Tohoku Math. J.
60 (2008), 149–181

INFLECTION POINTS AND DOUBLE TANGENTS ON ANTI-CONVEX CURVES IN THE REAL PROJECTIVE PLANE

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(Received July 5, 2006, revised August 3, 2007)

Abstract. A simple closed curve in the real projective plane is called *anti-convex* if for each point on the curve, there exists a line which is transversal to the curve and meets the curve only at that given point. Our main purpose is to prove an identity for anti-convex curves that relates the number of independent (true) inflection points and the number of independent double tangents on the curve. This formula is a refinement of the classical Möbius theorem. We also show that there are three inflection points on a given anti-convex curve such that the tangent lines at these three inflection points cross the curve only once. Our approach is axiomatic and can be applied in other situations. For example, we prove similar results for curves of constant width as a corollary.

Introduction. Let P^2 denote the real projective plane. We assume curves to be parameterized and C^1-regular. A simple closed curve in P^2 is said to be *anti-convex* or *satisfying the Barner condition* if for each point p on the curve, there exists a line which is transversal to the curve and meets the curve only at p. This condition is the $n = 2$ case of a condition introduced by Barner in [3] for simple closed curves in the real projective n-space P^n for $n \geq 2$. An anti-convex curve is automatically not contractible.

Let γ_1 and γ_2 be two arcs in some affine plane $A^2 \subset P^2$. We say that γ_1 *crosses* γ_2 in a closed arc α if α is a maximal common arc of γ_1 and γ_2 and there is an open subarc $\tilde{\alpha}$ of γ_1 containing α such that the two components of $\tilde{\alpha} - \alpha$ do not lie on the same side of γ_2 (but might not be disjoint from γ_2). The arc α can of course consist of a single point. If γ_1 meets γ_2 transversally in a point p, then γ_1 of course crosses γ_2 in p. Examples of crossing curves are shown in Figure 1.

An inflection point p of a curve γ will be called a *true inflection point* if the tangent line of γ at p crosses γ in an arc containing p. Two inflection points are called *independent* if they are not contained in an arc of γ consisting of true inflection points. (The inflection points on the curve γ_2 on the right in Figure 1 are not independent. On the other hand, the three inflection points in Figure 4 are independent.) We will denote the maximal number of independent true inflection points on γ by $i(\gamma)$.

2000 *Mathematics Subject Classification.* Primary 53A20; Secondary 53C75.

The first author was supported in part by the DFG-Schwerpunkt *Globale Differentialgeometrie*.

The second author was partly supported by the Grant-in-Aid for Scientific Research (B), Japan Society for the Promotion of Science.

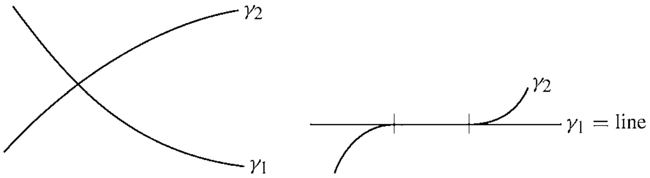

FIGURE 1. Crossing curves.

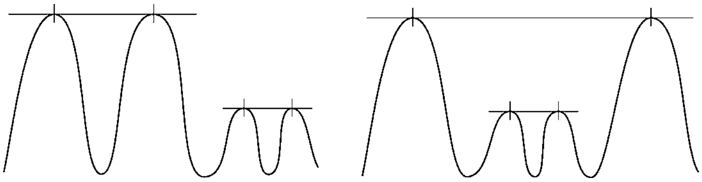

FIGURE 2. Two types of independent double tangent.

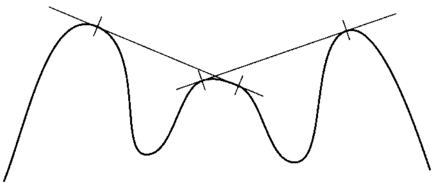

FIGURE 3. Dependent double tangents.

A *double tangent* of a curve γ is roughly speaking a line L that is tangent to γ at the endpoints of a nontrivial arc α of γ contained in an affine plane $A^2 \subset P^2$ in such a way that α is locally around its endpoints on the same side of $L \cap A^2$. (A precise definition will be given in Section 4.) We call α a *double tangent arc*. A set of double tangent arcs $\alpha_1, \dots, \alpha_k$ is said to be *independent* if any two of the arcs are either disjoint or one is a subarc of the other; see Figures 2 and 3.

We will denote the number of elements in a maximal set of independent double tangent arcs by $\delta(\gamma)$. It will follow from Theorem A, which we now state, that $\delta(\gamma)$ is independent of the choice of a maximal set of independent double tangent arcs on γ.

THEOREM A. *Let γ be a C^1-regular anti-convex curve in P^2 which is not a line. If the number $i(\gamma)$ of independent true inflection points on γ is finite, then so is the number $\delta(\gamma)$ of elements in a maximal set of independent double tangents, and*

$$i(\gamma) - 2\delta(\gamma) = 3 \tag{$*$}$$

holds. In particular, the number $\delta(\gamma)$ does not depend on the choice of a maximal set of independent double tangents if $i(\gamma)$ is finite.

Formula ($*$) is reminiscent of the Bose formula for simple closed curves in the Euclidean plane saying that $s - t = 2$, where s is the number of inscribed osculating circles and t is the number of triple tangent inscribed circles. This formula was proved for convex curves by Bose in [4] and in the general case by Haupt in [8]. Our method to prove Theorem A will be similar to the one used by the second author to prove the Bose formula in [20]. The authors do not know whether Formula ($*$) holds for non-contractible simple closed curves which are not necessarily anti-convex.

There is a well-known formula for generic closed curves in the affine plane A^2 due to Fabricius-Bjerre relating the numbers of double points, inflection points, and double tangents; see [5]. When the curves have no inflection points, Ozawa [13] gave a sharp upper bound on the number of double tangents. Formulas for real algebraic curves in P^2 go at least back to Klein; see the paper [21] of Wall.

We will also prove the following theorem.

THEOREM B. *Let γ be a C^2-regular anti-convex curve in P^2 which is not a line. Then γ has at least three inflection points with the property that the tangent lines at these inflection points cross γ only once.*

The theorem is optimal. An inflection point p is called *clean* if the tangent line at p meets the curve in a connected set. A clean inflection point is a typical example of an inflection point as in Theorem B. An anti-convex curve is called *generic* if it meets each line in at most finitely many components. If the curve γ in Theorem B is generic, one can improve the conclusion and prove that γ has even three clean inflection points. (See Theorem 2.10.) The noncontractible branch of a regular cubic in P^2 has three clean inflection points. Möbius proved that a simple closed noncontractable curve in P^2 has at least three (true) inflection points. Several proofs of this result are known; see [9], [7] and [14]. One can show with examples that none of these has to be a clean inflection point; see Figure 4.

A similar result is proved in [18] and [19] for clean sextactic points on a strictly convex curve in the affine plane. It says that such a curve has *three inscribed osculating conics and three circumscribed osculating conics*. It should also be remarked that the Tennis Ball Theorem ([1] and [2]), the theorem of Segre on space curves in [15], and the refinement of the

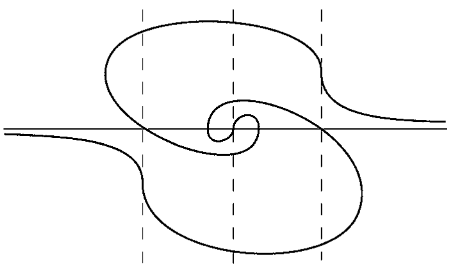

FIGURE 4. A simple closed curve with no clean inflection points.

Four-Vertex Theorem in [17] can be considered as generalizations of the Möbius Theorem; see [17].

In the proof of Theorem B we use an approach that goes back to Kneser's proof of the four vertex theorem; see [10], [20] and also [17], [16]. (A further development of this approach is also crucial in the proof of Theorem A.)

The theorems will be proved in later sections. Here we would like to explain some of the basic ideas in the proofs. Let $\hat{\pi} : S^2 \to P^2$ be the universal covering of P^2. Since γ is not contractible, it lifts to a simple closed curve $\hat{\gamma}$ that double covers γ. Through every point p on $\hat{\gamma}$, there is a great circle $\hat{L}_p$ on S^2 (which is the double cover of the line L_p) that meets $\hat{\gamma}$ only at p and the antipodal point $T(p) = -p$. The parametrization of $\hat{\gamma}$ and the orientation of S^2 give us a tangent vector field and a normal vector field along $\hat{\gamma}$ respectively. We will assume the normal direction to be on the left side of the curve. We define the positive rotation direction along the curve by rotating the normal vector towards the tangent vector. Notice that the positive rotation direction is the clockwise direction. Let us now rotate the circle $\hat{L}_p$ around p as far as possible in the positive direction through circles which only meet $\hat{\gamma}$ at p and $T(p)$. We denote the limiting great circle by C_p. There are two possibilities. The first is that $\hat{\pi}(C_p)$ only meets γ in one component. Then p is a clean inflection point. The other possibility is that $\hat{\pi}(C_p)$ meets γ in more than one component; see Figure 5. In this case p may or may not be an inflection point, but it is of course not a clean inflection point. We define a closed subset $F(p)$ by setting

$$F(p) = C_p \cap \hat{\gamma} \,. \tag{1}$$

We identify S^1 with the image of the curve $\hat{\gamma}$ and introduce on S^1 a cyclic order that agrees with the orientation of the curve. We will first assume that no line meets γ at infinitely many points and then discuss the general case. If p in S^1 is not an inflection point, we let δ denote the distance from p to the next point $q \in F(p)$ in (p, Tp), where (a, b) denotes the interval from $a \in S^1$ to $b \in S^1$ with respect to the cyclic order of S^1 and $F(p)$ is defined in Equation (1). Let p_1 be the midpoint of the interval $[p, q]$. The subset $F(p_1)$ lies in the interval $[p, q] \cup [Tp, Tq]$. If p_1 is not a clean inflection point we let δ_1 denote the distance to the point q_1 closest to p_1 in $F(p_1) \cap (p_1, Tp_1)$. Notice that $\delta_1 \leq \delta/2$. Iterating this process, we either arrive at a point p_n which is a clean inflection point, or get a sequence (p_n) that converges to a clean inflection point. As we will see in Section 2, this approach leads to the

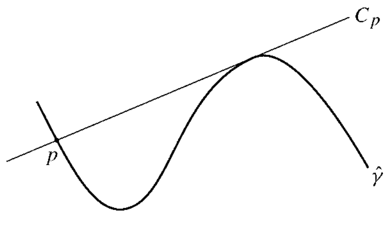

FIGURE 5. The projection of limiting great circle.

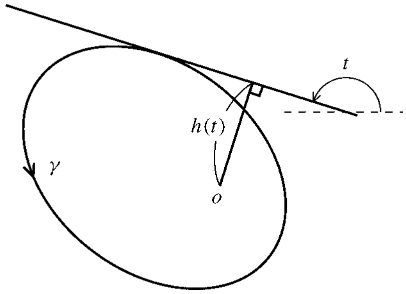

FIGURE 6. The supporting function.

existence of at least three inflection points. In the proof of Theorem B we only use a few axiomatic properties of the family $\{F(p)\}_{p\in S^1}$ of closed subsets in S^1. It can therefore be applied to different situations.

In Section 5, we apply the above method to convex curves of constant width.

Let γ be a strictly convex curve in $\boldsymbol{R}^2$. For each $t \in [0, 2\pi)$, there is a unique oriented tangent line $L(t)$ of the curve which makes angle t with the x-axis. Let $h(t)$ be the distance between a fixed point o in the open domain bounded by γ and the line $L(t)$; see Figure 6. Note that t gives a parametrization of the strictly convex curve γ, which we will use from now on. The function h is called the *supporting function of the curve γ with respect to o.* A strictly convex curve has *constant width d* if and only if $h(t) + h(t + \pi) = d$ holds.

