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# Influence of End Constraint in the Testing of Anisotropic Bodies

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One of the most elementary concepts in elasticity theory is that of a uniform state of stress. Producing such a state of stress in the laboratory, however, is not a trivial task. A common experiment in composite mechanics—the tension test of off-angle composites—is discussed in this paper and the influence of end constraint on the uniform stress field is investigated. Analytical and experimental evidence is presented to show the serious effects caused by conventional clamping devices.

## INTRODUCTION

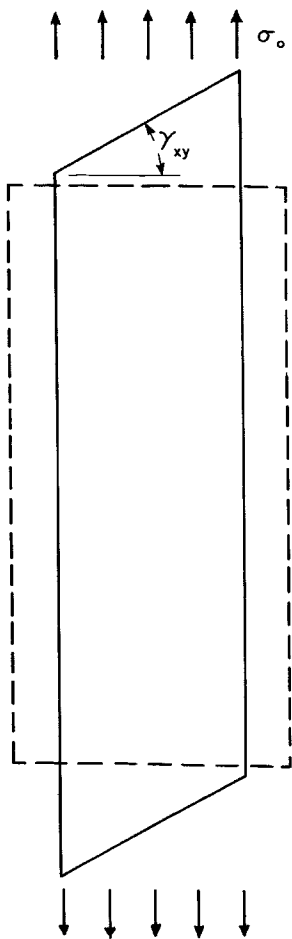
ONE of the more common experiments currently being utilized to characterize composite materials is the tension test of off-angle specimens, i.e., specimens in which the unidirectional filaments are neither parallel nor perpendicular to the direction of the applied tensile force. In order to interpret the data from such experiments, it is assumed that uniform states of stress and strain exist within the gage section. In this paper, we shall present analytical and experimental evidence which indicates that conventional modes of end constraint induce severe perturbations in the stress and strain fields. Owing to the restraint caused by clamping devices, significant shear and bending effects are present.

Consider the off-angle composite specimen (or simply, a homogeneous anisotropic material) under uniform normal stress  $\sigma_0$  as shown in Figure 1. The deformed configuration is indicated by solid lines. Of particular concern is the shear strain

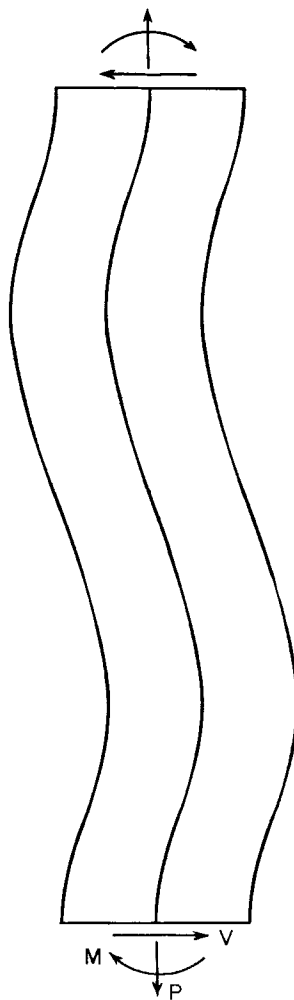
$$\gamma_{xy} = S_{16}\sigma_0 \quad (1)$$

where  $S_{16}$  is the shear coupling compliance, which causes the bar to

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**Figure 1.** Uniform state of stress.



**Figure 2.** Effect of clamped ends.

distort into a parallelogram. Suppose, however, that the ends of the bar are constrained to remain horizontal, a condition which approximates the effect of clamped ends. As shown in Figure 2, the application of constant end displacements induces shearing forces and bending couples at the ends of the bar, which result in the non-uniform deformation shown in the figure.

