

## *Inflow Control on Expressway Considering Traffic Equilibria*

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(Received February 14, 1986)

### SYNOPSIS

When expressway and roads coexist, it is necessary to establish a reasonable traffic share between them. It may be practiced by the regulation of toll-rate of expressway. But at an occasional traffic congestion, the reasonable share is disturbed, so that some traffic control means should be taken. In this paper, we deal how to control inflows on expressway, from a viewpoint of the optimal share between expressway and roads. The minimization of total travel cost in a system is aimed under traffic equilibrium conditions. The problem is formed as a two-stage programming model, and a simple example solving the problem is showed.

### 1. INTRODUCTION

So far, the traffic control on expressway mainly aims how to dissolve speedily the traffic congestion caused by an accident. But nowadays, natural traffic congestions frequently occur on expressway with the increase of traffic demand. Accordingly, it becomes to be necessary to control inflows on expressway.

When inflows are limited at interchanges, the traffic flow on a parallel road increaes, and traffic congestion on it becomes intensified. For this reason, the limitation of inflows on express-

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way may not be allowed, if traffic congestion gets worse. It may be allowed only if the social benefit is improved through the control. We aim the minimization of total travel cost in a network instead of the promotion of social benefit.

By the way, the direct control of inflows is not practical under present facilities of expressway and operations of traffic in our nation. A possible way is to control inflows indirectly, through the control of the number of open booths. On the other hand, expressway users may decide their behaviors on informations offered by the traffic operator of expressway. In this case, users will act such that their utility is maximized or their loss is minimized. So that, the traffic operator can decide the optimal control strategy, taking into account user's behavior. This can be considered as a two-person game, in which the traffic operator have the initiative and can decide the optimum strategy ahead of users.

## 2. FORMULATION

Now we consider a network which consists of an expressway, such roads that are substitutive for expressway and cross connections with expressway. As traffic demands, we take into account only those concerned with expressway. Other demands are treated as fixed values. In order to simplify, a kind of vehicles is assumed.

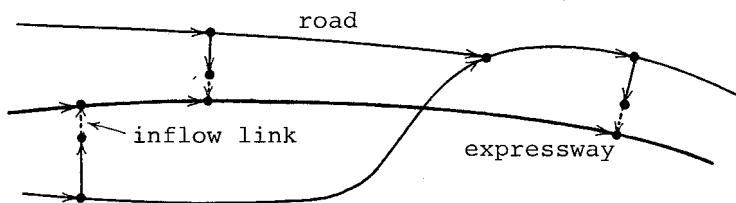


Fig.1 An Example of Expressway and Road Network

Let denote

$S_{ij}$ :demand from  $i$  to  $j$ ,

$x_{ijp}$ :flow on route  $p$  from  $i$  to  $j$ ,

$r_{ijp}^k=1$ :if link  $k$  is contained in route  $p$  from  $i$  to  $j$

0:otherwise,

$c_{ijp}$ :toll rate on route  $p$  from  $i$  to  $j$ .

We divide the set of link into two sets. One is the set of inflow links to expressway and the other is the set of other ordinary links. The former is represented by  $L_i$  and the latter is by  $L_o$ . It is assumed that the travel time on a ordinary link  $k$  is a monotone increasing function of link flow, denoted by  $T_k=f_k(X_k)$ . On the other hand, we assume that the travel time on inflow link  $k$  is a function of both capacity  $y_k$  and flow  $X_k$ , denoted by  $T_k=g_k(y_k, X_k)$ .

Inflow link capacity depends on the number of open booths, but we assume it as a continuous variable for convenience. The travel time on a inflow link represents the delay caused by queues at toll-gate. It should be monotone increasing for flow and monotone decreasing for capacity. The flows which enter from interchanges out of control area are treated as fixed values. The fixed flows on link  $k$  is denoted by  $X_{k0}$ .

Now we assume any user acts such that his travel cost is minimized. As travel cost, we take the sum of the value of travel time and toll-rate on the route. Then the distribution of flows, on a network under given inflow link capacities, is obtained through traffic equilibrium conditions. That is the solution of the following optimization problem(1), (2).

$$\begin{aligned} & \text{minimize } J(\mathbf{y}, \mathbf{x}) \\ & = v \sum_{k \in L_n} \int_0^{X_k} f_k(X) dX + v \sum_{k \in L_o} \int_0^{X_k} g_k(y_k, X) dX + \sum_{ijp} x_{ijp} c_{ijp} \end{aligned} \quad (1)$$

for  $\mathbf{x}$ , subject to

$$X_k = \sum_{ijp} x_{ijp} \cdot r_{ijp}^k + X_{k0} \quad \forall k \quad (2)$$

$$\sum_p x_{ijp} = S_{ij} \quad \forall ij \quad (3)$$

$$x_{ijp} \geq 0 \quad \forall ij, p \quad (4)$$

Where  $v$  is the average time value.