We now fix a curve γ of constant width d. For each point p on the curve, there exists a unique circle Γ_p of width d such that Γ_p is tangent to γ at p, that is, Γ_p and γ meet at p with multiplicity at least two. Since Γ_p is the best approximation of γ at p among the circles of width d, we call Γ_p the *osculating d-circle at p.* Generically, the osculating d-circle of γ at p does not cross γ at p.

We will prove the following theorem in Section 5.

THEOREM C. *Let γ be a C^3-regular strictly convex curve of constant width d, which is not a circle. Then there exist at least three osculating d-circles which cross γ exactly twice, both times tangentially. Moreover, these circles coincide with the osculating circles* (*in the usual sense*) *at each of their crossing points on γ.*

The above theorem is a refinement of the fact that there are six distinct points on γ whose osculating circles have radius $d/2$. (Basic properties of curves of constant width can be found in [22].) In Figure 7 we indicate the three osculating circles of diameter d of the curve of constant width whose supporting function is $(d/2) + \sin 3t$.

We will also prove a formula analogous to the one in Theorem A for curves of constant width in Section 5.

The authors would like to thank the referee for the careful reading and the valuable comments.

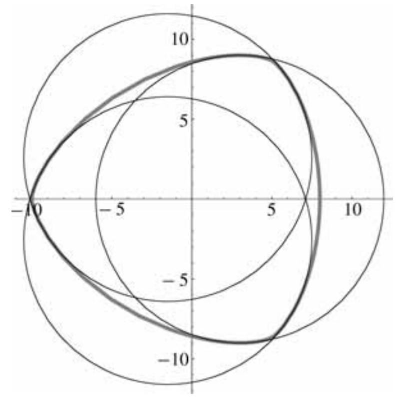

FIGURE 7. The three osculating circles.

1. Intrinsic line systems. In this section, we shall derive some basic properties of the family of closed subsets $\{F(p)\}_{p\in S^1}$ defined in Equation (1) in Introduction. We shall then use these properties to define what we will call an 'intrinsic line system'.

Let $\gamma : P^1 \to P^2$ be a C^1-regular anti-convex curve in P^2, where P^1 is a closed circle considered as a projective line. We assume that the image of γ is not a line in P^2. Let $\hat{\pi} : S^2 \to P^2$ and $\pi : S^1 \to P^1$ be the canonical covering projections. Then there exists a simple closed curve $\hat{\gamma} : S^1 \to S^2$ such that

$$\hat{\pi} \circ \hat{\gamma} = \gamma \circ \pi .$$

Moreover, for each point p on $\hat{\gamma}$, there exists a great circle $\hat{L}_p$ on S^2 such that $\hat{\pi}(\hat{L}_p) = L_{\pi(p)}$. By rotating $\hat{L}_p$ in the clockwise direction through great circles that only meet $\hat{\gamma}$ at p and the antipodal point Tp, we arrive at the limiting great circle C_p as in Introduction. Let $D_{\hat{\gamma}}$ be the domain on the left hand side of $\hat{\gamma}$. We orient $\hat{L}_p$ such that it passes into $D_{\hat{\gamma}}$ after going through p. The orientation of the great circle $\hat{L}_p$ induces an orientation on the limiting great circle C_p.

If C is an oriented great circle, we denote by $H^+(C)$ (resp. $H^-(C)$) the closed hemisphere on the left (resp. right) hand side of C. By applying a suitable diffeomorphism to S^2, we can map $\hat{\gamma}$ onto the equator and $D_{\hat{\gamma}}$ on the upper hemisphere. If we compose this with the stereographic projection into the plane, $\hat{\gamma}$ and $H^+(\hat{L}_p)$ look as in Figure 8. Though $\hat{\gamma}$ may not be star-shaped in general, we shall frequently use this kind of sketches of $\hat{\gamma}$ to simplify the figures.

The following assertion is obvious.

PROPOSITION 1.1. *The arc of* $\hat{\gamma} : S^1 \to S^2$ *from* p *to* Tp (*resp. from* Tp *to* p) *lies in* $H^-(\hat{L}_p)$ (*resp.* $H^+(\hat{L}_p)$).

PROPOSITION 1.2. *The limiting great circle* C_p *has the following properties*:

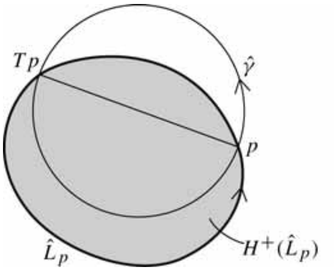

FIGURE 8.

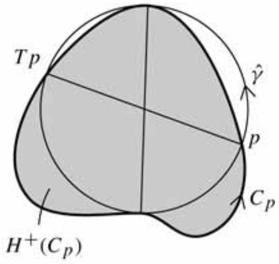

FIGURE 9.

(a) *The arc of $\hat{\gamma}$ from p to Tp (resp. from Tp to p) lies in $H^-(C_p)$ (resp. $H^+(C_p)$).*

(b) *The set $F(p)$ has at least three connected components, if C_p is not the tangent line of $\hat{\gamma}$ at p.*

PROOF. Since C_p is the limit of circles like $\hat{L}_p$, Property (a) follows from Proposition 1.1. To prove (b), we suppose that C_p is not a tangent line of $\hat{\gamma}$ at $p \in S^1$. Then C_p meets γ transversally at p and Tp. Hence, if C_p meets $\hat{\gamma}$ only at these two points, one can rotate it slightly in positive direction through curves that are transversal to $\hat{\gamma}$ in p and Tp and meet $\hat{\gamma}$ only at these two points. This contradicts the definition of C_p. Thus there exits a point q in $F(p) = C_p \cap \hat{\gamma}$ which is distinct from both p and Tp. Since $\hat{\gamma}$ is not a great circle, p and Tp belong to different connected components of $F(p)$. Since both C_p and $\hat{\gamma}$ are symmetric with respect to T, it follows that C_p is neither a tangent line at p nor at Tp. If q is in the same connected component of $F(p)$ as p (or Tp), C_p contains the segment of $\hat{\gamma}$ between p and q (or Tp and q), which implies that C_p must be the tangent line at p (resp. Tp), a contradiction. □

Conversely, we have the following

PROPOSITION 1.3. *If a great circle C through p and Tp satisfies the following two properties, then C coincides with C_p.*

(a) *The arc of $\hat{\gamma}$ from p to Tp (resp. from Tp to p) lies in $H^-(C)$ (resp. $H^+(C)$).*

(b) *C is tangent to $\hat{\gamma}$ at all points in $C \cap \hat{\gamma}$ different from p and Tp. Moreover, if C is not tangent to $\hat{\gamma}$ at p and Tp, then $C \cap \hat{\gamma}$ contains a point different from p and Tp.*

PROOF. Since C is tangent to $\hat{\gamma}$ at all points in $C \cap \hat{\gamma}$ different from p and Tp, we can rotate C slightly in negative direction into a great circle which meets $\hat{\gamma}$ transversally in p and Tp and does not have any further points with it in common. It now follows from the definition of C_p that $C = C_p$. □

We will denote by $F_0(p)$ the connected component of $F(p) = C_p \cap \hat{\gamma}$ containing p for each point p on S^1.

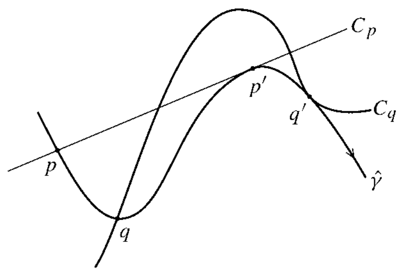

FIGURE 10.

PROPOSITION 1.4. *Suppose that $\gamma : P^1 \to P^2$ is an anti-convex curve which is not a line and meets each line in P^2 in at most finitely many connected components. Then the corresponding family $\{F(p)\}_{p\in S^1}$ of subsets of S^1 satisfies the following properties*:

(L1) *$p \in F(p)$.*

(L2) *$F(p)$ is a closed proper subset of S^1 and has finitely many connected components.*

(L3) *If $q \in F(p)$, then $Tq \in F(p)$ where $T : S^1 \to S^1$ is the restriction of the antipodal map on S^2 to $\hat{\gamma}$.*

(L4) *Suppose $p' \in F(p)$ and $q' \in F(q)$ satisfy*

$$p \le q \le p' \le q'(\le Tp)$$

or

$$p \ge q \ge p' \ge q'(\ge Tp)\,,$$

where $\ge$ and $\le$ denote the cyclic order of S^1. Then $F(p) = F(q)$.

(L5) *If $\pi(F(p)) = \pi(F_0(p))$, then $\pi(F(Tp)) \ne \pi(F_0(Tp))$, where $\pi : S^1 \to P^1$ denotes the canonical projection.*

(L6) *$q \in F_0(p)$ if and only if $F(p) = F(q)$.*

(L7) *Let (p_k) be a sequence in S^1 that converges to an element p in S^1, and let (s_k) be another sequence in S^1 such that $s_k \in F(p_k)$ and $\lim s_k = s$. Then $s \in F(p)$.*

PROOF. (L1) is obvious. (L2) is a trivial consequence of the assumption that γ and a line meet in at most finitely many connected components. (L3) follows from the fact that $\hat{\gamma}$ and $\hat{L}_p$ are both symmetric with respect to the antipodal map T.

We now prove (L4). If C_p and C_q are great circles which meet at two points which are not antipodal, then C_p must be equal to C_q. Suppose $p' \in F(p)$ and $q' \in F(q)$, and $p \le q \le p' \le q'(\le Tp)$ or $p \ge q \ge p' \ge q'(\ge Tp)$ holds. Then the subarc of C_q between q and q' must meet C_p twice. One is between p and p', and the other is between p' and Tp on C_p. (See Figure 10 for the case $p \le q \le p' \le q'$.) Thus $C_p = C_q$ holds.

Now we prove (L5). If $\pi(F(p)) = \pi(F_0(p))$, then $F(p)$ consists of two connected components. By Proposition 1.2 (b), C_p is a tangent line at p. The great circle C_{Tp} coincides with the great circle which we get by rotating C_p in negative direction through great circles meeting $\hat{\gamma}$ only at p and Tp until it hits $\hat{\gamma}$. The great circles C_p and C_{Tp} cannot coincide,

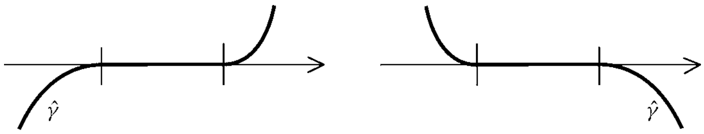

FIGURE 11. Negatve inflection and positive inflection.

since γ is not a line. It follows that C_{Tp} is not tangent to $\hat\gamma$ at p and hence also not at Tp. By Proposition 1.2 (b), $F(Tp)$ contains at least three components, two of which consist of p and Tp, since the intersection between C_{Tp} and $\hat\gamma$ is transversal at these points. Hence $\pi(F(Tp))$ is not connected and we see that $\pi(F(Tp)) \neq \pi(F_0(Tp))$.

We now prove (L6). Suppose $q \in F_0(p)$. We may assume that $q \neq p$. Then $F_0(p)$ is a closed interval and C_p must be the tangent line both at p and q. It follows that C_p must be equal to the great circle C_q by Proposition 1.3. This implies $F(p) = F(q)$. Now we assume that $F(p) = F(q)$. We let A denote the set of points r in $F(p) = F(q)$ such that the tangent great circle of $\hat\gamma$ at r contains $F(p) = F(q)$ and r is not a true inflection point. Let B denote the complement of A in $F(p) = F(q)$. By Proposition 1.2 the set B coincides with $F_0(p) \cup T(F_0(p)) = F_0(q) \cup T(F_0(q))$. Now note that a set $T(F_0(r))$ cannot coincide with a set $F_0(s)$ for any r and s in S^1, since the curve $\hat\gamma$ crosses C_r from right to left in $F_0(r)$ and C_s from left to right in $F_0(s)$; see Figure 11.