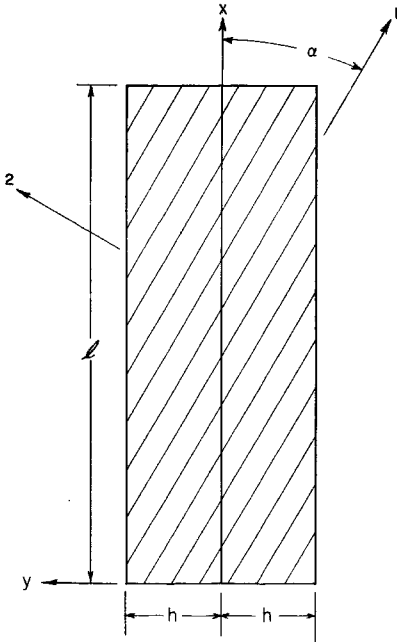
**ANALYTICAL SOLUTION**

In order to gain insight as to the influence of the constraint produced by gripping the ends of a tensile specimen, we shall simulate these displacement boundary conditions and solve the appropriate boundary value problem in the linear theory of elasticity (see Figure 3). Since composite specimens are thin members, we assume that the bar is in a state of plane stress in the  $xy$  plane. We also consider a macroscopically homogeneous material, which is consistent with the nature of the response to be studied. In Figure 3 the axes of material symmetry are at an angle  $\alpha$  with the  $x$  and  $y$  coordinate axes, and  $xy$  is a plane of material symmetry. Hence, the governing equations which must be satisfied in the orthotropic medium [1] are the equations of equilibrium,

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} &= 0 \end{aligned} \tag{2}$$

the strain-displacement relations,

$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{3}$$



**Figure 3.** Specimen geometry.

and the constitutive relations,

$$\begin{aligned}\epsilon_x &= S_{11}\sigma_x + S_{12}\sigma_y + S_{16}\tau_{xy} \\ \epsilon_y &= S_{12}\sigma_x + S_{22}\sigma_y + S_{26}\tau_{xy} \\ \gamma_{xy} &= S_{16}\sigma_x + S_{26}\sigma_y + S_{66}\tau_{xy}\end{aligned}\tag{4}$$

where  $S_{ij}$  are the compliance coefficients with respect to the  $xy$  coordinate system, four of these being independent. Eliminating the strain and displacement components from eqs. (3) and (4) and using (2), we obtain the following stress compatibility equation:

$$(2S_{12} + S_{66}) \frac{\partial^2 \sigma_x}{\partial x^2} + S_{11} \frac{\partial^2 \sigma_x}{\partial y^2} - 2S_{16} \frac{\partial^2 \sigma_x}{\partial x \partial y} - 2S_{26} \frac{\partial^2 \sigma_y}{\partial x \partial y} + S_{22} \frac{\partial^2 \sigma_y}{\partial x^2} = 0\tag{5}$$

Therefore, in the stress formulation, the governing equations are (2) and (5).

If an end surface of the bar in Figure 3, say  $x = 0$ , is supported by a rigid clamp, the boundary conditions on this end can be expressed by

$$v(0, y) = \frac{\partial u}{\partial y}(0, y) = 0\tag{6}$$

while on the surfaces  $y = \pm h$ , the prescribed boundary conditions are given by

$$\sigma_y(x, \pm h) = \tau_{xy}(x, \pm h) = 0\tag{7}$$

since these edges are free surfaces.

The boundary value problem described by the solution of eqs. (2) and (5) which satisfies (7) and the end conditions corresponding to (6) is a very complicated one, and can probably only be solved by numerical methods. Furthermore, photographs shown later illustrate that the displacements at a clamped end do not satisfy eqs. (6) in an actual experiment. Rather, the specimen tends to be pulled out of the clamp due to the Poisson contraction in the thickness direction. In view of this, we shall seek a solution that satisfies eqs. (2), (5), and (7), and replace eqs. (6) by boundary conditions on the center line, i.e.,

$$v(0, 0) = \frac{\partial u}{\partial y}(0, 0) = 0\tag{8}$$

$$v(\ell, 0) = \frac{\partial u}{\partial y}(\ell, 0) = 0$$

and

$$\begin{aligned}u(0, 0) &= 0 \\ u(\ell, 0) &= \epsilon_0 \ell\end{aligned}\tag{9}$$

where  $\epsilon_0$  (center-line strain) is a constant which is directly proportional to the magnitude of the applied axial force.

