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Inflections of Toric Varieties

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Let $V = \{m_0, \dots, m_t\}$ be a set of distinct lattice points in $\mathbb{Z}^n_{\geq 0}$ with $m_0 = \vec{0}$. Associated with V is an affine monomial map

$$\begin{aligned} v\colon \mathbb{C}^n &\to \mathbb{C}^{t+1}, \\ x &\mapsto (1, x^{m_1}, \dots, x^{m_t}), \end{aligned}$$

where x^{m_i} stands for the monomial $x_1^{m_{i1}} x_2^{m_{i2}} \cdots x_n^{m_{in}}$. (The ordering of the lattice points will not be important. The lattice point $m_0 = \vec{0}$ is included, anticipating the move to projective space.) As will be described carefully in Section 1, the span of the derivatives of v up to order k at a point p determines the osculating space of order k at p. If the dimension of this osculating space is smaller than expected then we say that v is inflected at p. In this paper, we show how inflection points are related to the lattice points V and use this information to characterize toric varieties with certain extreme inflectional behavior.

The following two theorems are examples of previous work in which varieties are characterized by their inflectional behavior.

Theorem 0.1 [FKPT]. *Let $t = \binom{n+k}{k} - 1$, and let $X \subset \mathbb{P}^t$ be a smooth projective n-fold whose k*th *osculating space is all of $\mathbb{P}^t$ at all points of X; then X is isomorphic to $\mathbb{P}^n$ embedded via the k-fold Veronese mapping.*

Theorem 0.2 [BPT]. *Let $t \geq 2$, and let $X \subset \mathbb{P}^{2t+1}$ be a smooth projective surface not contained in a hyperplane such that the dimension of its k*th *osculating space is $2k$ at all points of X and for all $k \leq t$. Then X is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ embedded via all global sections of $\mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(t)$, so X is a rational normal scroll of degree $2t$.*

These two theorems are proved using sophisticated machinery (in the former case, a result of Mori characterizing projective space as the only variety with ample tangent bundle; in the latter, adjunction theory). However, in all cases, the varieties and embeddings turn out to be toric. As might be expected, if we are willing to restrict our attention to toric mappings then we can establish these theorems by fairly easy combinatorics characterizing polytopes with certain properties.

In Section 1, we show that the dimensions of the osculating spaces of v are given by the Hilbert function of the set of lattice points, V. In Section 2 we discuss

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extending the monomial mapping to a mapping of a toric variety into projective space. In Section 3, our main result is to describe toric varieties of dimensions 2 and 3 embedded in projective space so that the osculating spaces up to a certain order are as large as possible at all points of the variety and strictly smaller than possible for higher orders (cf. Theorem 3.2 and Theorem 3.5). In the case of dimension 2, we show that the variety must be the projective plane (embedded via a Veronese), a Hirzebruch surface, or one of three exceptions. The exceptional cases are nonruled varieties whose second-order osculating spaces have dimension strictly less than 5 at all points. One of these cases was first noticed in [T], and the other two are new.

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1. Inflections of Affine Monomial Maps

For $a \in \mathbb{Z}^n_{\geq 0}$, we denote the ath partial derivative of v in the following way:

$$v_a := \frac{1}{a!}\frac{\partial^{|a|}v}{\partial x_1^{a_1}\cdots\partial x_n^{a_n}},$$

where $|a| = \sum_{i=1}^n a_i$ and $a! = a_1!\cdots a_n!$. To study the inflections of v, define for each integer $k \geq 0$ the matrix of *k-jets of v*

$$J_k v := (v_a)_{0\leq |a| \leq k}$$

whose rows are the partial derivatives of v up to order k, written in any order. The kth *osculating space of v* at the point $p \in \mathbb{C}^n$, $\operatorname{Osc}_k v(p)$, is the span of the vectors $v_a(p)$ for $1 \leq |a| \leq k$, translated out to $v(p)$. Hence, $\operatorname{Osc}_1 v(p)$ is the tangent space for v at p, and $\operatorname{Osc}_{k+1} v(p)$ is determined by the first-order infinitesimal motions of $\operatorname{Osc}_k v(p)$. Since $\vec{0}$ is included in V, it follows that v is linearly independent from the v_a with $|a| > 0$. Hence, $\operatorname{rk} J_k v(p) = 1 + \dim(\operatorname{Osc}_k v(p))$. If this rank is not as large as possible—that is, if it is less than $\binom{n+k}{k}$—then we say that p is an *inflection point*. In general, the rank of $J_k v$ will have a generic value—which might be less than maximal, and so v will be inflected everywhere—but might drop below this generic value at special points. We call these special points *proper inflections* or just *inflections*, again. (For an arbitrary mapping $v\colon X \to \mathbb{P}^t$ of a smooth variety into projective space, one may define osculating spaces similarly after taking local coordinates on X and lifting to $\mathbb{C}^{t+1}$. It is a standard result that the osculating spaces are independent of the choice of local coordinates, and it is clear that dimensions of the osculating spaces do not change after an affine change of coordinates in the target space $\mathbb{C}^t$. We are choosing to avoid the machinery of principal parts or jet bundles as an unnecessary complication for the purposes of this paper.)

As a further measure of inflection, let $\mathcal{F}_k^i v$ denote the ith Fitting ideal of $J_k v$, the ideal generated by the determinants of $i \times i$ minors of $J_k v$. Then there exist inclusions

$$\begin{array}{ccc} \mathcal{F}_k^{i+1} v & \subset & \mathcal{F}_k^i v \\ \cup & & \cup \\ \mathcal{F}_{k-1}^{i+1} v & \subset & \mathcal{F}_{k-1}^i v. \end{array}$$

The rank of $J_k v(p)$ is the largest r such that $\mathcal{F}_k^r v(p) = (1)$. Note that the Fitting ideals are monomial ideals (this can be easily seen by direct computation, or by appealing to the natural torus action on $\mathbb{C}^n$). This means that we can realize all possible ranks for $J_k v(p)$ by looking at only those p whose coordinates are zeros and ones. In other words, the rank of $J_k v(p)$ only depends upon the smallest coordinate flat, $\{x_{i_1} = \cdots = x_{i_s} = 0\}$, to which p belongs.

Main Question. What is the relation between V (i.e., the set of lattice points serving as exponents for v) and the inflections of v?

To start, we can evaluate a partial derivative of a monomial x^ℓ:

$$\begin{aligned} x_a^\ell &= \frac{1}{a!} \frac{\partial^{|a|} x^\ell}{\partial x^{a_1} \cdots \partial x^{a_n}} \\ &= \binom{\ell_1}{a_1} \cdots \binom{\ell_n}{a_n} x^{\ell - a} \\ &= \binom{\ell}{a} x^{\ell - a}, \end{aligned} \tag{1}$$

where the multinomial coefficient is defined to be zero if $a_i > \ell_i$ for some i. An easy consequence is that

$$\operatorname{rk} J_k v(0) = |\{m_i \in V \mid |m_i| \leq k\}|. \tag{2}$$

Remark. The affine version of Theorem 0.1 follows: If the kth osculating spaces of $v\colon \mathbb{C}^n \to \mathbb{C}^{t+1}$ all have dimension t, then (2) says that $t + 1 = |\{m_i \in V \mid |m_i| \leq k\}|$. If $t = \binom{n+k}{k} - 1$, we are forced to take $V = \{m \in \mathbb{Z}^n_{\leq 0} \mid |m| \leq k\}$.

Having determined the smallest possible rank, we now determine the largest.

Proposition 1.1. *The generic rank of $J_k v$ is*

$$\operatorname{rk} J_k v(1, \ldots, 1) = H_V(k),$$

where H_V is the affine Hilbert function of V. That is, $H_V(k)$ is the codimension in the linear space of polynomials in n variables and of degree $\leq k$ of those polynomials that are satisfied by the lattice points $V \subset \mathbb{Z}^n \subset \mathbb{C}^n$.

Proof. From (1), the column of $J_k v(1,\dots,1)$ corresponding to the monomial x^{m_i} has the form

$$\binom{m_i}{a}_{0\le|a|\le k} = \left[\binom{m_{i,1}}{a_1}\cdots\binom{m_{i,n}}{a_n}\right]_{0\le|a|\le k}.$$

But $\left\{\binom{x_1}{a_1}\cdots\binom{x_n}{a_n}\right\}_{0\le|a|\le k}$ forms a basis for the space of polynomials of degree $\le k$ in $x_1,\dots,x_n$. Therefore, the linear relations among the rows of $J_k v(1,\dots,1)$ correspond to polynomials of degree $\le k$ passing through V. $\square$

Hence, finding a monomial map whose osculating spaces have fixed generic dimensions is the same as finding a set of lattice points with a certain Hilbert function. (For the extension of this result to toric mappings, see Remark 2.3.)

In what follows, we will need to know only the generic rank of $J_k v$ and its rank at the origin; but for completeness we will extend Proposition 1.1 to determine the rank at all points. As noted earlier, is suffices to consider only points whose coordinates consist of zeros and ones. We will use the following notation. Given $\alpha \subset \{1,\dots,n\}$, let $p_\alpha \in \mathbb{C}^n$ be the point whose ith coordinate is 0 if $i \in \alpha$ and 1 otherwise. Thus, the generic value of $\operatorname{rk} J_k v$ along the flat $\{x_i = 0 \mid i \in \alpha\}$ is determined by the rank at the point p_α. Let Π denote the lattice of integer points in this flat. To calculate the rank at p_α, we will take slices of the lattice that are parallel to Π. For each positive lattice point in the space normal to Π (i.e., for each $a \in \Pi^\perp := \{b \in \mathbb{Z}^n_{\ge 0} \mid b_i = 0,\ i \notin \alpha\}$), we take the slice through a, $\Pi_a := \Pi + a := \{b \in \mathbb{Z}^n_{\ge 0} \mid b_i = a_i,\ i \in \alpha\}$. By forgetting the components they have in common, we can think of the exponents of v that lie in a particular slice Π_a as defining a monomial map from the smaller space $\Pi_a \otimes \mathbb{C} \cong \mathbb{C}^{n-|\alpha|}$. Applying Proposition 1.1 to determine the rank of the $(k-|a|)$-jets of this new map and then summing over the first $k+1$ slices gives the result. To formalize this, let π be the orthogonal projection of $\mathbb{C}^n$ onto the slice $\Pi_a \otimes \mathbb{C} \cong \mathbb{C}^{n-|\alpha|}$ and consider the exponents for the new monomial map just described:

$$V_a := \pi((V \cap \Pi_a) \cup \{a\}).$$

We have forced V_a to include the origin, by adding $\{a\}$, in order to apply Proposition 1.1. To compensate, define

$$\delta_V(a) := \begin{cases} 0 & \text{if } a \in V, \\ 1 & \text{otherwise.} \end{cases}$$