Finally, we prove (L7). We may assume that s is neither p nor Tp. After replacing (p_k) by a subsequence if necessary, we may also assume that C_{p_k} converges to a great circle C. Since C_{p_k} satisfies Properties (a) and (b) in Proposition 1.3 for all k, so does C, and it follows that $C = C_p$ holds. Hence $s \in F(p)$. □

DEFINITION 1.5. We call a family $\{F(p)\}_{p\in S^1}$ of closed subsets of S^1 an *intrinsic line system* if it satisfies Properties (L1) through (L7) in Proposition 1.4.

This definition is an analogue of the somewhat simpler intrinsic circle systems, see [20] and [17], which are useful, e.g., in proving the existence of two inscribed (resp. circumscribed) osculating circles of a given simple closed C^2-regular curve in the Euclidean plane.

An inflection point of a curve $\hat\gamma$ is called *positive* if the tangent great circle crosses $\hat\gamma$ from right to left, and *negative* if the tangent great circle crosses $\hat\gamma$ from left to right. Since the sign of an inflection point is reversed by the antipodal map, the notion is meaningful for $\hat\gamma$ but not for γ.

DEFINITION 1.6. Let $\{F(p)\}_{p\in S^1}$ be an intrinsic line system. A point $p \in S^1$ satisfying

$$\pi(F(p)) = \pi(F_0(p)) \qquad (\text{resp. } \pi(F(Tp)) = \pi(F_0(Tp)))$$

is called a *positive c-inflection point* (resp. a *negative c-inflection point*).

We defined clean inflection points on anti-convex curves γ in Introduction. A clean inflection point corresponds to a c-inflection point of the intrinsic line system associated to γ.

2. Clean inflection points. In this section we prove Theorem B in Introduction. The crucial point is that we only use properties (L1)–(L6) of intrinsic line systems to prove the theorem under the assumption that γ meets each line in at most finitely many components. We call such γ *generic*. It is only in the last step where we remove this assumption by approximating a given non-generic anti-convex curve by a generic one.

LEMMA 2.1. *Let $p \in S^1$. Suppose that $q \in F(p) \cap (p, Tp)$. Let r be a point in (p, q). Suppose that r is not contained in $F_0(p)$. Then*

$$\pi(F(r)) \subset \pi((p, q))$$

holds.

PROOF. Suppose that $\pi(F(r))$ contains an element $a \notin \pi((p, q))$. Let $\{\hat{a}^+, \hat{a}^-\}$ be the preimage of a under π. Without loss of generality, we may assume that $\hat{a}^+ \in (p, Tp]$. Since $a \notin \pi((p, q))$, we have $\hat{a}^+ \in (q, Tp]$. Hence we have the inequality

$$p < r < q \leq \hat{a}^+ \leq Tp\,.$$

By (L4), we have $F(p) = F(r)$. In particular, $r \in F_0(p)$ by (L6), which is a contradiction. $\square$

With similar arguments we can prove the following lemma.

LEMMA 2.2. *Let $p \in S^1$. Suppose that $q \in F(p) \cap (Tp, p)$. Let r be a point in (q, p). Suppose that r is not contained in $F_0(p)$. Then*

$$\pi(F(r)) \subset \pi((q, p))$$

holds.

Next we prove the following lemma.

LEMMA 2.3. *Let $p \in S^1$. Suppose that $q \in F(p) \cap (p, Tp)$ and $(p, q) \cap F_0(p) = \emptyset$. Let r be the midpoint of (p, q). Then at least one of the following three cases occurs*:

(i) *r is a positive c-inflection point.*

(ii) *There exist $p_1, q_1 \in F(r) \cap (r, q)$ such that $p_1 \in F_0(r)$ and $(p_1, q_1) \cap F_0(r) = \emptyset$.*

(iii) *There exist $p_1, q_1 \in F(r) \cap (p, r)$ such that $p_1 \in F_0(r)$ and $(q_1, p_1) \cap F_0(r) = \emptyset$.*

PROOF. Assume that r is not a positive c-inflection point. Then there exists a point $b \in \pi(F(r))$ such that $b \notin \pi(F_0(r))$. Let $\{q_1, Tq_1\}$ be the points such that $\pi(q_1) = b$. Since $(p, q) \cap F_0(p) = \emptyset$, we have $r \notin F_0(p)$. Thus by Lemma 2.1, we have $b \in \pi(F(r)) \subset \pi((p, q))$. So we may assume that $q_1 \in (p, q)$ without loss of generality. Since $b \notin \pi(F_0(r))$, we have $q_1 \notin F_0(r)$. There are two possibilities, one being $q_1 \in (r, q)$ and the other being $q_1 \in (p, r)$.

First, we consider the case $q_1 \in (r, q)$. Since $F_0(r)$ is a proper subset of S^1, it is a linearly ordered set with respect to the restriction of the cyclic order of S^1 and one can define its supremum and infimum. We set

$$p_1 := \sup(F_0(r))\,.$$

Since $F_0(r) \subset (p,q)$ and $r \in F_0(r)$, it holds that $p_1 \in [r,q]$. On the other hand, since $q_1 \notin F_0(r)$ and $q_1 \in (r,q)$, we have

$$r \leq p_1 < q_1 < q\,.$$

This is Case (ii).

Next, we consider the case $q_1 \in (p,r)$. We set

$$p_1 := \inf(F_0(r))\,.$$

Since $F_0(r) \subset (p,q)$ and $r \in F_0(r)$, it holds that $p_1 \in [p,r]$. On the other hand, since $q_1 \notin F_0(r)$ and $q_1 \in (p,r)$, we have

$$r \geq p_1 > q_1 > p\,.$$

This is Case (iii). □

Similarly, we get the following lemma.

LEMMA 2.4. *Let $p \in S^1$. Suppose that $q \in F(p) \cap (Tp, p)$ and $(q, p) \cap F_0(p) = \emptyset$. Let r be the midpoint of (q, p). Then at least one of the following three cases occurs*:

(i) *r is a positive c-inflection point.*

(ii) *There exist $p_1, q_1 \in F(r) \cap (r, p)$ such that $p_1 \in F_0(r)$ and $(p_1, q_1) \cap F_0(r) = \emptyset$.*

(iii) *There exist $p_1, q_1 \in F(r) \cap (q, r)$ such that $p_1 \in F_0(r)$ and $(q_1, p_1) \cap F_0(r) = \emptyset$.*

We use Lemma 2.3 and Lemma 2.4 to prove the following proposition.

PROPOSITION 2.5. *Let $p \in S^1$. Suppose that $q \in F(p) \cap (p, Tp)$ and $(p,q) \cap F_0(p) = \emptyset$. Then there exists a positive c-inflection point s in (p,q) such that $\pi(F(s)) \subset \pi((p,q))$.*

PROOF. Suppose that there are no positive c-inflection points in (p,q). Let δ be the length of the interval (p,q). Let r denote the midpoint of the interval (p,q). By Lemma 2.3 or Lemma 2.4, there are two points $p_1, q_1 \in (p,q)$ satisfying the following properties:

(1) $q_1 \in F(r)$ and $p_1 \in F_0(r)$.

(2) $(p_1, q_1) \cap F_0(r) = \emptyset$ if $q_1 > p_1$ and $(q_1, p_1) \cap F_0(r) = \emptyset$ if $q_1 < p_1$.

(3) The length of the interval between the two points p_1 and q_1 is less than or equal to $\delta/2$.

Since $p_1 \in F_0(r)$, we have $F(r) = F(p_1)$ by (L6). So we have:

(1′) $q_1 \in F(p_1)$.

(2′) $(p_1, q_1) \cap F_0(p_1) = \emptyset$ if $q_1 > p_1$ and $(q_1, p_1) \cap F_0(p_1) = \emptyset$ if $q_1 < p_1$.

We can repeat this argument replacing $\{p,q\}$ by $\{p_1, q_1\}$. Applying Lemma 2.3 and Lemma 2.4 inductively, we find sequences (p_n) and (q_n) satisfying the following properties:

(a) p_n lies in the interval beteen p_{n-1} and q_{n-1}, and $q_n \in F(p_n)$.

(b) $(p_n, q_n) \cap F_0(p_n) = \emptyset$ if $q_n > p_n$ and $(q_n, p_n) \cap F_0(p_n) = \emptyset$ if $q_n < p_n$.

(c) The length of the interval between the two points p_n and q_n is less than or equal to $\delta/2^n$.

It follows from Lemma 2.1 and Lemma 2.2 that

$$\pi(F(p_n)) \subset \pi(p_{n-1}, q_{n-1})\,.$$

In particular, the length of $\pi(F(p_n))$ is less than $\delta/2^{n-1}$. We set

$$y = \lim p_n = \lim q_n\,.$$

The limit y lies between p_n and q_n for all n.

We now prove that $\pi(F(y)) = \{\pi(y)\}$. Suppose that $\pi(F(y))$ does not consist only of $\pi(y)$. Then there is a point $z \in F(y)$ such that $Ty > z > y$. For sufficiently large n, we either have

$$Tp_n > z > q_n > y > p_n$$

or

$$Ty > Tq_n > z > p_n > y\,.$$

In both cases (L4) implies that $F(y) = F(p_n)$. In particular, $y \in F_0(p_n)$, which contradicts $(q_n, p_n) \cap F_0(p_n) = \emptyset$. Thus we can conclude that $\pi(F(y)) = \{\pi(y)\}$, which implies that y is a positive c-inflection point. This is a contradiction. Hence there is a positive c-inflection point s in (p, q). By Lemma 2.1, we have $\pi(F(s)) \subset \pi((p, q))$. □

By reversing the orientation of S^1, Proposition 2.5 implies the following

PROPOSITION 2.6. *Let $p \in S^1$. Suppose that $q \in F(p) \cap (Tp, p)$ and $(q, p) \cap F_0(p) = \emptyset$. Then there exists a positive c-inflection point s in (q, p) such that $\pi(F(s)) \subset \pi((q, p))$.*

COROLLARY 2.7. *Let $p \in S^1$. Suppose that $q \in F(p) \cap (p, Tp)$ and $q \notin F_0(p)$. Then there exists a positive c-inflection point s in (p, q) such that $\pi(F(s)) \subset \pi((p, q))$ and $F(s) \cap F_0(p) = \emptyset$.*

PROOF. We set

$$p' = \sup F_0(p)\,.$$

Since $q \notin F_0(p)$ and $F_0(p') = F_0(p)$, we have

$$q > p' \geq p\,, \qquad (p', q) \cap F_0(p') = \emptyset\,.$$

Applying Proposition 2.5 to the pair (p', q), we find a positive c-inflection point s in $(p', q) \subset (p, q)$. We have $F(s) \cap F_0(p) = \emptyset$, since $\pi(F(s)) \subset \pi((p', q))$. □

Similarly, we get the following corollary.

COROLLARY 2.8. *Let $p \in S^1$. Suppose that $q \in F(p) \cap (Tp, p)$ and $q \notin F_0(x)$. Then there exists a positive c-inflection points in (q, p) such that $\pi(F(s)) \subset \pi((q, p))$ and $F(s) \cap F_0(p) = \emptyset$.*

Applying Corollary 2.7 and Corollary 2.8, we get the following

COROLLARY 2.9. *Suppose that $q \in F(p)$ satisfies $q \neq Tp$ and $q \notin F_0(p)$. Let J be the open interval bounded by p and q. Then there exists a positive c-inflection point s in J such that $\pi(F(s)) \subset \pi(J)$ and $F(s) \cap F_0(p) = \emptyset$.*

Theorem B in Introduction is a consequence of the following theorem if the curve γ meets each line in at most finitely many components.