Since the shearing force in the bar is independent of  $x$ , we assume a solution for  $\tau_{xy}$  of the form

$$\tau_{xy} = f(y) \tag{10}$$

where  $f(y)$  is an arbitrary function of  $y$  alone. Substituting eq. (10) into (2) and integrating, we find that

$$\begin{aligned} \sigma_x &= -xf'(y) + g(y) \\ \sigma_y &= h(x) \end{aligned} \tag{11}$$

where  $h(x)$  and  $g(y)$  are arbitrary functions of the respective variables. Putting eqs. (11) into the compatibility equation (5) and using (7) yields

$$\begin{aligned} f(y) &= C_0(y^2 - h^2) \\ g(y) &= -2 \frac{S_{16}}{S_{11}} C_0 y^2 + C_1 y + C_2 \\ h(x) &= 0 \end{aligned} \tag{12}$$

where  $C_0, C_1, C_2$  are constants. Thus the stress and strain components are given by

$$\begin{aligned} \sigma_x &= -2C_0xy - 2 \frac{S_{16}}{S_{11}} C_0 y^2 + C_1 y + C_2 \\ \sigma_y &= 0 \\ \tau_{xy} &= C_0(y^2 - h^2) \end{aligned} \tag{13}$$

and

$$\begin{aligned} \epsilon_x &= S_{11}(-2C_0xy + C_1 y + C_2) - S_{16}C_0(y^2 + h^2) \\ \epsilon_y &= S_{12}\left(-2C_0xy - 2 \frac{S_{16}}{S_{11}} C_0 y^2 + C_1 y + C_2\right) + S_{26}C_0(y^2 - h^2) \end{aligned} \tag{14}$$

$$\gamma_{xy} = S_{16}\left(-2C_0xy - 2 \frac{S_{16}}{S_{11}} C_0 y^2 + C_1 y + C_2\right) + S_{66}C_0(y^2 - h^2)$$

Integrating the strain-displacement relations (3) after inserting eqs. (14) gives the following displacement functions:

$$\begin{aligned} u &= -S_{16}C_0x(y^2 + h^2) + S_{11}x(C_2 + C_1y - C_0xy) + C_5 \\ &+ (S_{16}C_2 - S_{66}C_0h^2 - C_3)y + \frac{S_{16}C_1y^2}{2} + \frac{C_0y^3}{3} \left(S_{12} + S_{66} - \frac{2S_{16}^2}{S_{11}}\right) \\ v &= S_{12}y\left(\frac{-2S_{16}}{3S_{11}} C_0 y^2 + \frac{C_1y}{2} + C_2 - C_0xy\right) + \frac{S_{26}C_0y}{3} (y^2 - 3h^2) \\ &+ C_4 + C_3x - \frac{S_{11}C_1x^2}{2} + \frac{S_{11}C_0x^3}{3} \end{aligned} \tag{15}$$

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By use of boundary conditions (8) and (9), the various constants are determined as

$$\begin{aligned}
 C_0 &= \frac{6S_{16}\epsilon_0}{6h^2(S_{11}S_{66} - S_{16}^2) + S_{11}^2 \ell^2} \\
 C_1 &= C_0 \ell \\
 C_2 &= \frac{C_0}{6S_{16}} (6S_{66}h^2 + S_{11}\ell^2) \\
 C_3 &= \frac{C_0 S_{11} \ell^2}{6} \\
 C_4 &= C_5 = 0
 \end{aligned}
 \tag{16}$$

which completes the solution. We observe that the stress and strain components on the center line  $y = 0$  assume constant values. In particular, we see that

$$\begin{aligned}
 \sigma_x(x, 0) &= C_2 \\
 \epsilon_x(x, 0) &= S_{11}C_2 - S_{16}C_0h^2
 \end{aligned}
 \tag{17}$$