PROPOSITION 1.2. *With notation as before,*

$$\operatorname{rk} J_k v(p_\alpha) = \sum_{a\in\Pi^\perp,\,|a|\le k} (H_{V_a}(k-|a|) - \delta_V(a)).$$

Proof. For each $a \in \Pi^\perp$ with $|a| \le k$, let B_a be the matrix whose rows are the partial derivatives v_b for $b \in \Pi_a$ such that $|b| \le k$. Hence, up to a re-arrangement of rows, $J_k v$ consists of the blocks of rows, B_a. Using (1) to take the partial derivative of a monomial gives

$$\begin{aligned} x_a^m(p_\alpha) \neq 0 &\iff m_i = a_i \text{ for } i \in \alpha \\ &\iff m \in \Pi_a. \end{aligned} \tag{$*$}$$

Hence, each column of $J_k v(p_\alpha)$ will be nonzero along at most one of the blocks, B_a; therefore,

$$\operatorname{rk} J_k v(p_\alpha) = \sum_{a \in \Pi^\perp, |a| \leq k} \operatorname{rk} B_a(p_\alpha).$$

Consider the map $\tilde{v}$ associated with the exponents V_a. It follows from $(*)$ that the nonzero columns of $B_a(p_\alpha)$ come from those components of v of the form x^m with $m \in \Pi_a$. Hence, disgarding the columns of zeros, $B_a(p_\alpha)$ is $J_{k-|a|}\tilde{v}(1, \dots, 1)$ provided $a \in V$. The rank of $B_a(p_\alpha)$ is then given by Proposition 1.1. If $a \notin V$, then it has been added to from V_a. So $\tilde{v}$ has an "extra" component, and $J_{k-|a|}\tilde{v}(1, \dots, 1)$ has an "extra" row, linearly independent from the others. Thus, in this case, $\operatorname{rk} B_a(p_\alpha) = \operatorname{rk} J_{k-|a|}\tilde{v}(1, \dots, 1) - 1$. This accounts for the presence of δ_V. $\square$

REMARK 1.3. In the case $p_\alpha = \vec{0}$, we have $\Pi = \{\vec{0}\}$, $\Pi^\perp = \mathbb{Z}^n_{\geq 0}$, $\Pi_a = \{a\}$, and

$$H_{V_a}(k) - \delta_V(a) := \begin{cases} 1 & \text{if } a \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, Proposition 1.2 reduces to (2).

2. Inflections of Toric Varieties

This section introduces the notation we use to describe toric varieties and defines inflections for toric mappings. As general references, we use [F; O]. Let X be an n-dimensional toric variety associated with a fan Δ in an n-dimensional lattice $N \cong \mathbb{Z}^n$. We will use the notation in [C]: $M = \operatorname{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ is the dual lattice; $\Delta(k)$ is the set of k-dimensional cones of Δ; if $\sigma \in \Delta(k)$, then $\sigma(\ell)$ denotes the ℓ-dimensional cones contained in σ; for each $\rho \in \Delta(1)$, let n_ρ be the generator of $\rho \cap N$ and D_ρ the associated T-invariant Weil divisor; and the set of such D_ρ is a basis for the free abelian group of T-Weil divisors, $\mathbb{Z}^{\Delta(1)}$. To describe the homogeneous coordinate ring of X introduced in [C], recall the exact sequence

$$\begin{gathered} 0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Delta(1)} \longrightarrow A_{n-1}(X) \longrightarrow 0, \\ m \longmapsto D_m = \sum_\rho \langle m, n_\rho \rangle D_\rho, \end{gathered} \tag{3}$$

where $A_{n-1}(X)$ is the group of Weil divisors modulo rational equivalence and the map $\mathbb{Z}^{\Delta(1)} \to A_{n-1}(X)$ sends a divisor to its class. For each $\rho \in \Delta(1)$, let x_ρ be a variable. There is a 1–1 correspondence between T-Weil divisors and monomials in the x_ρ, namely, $D = \sum_\rho a_\rho D_\rho \in \mathbb{Z}^{\Delta(1)}$ corresponds with $x^D = \prod_\rho x_\rho^{a_\rho}$. The homogeneous coordinate ring of X is $S = \mathbb{C}[x_\rho \mid \rho \in \Delta(1)]$ with grading given by the class group $A_{n-1}(X)$. This means that two monomials x^D and x^E have the same degree if $[D] = [E]$ in $A_{n-1}(X)$. For each T-Weil divisor D, there exists a coherent sheaf $\mathcal{O}_X(D)$ as well as the polyhedron

$$P(D) = \{m \in M \otimes \mathbb{R} \mid \langle m, n_\rho \rangle \geq -a_\rho \ \forall \rho \in \Delta(1)\},$$

whose elements may be thought of as global sections of $\mathcal{O}_X(D)$.

For the rest of this section, assume that X is smooth and projective; hence, each $\mathcal{O}_X(D)$ is a line bundle. Let $v\colon X \to \mathbb{P}^t$ be a toric mapping—that is, determined by globally generating sections $m_0, \dots, m_t$ of a T-line bundle $\mathcal{O}_X(D)$. We identify each m_i with a point in the polytope, $P(D)$. The fact that these m_i globally generate $\mathcal{O}_X(D)$ means that they include the vertices of $P(D)$. In homogeneous coordinates we have

$$\begin{aligned} v\colon X &\to \mathbb{P}^t, \\ x &\to (x^{D_{m_0}+D}, \dots, x^{D_{m_t}+D}), \end{aligned} \tag{4}$$

where $D_m = \sum_\rho \langle m, n_\rho \rangle D_\rho$ as before. Each $[D_{m_i}] = 0$ in $A_{n-1}(X)$, so v is homogeneous of degree D.

We define inflections for v by restricting it to the natural affine subsets of X. The variety X comes from gluing the $X_\sigma \cong \mathbb{C}^n$ for each maximal cone $\sigma \in \Delta(n)$. For each such σ, the 1-dimensional cones $\rho \in \sigma(1)$ correspond to variables x_ρ in the homogeneous coordinate ring. Setting the remaining $x_\rho = 1$ in (4) determines an affine map, $v^\sigma : X_\sigma \to \mathbb{C}^{t+1}$, of the sort considered in Section 1. This map is the restriction of v to X_σ, lifted to $\mathbb{C}^{t+1}$. The lattice points determining this affine map can be determined from the $m_i \in P(D)$ by translating the vertex of $P(D)$ corresponding to σ to the origin and writing the m_i with respect to the basis for the lattice, $\{n_\rho \mid \rho \in \sigma(1)\}$. Restricting v to another maximal affine subset amounts to choosing another vertex of $P(D)$ and writing the m_i with respect to the corresponding basis for the lattice (cf. Example 2.4).

A point $p \in X_\sigma$ is an *inflection point* for v if it is an inflection point for v^σ. To show that this definition is independent of the choice of σ, we define the matrix of *homogeneous k-jets,* taking derivatives up to order k with respect to the homogeneous coordinates:

$$J_k^h v := (v_a)_{a \in \mathbb{Z}_{\geq 0}^{|\Delta(1)|}, |a| \leq k}.$$

PROPOSITION 2.1. *For each $\sigma \in \Delta(n)$, the $\mathbb{C}[x_\rho \mid \rho \in \sigma(1)]$-span of the rows of the matrix of homogeneous jets restricted to X_σ,*

$$J_k^h v\big|_\sigma := J_k^h v\big|_{x_\rho = 1,\, \rho \notin \sigma(1)},$$

is the same as that of the affine jets on X_σ,

$$J_k v^\sigma := (v_a^\sigma)_{a \in \mathbb{Z}_{\geq 0}^n, |a| \leq k}.$$

Proof. One direction is obvious, since the rows of the latter matrix are a subset of the rows of the former. To show the opposite inclusion, we use "Euler formulas" for X to rewrite the derivatives of homogeneous coordinates in terms of derivatives of affine coordinates. As shown in [BC], there is one Euler formula for each element $\phi \in \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{Z})$. If $f \in S$ is a homogeneous polynomial of degree $[E]$, then the Euler formula corresponding to ϕ is

$$\sum_{\rho \in \Delta(1)} \phi([D_\rho]) x_\rho f_{x_\rho} = \phi([E]) f.$$

Since X is smooth, $\{n_\rho\}_{\rho\in\sigma(1)}$ is a basis for the lattice N. It follows from (3) that $[D_\rho]_{\rho\notin\sigma(1)}$ forms a basis for $A_{n-1}(X)$. Hence, for each $\rho \notin \sigma(1)$, there is an element $\phi \in \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{Z})$ such that $\phi([D_\rho]) = 1$ and $\phi([D_\mu]) = 0$ for all other $\mu \notin \sigma(1)$. Thus, for any polynomial f of degree $[E]$, after setting $x_\rho = 1$ we can solve for f_{x_ρ} using the Euler formula corresponding to ϕ to obtain

$$f_{x_\rho} = \phi([E])f - \sum_{\mu\in\sigma(1)} \phi([D_\mu])x_\mu f_{x_\mu}.$$

In this way, a derivative of any order with respect to the homogeneous variables can be reduced to an expression involving only the affine coordinates. □

As an immediate consequence, we see that our definition of inflection does not depend on a choice of X_σ.

COROLLARY 2.2. *The Fitting ideals of $J_k^h v\big|_\sigma$ and $J_k v^\sigma$ are the same. In particular,* $\mathrm{rk}\, J_k^h v(p) = \mathrm{rk}\, J_k v^\sigma(p)$ *for all $p \in X_\sigma$.*

REMARK 2.3. Note that if $V = \{m_0, \dots, m_t\} \subset P(D)$ is the set of monomials defining v, then the dimension of the kth osculating space at a generic point of X is again given by the Hilbert function, $H_V(k)$. This follows from Proposition 1.1 because the set of monomials defining any restriction of v to a maximal affine open set, X_σ, differs from V by an affine change of coordinates, which does not affect H_V.

EXAMPLE 2.4 (Togliatti's Del Pezzo). Consider the affine map

$$\mathbb{C}^2 \to \mathbb{C}^6,$$
$$(x, y) \mapsto (1, x, y, x^2y, xy^2, x^2y^2),$$

defined by monomials whose exponents form the vertices of the hexagon pictured in Figure 1.

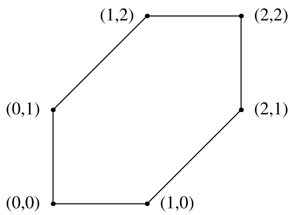

Figure 1

The mapping naturally extends to a mapping of the toric surface X (with fan Δ) determined by the inward normals of the hexagon (see Figure 2).