THEOREM 2.10. *Let $\{F(p)\}_{p \in S^1}$ be an intrinsic line system. Then there exist three positive c-inflection points s_1, s_2, s_3 in S^1 such that*

$$s_1 < Ts_3 < s_2 < Ts_1 < s_3 < Ts_2 (< s_1) .$$

Moreover, the sets $F(s_1)$, $F(s_2)$, $F(s_3)$ are mutually disjoint.

PROOF. Take a point p which is not a c-inflection point. Then there exists a point $q \in F(p)$ such that $q \notin F_0(p)$. By Corollary 2.9, there is a c-inflection point s_1 between p and q. By (L5), we have $\pi(F(Ts_1)) \neq \pi(F_0(Ts_1))$. Then there exists a point $u \in (s_1, Ts_1)$ such that $u \in F(Ts_1)$ but $u \notin F_0(Ts_1)$. Then, by Corollary 2.9, we find a c-inflection point s_2 on $(u, Ts_1) \subset (s_1, Ts_1)$. Notice that $Tu \in F(Ts_1)$ and $Tu \notin F_0(Ts_1)$. Hence we find another positive c-inflection point s_3 on $(Ts_1, Tu) \subset (Ts_1, s_1)$ by Corollary 2.9. The sets $F(s_3)$ and $F(s_2)$ are disjoint, since $F(s_2) \subset (u, Ts_1)$ and $F(s_3) \subset (Ts_1, Tu)$.

Suppose that $F(s_2) \cap F(s_1) \neq \emptyset$. Since $F(s_2) = F_0(s_2)$ and $F(s_1) = F_0(s_1)$, we have $F(s_2) = F(s_1)$ by (L6). Then $Ts_1 \in F(s_2)$, contradicting $F(s_2) \subset (u, Ts_1)$. Thus $F(s_2) \cap F(s_1) = \emptyset$. Similarly, we show $F(s_3) \cap F(s_1) = \emptyset$. □

Until now, we have assumed that γ meets each line in at most finitely many components. We now prove Theorem B in the general case, using that such curves are generic in the set of anti-convex curves. In the proof we will need that the curve γ is C^2. So far we only used that it is C^1.

PROOF OF THEOREM B. Let γ be an arbitrary anti-convex curve on P^2 that we assume to be π-periodic, that is, $\gamma(t) = \gamma(t + \pi)$ for $t \in \boldsymbol{R}$. A point $p \in \boldsymbol{R}^3 \setminus \{0\}$ determines a point $[p]$ in P^2, where $[p]$ denotes the line in $\boldsymbol{R}^3$ spanned by p. There is a π-anti-periodic C^2-regular map $F : \boldsymbol{R} \to \boldsymbol{R}^3$ such that

$$\gamma(t) = [F(t)] \in P^2 ,$$

where a map $F(t)$ is called *π-anti-periodic* if it satisfies $F(t + \pi) = -F(t)$ for all $t \in \boldsymbol{R}$. The map F has the Fourier series expansion

$$F(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(2n + 1)t + b_n \sin(2n + 1)t) ,$$

where $a_0, a_1, b_1, \ldots$ are vectors in $\boldsymbol{R}^3$, and this series converges uniformly to $F(t)$. We set

$$F_N(t) = a_0 + \sum_{n=1}^{N} (a_n \cos(2n + 1)t + b_n \sin(2n + 1)t) .$$

One can easily show that $\gamma_N(t) = [F_N(t)]$ is also anti-convex regular curve for sufficiently large N, since γ is C^2. Moreover γ_N is generic, that is, it meets each line at at most finitely many components, since it is real analytic. We set

$$\hat{\gamma}_N(t) = \frac{F_N(t)}{|F_N(t)|} : \boldsymbol{R} \to S^2 .$$

Then the spherical curve $\hat{\gamma}_N$ is a double cover of the curve γ_N in P^2. By Theorem 2.10, there exists three positive c-inflection points $s_1(N), s_2(N), s_3(N)$ on $\hat{\gamma}_N(t)$ such that

$$0 \le s_1(N) < s_3(N) - \pi < s_2(N) < s_1(N) + \pi < s_3(N) < s_2(N) + \pi < 2\pi .$$

By taking a subsequence, we may assume that $s_j(N)$ converges to s_j for $j = 1, 2, 3$. Since $\hat{\gamma}$ is not a great circle, positive c-inflection points do not accumulate to negative c-inflection points. Thus we have

$$0 \le s_1 < s_3 - \pi < s_2 < s_1 + \pi < s_3 < s_2 + \pi < 2\pi .$$

These six points may not be clean inflection points. However, the tangent great circles at these six points topologically cross $\hat{\gamma}$ exactly twice. Hence the corresponding tangent lines of γ only cross γ once. $\square$

3. Further properties of intrinsic line systems. In this section we derive some properties of intrinsic line systems, which will be used in the next section to prove Theorem A in the introduction. Throughout this section we will assume that an intrinsic line system $\{F(p)\}_{p \in S^1}$ is given.

For a point $p \in S^1$, we set

$$Y(p) := F(p) \setminus (F_0(p) \cup T F_0(p)) ,$$

$$Y^+(p) := Y(p) \cap [p, Tp] , \quad Y^-(p) := Y(p) \cap [Tp, p] ,$$

$$F^+(p) := Y^+(p) \cup F_0(p) , \quad F^-(p) := Y^-(p) \cup T(F_0(p)) .$$

For example, in the case of Figure 12, we have

$$F_0(p) = \{p\} , \quad Y^+(p) = \{q_1, q_2, q_3\} , \quad Y(p) = \{q_1, q_2, q_3, Tq_1, Tq_2, Tq_3\} .$$

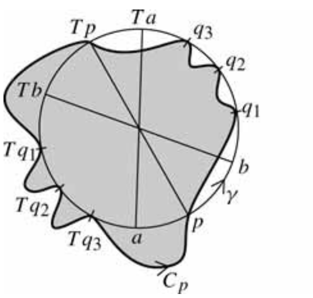

FIGURE 12. Definition of $Y(p)$.

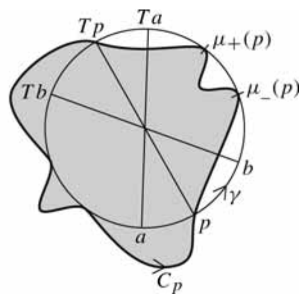

FIGURE 13. Definition of $\mu_\pm(p)$.

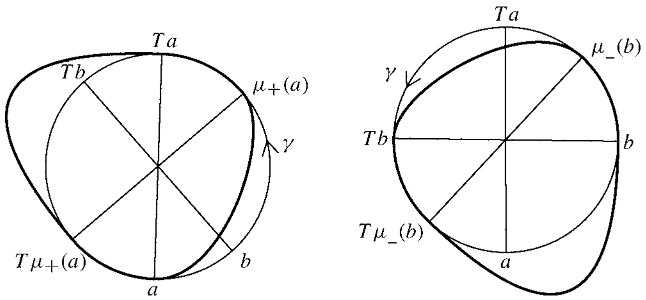

FIGURE 14. Definitions of $\mu_-(a)$ and $\mu_+(b)$.

DEFINITION 3.1. An open interval (a, b) is said to be *admissible* if $b \in (a, Ta)$ and there are no positive c-inflection points in (a, b).

Let (a, b) be an admissible interval. Then $Y^+(p)$ is non-empty for all $p \in (a, b)$. So we set (see Figure 13)

$$\mu_-(p) := \inf_{(p,Tp)} Y^+(p), \quad \mu_+(p) := \sup_{(p,Tp)} Y^+(p)$$

for $p \in (a, b)$. For example,

$$\mu_-(p) = q_1, \quad \mu_+(p) = q_3$$

holds in the case of Figure 12. Moreover, we set

$$\mu_-(a) := \begin{cases} \inf_{[a,Ta]} Y^+(a) & \text{if } a \text{ is not a positive c-inflection point}, \\ \inf_{[a,Ta]} TF_0(a) & \text{if } a \text{ is a positive c-inflection point}, \end{cases}$$

$$\mu_+(b) := \begin{cases} \sup_{[b,Tb]} Y^+(b) & \text{if } b \text{ is not a positive c-inflection point}, \\ \sup_{[b,Tb]} F_0(b) & \text{if } b \text{ is a positive c-inflection point}. \end{cases}$$

Figure 14 explains the definitions of $\mu_-(a)$ and $\mu_+(b)$ when a and b are c-inflection points and neither $F_0(a)$ nor $F_0(b)$ reduces to a point. The left and the right of the figure correspond to the definition of $\mu_-(a)$ and $\mu_+(b)$ when a and b are positive c-inflection points, respectively.

These definitions have analogues in the theory of intrinsic circle system; see p. 190 in [20] by the second author. The results in this section correspond to Lemma 1.3, Theorem 1.4 and Theorem 1.6 in [20].

REMARK 3.2. Let S^1_{rev} be the circle S^1 with the reversed orientation. Then $\{F(p)\}_{p \in S^1_{\text{rev}}}$ gives another intrinsic line system. An admissible interval (a, b) of $\{F(p)\}_{p \in S^1}$ corresponds to the admissible interval (b, a) of $\{F(p)\}_{p \in S^1_{\text{rev}}}$, and $\mu_-(p)$ for $p \in (a, b)$ with respect to $\{F(p)\}_{p \in S^1}$ coincides with $\mu_+(p)$ with respect to $\{F(p)\}_{p \in S^1_{\text{rev}}}$.

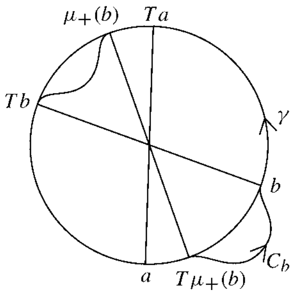

FIGURE 15. The case $\mu_+(b) \geq Ta$.

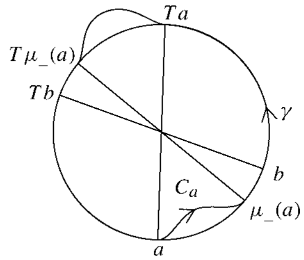

FIGURE 16. The case $\mu_-(a) \leq b$.

LEMMA 3.3. *Let (a, b) be an admissible interval. Then we have the inequalities*

$$b \leq \mu_+(p) < Ta$$

for all $p \in (a, b]$ and

$$b < \mu_-(p) \leq Ta$$

for all $p \in [a, b)$.

PROOF. We first assume that $p \in (a, b)$. Then p is not a positive c-inflection point and $Y^+(p)$ is non empty. We fix $q \in Y^+(p)$ arbitrarily. Then, by Corollary 2.9, there is a positive c-inflection point r on (p, q). Since (a, b) is an admissible arc, we have $q > r > b$. Suppose that

$$(Tb > Tp) > q \geq Ta .$$

Then we have

$$b > p > Tq \geq a .$$

Since $Tq \in Y^-(p)$, there is a positive c-inflection point on $(Tq, p) \subset (a, b)$ by Corollary 2.9, which contradicts the fact that (a, b) is an admissible arc. Thus we have $Ta > q$, which implies $q \in (b, Ta)$. Since q is arbitrary, we have

$$b < \mu_-(p) \leq \mu_+(p) < Ta$$

for all $p \in (a, b)$.

Next, we consider the case $q = b$. If b is not a positive c-inflection point, then $\mu_+(b) \in Y^+(b)$ and the above arguments yield $b < \mu_+(b) < Ta$. So we assume that b is a positive c-inflection point. Then $b \leq \mu_+(b)$ holds by definition. Suppose now that $\mu_+(b) \geq Ta$. (See Figure 15.) Then $T(\mu_+(b)) \notin F_0(b)$ and $\mu_+(b) \neq Tb$. Therefore there is a positive c-inflection point between $(T(\mu_+(b)), b)$ by Corollary 2.9, which is a contradiction, since $T(\mu_+(b)) \in (a, b)$ and (a, b) is admissible. Thus we have $\mu_+(b) < Ta$.

