth (2000) “A strategic market game with active bankruptcy”. *Journal of Mathematical Economics* **34**, 359–396.

Karatzas, I., M. Shubik & W. Sudderth (1994) “Construction of stationary Markov equilibria in a strategic market game”. *Mathematics of Operations Research* **19**, 975–1006.

Karatzas, I., M. Shubik & W. Sudderth (1997) “A strategic market game with secured lending”. *Journal of Mathematical Economics* **28**, 207–247.

Karatzas, I., M. Shubik & W. Sudderth & J. Geanakoplos (2006) “The Inflationary Bias of Real Uncertainty and the Harmonic Fisher Equation”. *Economic Theory* **28**, 481–512.

Karatzas, I. & W. Sudderth (2008) “Two characterizations of optimality in dynamic programming.” Preprint.

Langen, H.J. (1981) “Convergence of dynamic programming models”. *Mathematics of Operations Research* **6**, 493–512.

Rieder, U. (1976) “On optimal policies and martingales in dynamic programming”. *Journal of Applied Probability* **3**, 507–518.

Stokey, N.L. & R.E. Lucas (1989) *Recursive Methods in Economic Dynamics.* Cambridge: Harvard University Press.

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