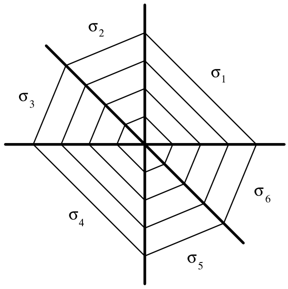

Figure 2

The maximal cones are labeled $\sigma_1, \ldots, \sigma_6$. The 1-dimensional cones have generators $(1,0)$, $(0,1)$, $(-1,1)$, $(-1,0)$, $(0,-1)$, and $(1,-1)$ to which correspond the homogeneous coordinates $x_1, \ldots, x_6$, respectively. Choosing the dual basis for M and $D_3, \ldots, D_6$ as a basis for $A_1(X)$, the exact sequence (3) becomes

$$0 \to \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ -1 & 0 \\ 0 & -1 \\ 1 & -1 \end{pmatrix}} \mathbb{Z}^6 \xrightarrow{\begin{pmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}} \mathbb{Z}^4 \to 0.$$

Choosing $D = D_3 + 2D_4 + 2D_5 + D_6$, the polytope $P(D)$ is the convex hull of the exponents with which we started. The extension of the mapping to X is given in homogeneous coordinates (cf. (4)) by

$$\begin{aligned} v\colon X &\to \mathbb{P}^5, \\ (x_1, \ldots, x_6) &\mapsto (x_3x_4^2x_5^2x_6,\ x_1x_4x_5^2x_6^2,\ x_2x_3^2x_4^2x_5, \\ &\qquad\qquad x_1^2x_2x_5x_6^2,\ x_1x_2^2x_3^2x_4,\ x_1^2x_2^2x_3x_6). \end{aligned}$$

Our original map is the restriction of v to X_{σ_1} (lifted to $\mathbb{C}^6$) obtained by setting $x_3 = \cdots = x_6 = 1$. To restrict v to X_{σ_2}, we set $x_1 = x_4 = x_5 = x_6 = 1$:

$$\begin{aligned} v^{\sigma_2}\colon \mathbb{C}^2 &\to \mathbb{C}^6, \\ (x_2, x_3) &\mapsto (x_3, 1, x_2x_3^2,\ x_2,\ x_2^2x_3^2,\ x_2^2x_3). \end{aligned}$$

The exponents for this affine restriction come from the original exponents by the following affine change of coordinates: translate the vertex $(1,0)$ to the origin and use the first lattice points lying along the two edges of $P(D)$ emanating from this vertex as a basis for the lattice. Using Proposition 1.1, Proposition 1.2, and (2) (or by direct calculation), we find the dimensions of the osculating spaces. By symmetry, we need only consider v^{σ_1}—the affine map with which we started—and by the discussion in Section 1 we need only consider the points $(0,0)$, $(1,0)$, and $(1,1)$ (see Table 1).

The first osculating spaces are all 2-dimensional since v is an embedding. We would expect the generic second osculating space to have dimension 5. However,

Table 1 Dimensions of the osculating spaces of Togliatti's Del Pezzo

k	$\mathrm{Osc}_k\, v^{\sigma_1}(0,0)$	$\mathrm{Osc}_k\, v^{\sigma_1}(1,0)$	$\mathrm{Osc}_k\, v^{\sigma_1}(1,1)$
1	2	2	2
2	2	4	4
3	4	5	5
≥ 4	5	5	5

the six exponents happen to lie on a conic, so the dimension is 4. The exponents place independent conditions on higher-degree curves. By (2), the dimension of the kth osculating space at the origin is found by counting the number m of exponents, with $|m| \leq k$.

The unusual inflectionary behavior in this example was first noticed in [T]. It is a special projection—the exponent $(1,1) \in P(D)$ is not included—of the Del Pezzo surface of degree 6 in $\mathbb{P}^6$.

3. Characterization of Toric Varieties with Special Inflectionary Behavior

Using the notation of the previous section, let X be a smooth, projective, toric n-fold, and let $v\colon X \to \mathbb{P}^t$ be a toric embedding determined by sections of a line bundle $\mathcal{O}_X(D)$. We identify the sections with a set of lattice points $V \subset P(D) \subset M \cong \mathbb{Z}^n$. Assume that v spans $\mathbb{P}^t$. In this section, we will characterize v with osculating spaces of certain dimensions. The idea is to apply the results in Section 1 relating the dimensions with Hilbert functions of lattice points to determine $P(D)$ and V, from which X and v can be reconstructed (cf. [F, Sec. 3.4]).

Veronese

First, we have the toric version of Theorem 0.1.

THEOREM 3.1. *Let* $t = \binom{n+k}{k} - 1$ *and suppose that* $\mathrm{Osc}_k\, v(p) = \mathbb{P}^t$ *for all* $p \in X$. *Then* $X = \mathbb{P}^n$, *and* v *is the* k*-fold Veronese embedding.*

Proof. Translate any vertex of $P(D)$ to the origin and take the first lattice points along the 1-dimensional faces emanating from the vertex as a basis for the lattice (possible since X is smooth). As discussed previously, we can think of the points of V (after the affine change of coordinates) as defining an affine map as in Section 1, the restriction of v to a maximal affine subset of X. According to (2), we must include the lattice points $m \in \mathbb{Z}^n_{\geq 0}$ with $|m| \leq k$. However, since there are $t+1 = \binom{n+k}{k}$ of these, there can be no other lattice points in V. So, up to an affine change of coordinates, $P(D)$ and V (and hence X and v) are determined. $\square$

Surfaces

Consider now the case where X is a surface and, at all points of X, the osculating spaces for v are as large as possible up through order $s-1$ and strictly smaller than possible for order s.

THEOREM 3.2. *Suppose that* $\dim(\mathrm{Osc}_k\, v(p)) = \binom{2+k}{k} - 1$ *for* $k = 1, \ldots, s-1$ *and that* $\dim(\mathrm{Osc}_s\, v(p)) < \binom{2+s}{s} - 1$ *for all* $p \in X$. *Up to an isomorphism in the category of toric mappings to projective space, the following are the only possibilities.*

(1) X *is* $\mathbb{P}^2$, v *is the* $(s-1)$th *Veronese embedding, and* $P(D)$ *is as shown in Figure 3*; V *consists of all lattice points in* $P(D)$.

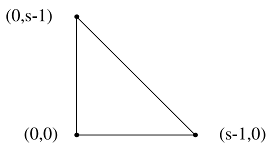

Figure 3

(2) X *is a Hirzebruch surface* $\mathbb{F}_b$ *(including the case* $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$), *and* $P(D)$ *is as shown in Figure 4 for integers* $a \geq s-1$ *and* $b \geq 0$; V *can be any subset of the lattice points of* $P(D)$ *containing all points "out to level* $s-1$," *as described in the beginning of the proof.*

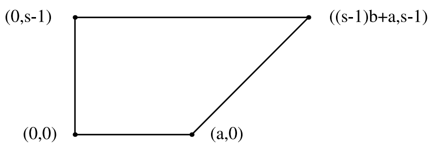

Figure 4

(3) $s = 2$, *the surface* X *is* $\mathbb{P}^1 \times \mathbb{P}^1$ *blown up at two T-fixed points,* v *is a special projection to* $\mathbb{P}^5$ *of the Del Pezzo surface of degree* 6 *in* $\mathbb{P}^6$ (*cf. Example 2.4*), *and* $P(D)$ *is as shown in Figure* 5; V *consists of the vertices of* $P(D)$.

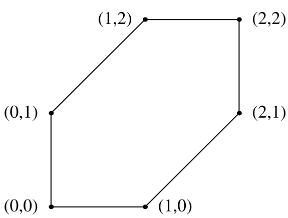

Figure 5

(4) $s = 2$, *the surface X is the blow-up of the preceding example in two T-fixed points, υ embeds X as a surface of degree* 14 *in* $\mathbb{P}^7$, *and $P(D)$ is as shown in Figure 6; V consists of the vertices of $P(D)$.*

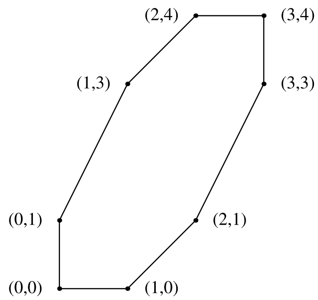

Figure 6

(5) $s = 2$, *the surface X is the blow-up of the preceding example in four T-fixed points, υ embeds X as a surface of degree* 48 *in* $\mathbb{P}^{11}$, *and $P(D)$ is as shown in Figure 7*; *V consists of the vertices of $P(D)$.*

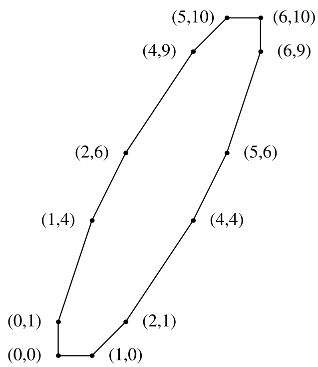

Figure 7

Proof. Translate a vertex of $P(D)$ to the origin. Since X is smooth, the generators of the two edges emanating from the vertex must form a $\mathbb{Z}$-basis for the lattice M. With respect to this basis, the condition that $\operatorname{rk} J_{s-1}\upsilon(p) = \binom{1+s}{s-1}$ for all p implies, by (2) from Section 1, that the sections determining υ must include all those corresponding to lattice points (a, b) with $a + b \leq s - 1$. We say that V includes all points *out to level* $s - 1$ with respect to the chosen vertex. The fact that $\operatorname{rk} J_s\upsilon(p) < \binom{2+s}{s}$ for all p means that the lattice points determining υ must satisfy a polynomial of degree s in two variables, say F. This reasoning allows us to construct the possibilities for $P(D)$ and the lattice points, V.

So far, we know that V (after the affine change of coordinates just described) must include (a, b) such that $a + b \leq s - 1$. To construct $P(D)$, go out along one of the edges emanating from the vertex $(0, 0)$ to the next vertex, say $(a, 0)$, where $a \geq s - 1$. Since X is smooth, the next edge emanating from this vertex passes through a lattice point of the form $(b, 1)$ (see Figure 8). We divide the problem into four cases as follows.

Figure 8

Case (i): $b > a$, $s > 2$. In this case, including all points out to level $s - 1$ with respect to the vertices $(0, 0)$ and $(a, 0)$ implies that V must include $(b - 1, 1)$ as well as $(b, 1)$. This means that there are at least $s + 1$ elements of V along the line $y = 1$. Hence, the curve of degree s containing V must have a linear component: $F = (y - 1)G$ for some polynomial G of degree $s - 1$ passing through the points V not on the line $y = 1$. However, there are at least s points of V lying along the x-axis. Thus, $G = yH$, where H has degree $s - 2$ and passes through the remaining lattice points. Continuing this reasoning now for the lines $y = 2$, $y = 3, \ldots$ shows that v is a mapping of a Hirzebruch surface as described in the statement of the theorem.

Case (ii): $a > s - 1$, $s \geq 2$. The analysis here is similar to that for case (i). Since we must have points out to level $s - 1$ with respect to the vertex $(0, 0)$, there are more than s elements of V along the x-axis and so y must be a factor of F. Continue, showing that $y - 1, \ldots, y - (s - 1)$ are factors. The only possibility (again) is the Hirzebruch surface.