Finally, we consider the case $q = a$. If a is not a positive c-inflection point, $\mu_-(a) \in Y^+(a)$ and the above arguments yield $b < \mu_-(a) < Ta$. So we assume that a is a positive c-inflection point. Then $\mu_-(a) \leq Ta$ holds by definition. Suppose now that $\mu_-(a) \leq b$. (See Figure 16.) Then $\mu_-(a) \neq Ta$. Since $\mu_-(a) \notin F_0(a)$, there is a positive c-inflection point between $(a, \mu_-(a))$ by Corollary 2.9, which is a contradiction, since $T(\mu_-(a)) \in (a, b)$ and (a, b) is admissible. Thus we have $b < \mu_-(b)$. $\square$

PROPOSITION 3.4. *Let (a, b) be an admissible interval. Then we have the inequalities*

$$(b \leq)\mu_+(b) \leq \mu_+(p)\,,$$
$$\mu_-(p) \leq \mu_-(a)(\leq Ta)$$

for all $p \in (a, b)$.

PROOF. In the previous lemma, we already proved that

$$b < \mu_+(p)$$

for all $p \in (a, b)$. Suppose now that $\mu_+(p) \in (b, \mu_+(b))$. Applying Lemma 3.3 to (p, b), we get $b \leq \mu_+(b) < Tp$. Thus

$$p < b < \mu_+(p) < \mu_+(b)(< Tp)$$

holds. Since $p, \mu_+(p) \in F^+(p)$, we have $F(b) = F(p)$ by (L4). Thus b is like p not a positive c-inflection and

$$\mu_+(b) = \mu_+(p)\,,$$

contradicting the assumption $\mu_+(p) < \mu_+(b)$. So we have $\mu_+(p) \geq \mu_+(b)$.

By Lemma 3.3, we have $\mu_-(p) < Ta$. Now we suppose

$$\mu_-(a) < \mu_-(p) < Ta\,.$$

Applying Lemma 3.3 to (a, p), we get $p < \mu_-(a)$. Thus

$$p < \mu_-(a) < \mu_-(p) < Ta(< Tp)$$

holds. Since $p, \mu_-(p) \in F^+(p)$, we have $F(a) = F(p)$ by (L4). Then a is like p not a positive c-inflection point. Thus we have $\mu_-(a) = \mu_-(p)$, contradicting the assumption $\mu_-(a) < \mu_-(p)$. So we have $\mu_-(p) \leq \mu_-(a)$. $\square$

COROLLARY 3.5 (Monotonicity Lemma). *Let (a, b) be an admissible arc and $p, q \in (a, b)$. Suppose that $p < q$. Then we have*

$$\mu_-(p) \geq \mu_-(q)\,, \qquad \mu_+(p) \geq \mu_+(q)\,.$$

Moreover $\mu_-(p) > \mu_+(q)$ holds when $F(p) \neq F(q)$, and $\mu_-(a) > \mu_+(b)$ if there are points p and q in (a, b) such that $F(p) \neq F(q)$.

PROOF. The first two inequalities follow directly from Proposition 3.4.

We now prove that $\mu_-(p) > \mu_+(q)$ when $F(p) \neq F(q)$. Assume that $F(p) \neq F(q)$ and $\mu_-(p) \leq \mu_+(q)$. By Proposition 3.4 we have

$$(a <)p < q < \mu_-(p) \leq \mu_+(q) < Ta\,,$$

which implies by (L4) that $F(p) = F(q)$, a contradiction. Hence $\mu_-(p) > \mu_+(q)$.

Finally, we prove the inequlity $\mu_-(a) > \mu_+(b)$ under the assumption that there are points $p, q \in (a, b)$ such that $p < q$ and $F(p) \neq F(q)$. It follows from Proposition 3.4 and the inequality we have just proved that

$$\mu_+(b) \leq \mu_+(q) < \mu_-(p) \leq \mu_-(a)\,,$$

which proves the claim. □

PROPOSITION 3.6 (Semi-continuity). *Let (a, b) be an admissible arc. Then*

$$\lim_{x\to a+0} \mu_-(x) = \mu_-(a)\,, \qquad \lim_{x\to b-0} \mu_+(x) = \mu_+(b)\,.$$

PROOF. We prove the first formula. The second formula can be proved similarly. (See Remark 3.2.) When there is a point $p \in (a, b)$ such that $p \in F_0(a)$, the assertion is obvious. So we may assume that $(a, b) \cap F_0(a) = \emptyset$. Let (r_n) be a strictly decreasing sequence in (a, b) converging to a. There are points p_n and q_n in the interval (a, r_n) such that $F(p_n) \neq F(q_n)$, since otherwise the closed set $F(q)$ would contain the interval $[a, r_n]$ for all $q \in (a, r_n)$ and it would follow that $[a, r_n] \subset F_0(a)$. Hence, by Proposition 3.4 and Corollary 3.5, we have that

$$\mu_+(b) < \mu_-(r_n) < \mu_-(r_{n+1}) < \mu_-(a)$$

holds. So the sequence $\mu_-(r_n)$ has a limit s. Since $\mu_-(r_n) \in F(r_n)$, (L7) implies that

$$s \in F(a)\,.$$

Since $b < \mu_+(b)$, we have $b \leq s \leq \mu_-(a)$. Since $(a, b) \cap F_0(a) = \emptyset$, we have that $(a, \mu_-(a))$ is disjoint from the set $F(a)$. Thus we have $s = \mu_-(a)$, since $s \in F(a)$. □

THEOREM 3.7. *Let (a, b) be an admissible arc. Then, for any $q \in (\mu_+(b), \mu_-(a))$, there exists a point $p \in (a, b)$ such that*

$$\mu_-(p) \leq q \leq \mu_+(p)\,.$$

PROOF. We set

$$B_q := \{x \in (a, b)\,;\ \mu_+(x) \leq q\}\,.$$

By Proposition 3.6 we have that $\lim_{x\to b-0} \mu_+(x) = \mu_+(b) + 0$. Thus a point $x \in (a, b)$ sufficiently close to b belongs to B_q. Since B_q is non-empty, we can set

$$p := \inf_{[a,b]}(B_q)\,.$$

Since $\mu_-(a) > q$, we have $p \in (a, b)$. By the definition of p, there exists a sequence (r_n) in B_q such that $\lim_{n\to\infty} r_n = p + 0$. By the definition of B_q, we have

$$\mu_-(r_n) \leq \mu_+(r_n) \leq q\,.$$

Since $\lim_{n\to\infty} \mu_-(r_n) = \mu_-(p)$ by Proposition 3.6, we have

$$\mu_-(p) \leq q\,.$$

On the other hand, let (s_n) be a sequence such that $\lim_{n\to\infty} s_n = p - 0$. By the definition of B_q, we have $q < \mu_+(s_n)$. Since $\lim_{n\to\infty} \mu_+(s_n) = \mu_+(p)$, we have $q \leq \mu_+(p)$. □

4. Double tangents. We will assume throughout this section that $\gamma : P^1 \to P^2$ is an anti-convex C^1-regular curve whose number $i(\gamma)$ of true inflection points is finite. It follows from the last assumption that *each line in P^2 meets the curve γ in at most finitely many components.*

LEMMA 4.1. *Let $\gamma : P^1 \to P^2$ be an anti-convex curve. Suppose that γ meets a line L at $\gamma(a)$ and $\gamma(b)$, and denote one of the closed intervals on P^1 bounded by a and b by $[a, b]$. Then one of the two closed line segments L_1 and L_2 on L bounded by $\gamma(a)$ and $\gamma(b)$, say L_1, has the property that $\gamma([a, b]) \cup L_1$ lies in an affine plane and $\gamma([a, b]) \cup L_2$ is not homotopic to a point. The curve $\gamma([a, b]) \cup L_1$ bounds a contractible domain having acute interior angles at $\gamma(a)$ and $\gamma(b)$ if it is free of self-intersections.*

We call L_1 the *chord with respect to the interval* $[a, b]$ and denote it by $\overline{\gamma(a)\gamma(b)}$.

PROOF. We choose a point $c \notin [a, b]$. Then there is a line L_c which meets γ only at $\gamma(c)$. Then L_c meets L at one point which we assume to be on the line segment on L bounded by $\gamma(a)$ and $\gamma(b)$ that we denote by L_2. Then $\gamma([a, b]) \cup L_1$ lies in the affine plane $P^2 \backslash L_c$.

Since L is not null-homotopic, either $\gamma([a, b]) \cup L_1$ or $\gamma([a, b]) \cup L_2$ is not null-homotopic. So $\gamma([a, b]) \cup L_2$ is not homotopic to a point.

Assume $\gamma([a, b]) \cup L_1$ is free of self-intersections and let D denote the contractible domain in the affine plane bounded by $\gamma([a, b]) \cup L_1$. If its interior angle at $\gamma(a)$ or $\gamma(b)$ is not acute, any line passing through the point meets γ, which contradics the anti-convexity of γ. □

The following assertion is one of the fundamental properties of anti-convex curves.

PROPOSITION 4.2. *Let $\gamma : P^1 \to P^2$ be an anti-convex curve. Let $[a, b]$ be a closed interval on P^1 and suppose $\gamma([a, b])$ meets a line L in an affine plane A^2 at $\gamma(t_1)$, $\gamma(t_2), \ldots,$ $\gamma(t_n)$ with*

$$a = t_1 < t_2 < \cdots < t_n = b\,.$$

Then

$$\gamma(t_1)\,,\ \gamma(t_2)\,, \ldots,\ \gamma(t_n)$$

lie on $\overline{\gamma(a)\gamma(b)}$ in this order.

PROOF. Assume that the claim is not true. Then there is a smallest i such that $\gamma(t_i)$ lies on $\overline{\gamma(a)\gamma(t_{i-1})}$. Then any line passing through $\gamma(t_i)$ must meet $\gamma((t_1, t_i))$, which contradicts the anti-convexity of γ. □

By Lemma 4.1, $\gamma([a, b])$ and the chord $\overline{\gamma(a)\gamma(b)}$ lie in an affine plane A^2. We define a new curve $\gamma_1 : P^1 \to P^2$ by setting

$$\gamma_1(t) := \begin{cases} \dfrac{\gamma(b)(t-a) + \gamma(a)(t-b)}{b-a} & \text{for } t \in [a, b]\,, \\ \gamma(t) & \text{for } t \notin (a, b)\,, \end{cases}$$

which is the curve one gets by replacing $\gamma([a, b])$ by $\overline{\gamma(a)\gamma(b)}$. Notice that the vector operations in the definition of γ_1 depend on the affine plane A^2. We call γ_1 the *reduction of* γ *with respect to the interval* $[a, b]$.

An interval $[a, b]$ on P^1 is called an *inflection interval* if a is a true inflection point and $\gamma([a, b])$ is the connected component of $\gamma \cap L_a$, where L_a is the tangent line of γ at a.

DEFINITION 4.3. Let $\gamma : P^1 \to P^2$ be an anti-convex curve. A nonempty proper open subinterval (a, b) on P^1 is called a *double tangent interval* if

(1) the chord $\overline{\gamma(a)\gamma(b)}$ is tangent to γ at $\gamma(a)$ and $\gamma(b)$,

(2) there is a point in $\gamma([a, b])$ which is not contained in $\overline{\gamma(a)\gamma(b)}$,

(3) $[a, b]$ is not an inflection interval of γ_1, where γ_1 is the reduction of γ with respect to the interval $[a, b]$.

By Lemma 4.1, the following assertion is obvious.

COROLLARY 4.4. *If* (a, b) *is a double tangent interval of an anti-convex curve* γ, *then the orientations of the tangent lines of* γ *at* $\gamma(a)$ *and* $\gamma(b)$ *induce the same direction on* $\overline{\gamma(a)\gamma(b)}$.