Suppose that the tension test of an off-angle composite is used as the basis for determining  $E_{11}$ , the composite modulus of elasticity in the  $x$  direction. If the effects noted here are not taken into account, this modulus will be erroneously recorded as  $E_{11}^*$ , where

$$E_{11}^* = \frac{\sigma_x(x, 0)}{\epsilon_x(x, 0)}
 \tag{18}$$

However, eqs. (17) and (18) show that

$$E_{11}^* = \frac{1}{S_{11}} \left( \frac{1}{1 - \eta} \right)
 \tag{19}$$

where

$$E_{11} = \frac{1}{S_{11}}
 \tag{20}$$

and

$$\eta = \frac{6S_{16}^2}{S_{11} \left( 6S_{66} + S_{11} \frac{\ell^2}{h^2} \right)}
 \tag{21}$$

In other words,  $\eta$  is a measure of the error in the observed modulus. The value of  $\eta$  can be quite large for certain values of  $\alpha$  (see Figure 3) in highly anisotropic composites like boron-epoxy and graphite-epoxy, but for materials like glass-epoxy, it is quite small. Using data

and transformation curves presented by Tsai [2] on boron-epoxy composites for a fiber volume fraction of .65,  $\alpha = 30^\circ$ , and  $l/w = 2$ , the value of  $\eta$  is found to be .33, whereas for  $l/w = 6$ ,  $\eta = .07$ ; these represent errors in the observed value of  $E_{11}$  of 50% and 7%, respectively, according to eq. (19). For large values of  $l/w$ , the value of  $\eta$  approaches zero.

The numerical values presented in the previous paragraph are given as an estimate of the error produced by end constraint. We must recall that eq. (19) is based upon an approximate version of the displacement boundary conditions at the clamped ends, *i.e.*, we have assumed restraint at one point (on the center line) at each end. Although subsequent photographs indicate that this assumption is reasonable, it appears that bending effects would be more pronounced if a finite width of the specimen is restrained. It might seem that errors can be reduced by allowing an end clamp to rotate, which is a common practice in conventional testing machines. The major factor, however, is the clamping or gripping *per se*, rather than the orientation of the end fixture, so that rotation of an end has little effect on the strain field. We shall return to these points in the discussion of our experimental results.

We may also express our results in terms of the applied axial force  $P$  as shown in Figure 2, rather than using the parameter  $\epsilon_0$ , since

$$P = t \int_{-h}^h \sigma_x dy = 2th \left( C_2 - \frac{2S_{16}C_0h^2}{3S_{11}} \right) \quad (22)$$

where  $t$  is the thickness of the bar and  $C_0$  and  $C_2$  are given by eqs. (16).

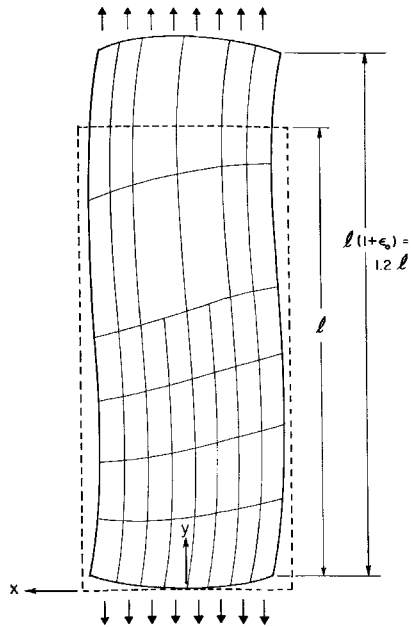
It is interesting to determine the solution of the present problem as  $l$  becomes very large. Consideration of eqs. (13)-(16) as  $l \rightarrow \infty$  yields

$$\begin{aligned} u &\rightarrow \epsilon_0 x + \frac{6S_{16}\epsilon_0 y}{S_{11}} \left( \frac{x}{l} - \frac{x^2}{l^2} \right) \\ v &\rightarrow \frac{S_{12}}{S_{11}} \epsilon_0 y + \frac{S_{16}\epsilon_0 x}{S_{11}} \left( 1 - 3 \frac{x}{l} + 2 \frac{x^2}{l^2} \right) \\ \epsilon_x &\rightarrow \epsilon_0, \quad \epsilon_y \rightarrow \frac{S}{S_{11}} \epsilon_0, \quad \gamma_{xy} \rightarrow \frac{S_{16}}{S_{11}} \epsilon_0 \end{aligned} \quad (23)$$

Theoretically then, the strain components correspond to the uniform state of stress in Figure 1, although the displacement field does not. This paradox is due to the disappearance of certain displacement gradients as  $l \rightarrow \infty$ . Hence, the stress field approaches uniformity with increasing length.