Case (iii): $a = s - 1$, $s > 2$. Given cases (i) and (ii), we may assume that each edge of $P(D)$ contains exactly s lattice points and that $b \leq a$. The edge emanating from $(a, 0)$ is forced to connect with the vertex lying on the y-axis, $(0, s - 1)$. The only possibilities are $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ (if $b = a$) and the Veronese (if $b < a$).

Case (iv): $s = 2$. This last case is more difficult. First, given the foregoing cases and that the toric variety is smooth, we can assume that—besides $(0, 0)$, $(1, 0)$, and $(0, 1)$—the set V contains lattice points of the form $(b, 1)$ and $(1, c)$ with $b, c > 1$. Fixing b and c, there is a unique conic passing through these points:

$$Q = (c - 1)x(x - 1) + (b - 1)y(y - 1) - (b - 1)(c - 1)xy.$$

The lattice points on Q are easy to describe. It happens that if p is a lattice point on Q then the horizontal line through p (if not tangent) meets in another lattice point, and similarly for vertical lines. Starting with $(0, 0)$, the horizontal and vertical lines meet Q in $(1, 0)$, $(0, 1)$, respectively. Repeating for the points $(1, 0)$ and $(0, 1)$ gives the points $(1, c)$ and $(b, 1)$, respectively, and so on. In this way, we get all of the lattice points on Q (see Figure 9). We want to show that to build $P(D)$

starting with the initial five points requires that we take all the lattice points on Q; hence, b and c must be such that Q is an ellipse (since X is projective, $P(D)$ must have a finite number of vertices, for instance).

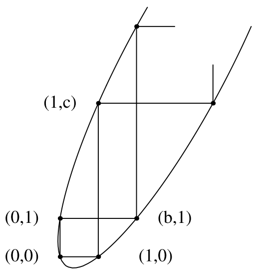

Figure 9

Translating any lattice point of Q to the origin, we now show that the lattice points on Q that are adjacent to that point will form a basis for the lattice. Hence, thinking of the lattice point as vertex of a potential $P(D)$, the smoothness condition is satisfied. Consider consecutive lattice points, (x_1, y_1), (x_2, y_2), and (x_3, y_3) along Q (Figure 10) and assume the smoothness condition is satisfied at (x_2, y_2):

$$\det\begin{pmatrix} x_3 - x_2 & y_3 - y_2 \\ x_1 - x_2 & y_1 - y_2 \end{pmatrix} = 1.$$

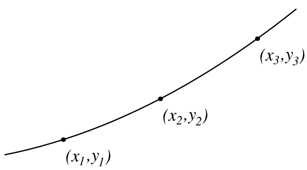

Figure 10

It is easy to check that the intersection of a vertical line through any point (x, y) on Q meets Q again at $(x, 1 + (c - 1)x - y)$ and that the intersection of a horizontal line through (x, y) meets Q again at $(1 + (b - 1)y - x, y)$. Drawing the vertical lines through the (x_i, y_i) gives a sequence $(x_i, 1 + (c - 1)x_i - y_i)$ with $i = 1, 2, 3$ of consecutive lattice points on Q for which the smoothness condition is still satisfied:

$$\det\begin{pmatrix} x_3 - x_2 & (c-1)(x_3 - x_2) - (y_3 - y_2) \\ x_1 - x_2 & (c-1)(x_1 - x_2) - (y_1 - y_2) \end{pmatrix}$$
$$= \det\begin{pmatrix} x_3 - x_2 & -(y_3 - y_2) \\ x_1 - x_2 & -(y_1 - y_2) \end{pmatrix} = -1.$$

Similar reasoning holds for horizontal lines. Since any sequence of consecutive lattice points on Q comes from repeatedly intersecting with horizontal or vertical lines starting with the sequence $(b,1),\ (1,0),\ (0,0)$—for which the smoothness condition holds—the smoothness condition is satisfied for any three consecutive points along Q, as claimed.

Let $p_0,\ p_1$, and p_2 be adjacent lattice points of $P(D)$ lying consecutively on Q. Thinking of p_2 as the vertex, the lattice points p_3 for which $p_3 - p_2$ and $p_1 - p_2$ form a basis for the lattice lie on a line passing through p_0. This line intersects Q in at most one other point; hence, it must lie consecutively along Q with p_1 and p_2. This shows that, in order to build $P(D)$ starting from our initial five points, we must take consecutive lattice points along Q. The construction can work only when Q is an ellipse (i.e., when the discriminant of Q is less than 0). This means that $(b-1)(c-1) < 4$, which implies $(b,c) \in \{(2,2),(2,3),(3,2),(2,4),(4,2)\}$. The case of $b = c = 2$ gives Togliatti's surface, (3); $b = 2$ and $c = 3$ give (4) (isomorphic to $b = 3$ and $c = 2$); and $b = 2$ and $c = 4$ give (5) (isomorphic to $b = 4$ and $c = 2$). □

REMARK 3.3. Restricting to the category of toric varieties, Theorem 0.2 is easily established as a corollary to Theorem 3.2. Let $t \geq 2$, and let $v\colon X \to \mathbb{P}^{2t+1}$ be an embedding with $\dim(\mathrm{Osc}_k\, v(p)) = 2k$ for $k \leq t$ at all points $p \in X$. Apply Theorem 3.2 with $s = 2$ to narrow the possibilities. We then look for polytopes P and a subset of lattice points V, including the vertices of P, satisfying two properties:

(1) translating any vertex of P to the origin, and choosing the adjacent lattice points as a basis for the lattice, the number of $(a,b) \in V$ with $a + b \leq k$ is $2k+1$ for $k \leq t$ (cf. Section 1, (2));

(2) $H_V(k) = 2k$ for $k \leq t$ (Proposition 1.1, Remark 2.3).

It is straighforward to check that the only possibility is as stated in Theorem 0.2.

The following conjecture can be similarly established for toric varieties.

CONJECTURE 3.4 [PT]. *Let $t \geq 2$, and let $X \subset \mathbb{P}^{2t+2}$ be a smooth projective surface not contained in a hyperplane and such that the dimension of its kth osculating space is $2k$ at all points of X and for all $k \leq t$. Then X is isomorphic to a $\mathbb{P}^1$-bundle, the Hirzebruch surface $\mathbb{F}_1$ embedded as a scroll via the natural map*

$$X = \mathbb{F}_1 \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(t) \oplus \mathcal{O}_{\mathbb{P}^1}(t+1)) \\ \to \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(t)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(t+1))) \cong \mathbb{P}^{2t+2}.$$

In fact, it turns out that the conjecture does not hold outside of the toric setting. R. Piene and H. Tai (personal communication) have found special (nontoric) projections of Veronese embeddings that satisfy the hypotheses of the conjecture. Unlike our toric examples, the hypotheses hold for these projections with $k \leq t+1$, not just $k \leq t$.

These counterexamples to the conjecture are at least projections of toric varieties, and one might get the impression that placing many restrictions on the osculating spaces of an embedding at least forces the variety to be rational. However, Dye has given examples of embeddings of irrational varieties with very special osculating spaces [D], including a nonruled surface in $\mathbb{P}^5$ whose second osculating spaces are all of dimension ≤ 4.

3-Folds

We now consider the same problem for 3-folds. Assume that, at all points of the 3-fold X, the osculating spaces are as large as possible up through order $s-1$ and are strictly smaller than possible for order s. As in the proof of Theorem 3.2, we talk about lattice points of a polytope having a certain "level" with respect to a vertex (cf. V.3 in the proof of Theorem 3.5).

THEOREM 3.5. *Suppose that* $\dim(\mathrm{Osc}_k\, v(p)) = \binom{3+k}{k} - 1$ *for* $k = 1, \ldots, s-1$, *and that* $\dim(\mathrm{Osc}_s\, v(p)) < \binom{3+s}{s} - 1$ *for all* $p \in X$. *Up to an isomorphism in the category of toric mappings to projective space, the following are the only possibilities.*

(1) *X is $\mathbb{P}^3$, v is the* $(s-1)$th *Veronese embedding, and $P(D)$ is the tetrahedron shown in Figure 11; V consists of all lattice points in $P(D)$.*

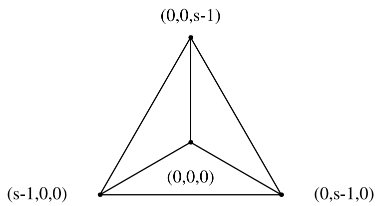

Figure 11

(2) *X is an equivariant fiber bundle over $\mathbb{P}^1$ with fiber equal to one of the toric varieties appearing in Theorem 3.2. The polytope $P(D)$ is a truncated cylinder over one of the polygons in Theorem 3.2. Hexagons, octagons, and dodecagons are allowed for the base of the cylinder only in the case $s = 2$ (cf. Remark 3.6). V is any subset of the lattice points in $P(D)$ containing all points out to level $s-1$.*

(3) *X is a $\mathbb{P}^1$-bundle of the form $\mathbb{P}(\mathcal{O}_S(E) \oplus \mathcal{O}_S(E'))$ for some ample divisors E, E' on an arbitrary smooth toric surface S (cf. Remark 3.6), and $P(D)$ has the form shown in Figure 12. The shaded polygon (in the plane $z = 0$) represents $P(E)$ and the dashed polygon (in the plane $z = 1$) represents $P(E')$; the top face (in the plane $z = s-1$) also has the form $P(E'')$ for a divisor E'' on S. The set V is any subset of the lattice points in $P(D)$ containing all points out to level $s-1$.*

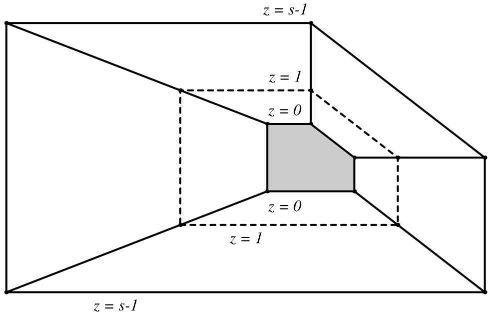

Figure 12

(4) $s = 2$, X *is* $\mathbb{P}^3$ *blown up in four points, and* $P(D)$ *is the truncated tetrahedron shown in Figure 13*; V *consists of the vertices of* $P(D)$.

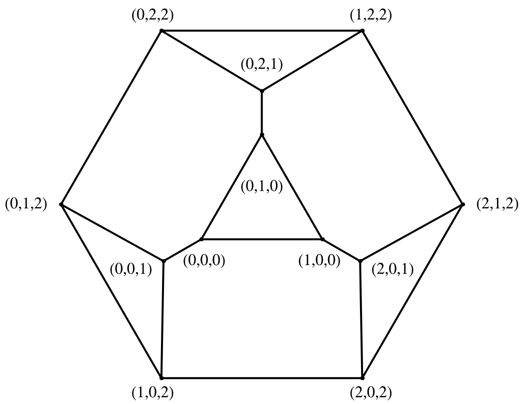

Figure 13

(5) $s = 2$ *and* $P(D)$ *is the join of two hexagons sharing an edge* (*see Figure 14*); V *consists of the vertices of* $P(D)$.