REMARK 4.5. If (a, b) is a double tangent interval, then the same cannot be true for $(b, a) = P^1 \setminus [a, b]$. In fact, the reduction γ_2 of γ with respect to the interval $[b, a]$ has $[b, a]$ as an inflection interval which violates Property (3) in Definition 4.3. This phenomenon is explained in Figure 17, where the two sketches indicate the same curve γ in different affine planes.

DEFINITION 4.6. Let $\gamma : P^1 \to P^2$ be an anti-convex curve. Two double tangent intervals (a_1, b_1) and (a_2, b_2) are called *independent* if they are disjoint or if the closure of one is contained in the other.

We now begin the proof of Theorem A in Introduction.

PROOF OF THEOREM A. To prove Formula $(*)$, we will start with a double tangent interval (a, b) and introduce the following reductions of γ. We let γ_1 be the reduction of γ with respect to the double tangent interval $[a, b]$, and γ_2 be the reduction of γ with respect to the interval $[b, a]$; see Figure 18. By our construction, γ_1 and γ_2 are both C^1-regular curves in P^2.

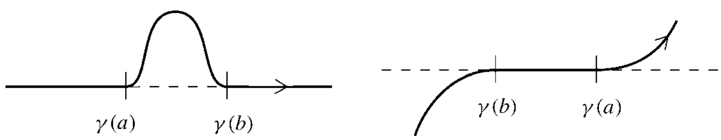

FIGURE 17. The same curve γ in different affine planes.

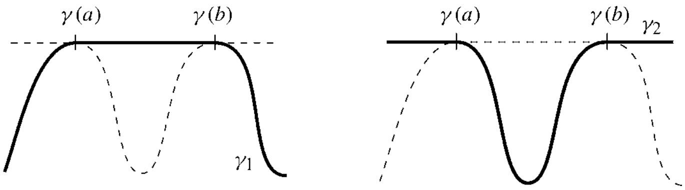

FIGURE 18. γ_1 and γ_2.

We now bring a couple of lemmas and propositions that will be needed to finish the proof of Theorem A.

LEMMA 4.7. *The curves γ_1 and γ_2 are both without self-intersections.*

PROOF. We will prove the claim for γ_1. Suppose that $\gamma(P^1 \setminus [a, b])$ meets the chord $\overline{\gamma(a)\gamma(b)}$ at $\gamma(c)$. By Proposition 4.2, the points $\gamma(a)$, $\gamma(b)$, $\gamma(c)$ must lie on the segment $\overline{\gamma(a)\gamma(c)}$ in this order, since $a < b < c$. This is a contradiction. It follows that γ_1 does not have self-intersections. One can similarly prove that γ_2 does not have self-intersections. □

The following is a key to prove Formula $(*)$.

PROPOSITION 4.8. *The curves γ_1, γ_2 are both anti-convex and the identity*

$$i(\gamma) = i(\gamma_1) + i(\gamma_2) - 1 \tag{2}$$

holds.

PROOF. We first show that γ_1 is anti-convex. We may assume that $\gamma([a, b])$ lies in an affine plane A^2. For a point $x \in P^2$, the pencil of lines passing through x is a projective line in the dual space of P^2 that we denote by $P^1(x)$. For a point $t \in P^1$, we define a subset $\mathcal{B}_\gamma(t)$ of $P^1(\gamma(t))$ such that each line L in $\mathcal{B}_\gamma(t)$ meets γ only at p and L is transversal to the tangent line at p. Since $\gamma(t)$ is an anti-convex curve, $\mathcal{B}_\gamma(t)$ is non-empty for all $t \in P^1$. One can easily prove that $\mathcal{B}_\gamma(t)$ is an open interval in $P^1(x)$. We will call $\mathcal{B}_\gamma(t)$ the *Barner set* of γ.

We have that $\mathcal{B}_\gamma(t)$ is contained in the Barner set $\mathcal{B}_{\gamma_1}(t)$ of γ_1 for every $t \notin [a, b]$, since no line $L \in \mathcal{B}_\gamma(t)$ can meet the chord $\overline{\gamma(a)\gamma(b)}$. So it is sufficient to show that $\mathcal{B}_{\gamma_1}(t)$ is not empty for $t \in (a, b)$. Suppose $\gamma : P^1 \to P^2$ meets the chord $\overline{\gamma(a)\gamma(b)}$ at

$$a = t_1 < t_2 < \cdots < t_n = b\,.$$

By Proposition 4.2, we have that

$$\gamma(t_1)\,,\ \gamma(t_2)\,,\ldots,\ \gamma(t_n)$$

lie on $\overline{\gamma(a)\gamma(b)}$ in this order.

Suppose now that there exists a point $x \in \overline{\gamma(a)\gamma(b)}$ such that the Barner set of γ_1 at x is empty. Then there exists a positive integer i where $1 \le i \le n-1$ such that $x \in \overline{\gamma(t_i)\gamma(t_{i+1})}$ and $x \neq \gamma(t_i), \gamma(t_{i+1})$.

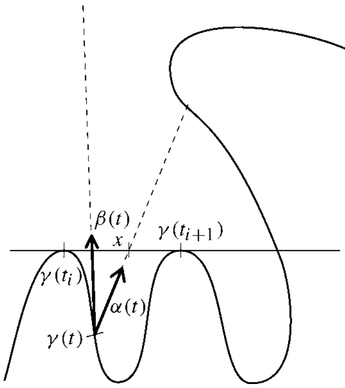

FIGURE 19.

We now set

$$I := [t_i, t_{i+1}] .$$

In the following argument we work in A^2 that we equip with the orientation such that $\overline{\gamma(t_i)\gamma(t_{i+1})}$ lies on the left hand side of $\gamma(I)$ as in Figure 19. We define continuous non-vanishing vector fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ along $\gamma|_I$ as follows:

(1) Both $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ point the left hand side of γ for every $t \in I$,
(2) $\boldsymbol{\alpha}(t)$ lies on the chord $\overline{\gamma(t)x}$,
(3) $\boldsymbol{\beta}(t)$ generates a line in $\mathcal{B}_\gamma(t)$.

We set

$$I_L := \{t \in I\ ;\ \boldsymbol{\alpha}(t), \boldsymbol{\beta}(t) \text{ is a positive frame}\} ,$$
$$I_R := \{t \in I\ ;\ \boldsymbol{\alpha}(t), \boldsymbol{\beta}(t) \text{ is a negative frame}\} ,$$

that is, I_L (resp. I_R) consists of those t with the property that the Barner direction $\boldsymbol{\beta}(t)$ is on the left of (resp. right of) $\overline{\gamma(t)x}$.

Notice that $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ are linearly independent for all $t \in I$, since the Barner set of γ_1 at x is empty. Hence the sign of $\det(\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t))$ is either positive or negative, implying that either I_L or I_R is empty. By Corollary 4.4 the tangent lines of γ at $\gamma(a)$ and $\gamma(b)$ induce the same direction on $\overline{\gamma(a)\gamma(b)}$. Hence it follows that $t_i \in I_L$ and $t_{i+1} \in I_R$, and thus that neither I_L nor I_R is empty. This is a contradiction and we can conclude that the Barner set of γ_1 at a point $x \in \overline{\gamma(a)\gamma(b)}$ is not empty. This finishes the proof that γ_1 is anti-convex. The proof that γ_2 is anti-convex is analogous.

Next we prove Formula (2). Let I_1 and I_2 be the number of independent inflection points of γ on $S^1 \setminus [a, b]$ and $[a, b]$, respectively. By definition, it is obvious that

$$i(\gamma_2) = I_2 + 1 . \tag{3}$$

In fact, $[b, a]$ is an additional inflection interval on γ_2. This phenomenon was explained in Remark 4.5 and Figure 17 above. On the other hand, we have

$$i(\gamma_1) = I_1 . \tag{4}$$

By (3) and (4), we hence have

$$i(\gamma_1) + i(\gamma_2) = I_1 + I_2 + 1 = i(\gamma) + 1 ,$$

which proves (2). □

COROLLARY 4.9. *If $i(\gamma) = 3$, then there are no double tangent intervals on γ.*

PROOF. Suppose that there is a double tangent interval. Then we can consider the anti-convex curves γ_1 and γ_2 as in Proposition 4.8. Since both $i(\gamma_1)$ and $i(\gamma_2)$ are at least 3 by Theorem 2.10, we have

$$i(\gamma) = i(\gamma_1) + i(\gamma_2) - 1 \geq 3 + 3 - 1 = 5 ,$$

which contradicts $i(\gamma) = 3$. □

We are assuming in this section that the number $i(\gamma)$ is finite. This has a consequence for number of elements in a set consisting of independent double tangent intervals as the next corollary shows.

COROLLARY 4.10. *The number of elements in a set of independent double tangent intervals is finite.*

PROOF. We assume that this number is infinite. Let n be an arbitrary positive integer. Then we can find independent double tangent intervals $(a_1, b_1), (a_2, b_2), \dots, (a_n, b_n)$. We order the intervals such that (a_i, b_i) does not contain (a_j, b_j) for $i < j$. We can associate to (a_1, b_1) two anti-convex curves $\gamma_1^{(1)}$ and $\gamma_2^{(1)}$ as was done before Lemma 4.7. Then we use the same construction to associate to (a_2, b_2) and $\gamma_1^{(1)}$ two new anti-convex curves $\gamma_1^{(2)}$ and $\gamma_2^{(2)}$. In this way we can get a finite sequence of pairs of anti-convex curves $\gamma_1^{(k)}$ and $\gamma_2^{(k)}$ for $k = 1, \dots, n$. By Proposition 4.8 we have

$$i(\gamma) = i(\gamma_1^{(n)}) - n + \sum_{k=1}^{n} i(\gamma_2^{(k)}) .$$

Since $i(\gamma_1^{(k)}), i(\gamma_2^{(k)}) \geq 3$, we have $i(\gamma) \geq 3 - n + 3n = 3 + 2n$. Since n is arbitrary, this contradicts the fact that $i(\gamma)$ is finite. □

The proof of the next proposition relies on the results of Section 3.

PROPOSITION 4.11. *If there are no double tangent intervals on γ, then $i(\gamma) = 3$ holds.*

Let $\hat{\gamma} : S^1 \to S^2$ be the lift of γ to a closed curve on S^2. We will need the following lemma in the proof of the proposition.

LEMMA 4.12. *Let (a, b) be an admissible interval on S^1 in the sense of Definition 3.1. Suppose that there are no double tangent intervals on γ. Then there are no true inflection points on $(\mu_+(b), \mu_-(a))$.*

PROOF. Let $\{F(p)\}_{p\in S^1}$ be the intrinsic line system associated to the lift $\hat{\gamma}$. Suppose that there is a true inflection point $c \in (\mu_+(b), \mu_-(a))$. By Theorem 3.7, there exists a point $p \in (a, b)$ such that

$$\mu_-(p) \le c \le \mu_+(p)\,.$$

Since c is a true inflection point, the limiting great circle C_p cannot pass through $\hat{\gamma}(c)$. This implies that there is a double tangent interval on γ. This contradiction proves the claim. □

PROOF OF PROPOSITION 4.11. By Theorem 2.10, there are at least three positive c-inflection intervals $[a_1, a_2]$, $[b_1, b_2]$ and $[c_1, c_2]$ on S^1, some of which may of course reduce to points. We assume that

$$a_1 \le a_2 < b_1 \le b_2 < c_1 \le c_2$$

and that there are no positive c-inflection points on (a_2, b_1) and (b_2, c_1).