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**Figure 4.** Deformed bar, analytical solution.

Another interesting case arises when  $S_{16} = 0$ . In this case, we see from eqs. (16) that

$$C_0 = C_1 = C_3 = 0 \tag{24}$$

$$C_2 = \frac{\epsilon_0}{S_{11}}$$

and from eqs. (13) and (14), we find that the stress and strain fields are uniform.

The deformed shape of a “tensile” specimen, as predicted by eqs. (15), is drawn to scale in Figure 4. The compliance coefficients used in the calculations were (in.<sup>2</sup>/#):

$$\begin{array}{ll} S_{11} = .00080 & S_{26} = -.000098 \\ S_{12} = -.00043 & S_{22} = .00142 \\ S_{16} = -.00098 & S_{66} = .00204 \end{array} \tag{25}$$

which corresponds to the material discussed in the next section for an angle  $\alpha = 30^\circ$ . In order to clearly illustrate the response characteristics, the longitudinal strain  $\epsilon_0$  is taken as .20 in the figure.

### EXPERIMENTAL RESULTS

In order to demonstrate the response discussed in the previous

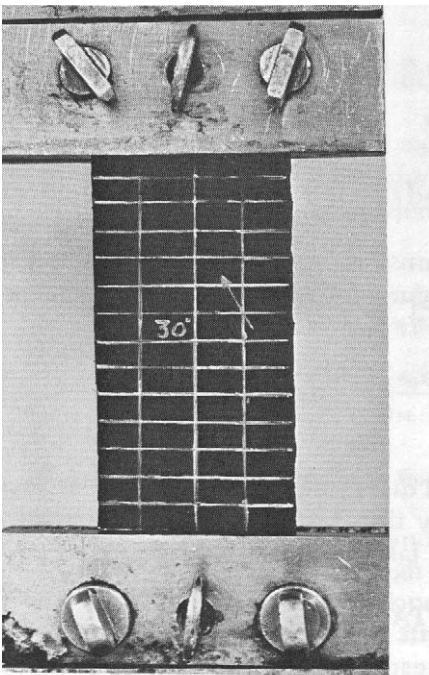
section, it is convenient to utilize a relatively soft material, capable of sustaining deformations which can be detected visually. The material selected was a nylon-reinforced rubber which has the following properties (in.<sup>2</sup>/#):

$$\begin{aligned} s_{11} &= .000088 & s_{22} &= .0013 \\ s_{12} &= -.000044 & s_{66} &= .0036 \end{aligned} \quad (26)$$

where  $s_{ij}$  are the compliances with respect to the axes of material symmetry, with the 1 direction being oriented along the fibers as shown in Figure 3. These coefficients were determined by the method described in [3]. In order to relate the elastic response of this material to more conventional composites, it may be helpful to note the modulus ratios

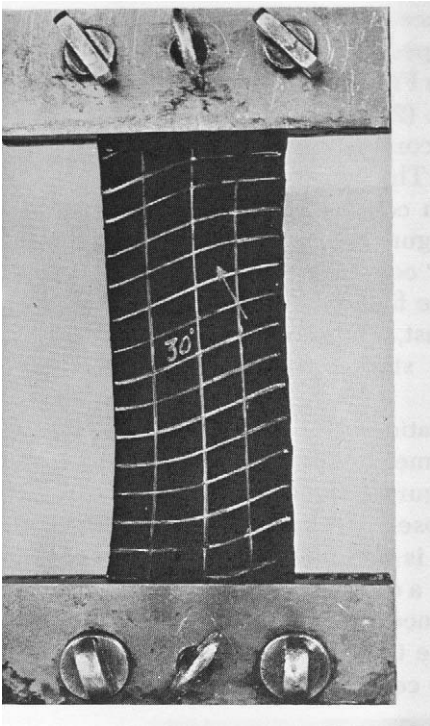
$$\frac{E_L}{E_T} = \frac{s_{22}}{s_{11}} = 15, \quad \frac{E_L}{G_{LT}} = \frac{s_{66}}{s_{11}} = 40 \quad (27)$$