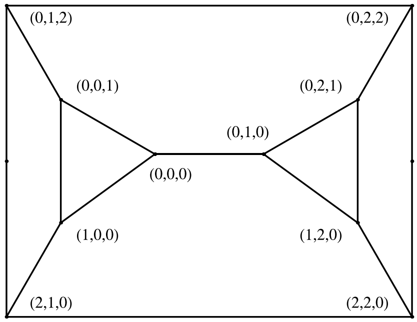

Figure 14

(6) *$s = 2$, X is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ blown up in six points, and $P(D)$ is the truncated octahedron shown in Figure 15; V consists of the vertices of $P(D)$.*

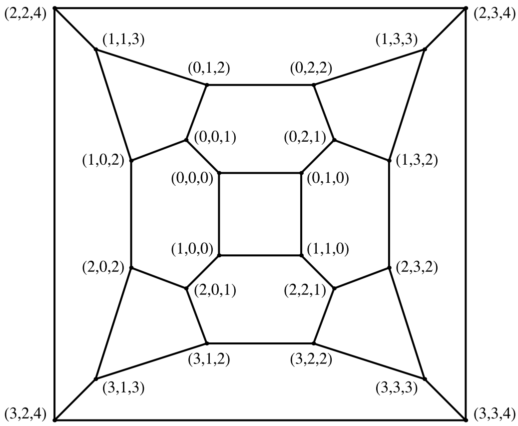

Figure 15

(7) *$s = 2$ and $P(D)$ is constructed from eight dodecagons, six octagons, and twenty-four triangles (see Figure 16); V consists of the vertices of $P(D)$.*

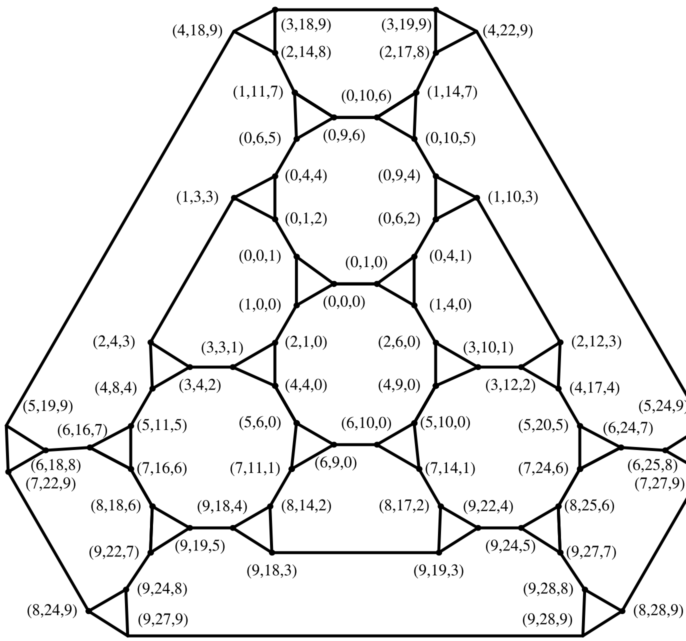

Figure 16

(8) *$s = 2$ and $P(D)$ is constructed from four dodecagons, four hexagons, and twelve triangles* (*see Figure 17*)*; V consists of the vertices of $P(D)$.*

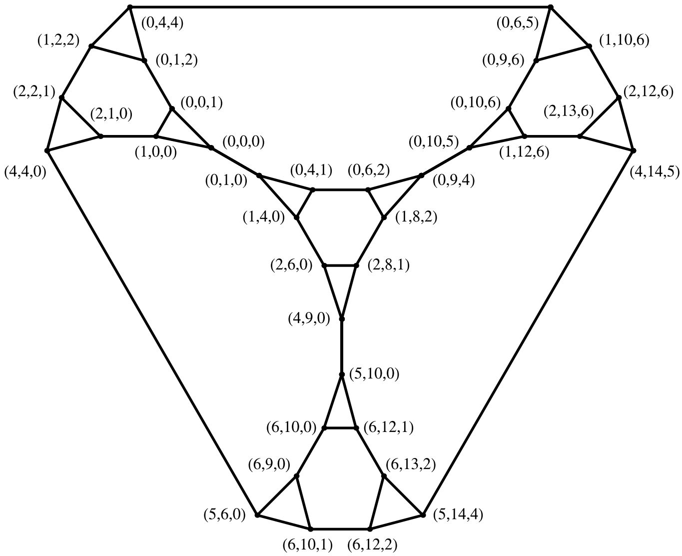

Figure 17

(9) *$s = 2$ and $P(D)$ is the join of two dodecagons sharing an edge* (*see Figure 18*)*; V consists of points out to level 2 from the vertices of $P(D)$ and any number of the lattice points along the edges joining the dodecagons.*

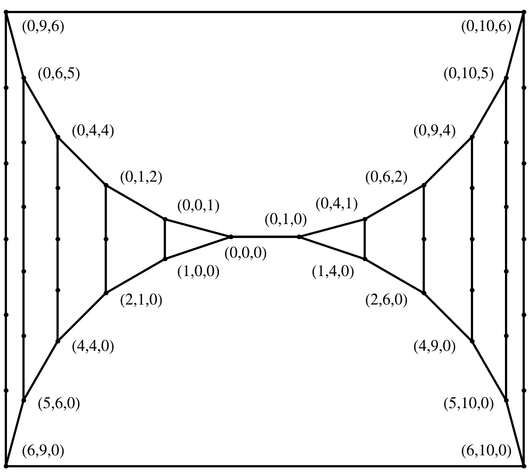

Figure 18

(10) $s = 2$ *and* $P(D)$ *is constructed from two dodecagons, six octagons, and twelve triangles* (*see Figure 19*); V *consists of the vertices of* $P(D)$.

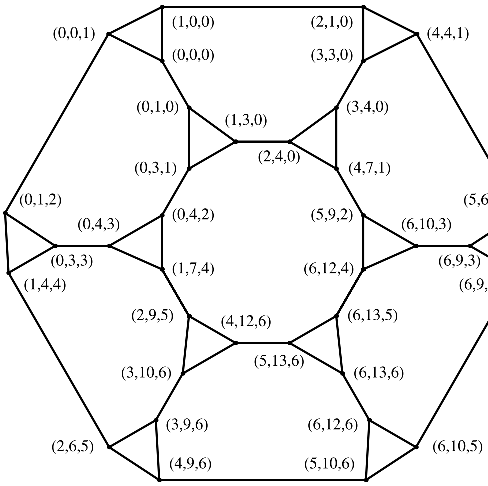

Figure 19

(11) $s = 2$ *and* $P(D)$ *is constructed from six octagons and eight triangles* (*see Figure 20*); V *consists of the vertices of* $P(D)$.

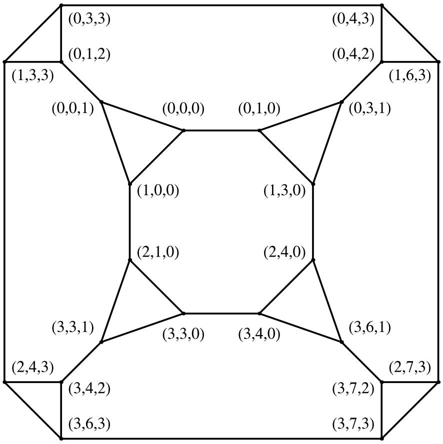

Figure 20

(12) $s = 2$ *and* $P(D)$ *is constructed from three octagons, two hexagons, and six triangles* (*see Figure 21*); V *consists of the vertices of* $P(D)$.

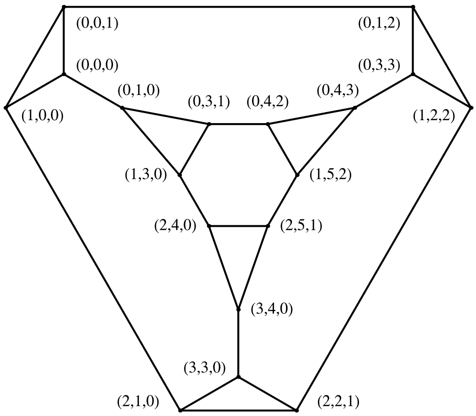

Figure 21

(13) *$s = 2$ and $P(D)$ is the join of two octagons sharing an edge* (*see Figure 22*); *V consists of points out to level 2 from the vertices of $P(D)$ and any number of the lattice points along the edges joining the octagons.*

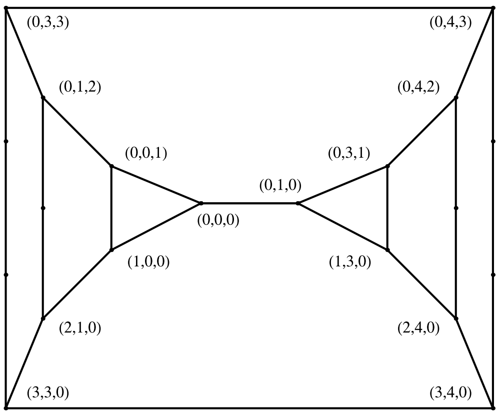

Figure 22

(14) *$s = 2$ and $P(D)$ is constructed from one dodecagon, three octagons, four hexagons, three quadrilaterals, and six triangles* (*see Figure 23*); *V consists of the vertices of $P(D)$.*

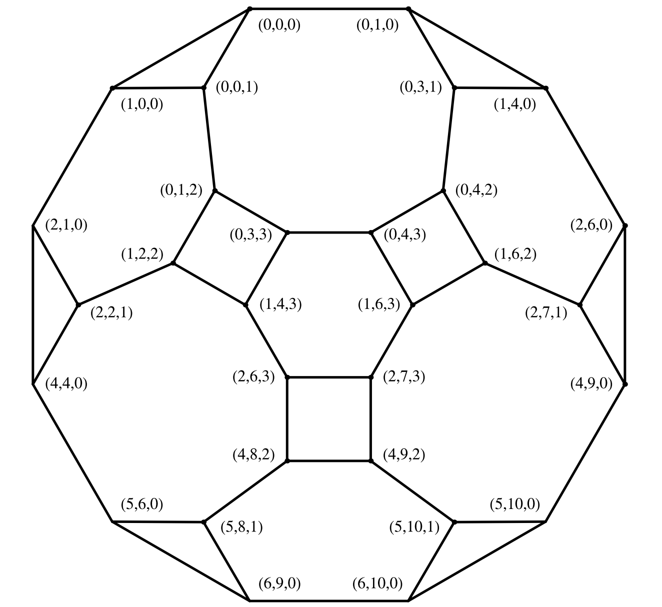

Figure 23

(15) *$s = 2$ and $P(D)$ is constructed from one dodecagon, seven hexagons, and six triangles* (*see Figure 24*); *V consists of the vertices of $P(D)$.*