On the other hand, we assume that the traffic operator controls the number of open booths such that user's total travel cost is minimized, taking into account user's route choice behaviors. As travel cost, we take the total value of time loss and toll-rate on a network. Then, user's total travel cost is represented by

$$F(\mathbf{y}, \mathbf{x}) = v \left\{ \sum_{k \in L_o} X_k f_k(X_k) + \sum_{k \in L_i} X_k g_k(y_k, X_k) \right\} + \sum_{ijp} x_{ijp} c_{ijp} \quad (5)$$

where,

$$X_k = \sum_{ijp} x_{ijp} \cdot r_{ijp}^k + X_{k0} \quad \forall k \quad (6)$$

Inflow link capacity  $y_k$  should be under less than the supremum  $\bar{y}_k$ , which is equivalent to the capacity when all booths are opened, so that

$$0 \leq y_k \leq \bar{y}_k \quad k \in L_i \quad (7)$$

The traffic operator can decide on the optimum capacity of inflow links, taking into account user's route choice behavior. On the other hand, user's can not know the strategy of traffic operator. They merely decide their behaviors on informations provided by the operator. So that the traffic operator has an initiative in this two-person game, and can take the optimum strategy ahead of users. Whence, this problem can be formed as the following two-level optimization problem<sup>(3)</sup>.

$$F(\mathbf{y}^*, \hat{\mathbf{x}}(\mathbf{y}^*)) = \min_{\mathbf{y}} F(\mathbf{y}, \hat{\mathbf{x}}(\mathbf{y})) \quad (8)$$

sub. to

$$0 \leq y_k \leq \bar{y}_k \quad k \in L_i \quad (9)$$

$$J(\mathbf{y}, \hat{\mathbf{x}}(\mathbf{y})) = \min_{\mathbf{x}} J(\mathbf{y}, \mathbf{x}) \quad (10)$$

sub. to

$$x_k = \sum_{ijp} x_{ijp} \cdot r_{ijp}^k + x_{k0} \quad \forall k \quad (11)$$

$$\sum_p x_{ijp} = S_{ij} \quad \forall ij \quad (12)$$

$$x_{ijp} \geq 0 \quad \forall ij, p \quad (13)$$

Where,  $\mathbf{y}^*$  is the optimal solution for inflow link capacities, and  $\hat{\mathbf{x}}(\mathbf{y})$  is the parametric optimal solution of the inferior problem corresponding to given inflow link capacities  $\mathbf{y}$ . Clearly  $\hat{\mathbf{x}}(\mathbf{y})$  shows one-to-one correspondence, so that if the inferior problem is feasible, an equilibrium solution, i.e. the stackelberg solution exists.

This two-level optimization problem may be solved by the following algorithm.

- (1) Select an feasible initial value  $\mathbf{y}^{(1)}$ , and set  $n=1$ .
- (2) Solve the inferior problem for  $\mathbf{x}$  under given capacities  $\mathbf{y}^{(n)}$ .
- (3) Compute the gradient of objective function of the superior problem, using the optimality conditions of the inferior problem.
- (4) Solve the superior problem, using the gradient. Write  $\mathbf{y}^{(n)}$  for the optimal solution. If  $\|\mathbf{y}^{(n)} - \mathbf{y}^{(n-1)}\| < \epsilon$ , then stop. Otherwise, set  $n=n+1$ , and go to step (2).

3. EXAMPLE

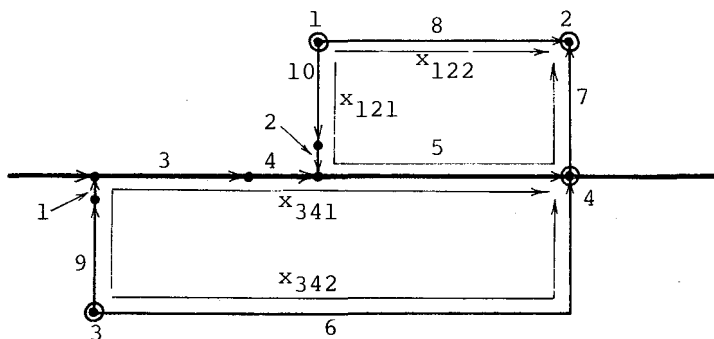


Fig.2 Network for Example

Now, we consider a simple network illustrated in Fig.2. Link 1 and 2 are inflow links and the other are ordinary links. We take into consideration traffic demands only from node 3 to 4 and from node 1 to 2. Flows with respect to other OD pairs are fixed. For each OD pairs, two routes by expressway and by road are considered respectively. If these routes are balanced in travel cost, the equilibrium conditions for the inferior problem are represented by

$$x_{121} + x_{122} = S_{12} \quad (14)$$

$$x_{341} + x_{342} = S_{34} \quad (15)$$

$$v(T_2 + T_5 + T_7 + T_{10}) + c_{121} = vT_8 \quad (16)$$

$$v(T_1 + T_3 + T_4 + T_5 + T_9) + c_{341} = vT_6 \quad (17)$$

These equations may be considered as simultaneous equations of route flows, so that implicit functions

$$\begin{aligned} x_{121} &= h_1(y_1, y_2) \\ x_{122} &= h_2(y_1, y_2) \\ x_{341} &= h_3(y_1, y_2) \\ x_{342} &= h_4(y_1, y_2) \end{aligned} \quad (18)$$

exist. The partial differentials of them are expressed as follows.