Figure 5 is a photograph of the initial (undeformed) configuration of a specimen with fibers at an angle  $\alpha$  of  $30^\circ$  with the vertical in an Instron testing machine. In Figure 6, a longitudinal strain of .20 is applied to the specimen, which is supported by rigid clamps (no rotation). The character of the response is quite similar to that given

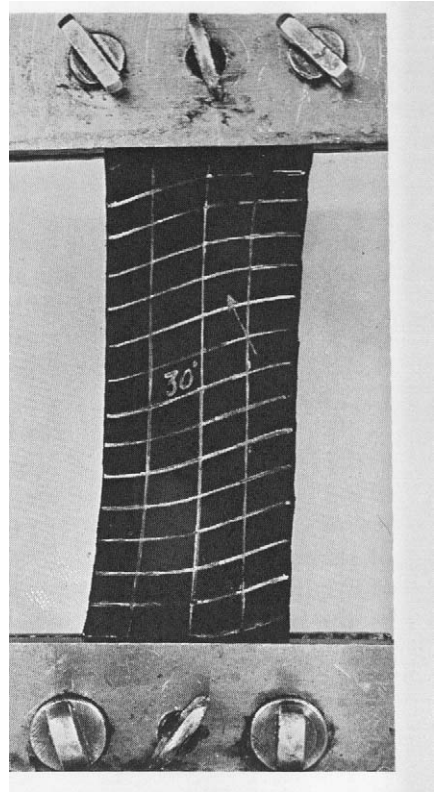


**Figure 5.** Undeformed  $30^\circ$  specimen.

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**Figure 6.** Deformed 30° specimen: rigid clamps.



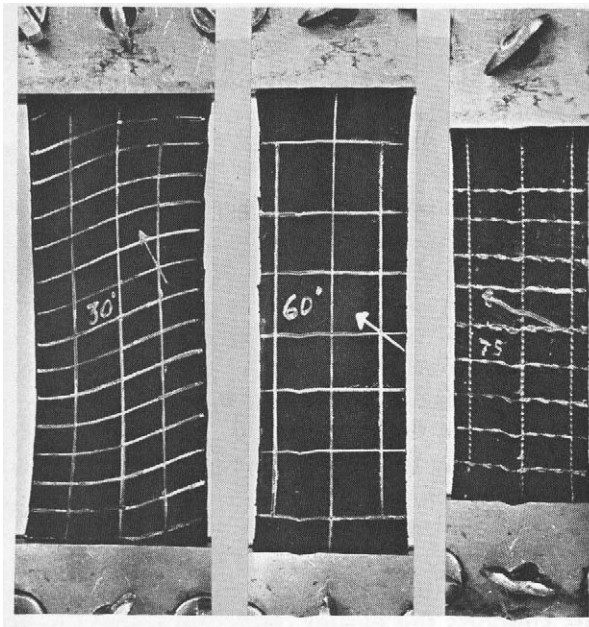
**Figure 7.** Deformed 30° specimen: rotating clamp.

by the analytical solution (Figure 4), but an exact correlation is not possible due to the large strains imposed. Specimens subjected to smaller strains exhibit fair agreement with the analytical solution for a given value of  $\epsilon_0$ , however the analytical solution tends to underestimate the magnitude of the shear strain. One can observe the specimen pulling out from the clamps in the regions of high tensile stress. The applied loads are not shown in the various figures in this section since we do not feel that the load recorded on the dial indicator is accurate owing to the large bending moment and shearing force acting at the clamps.