By Lemma 4.12, there are no inflection points on (c_2, Tb_1), since (b_2, c_1) is an admissible arc and $\mu_+(c_1) = c_2$, $\mu_-(b_2) = Tb_1$. Since $\pi((c_2, Tb_1)) = \pi((Tc_2, b_1))$, there are no inflection points on

$$A := (c_2, Tb_1) \cup (Tc_2, b_1)\,. \tag{5}$$

There are also no positive c-inflection points on $[a_2, b_1]$. Applying Lemma 4.12 to the interval (a_2, b_1), we conclude that there are no inflection points on

$$C := (b_2, Ta_1) \cup (Tb_2, a_1)\,. \tag{6}$$

In particular, there are no positive c-inflection points on

$$(c_2, a_1) = (c_2, Tb_1) \cup (Tb_1, Tb_2) \cup (Tb_2, a_1)\,.$$

Applying Lemma 4.12 to the interval (c_2, a_1), we conclude that there are no inflection points on

$$B := (a_2, Tc_1) \cup (Ta_2, c_1)\,. \tag{7}$$

Now it follows from (5), (6) and (7) that there are no inflection points on

$$S^1 \setminus ([a_1, a_2] \cup [Tc_1, Tc_2] \cup [b_1, b_2] \cup [Ta_1, Ta_2] \cup [c_1, c_2] \cup [Tb_1, Tb_2]) = A \cup B \cup C,$$

and hence that $i(\gamma) = 3$. □

We can now finish the proof of Theorem A. Let $\delta(\gamma)$ denote the number of elements in a maximal set of independent double tangent intervals. The number $\delta(\gamma)$ is finite by Corollary 4.10. It will follow from the proof that $\delta(\gamma)$ does not depend on the maximal set that was used to define it.

We prove Formula $(*)$ by induction on $i(\gamma)$. When $i(\gamma) = 3$, $(*)$ holds, since $\delta(\gamma) = 0$ by Corollary 4.9. So we assume that $(*)$ holds when $i(\gamma) \le n - 1$ and $n \ge 4$ and prove it for $i(\gamma) = n$. Since $i(\gamma) \geq 4$, there exists at least one double tangent interval $I = (a, b)$ by Proposition 4.11. There exist non-negative integers i and j such that

(1) $\{I, I_1, \dots, I_i, J_1, \dots, J_j\}$ is a maximal family of independent double tangent intervals.

(2) $I_1, \dots, I_i$ are subets of I,

(3) $J_1, \dots, J_j$ lie on $\subset P^1 \setminus (a, b)$.

Then we get two anti-convex curves γ_1, γ_2 with respect to $I = [a, b]$. By the induction assumption, $\delta(\gamma_1)$ and $\delta(\gamma_2)$ do not depend on the choice of the set of independent double tangent intervals. Since $\{I_1, \dots, I_i\}$ and $\{J_1, \dots, J_j\}$ are maximal sets of independent double tangent intervals on γ_1 and γ_2, respectively, we have

$$i + j + 1 = \delta(\gamma_2) + \delta(\gamma_2) + 1 .$$

By (2), we have

$$i(\gamma) - 2(i + j + 1) = (i(\gamma_1) - 2\delta(\gamma_1)) + (i(\gamma_2) - 2\delta(\gamma_2)) - 3 .$$

By the induction assumption,

$$i(\gamma_1) - 2\delta(\gamma_1) = i(\gamma_2) - 2\delta(\gamma_2) = 3 .$$

Thus we have

$$i(\gamma) - 2(i + j + 1) = 3 ,$$

which implies that the number $i + j + 1$ of the independent double tangent intervals is independent of the choice of $I, I_1, \dots, I_i, J_1, \dots, J_j$. Thus we have $\delta(\gamma) = i + j + 1$. This finishes the proof.

5. Anti-periodic functions and curves of constant width. Before giving a proof of Theorem C in Introduction, we explain some properties of periodic and anti-periodic functions, which we will need. We denote by $C^r(\boldsymbol{R})$, where $r = 1, 2, \dots, \infty$, the vector space of r times continuously differentiable real valued functions on $\boldsymbol{R}$. We define the following finite dimensional linear subspaces of $C^r(\boldsymbol{R})$:

$$\mathcal{A}_{2n+1} := \left\{ a_0 + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt) \, ; \, a_0, a_1, \dots, a_n, b_1, \dots, b_n \in \boldsymbol{R} \right\} ,$$

$$\mathcal{A}_{2n} := \left\{ \sum_{k=1}^{n} (a_k \cos(2k-1)t + b_k \sin(2k-1)t) \, ; \, a_1, \dots, a_n, b_1, \dots, b_n \in \boldsymbol{R} \right\} ,$$

where n is any natural number. Let f be a C^r-function and $m \leq r$ some natural number. For each point p on $\boldsymbol{R}$, there exists a unique function φ_p in $\mathcal{A}_m$ such that

$$f(p) = \varphi_p(p) , \ f'(p) = \varphi_p'(p) , \ f''(p) = \varphi_p''(p) , \ \dots , \ f^{(m-1)}(p) = \varphi_p^{(m-1)}(p) ,$$

namely, φ_p is the best approximation of f at p in $\mathcal{A}_m$. We call φ_p the *osculating function of order m* or *$\mathcal{A}_m$-osculating function at p*. In general, the m-th derivative $\varphi_p^{(m)}(p)$ at p is not equal to $f^{(m)}(p)$. If, however, $\varphi_p^{(m)}(p) = f^{(m)}(p)$ holds for p, then p is called a *flex of f of order m*.

Consider the following differential operators on $\boldsymbol{R}$:

$$L_{2n+1} := D(D^2+1)(D^2+2^2)\cdots(D^2+n^2)\,,$$
$$L_{2n} := (D^2+1)(D^2+3^2)\cdots(D^2+(2n-1)^2)\,,$$

where $D = d/dt$. Then $\mathcal{A}_m$ is the kernel of the operator L_m. The following proposition is proved in the appendix of [19], p. 135.

PROPOSITION 5.1. *A point p is a flex of f of order m if and only if $(L_m f)(p) = 0$.*

A function $f : \boldsymbol{R} \to \boldsymbol{R}$ is called *π-anti-periodic* if it satisfies $f(t+\pi) = -f(t)$. We now introduce the concept of clean flexes for 2π-periodic and π-anti-periodic functions.

DEFINITION 5.2. Let m be an integer that we first assume to be odd. Let f be a 2π-periodic C^r-function, where $r \geq m-1$. A point p is called a *clean flex of order m* if the set of zeros of the difference function $f - \varphi_p$ is connected in $\boldsymbol{R}/2\pi\boldsymbol{Z}$.

We next assume that m is an even integer and f a π-anti-periodic C^r-function, where $r \geq m-1$. Then a point p is called a *clean flex of order m* if the set of zeros of $f - \varphi_p$ is connected in $\boldsymbol{R}/\pi\boldsymbol{Z}$.

REMARK 5.3. One should notice that f does have only to be C^{m-1} in the definition of a *clean flex* of order m, but we needed C^m-regularity in the definition of a *flex* of order m. If f is C^m, then a clean flex of order m is a flex of order m in the sense of the former definition. It is crucial for many of our arguments to allow low differentiability. Examples for this are constructions like the reductions of curves with respect to an interval in Section 4.

In [19] the authors proved the following: *Let m be a positive odd integer and let f be a 2π-periodic C^{m-1}-function. Then f has at least $m+1$ clean flexes of order m in a period.*

In [19] only the case where m is odd is dealt with. One can expect that *a generic π-anti-periodic C^{m-1}-function has at least $m+1$ clean flexes in a period.* An indication for this is the fact that such a function of class C^m *has at least $m+1$ possibly not clean flexes*, as can be easily proved; see the appendix of [18]. In this section we give an affirmative answer for the problem if $m = 2$, and leave the general case as an open question. Our result is stated in the next theorem.

THEOREM 5.4. *Let $f : \boldsymbol{R} \to \boldsymbol{R}$ be a π-anti-periodic C^1-function not belonging to $\mathcal{A}_2$. Suppose that the zero set of $f - \psi$ is discrete for every ψ in $\mathcal{A}_2$. Then f has at least three clean flexes $t_1 < t_2 < t_3$ of order 2, where $t_3 < t_1 + \pi$, with the property that $f - \varphi_{t_1}$ and $f - \varphi_{t_3}$ change sign from negative to positive in t_1 and t_3, respectively, and $f - \varphi_{t_2}$ changes sign from positive to negative in t_2.*

The theorem is optimal, since $f(t) = \sin 3t$ has exactly three clean flexes at $t = 0, \pi/3, 2\pi/3$ in $[0, \pi)$; see Figure 20. The theorem implies the well-known existence of three (usual)

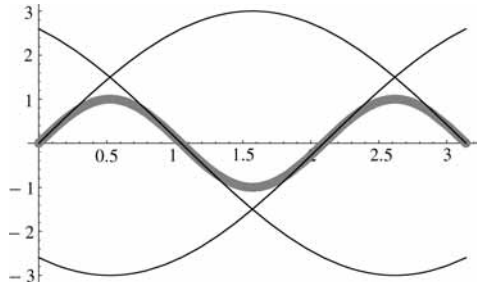

FIGURE 20. Three clean osculating functions for $\sin 3t$.

flexes of order 2, which can be proved by integration by parts; see [6] and the appendix of [19].

We start with some lemmas needed to prove Theorem 5.4. In the following f will be a π-anti-periodic C^1-function as in the theorem. For a point p we define a one-dimensional subspace V_p of $\mathcal{A}_2 = \{a \cos t + b \sin t\,;\ a, b \in \boldsymbol{R}\}$ by setting

$$V_p := \{\psi \in \mathcal{A}_2\,;\ \psi(p) = f(p)\}\,.$$

The osculating function φ_p at p belongs to V_p. For a given $s \in \boldsymbol{R}$, there is a unique $\psi \in V_p$ such that $\psi'(p) = s$, since $\mathcal{A}_2$ is the kernel of the operator L_2. We will denote this function by $\psi_{p,s}$. Thus we may write $V_p = \{\psi_{p,s}\,;\ s \in \boldsymbol{R}\}$. For sufficiently large s, the function $\psi_{p,s}$ has the following properties:

(1) $\psi_{p,s}(t)$ is greater than $f(t)$ on $(p, p+\pi)$ and

(2) $\psi_{p,s}(t)$ is less than $f(t)$ on $(p-\pi, p)$.

Let s_0 be the infimum over the set of real numbers s such that $\psi_{p,s}$ satisfies (1) and (2), and set

$$\psi_p := \psi_{p,s_0}\,.$$

We will call ψ_p the *limiting function of f at p*.

LEMMA 5.5. *The limiting function $\psi_p(t)$ of f at p has the following properties*:

(a) $\psi_p(t) \geq f(t)$ *for* $t \in (p, p+\pi)$.

(b) $\psi_p(t) \leq f(t)$ *for* $t \in (p-\pi, p)$.

(c) *If ψ_p is not the $\mathcal{A}_2$-osculating function of f at p, then there exists a point q on $(p, p+\pi)$ such that $\psi_p(q) = f(q)$.*

Conversely, a function $\psi \in V_p$ satisfying (a), (b) *and* (c) *must coincide with the limitting function at p.*

PROOF. The lemma is an analogue of Proposition 1.2 and Proposition 1.3, and follows directly from the definition of ψ_p. □

Now we identify

$$S^1 = \boldsymbol{R}/2\pi \boldsymbol{Z}\,, \quad P^1 = \boldsymbol{R}/\pi \boldsymbol{Z}\,,$$

and denote by

$$\pi : S^1 \to P^1$$

the canonical projection. We will consider f and the limiting functions ψ_p as functions on S^1. We now set

$$F(p) = \{t \in S^1 \, ; \; f(t) = \psi_p(t)\} .$$

LEMMA 5.6. *A point p is clean flex of order 2 if and only if $F(p)$ consists of exactly two points.*

PROOF. If p is a clean flex, then $\psi_p = \varphi_p$ by Property (c). Since the zero set of $f - \psi$ is discrete for every ψ in $\mathcal{A}_2$, $F(p)$ consists of exactly two points. Conversely, suppose that $F(p)$ consists of exactly two points. Then $\psi_p = \varphi_p$ by Lemma 5.5 and p is a clean flex. □

PROPOSITION 5.7. *Let f be as in Theorem 5.4. Then the associated family of closed subset $\{F(p)\}_{p\in S^1}$ is an intrinsic line system.*

PROOF. We have to show that Properties (L1) through (L7) in Proposition 1.4 are satisfied.