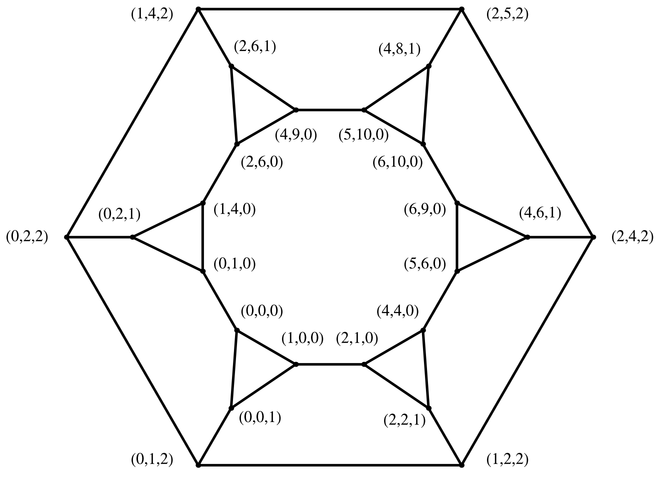

Figure 24

(16) *$s = 2$ and $P(D)$ is constructed from six octagons, eight hexagons, and twelve quadrilaterals* (*see Figure 25*); *V consists of the vertices of $P(D)$.*

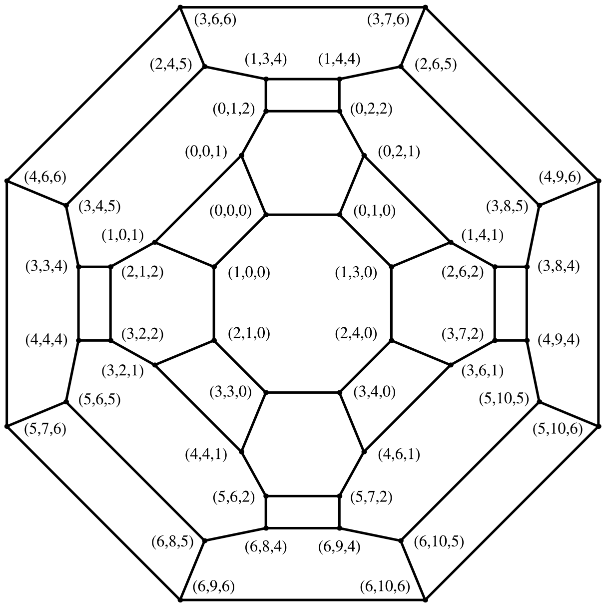

Figure 25

(17) *$s = 2$ and $P(D)$ is constructed from one octagon, four triangles, four hexagons, and a square* (*see Figure 26*); *V consists of the vertices of $P(D)$.*

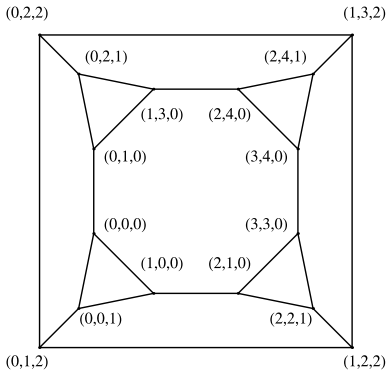

Figure 26

Proof. Arguing as at the beginning of the proof to Theorem 3.2, the problem is equivalent to finding all sets of lattice points V in $\mathbb{R}^3$ with convex hull a polytope P such that

(V.1) P is 3-valent (i.e., three edges emanate from each vertex);

(V.2) translating any vertex to the origin, the first lattice points along the three edges emanating from the vertex form a $\mathbb{Z}$-basis for the lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$ (ensuring that the associated toric variety is smooth);

(V.3) V includes all lattice points *out to level* $s-1$ with respect to each vertex: translating any vertex of P to the origin and letting x, y, z be the first lattice points along the three edges emanating from the vertex, the set V of lattice points must include those points corresponding to $ax + by + cz$ for a, b, c nonnegative integers with $a + b + c \leq s - 1$; and

(V.4) the lattice points in V must satisfy a nonzero polynomial in three variables with degree $\leq s$.

Let F be a (2-dimensional) face of P. The points of $F \cap V$, thought of as sitting in a 2-dimensional lattice in the plane supporting F, determine a mapping of a toric surface. We say that F is *proper* if, up to a change of coordinates of the 2-dimensional lattice, $F \cap V$ has one of the forms of Theorem 3.2 (for the same s): a certain triangle, a family of quadrilaterals, and a certain hexagon, octagon, or dodecagon (the last three are possibilities only when $s = 2$). We say that P is *proper* if each of its faces is proper; otherwise, P is *improper*. We divide the problem into four cases: (I) P proper, $s > 2$; (II) P proper, $s = 2$, but no octagons or dodecagons are allowed as faces of P; (III) P proper, $s = 2$, and P must include an octagon or dodecagon as a face; and (IV) P improper.

Case I: P proper, $s > 2$. Since P is proper and $s > 2$, each face of P must be a triangle or a quadrilateral. Euler's formula shows that the only such 3-valent polytopes are formed by: (i) four triangles (a tetrahedron); (ii) two triangles joined by three quadrilaterals (a truncated triangular cylinder); and (iii) six quadrilaterals (a box). We consider these cases separately.

(i) (four triangles) There is one possibility: a tetrahedron with s lattice points on each edge. In order to satisfy (V.3), the set V must contain all the lattice points on or in the tetrahedron. It is then easy to check that (V.1)–(V.4) are satisfied. This gives (1), the Veronese embedding.

(ii) (two triangles, three quadrilaterals) Since P is 3-valent, if two triangles meet then they must share an edge and P is forced to be a tetrahedron. Hence, in the present case, the two triangles do not touch. We may assume that the coordinate planes $x = 0$, $y = 0$, and $z = 0$ form three of the five supporting planes for P and that the bottom face (in the plane $z = 0$) is the triangle with vertices $(0, 0, 0)$, $(s - 1, 0, 0)$, and $(0, s - 1, 0)$, surrounded by quadrilaterals. The remaining triangle must also have s lattice points per side. Using the restriction from Theorem 3.2 on the quadrilateral faces, the vertices of this triangle must have the form $(0, 0, a)$, $(s-1, 0, b)$, $(0, s-1, c)$ for integers $a, b, c \geq s-1$. We may assume that a is no larger than b and c. Again using the restriction on the shapes of the quadrilateral faces, it follows that $b = (s-1)e + a$ and $c = (s-1)e' + a$ for some integers $e, e' \geq 0$. It is straightforward to check that the resulting polytope satisfies conditions (V.1) and (V.2). The lattice points in the base of the cylinder satisfy a polynomial of degree s in two variables, and all the lattice points in the polytope satisfy this same polynomial. Thus, any set of lattice points in the polytope meeting condition (V.3) also meets condition (V.4). There is no restriction on the height of the truncated cylinder. This example falls under (2) in the statement of the theorem.

(iii) (six quadrilaterals) We may take one vertex of the box, P, to be the origin and take the coordinate planes $x = 0$, $y = 0$, and $z = 0$ as three of the six supporting planes. We may take the vertices of the bottom of the box (in the plane $z = 0$) to be $(0, 0, 0)$, $(s-1, 0, 0)$, $(0, a, 0)$, and $(s-1, (s-1)e + a, 0)$ for integers $e \geq 0$ and $a \geq s-1$. Ruling out the case of a box with s lattice points per side, assume that the bottom is choosen so that $a > s-1$. The restriction from Theorem 3.2 on the shape of the quadrilateral in the plane $x = 0$ forces $(0, 0, s-1)$ to be a vertex of P and the remaining vertex in this plane to have the form $(0, (s-1)e' + a, s-1)$ for some integer e' such that $(s-1)e' + a \geq s-1$. Since the edge along the bottom connecting $(s-, 0, 0)$ and $(s-1, (s-1)e + a, 0)$ has length greater than $s-1$, the height of the face attached to the bottom along this edge must be $s-1$. This forces $z = s-1$ to be a supporting plane for P and the remaining vertex in the plane $y = 0$ to have the form $((s-1)e'' + s-1, 0, s-1)$ for some integer $e'' \geq 0$. To make the top of the box proper, we must take $(s-1)e' + a = s-1$ or $e'' = 0$. In the former case, the remaining vertex has the form $((s-1)(e''+1), (s-1)(e+1), s-1)$; in the latter, $(s-1, (s-1)(e+e') + a, s-1)$. In either case, we get a truncated cylinder over the face supported by $y = 0$.

It is straightforward to check that the resulting polytopes satisfy conditions (V.1) and (V.2). As in case (ii), any set of lattice points satisfying (V.3) will also satisfy (V.4). This example falls under (2).

Case II: P proper, $s = 2$, but no octagons or dodecagons are allowed as faces of P. In this case, P may have faces that are triangles, quadrilaterals, or hexagons. From Euler's formula, the possibilities for the number of triangles and quadrilaterals are the same as in Case I. However, given an acceptable number of triangular and quadrilateral faces, there are an infinite number of combinatorially different 3-valent polytopes that can be built by adding hexagons [G, pp. 253–281]. Only finitely many of these give rise to lattice points satisfying our conditions. We consider three cases, as before.

(i) (four triangles) As argued in Case I(i), if two triangles touch then P must be a tetrahedron; so suppose that no two triangles touch. Up to a change of coordinates of the lattice, we may assume that the polytope contains a triangle surrounded by three hexagons with coordinates as shown in Figure 27 (the coordinate planes are supporting planes). These lattice points sit on only one quadric,

$$Q = -x + x^2 - y + y^2 - z + z^2 + xy - xz - yz.$$

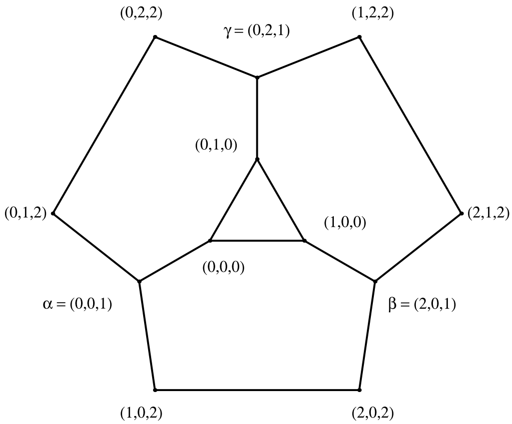

Figure 27

At each of the vertices labeled α, β, γ we must fit triangles, since a hexagon added there would not have vertices lying on the quadric (e.g., for a hexagon to sit at $\alpha = (0, 0, 1)$ it must also include the adjacent vertices $(1, 0, 2)$ and $(0, 1, 2)$). Lying in the plane determined by these vertices, the remaining vertices are forced to be $(2, 1, 4)$, $(1, 2, 4)$, and $(2, 2, 5)$, none of which lie on the quadric. Hence, the triangle with vertices α, $(1, 0, 2)$, and $(0, 1, 2)$ lies on the polytope and, similarly, there are triangles at vertices β and γ. Now, since the polytope is 3-valent, $z = 2$ must be a supporting face, which closes the polytope with a hexagon. The result is (4), a truncated tetrahedron. The vertices form a set of lattice points satisfying condition (V.3). They lie on a unique quadric Q and thus satisfy (V.4). None of the remaining lattice points from the polytope—the centers of the four hexagons—lie on Q. Finally, it is straightforward to check that (V.2) is satisfied.