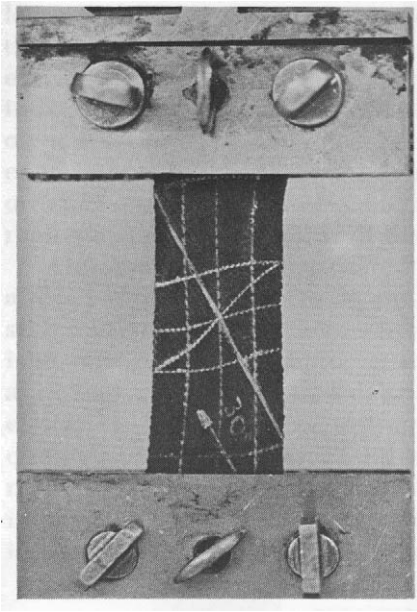
In Figure 7, the experiment is repeated, but in this case the upper clamp is allowed to rotate. Considerable bending is noted again—in fact, the strain distributions in Figures 6 and 7 are practically identical. This supports the earlier observation that the gripping restraint is the dominant factor in disturbing the uniform stress field.

As shown in [3], the compliance  $S_{16}$  vanishes when  $\alpha$  is approximately  $60^\circ$  for this material. A specimen of this configuration was deformed with  $\epsilon_0 = .20$  as shown in Figure 8(b). The resulting uniform state of strain, as predicted in eqs. (24) and (14), is quite evident. Of course, the constraint of lateral contraction in the grips gives the specimen a dogbone appearance. This constraint is accompanied by self-equilibrating lateral forces, in contrast to the type of constraint under discussion in this paper. Figure 8(c) shows the reversal of the direction of bending as the shear coupling compliance  $S_{16}$  changes sign, as predicted by eqs. (15). The fibers in Figure 8(c) are at  $75^\circ$  to the applied axial force. For contrast, Figure 6 is repeated as Figure 8(a). The corresponding uniform states of strain are depicted in [3], Figure 9.

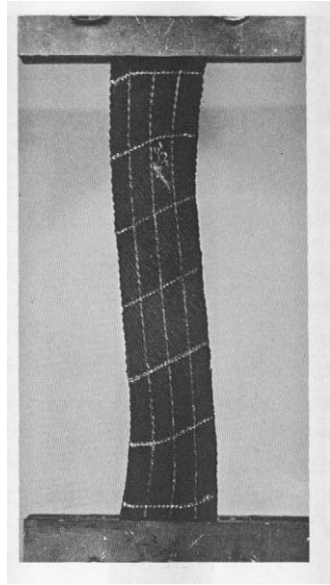
The effect of length to width ratio is illustrated in Figures 9, 10, and 11. The deformations of specimens having length to width ratios of 2, 4, and 6 are shown in these figures. The strain field in the central region of the bar in Figure 11 is closely approaching the uniform state of strain given by eqs. (23). This is verified further by comparison of Figures 11 and 12. In Figure 12, a uniform state of stress is induced by the method discussed in reference [3]. In Figures 11 and 12, the longitudinal strain  $\epsilon_0$  has the value 0.20. We can see that the strain fields are nearly equivalent in the central region of the bar.



**Figure 8** (a).  $30^\circ$  specimen. (b).  $60^\circ$  specimen. (c).  $75^\circ$  specimen.



**Figure 9.** 30° specimen,  $\frac{\ell}{w} = 2$ .



**Figure 10.** 30° specimen,  $\frac{\ell}{w} = 4$ .

## DISCUSSION AND CONCLUSIONS

We have shown that bending effects resulting from end constraint can produce serious consequences in the testing of off-angle composites. The analytical model presented includes the description of the various important response characteristics, and gives an approximate solution for the response. A more exact solution would entail a study of the displacement boundary conditions in an effort to simulate them more closely.

Although we have pointed out some difficulties which arise in performing and interpreting this experiment, we have not suggested an acceptable test method. One possibility is to study the length effect observed earlier in some depth to determine any theoretical or practical limitations on the generality of eqs. (23). As discussed earlier, it is quite possible that eq. (19) underestimates the error caused by gripping owing to the approximation employed to represent the displacement boundary conditions.