(L1) is obvious. (L2) follows from the fact that f does not belong to $\mathcal{A}_2$. (L3) holds, since f is π-anti-periodic. (L4) follows from the fact that the functions in $\mathcal{A}_2$ have at most one zero on $[0, \pi)$. (L6) is obvious since the zeros of $f - \psi (\psi \in \mathcal{A}_2)$ is descrete. (L7) holds, since the limit of a sequence of limiting functions is a limiting function. □

PROOF OF THEOREM 5.4. We have associated an intrinsic line system $\{F(p)\}_{p\in S^1}$ to f in Proposition 5.7. Now Theorem 5.4 implies that f has at least three clean flexes of order 2 and it is easy to see that they can be chosen as claimed in the theorem. □

DEFINITION 5.8. A nonempty proper open subinterval (a, b) on P^1 is called an $\mathcal{A}_2$-*double tangent interval of f* if there is a function φ in $\mathcal{A}_2$ such that

(1) the values of f and φ coincide in a and b,
(2) the derivatives of f and φ coincide in a and b,
(3) there is a point in $t \in (a, b)$ such that $\varphi(t) \neq f(t)$,
(4) the function $f - \varphi$ has either local maxima at both a and b, or local minima at both a and b.

If (a, b) is a double tangent interval, then the same cannot be true for $(b, a) = P^1 \setminus [a, b]$, since Condition (4) fails. (If we consider (a, b) to be an interval of $\boldsymbol{R}$, then $P^1 \setminus [a, b]$ corresponds to $(b, a + \pi)$.) The function φ in the definition of an $\mathcal{A}_2$-double tangent interval is uniquely determined. We will call it the *double tangent function* with respect to (a, b).

DEFINITION 5.9. Let $f : \boldsymbol{R} \to \boldsymbol{R}$ be an anti-periodic C^1-function. Then two $\mathcal{A}_2$-double tangents (a_1, b_1) and (a_2, b_2) are said to be *independent* if they are disjoint or if the closure of one is contained in the other.

Using the same method as in Section 4, we get the following:

THEOREM 5.10. *Let $f : \boldsymbol{R} \to \boldsymbol{R}$ be a π-anti-periodic C^1-function not belonging to $\mathcal{A}_2$. Suppose that the zero set of $f - \psi$ is discrete for every ψ in $\mathcal{A}_2$. Then the number $i(f)$ of flexes of order 2, and the number $\delta(\gamma)$ of elements in a maximal set of independent $\mathcal{A}_2$-double tangent intervals are both finite and $\delta(\gamma)$ is independent of the choice of the maximal set of double tangent intervals. Moreover,*

$$i(\gamma) - 2\delta(\gamma) = 3$$

holds.

PROOF. Let (a, b) be an $\mathcal{A}_2$-double tangent interval and φ the corresponding double tangent function in $\mathcal{A}_2$. Without loss of generality, we may asuume that $0 \le a < b < \pi$. Then we set

$$f_1(t) := \begin{cases} \varphi(t) & \text{for } t \in [a, b]\,, \\ f(t) & \text{for } t \in [0, \pi) \setminus [a, b]\,, \end{cases}$$

and extend f_1 to $\boldsymbol{R}$ as a π-anti-periodic function. Then f_1 is a C^1-function that we call the *reduction of f with respect to* $[a, b]$. Similarly, we set

$$f_2(t) := \begin{cases} f(t) & \text{for } t \in [a, b]\,, \\ \varphi(t) & \text{for } t \in [0, \pi) \setminus [a, b]\,, \end{cases}$$

and extend f_2 to $\boldsymbol{R}$ as a π-anti-periodic function. Then f_2 is a C^1-function that we call *the reduction of f with respect to* $[b, a]$. We now use the functions f_1 and f_2, as we used the reductions γ_1, γ_2 in Section 4, to prove Theorem A in Introduction by induction. □

Finally, we come to the applications of Theorem 5.4 and Theorem 5.10 to convex curves of constant width in the Euclidean plane $\boldsymbol{R}^2$.

We first describe the connection between strictly convex curves and periodic functions, where we mean by a *strictly convex curve* a convex curve with the property that the tangent lines at different points are different. Let o be a point in the open domain bounded by a strictly convex C^2-regular curve γ in $\boldsymbol{R}^2$. For each $t \in [0, 2\pi)$, there is a unique oriented tangent line $L(t)$ of the curve that makes angle t with the x-axis. Let $h(t)$ be the distance between o and the line $L(t)$. The C^1-function h is called the *supporting function of the curve γ with respect to o*. Set $\mathbf{e}(t) = (\cos t, \sin t)$ and $\mathbf{n}(t) = (-\sin t, \cos t)$. Then

$$\gamma(t) = h'(t)\mathbf{e}(t) - h(t)\mathbf{n}(t)$$

gives a parametrization of the curve γ. The following lemma follows immediately.

LEMMA 5.11. *Let γ_1 and γ_2 be two strictly convex C^2-regular curves having a common point o in their interior, let $h_1(t)$ and $h_2(t)$ be their supporting functions with respect to o, and let $\gamma_1(t)$ and $\gamma_2(t)$ be their parametrizations as above. Then the difference $h_2(t) - h_1(t)$ does not depend on the choice of the origin o. In particular, if γ_2 is a circle, then the point $\gamma_1(t)$ lies in the interior of γ_2 if and only if $h_2(t) - h_1(t) > 0$ holds.*

Note that a convex curve has constant width d if and only if $h(t) + h(t + \pi) = d$ holds.

Let γ be a C^2-regular strictly convex closed curve of constant width $d > 0$ and h its supporting function which is of class C^1. The function f_γ defined by

$$f_\gamma(t) = h(t) - \frac{d}{2}$$

is π-anti-periodic, since γ is of constant width. If γ is a circle of diameter d, the supporting function ψ can be written as

$$h(t) = \frac{d}{2} + b\cos t + c\sin t\,,$$

where $(c, -b)$ is the center of the circle.

For a point p on a curve γ of constant width d, there exists a unique circle Γ_p of width d such that Γ_p is tangent to γ at p, that is, Γ_p and γ meet at p with multiplicity at least two. Since Γ_p is the best approximation of γ at p by a circle of width d, we call Γ_p the *osculating d-circle at p*. When Γ_p meets γ with multiplicity higher than two at p, we call p a *d-inflection point*.

PROPOSITION 5.12. *Let γ be a C^3-regular convex curve of constant width d and h the supporting function of γ. Then the following properties are equivalent*:

(1) *A point $p = \gamma(t_0)$ is a d-inflection point.*

(2) $h''(t_0) + h(t_0) = d/2$.

(3) *The osculating d-circle Γ_p at p is an osculating circle in the usual sense*, *that is*, *the curvature radius of γ at p is $d/2$.*

PROOF. The supporting fuction h is a C^2-function because γ is C^3-regular. Since the radius of the osculating circle of γ at t is given by $r = h''(t) + h(t)$, the last two properties are equivalent. It is therefore sufficient to prove the equivalence of the first two properties.

Let

$$h_0 = \frac{d}{2} + b\cos t + c\sin t$$

be the supporting function of a circle Γ. Then Γ is the d-osculating circle at p if and only if

$$h_0(t_0) = h(t_0)\,, \quad h_0'(t_0) = h'(t_0)\,.$$

Moreover, Γ and γ meet with multiplicity higher than two at p if and only if the curvature radius of them coincide. Since the radius of the osculating circle of γ at t is given by $r = h''(t) + h(t)$, the circle Γ is a d-inflection point if and only if

$$r = h''(t_0) + h(t_0) = h_0''(t_0) + h_0(t_0) = \frac{d}{2}\,.$$

This proves that the first two properties are equivalent. □

If $\Gamma_p \cap \gamma$ consists of exactly two connected components, p is a d-inflection point which we will call a *clean d-inflection point*.

We can now prove Theorem C in Introduction as an application of Theorem 5.4.

PROOF OF THEOREM C. We first consider the special case that γ meets each circle at most finitely many points. As explained above, the supporting function h can be written in the form

$$h(t) = \frac{d}{2} + f_\gamma\,,$$

where f_γ is a π-anti-periodic function. Since we are assuming that γ meets each circle at at most finitely many points, $f_\gamma - \psi$ has a discrete zero set for every ψ in $\mathcal{A}_2$. Hence, by Theroem 5.4, the function f_γ has at least three clean positive flexes t_1, t_2, t_3 of order 2 on the interval $[0, \pi]$. By Proposition 5.1, $f_\gamma'' + f_\gamma$ vanishes at t_1, t_2 and t_3, that is,

$$h''(t) + h(t) = \frac{d}{2}$$

holds for $t = t_1, t_2, t_3, t_1 + \pi, t_2 + \pi$ and $t_3 + \pi$. By Proposition 5.12, these six points are d-inflection points. Moreover, since these six points are clean flexes, Lemma 5.11 implies that they clealy turn out to be six clean d-inflection points. Notice that the corresponding osculating d-circles meet γ exactly twice in t_i and $t_i + \pi$. This finishes the proof of the special case.

Next we consider the general case in which γ can meet circles infinitely many times. We consider the Fourier series expansion

$$h(t) = \frac{d}{2} + \sum_{n=1}^{\infty}(a_n \cos(2n+1)t + b_n \sin(2n+1)t)$$

of h and set

$$h_N(t) = \frac{d}{2} + \sum_{n=1}^{N}(a_n \cos(2n+1)t + b_n \sin(2n+1)t)\,.$$

Then the convex curve $\gamma_N(t)$ with the supporting function $h_N(t)$ is a regular curve of constant width for N sufficiently large. Moreover, γ_N meets each circle at at most finitely many points, since it is real analytic. Now we apply the same argumants as in the proof of Theroem B in Section 2 to find three distinct osculating d-circles as a limit of those of $\gamma_N(t)$. $\square$

One can of course define the double d-tangent intervals of a curve γ of constant width as the double tangent intervals of the corresponding function f_γ. We translate this into geometric properties of γ as follows.

A nonempty proper open interval (a, b) of $S^1 = \boldsymbol{R}/2\pi\boldsymbol{Z}$ is a *double tangent interval of* γ if there is a circle Γ which coincides with the osculating d-circles at $\gamma(a)$ and $\gamma(b)$, and has the property that there is a t in (a, b) such that $\gamma(t) \notin \Gamma$. We assume furthermore that Γ is locally, around $\gamma(a)$ and $\gamma(b)$, on the same side of γ. Notice that $(a+\pi, b+\pi)$ is a double tangent interval if (a, b) is such an interval.

Two double tangent intervals (a_1, b_1) and (a_2, b_2) are *independent* if they are not antipodal on S^1 and if they are disjoint or the closure of one is contained in the other.

Since $\mathcal{A}_2$-double tangent intervals correspond to the d-double tangent intervals, Theorem 5.10 implies the following theorem.

THEOREM 5.13. *Let γ be a convex C^3-regular curve of constant width d. Suppose that the curve meets each circle at at most finitely many points. Then the number $i(\gamma)$ of* *independent d-inflection points and the number $\delta(\gamma)$ of elements in a maximal set of independent d-double tangent intervals are both finite, and $\delta(\gamma)$ is independent of the choice of the maximal set used to define it. Moreover, we have the identity*

$$i(\gamma) - 2\delta(\gamma) = 3\,.$$

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