(ii) (two triangles, three quadrilaterals) Suppose that a triangle touches two quadrilaterals. Using 3-valency and properness, and considering the supporting planes of the polytope, we see that the only possibility is a truncated cylinder over the triangle.

Now suppose that a triangle touches exactly one quadrilateral and hence two hexagons. We may assume that the configuration is as shown in Figure 28. The supporting plane determined by points α, β, δ forces $\overline{\beta\varepsilon}$ to be an edge, and the supporting plane determined by $\beta, \gamma, \varepsilon$ forces $\overline{\gamma\phi}$ to be an edge. Thus, the polytope is what might be called the "join" of two hexagons sharing an edge, (5). It is easy to check that the smoothness property holds at each vertex. For instance, translating the vertex $\alpha = (0, 1, 2)$ to the origin, the adjacent lattice points—including the point $(1, 1, 1)$ on the edge $\overline{\alpha\delta}$—form a basis for the lattice:

$$\det\begin{pmatrix} (0,0,1)-\alpha \\ (0,2,2)-\alpha \\ (1,1,1)-\alpha \end{pmatrix} = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = 1.$$

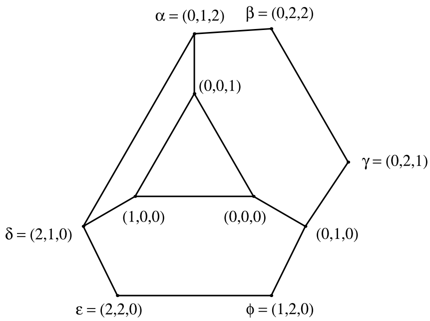

Figure 28

For the final possibility, if there is a triangle that does not touch a quadrilateral then we repeat the argument just given in (i) to get a truncated tetrahedron, (4).

(iii) (six quadrilaterals) There are three cases to consider. First, suppose that at least three quadrilaterals meet at a point. By considering supporting planes and using 3-valency, we are reduced to the case of no hexagons, Case I(iii). Second, suppose that no three quadrilaterals meet at a point yet there is set of two quadrilaterals meeting at a point. By 3-valency, the quadrilaterals that meet must share an edge. Since no three quadrilaterals meet, there is a hexagon at both vertices along this edge. It is straightforward to check that this forces the polytope to be a truncated hexagonal cylinder, an instance of (2). Finally, suppose that no two quadrilaterals meet. Since each quadrilateral is surrounded by hexagons having two lattice points on each edge, the quadrilaterals must also have exactly two lattice points on each edge. Starting with a square surrounded by hexagons, we proceed as in the latter part of Case II(i) and arrive at (6), a truncated octahedron as stated in the theorem.

Case III: P proper, $s = 2$, and P must include an octagon or dodecagon as a face. This case involves a long and tedious search, greatly facilitated by the use of a computer. The basic idea is that the points of V must lie on a quadric, and there are not too many choices for a quadric containing a dodecagon or an octagon. Trying to construct a polytope one face at a time soon determines the quadric completely and allows us to specify the possibilities. We will give an outline of the search as well as examples illustrating all of the techniques needed. The problem is divided into six cases: (i) two dodecagons meet, (ii) two octagons meet, (iii) a dodecagon and octagon meet, (iv) a dodecagon and hexagon meet, (v) an octagon and hexagon meet, and (vi) none of the above.

(i) (two dodecagons meet) First note that—taking any vertex of the dodecagon of Theorem 3.2 as the origin and the adjacent lattice points as a basis for the lattice—there are two possibilities: the original dodecagon or its flip about the diagonal, $y = x$. Thus, in our case, we consider two possible orientations for the dodecagons that meet. We may assume that one dodecagon sits in the plane $z = 0$

and includes the lattice points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(2,1,0)$ (the rest are determined). We may also assume that the second dodecagon sits in the plane $x = 0$ and includes the lattice points $(0,0,0)$, $(0,1,0)$, $(0,0,1)$. However, there are two possible orientations for the second dodecagon: it can include the vertex $(0,2,1)$ or its flip, $(0,1,2)$; the former case (see Figure 29) will be called orientation 1.

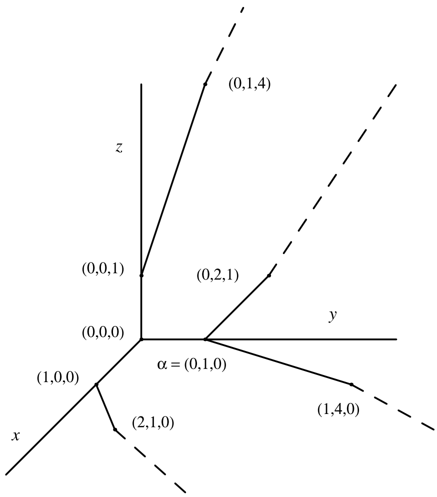

Figure 29

The quadrics containing these two dodecagons have the form

$$Q = 9ax^2 - 9axy + bxz + 3ay^2 - 3ayz + az^2 - 9ax - 3ay - az$$

for $a, b \in \mathbb{C}$. What are the possibilities for the remaining face meeting the dodecagons at vertex α? There are two possible dodecagons (differing by orientation) that could fit there, but it is straightforward to check that neither of these would have lattice points lying on Q. Similarly, there are two possible octagons, neither of which would lie on Q. The same holds for the unique hexagon that could fit there. Hence, a quadrilateral or a triangle must fit at α.

If a triangle fits at α, 3-valency would require that $(2,6,0)$, $(1,4,0)$, $(0,2,1)$, and $(0,4,4)$ be co-planar, but they are not. Thus, a quadrilateral is forced at α. The quadrilateral and the set V of exponents must contain the point $(1,5,1)$, forcing $b = 0$ in Q. A priori, there are many possibilities for the shape of the quadrilateral. Larger quadrilaterals would need to contain the next point out along either the edge containing $(0,2,1)$ and $(1,5,1)$ or the edge containing $(1,4,0)$ and $(1,5,1)$, namely: $(2,8,1)$ or $(1,6,2)$, respectively. However, it is easy to check that these points are not zeroes of $Q\big|_{b=0}$. Hence, the quadrilateral has vertices $(0,1,0)$, $(0,2,1)$, $(1,4,0)$, and $(1,5,1)$, as shown in Figure 30.

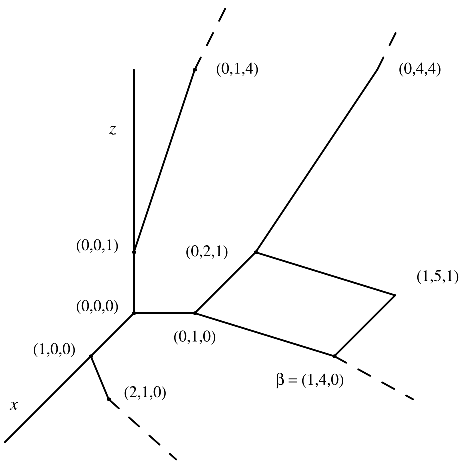

Figure 30

It is now easy to check with a computer that (a) a dodecagon must fit at vertex β, having orientation 1 with respect to the dodecagon it meets along the edge joining $(1,4,0)$ and $(2,6,0)$, and (b) a quadrilateral must fit at the origin, containing exactly four lattice points. By symmetry, the whole figure must consist of quadrilaterals and dodecagons, each quadrilateral surrounded by four dodecagons and each dodecagon surrounded by six quadrilaterals and six dodecagons. If the figure were to close up to give a polytope then consideration of Euler's formula would lead to a contradiction, as we now explain.

Let R be any 3-valent, 3-dimensional polytope, and let p_k be the number of faces of R having k edges. A simple consequence of Euler's formula relating the numbers of vertices, edges, and faces (taking 3-valency into account) is the following relation:

$$3p_3 + 2p_4 + p_5 = 12 + \sum_{k\geq 7}(k-6)p_k.$$

(In fact, there is a sort of converse. Eberhard's theorem [G, p. 254] states that, given any finite sequence of nonnegative integers p_k for $k \neq 6$, there is a 3-valent, 3-dimensional polytope, possibly containing hexagons, such that p_k is the number of faces with k edges.) In our case, the formula reads

$$2p_4 = 12 + 6p_{12}.$$

Combined with the additional fact that $4p_4 = 6p_{12}$ (a consequence of the arrangement of quadrilaterals and dodecagons), we obtain a contradiction. Thus, orientation 1 does not produce an acceptable polytope (although it produces interesting affine mappings).

Now consider orientation 2, shown in Figure 31. As in the case of orientation 1, there is a quadric Q, with two parameters, containing the vertices of the two dodecagons. One may check that either a hexagon, a quadrilateral, or a triangle must fit at vertex α. The quadrilateral and the hexagon lead to figures that do not close, as in the case of orientation 1 (using Eberhard's theorem again). So suppose that a triangle fits at α. The triangle does nothing to specialize the quadric Q, so we need to look at vertex β. A triangle is ruled out by 3-valency, but it turns out that a dodecagon, an octagon, a hexagon, or a quadrilateral are possibilities (i.e., consistent with Q) at β. The dodecagon leads to a figure that does not close, but the octagon, hexagon, and quadrilateral lead to polytopes for which (V.1)–(V.4) hold. These are (7), (8), and (9), respectively.

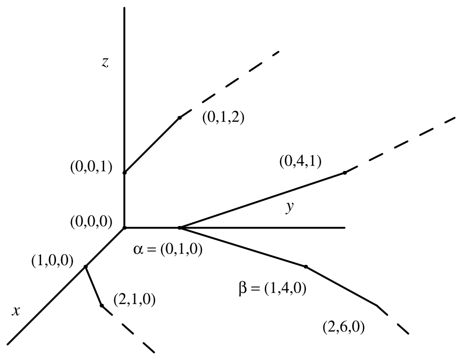

Figure 31

(ii) (two octagons meet) Proceeding as in (i), there are two orientations for the meeting octagons. One produces no examples; the other produces (10), (11), (12), and (13). As in (i), in the second orientation we fit a triangle at one of the vertices on the edge shared by the octagons and then find that a dodecagon, an octagon, a hexagon, or a quadrilateral can fit next to this triangle. Unlike (i), the dodecagon leads to an acceptable polytope.