An alternative to the off-angle tension test for elastic moduli is the use of angle-ply ( $\pm\alpha$ ) composites. However, the shear-coupling factors cannot be directly observed in such an experiment, and



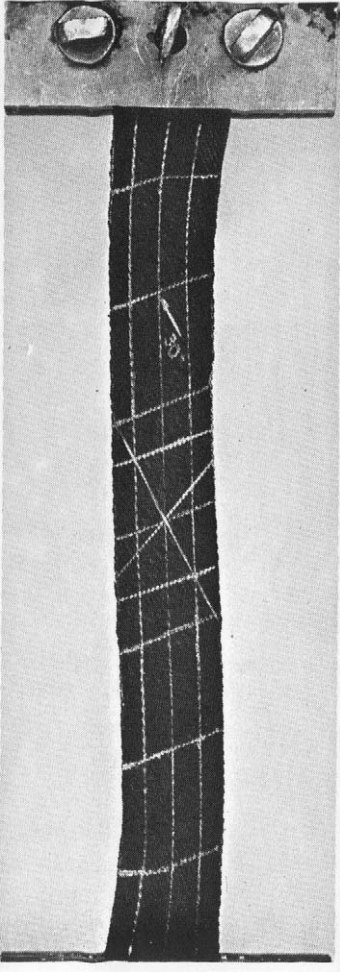


Figure 11. 30° specimen,  $\frac{l}{w} = 6$ .

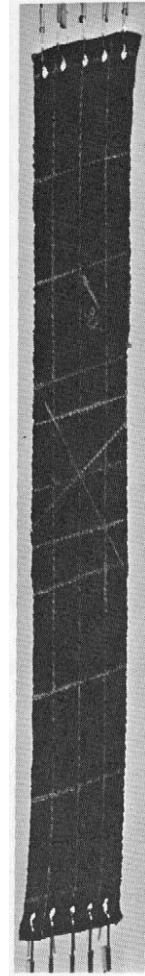


Figure 12. 30° specimen, uniform state of stress.

boundary layer effects near the free edges preclude an exact analytical description of the experiment.

An apparatus similar to that discussed by Halpin and Pagano [3] can be utilized to introduce a uniform state of stress in an off-angle composite. Although the design of this apparatus may need modification in the testing of hard materials such as structural composites, it appears to be the most promising method to induce uniform stress. This approach is obviously limited to the determination of elastic moduli, *i.e.*, it cannot be expected to yield reliable strength data. The

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latter problem can conceivably be solved by suitable modification of the ends of the test specimen. Considerable caution must be exercised in this regard since building up the ends of a test piece will induce similar effects to those observed in this paper, but probably on a smaller scale. It must be emphasized, however, that careful experimental verification of any such scheme must be undertaken in order to ensure the existence of a uniform state of strain, at least in a region away from the ends of the bar.

Although we have restricted our attention to a specific material and a particular experiment, the nature of the influence of end constraint is considerably more general. For example, any material which is macroscopically anisotropic, such as metals or polymers which are anisotropic because of their fabrication processes, are subject to these effects. In compression testing of anisotropic bodies, the consequences of end constraint tend to be more serious since these specimens are relatively short. Similar arguments can be advanced to illustrate the influence of gripping in torsion experiments as well as in the testing of plates and shells.

### NOMENCLATURE

$x, y$	= Cartesian coordinates.
$\sigma_x, \sigma_y, \tau_{xy}$	= Stress components.
$\epsilon_x, \epsilon_y$	= Normal strain components.
$\gamma_{xy}$	= Engineering shear strain.
$u, v$	= Displacement components in $x$ and $y$ directions.
$S_{11} \dots S_{66}$	= Compliance coefficients with respect to $x, y$ axes.
$s_{11} \dots s_{66}$	= Compliance coefficients with respect to material symmetry axes.
$\epsilon_0$	= Applied longitudinal strain.
$f(y), g(y), h(x)$	= Arbitrary functions.
$C_0 \dots C_5$	= Constants.
$w = 2h, l, t$	= Dimensions of specimen.
$E_{11}^{\circ}$	= Observed modulus of elasticity.
$\eta$	= Factor reflecting error in observed modulus of elasticity.
$E_L, E_T$	= Longitudinal and transverse moduli of elasticity.
$G_{LT}$	= Longitudinal-transverse shear modulus.
$\alpha$	= Angle of rotation.

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