$$\begin{bmatrix} \frac{\partial x_{121}}{\partial y_1} \\ \frac{\partial x_{122}}{\partial y_1} \\ \frac{\partial x_{341}}{\partial y_1} \\ \frac{\partial x_{342}}{\partial y_1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ T'_5 + T'_7 + T'_{10} & -T'_8 & T'_5 & 0 \\ T'_5 & 0 & T'_{1x} + T'_3 + T'_4 + T'_5 + T'_9 & -T'_6 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -T'_{1y} \end{bmatrix} \quad (19)$$

$$\begin{bmatrix} \frac{\partial x_{121}}{\partial y_2} \\ \frac{\partial x_{122}}{\partial y_2} \\ \frac{\partial x_{341}}{\partial y_2} \\ \frac{\partial x_{342}}{\partial y_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ T'_{2x} + T'_5 + T'_7 + T'_{10} & -T'_8 & T'_5 & 0 \\ T'_5 & 0 & T'_3 + T'_4 + T'_5 + T'_9 & -T'_6 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ -T'_{2y} \\ 0 \end{bmatrix} \quad (20)$$

Where,

$$T'_k = \frac{dT_k}{dx_k} \quad (k \in L_0), \quad T'_{kx} = \frac{\partial T_k}{\partial x_k} \quad T'_{ky} = \frac{\partial T_k}{\partial y_k} \quad (k \in L_1)$$

On the other hand, the total travel cost is represented by

$$\begin{aligned} F = & v \{ (X_{10} + x_{341})T_1 + (X_{20} + x_{121})T_2 + (X_{30} + x_{341})T_3 \\ & + (X_{40} + x_{341})T_4 + (X_{50} + x_{121} + x_{341})T_5 + (X_{60} + x_{342})T_6 \\ & + (X_{70} + x_{121})T_7 + (X_{80} + x_{122})T_8 + (X_{90} + x_{341})T_9 \\ & + (X_{100} + x_{121})T_{10} \end{aligned} \quad (21)$$

If implicit functions (18) are substituted into (21), total travel time  $F$  becomes a function of  $y_1$  and  $y_2$ , denoted by  $\tilde{F}(y_1, y_2)$ . The gradient of  $\tilde{F}$  is given by

$$\frac{\partial \tilde{F}}{\partial y_1} = X_1 T'_{1y} + A \cdot \begin{bmatrix} \frac{\partial x_{121}}{\partial y_1} \\ \frac{\partial x_{122}}{\partial y_1} \\ \frac{\partial x_{341}}{\partial y_1} \\ \frac{\partial x_{342}}{\partial y_1} \end{bmatrix} \quad (22) \quad \frac{\partial \tilde{F}}{\partial y_2} = X_2 T'_{2y} + A \cdot \begin{bmatrix} \frac{\partial x_{121}}{\partial y_2} \\ \frac{\partial x_{122}}{\partial y_2} \\ \frac{\partial x_{341}}{\partial y_2} \\ \frac{\partial x_{342}}{\partial y_2} \end{bmatrix} \quad (23)$$

where,

$$A = \begin{bmatrix} v(T_2 + X_2 T'_2 + T_5 + X_5 T'_5 + T_7 + X_7 T'_7 + T_{10} + X_{10} T'_{10}) + c_{121} \\ v(T_8 + X_8 T'_8) \\ v(T_1 + X_1 T'_1 + T_3 + X_3 T'_3 + T_4 + X_4 T'_4 + T_5 + X_5 T'_5 + T_9 + X_9 T'_9) + c_{341} \\ v(T_6 + X_6 T'_6) \end{bmatrix}^t \quad (24)$$

Then, we can compute the gradient of the evaluation function  $F$ , from (19), (20), (22) and (23).

The optimality conditions for the superior problem are as follows.

$$\begin{aligned} \frac{\partial \bar{F}}{\partial y_k} &\leq 0 && (\text{if } y_k = \bar{y}_k) \\ &= 0 && (\text{if } 0 < y_k < \bar{y}_k) \\ &\geq 0 && (\text{if } y_k = 0) \end{aligned} \quad (k=1,2) \quad (25)$$

If  $y_k < \bar{y}_k$  and  $\partial \bar{F} / \partial y_k < 0$ , then total travel cost may be decreased by expand of the inflow link capacity. On the other hand, if  $y_k > 0$ , and  $\partial \bar{F} / \partial y_k > 0$ , then we can not reduce total travel cost by increase of the inflow link capacity. In this case, we may take inflow restriction strategy at that interchange, through the restriction of the number of open booths.

#### 4. CONCLUDING REMARKS

An inflow control model on expressway was dealt on optimal traffic share between expressway and roads. It was assumed that any user acts reasonably, so as to minimize his travel cost under given conditions. This is valid, if informations with respect to traffic congestions and travel times are offered to users quickly and accurately.

The model proposed is still conceptional. There are some problems to use practically. One of them is how to calculate in case of large network. The other is how much does the control produce effects on total travel cost. These are the present pending questions.

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