(iii) (a dodecagon and octagon meet) Again, as in (i), there are two orientations. One produces no examples (giving a figure that does not close, as before); in the other, we can fit a quadrilateral or a triangle next to the meeting dodecagon and octagon. The quadrilateral leads to a figure that does not close, so we do not get an example. On the other hand, next to the triangle we can fit a dodecagon, an octagon, or a hexagon; the dodecagon and octagon lead to examples we have already seen, and the hexagon produces (14).

(iv) (a dodecagon and hexagon meet) There is only one orientation to consider. A dodecagon, a quadrilateral, or a triangle must fit next to the meeting dodecagon and hexagon. The dodecagon was already considered in (i) and did not lead to an example. The quadrilateral leads to a figure that does not close. Finally, the triangle yields several possibilities: a dodecagon, an octagon, a hexagon, or a quadrilateral can fit next to the triangle. The dodecagon and octagon then produce examples we have already seen. The hexagon produces (15). The quadrilateral does not produce an example.

(v) (an octagon and hexagon meet) This case is similar to (iv). An octagon, a quadrilateral, or a triangle can fit next to the meeting octagon and hexagon. The octagon was considered in (ii) and did not lead to an example. The quadrilateral gives (16). Next to the triangle, there can be a dodecagon (reducing to (iii)), an octagon (reducing to (ii)), a hexagon leading to (17), or a quadrilateral leading to no example.

(vi) (none of the above) We are left with the case of a dodecagon or octagon surrounded by triangles and quadrilaterals. Using 3-valency, one may check that there can be no triangle. We get a truncated cylinder over the dodecagon or octagon, instances of (2).

Case IV: P improper. Suppose we have V and P satisfying (V.1)–(V.4). Let $f(x, y, z)$ be a nonzero polynomial of degree $\leq s$ satisfied by the points of V, and let F be a face that is not proper. The lattice points in $F \cap V$, sitting in the plane supporting F (say, Π), determine a mapping of a toric surface. If f restricted to Π is nonzero, then this surface mapping would satisfy Theorem 3.2 and F would need to be proper. Hence, f contains the equation for Π as a factor. In fact, we will see that, up to an affine change of coordinates of the lattice in $\mathbb{R}^3$ and a constant factor, $f = \prod_{i=0}^{s-1}(z - i)$.

If $s > 2$, let $\tilde{V} = V \setminus (V \cap \Pi)$, let $\tilde{P}$ be the convex hull of the points of $\tilde{V}$, and let $\tilde{f}$ be the polynomial of degree $\leq s - 1$ obtained by removing the equation for Π from f. We now verify that $\tilde{V}$ and $\tilde{P}$ satisfy conditions (V.1)–(V.4) with s replaced by $s - 1$. Making an affine change of coordinates, assume for the moment that Π is defined by $z = 0$. Figure 32 labels consecutive vertices α_0, β_0, and γ_0 of F. The first lattice points on the edges leaving F from these vertices are α_1, β_1, and γ_1, and the vertices at the end of these edges are α, β, and γ.

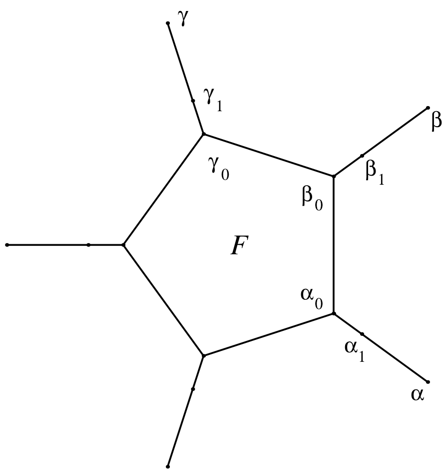

Figure 32

Let $\alpha_0 = (a, b, 0)$, $\gamma_0 = (c, d, 0)$, and $\beta_1 = (i, j, k)$. We may assume that $\beta_0 = (0, 0, 0)$. Since P satisfies (V.1), it follows that

$$\begin{vmatrix} a & b & 0 \\ c & d & 0 \\ i & j & k \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} k = \pm 1.$$

Hence, we may assume that $k = 1$ and, similarly, all the first lattice points on edges emanating from F lie in the plane $z = 1$. Thus, chopping off the face F leaves a new face $\tilde{F}$ in the plane $z = 1$. Also, note that none of these first lattice points are vertices of P (since $s > 2$), so the new polytope is combinatorially equivalent to P. Our computation shows that, after an affine change of coordinates, we can take the edges (of the truncated polytope) $\beta_1\alpha_1$, $\beta_1\gamma_1$, and $\beta_1\beta$ to lie along the coordinate axes. It is then easy to verify (V.1)–(V.3); of course, (V.4) is satisfied with $s - 1$ in place of s, using $\tilde{f}$.

Removing F from P gives the new polytope $\tilde{P}$. If $\tilde{P}$ is not proper and $s - 1 > 2$, then we can repeat the process of chopping off an improper face. Eventually we are reduced to lattice points V^* with convex hull the polytope P^*, combinatorially equivalent to P and satisfying (V.1)–(V.4) for some integer s^* in place of s. The polytope P^* is either (i) proper or (ii) improper with $s^* = 2$.

(i) (P^* proper) If P^* is proper, we have shown that (up to a change of coordinates) it must be: (1), a certain tetrahedron (the numbers here refer to the statement of the theorem); (2), a truncated cylinder over one of the polygons from Theorem 3.2; or, in the case $s^* = 2$, (4)–(17). To rule out each of these possibilities, imagine reversing the process of going from P to P^*. This would involve taking a face of P^*, whose supporting plane we can take to be $z = 0$, and extending the edges coming into this face down to a parallel plane, which we can take to be $z = -1$. The result is an intermediary polytope $\tilde{P}$, combinatorially equivalent to P and P^*, and improper. The polytope $\tilde{P}$ comes with a corresponding subset $\tilde{V}$ of lattice points such that (V.1)–(V.4) are satisfied for an $\tilde{s} = s^* + 1$.

In the case of the tetrahedron, (1), reversing the process would imply that P was proper, a contradiction. In the case of a truncated cylinder over one of the polygons from Theorem 3.2, reversing the process one step results in a $\tilde{P}$ that is either proper, combinatorially inequivalent to P^*, or for which $\tilde{V}$ necessarily violates (V.3). For instance, suppose P^* is a truncated cylinder over a quadrilateral (see Figure 33). If $\tilde{P}$ comes from moving the upper or lower faces out one unit, then $\tilde{V}$ could not satisfy (V.3) given that the base quadrilateral has not increased in size. The same argument holds for moving the left or right faces out one unit, or the back face in the case where $a = s^* - 1$. If a, b, and the height are large enough, then moving the front or back face out may produce a proper polytope—a contradiction. In the case where $s^* = 2$, $a = 1$, and $b > 0$, moving the front face out gives a polytope that is not combinatorially equivalent to P^*.

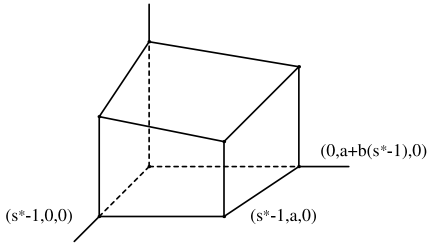

Figure 33

Finally, if P^* is one of the polytopes (4)–(17), reversing the process gives a polytope that is not combinatorially equivalent to P^*.

(ii) (P^* improper, $s^* = 2$) This case subsumes the case of $s = 2$. Let f^* be the quadric containing P^*, which is a factor of the original f. We may assume that $z = 0$ is a supporting plane of an improper face of P^*. By (V.2), the first lattice points along the edges emanating from the face in $z = 0$ lie in the plane $z = 1$. It follows that, up to a constant factor, $f^* = z(z-1)$. Going backwards from P^* to P, trying to add a face in a plane not parallel to $z = 0$ at any step would give rise to a polytope for which (V.3) could not hold. Hence, the edges emanating from the face in $z = 0$ terminate in the plane $z = s - 1$ and so, up to a constant factor, $f = \prod_{i=0}^{s-1}(z - i)$ as claimed; this determines P. Its shape is determined by the face in $z = 0$ and the convex hull of the lattice points of V lying in the plane $z = 1$. These two polygons give rise to the same toric surface S; that is, one polygon can be derived from the other by sliding each edge in a direction normal to that edge. Another way to say this is that the two polygons have the forms $P(E)$ and $P(E')$ for two ample divisors E, E' on the toric surface S (cf. Section 2). The toric variety determined by P is $\mathbb{P}(\mathcal{O}_S(E) \oplus \mathcal{O}_S(E'))$, giving (3). □

REMARK 3.6. In this remark, we describe more carefully the mappings in Theorem 3.5(2)—that is, those coming from truncated cylinders. In $\mathbb{R}^3$ with coordinates x, y, z, let P be a polygon sitting in the (x, y)-plane. Assume that P is a polygon allowed by Theorem 3.2 (i.e., it gives rise to a mapping of a surface with special osculating spaces). The cylinder over P is $C(P) = \{p + (0, 0, z) \mid p \in P,\ z \in \mathbb{R}\}$. To truncate $C(P)$ so that the corresponding toric 3-fold is smooth, fix a vector $(e, e', 1)$, where e, e' are arbitrary integers, and let Π be the plane normal to $(e, e', 1)$. The truncated cylinder corresponding to Π consists of the points of $C(P)$ on or between P and Π. The truncated cylinders in Theorem 3.5 are exactly those constructed in this way.

The 1-dimensional cones of the fan Δ for the 3-fold corresponding to one of these truncated cylinders are the 1-dimensional cones of the fan Δ'' for the toric surface corresponding to P (sitting in the (x, y)-plane in $\mathbb{R}^3$) and two more, generated by $(0, 0, 1)$ and $-(e, e', 1)$. The projection map $\mathbb{R}^3 \to \mathbb{R}$, forgetting the first two coordinates, maps Δ onto the fan $\Delta' \subset \mathbb{R}$ for $\mathbb{P}^1$. Let $\tilde{\Delta}'$ be the subfan of Δ with 1-dimensional cones generated by $(0, 0, 1)$ and $-(e, e', 1)$. Then

$$\Delta = \{\tilde{\sigma}' + \sigma'' \mid \tilde{\sigma}' \in \tilde{\Delta}',\ \sigma'' \in \Delta''\}.$$

Thus, according to [O, Prop. 1.33], the toric variety corresponding to the truncated cylinder is an equivariant fiber bundle over $\mathbb{P}^1$ with fiber isomorphic to the toric surface corresponding to P. The bundles appearing in Theorem 3.5, (9), can be analyzed similarly.

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