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Information Acquisition for Capacity Planning via Pricing and Advance Selling: When to Stop and Act?

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This paper investigates a capacity planning strategy that collects commitments to purchase before the capacity decision and uses the acquired advance sales information to decide on the capacity. In particular, we study a profit-maximization model in which a manufacturer collects advance sales information periodically prior to the regular sales season for a capacity decision. Customer demand is stochastic and price sensitive. Once the capacity is set, the manufacturer produces and satisfies customer demand (to the extent possible) from the installed capacity during the regular sales period. We study scenarios in which the advance sales and regular sales season prices are set exogenously and optimally. For both scenarios, we establish the optimality of a *control band* or a *threshold* policy that determines when to stop acquiring advance sales information and how much capacity to build. We show that advance selling can improve the manufacturer's profit significantly. We generate insights into how operating conditions (such as the capacity building cost) and market characteristics (such as demand variability) affect the value of information acquired through advance selling. From this analysis, we identify the conditions under which advance selling for capacity planning is most valuable. Finally, we study the joint benefits of acquiring information for capacity planning through advance selling *and* revenue management of installed capacity through dynamic pricing.

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1. Introduction

Capacity investments, such as construction of semiconductor fabrication or power plants, share four important characteristics. First, capacity investments are often very expensive and irreversible. The cost of unused capacity can be only partially recovered (if at all) by salvaging at a minimum value. Second, demand for the capacity is uncertain at the time of the capacity decision. Demand uncertainty is often quite significant because the capacity decision is taken well in advance of the sales season. Third, adjusting capacity during sales and production is often difficult or impossible. Hence, the amount of capacity built defines how much can be produced and sold during the sales season. Fourth, management often has some leeway about the timing of when to build capacity. The latest time to install capacity is the beginning of the sales period minus the construction leadtime necessary to build the capacity. Many strategic investments share these common characteristics (see, for example, Dixit and Pindyck 1994).

In an environment driven by demand uncertainty, a "build it and they will come" strategy requires a capacity provider to bear considerable risk in making the expensive capacity investment. In this paper, we explore a different strategy. We allow some customers to commit to buying prior to the capacity decision and the provider to build the capacity later; i.e., "let them come and build it later." In this paper, we refer to the capacity provider as the manufacturer because she also produces and delivers the final product.

Advance selling is a strategy that can help the manufacturer enhance her understanding of the market potential for her product and reduce demand uncertainty. By offering the product at a time preceding the regular sales period, the manufacturer can capture some of the market demand in advance and thereby moderates overall demand uncertainty. In addition, the amount of advance purchase commitments provides the manufacturer with information on the market demand potential of the product and enables her to plan capacity according to more accurate demand information. She also starts collecting revenue earlier. Advance selling strategy may be attractive to some customers as well. By committing earlier to purchase, customers reserve their products and therefore are not vulnerable to possible capacity shortages. Furthermore, customers may garner discounts for committing early (Neslin et al. 1995).

When the manufacturer postpones the capacity investment to acquire advance sales information, she *may* face an additional trade-off due to capacity building costs. There are always leadtimes associated with obtaining sites, equipment and other resources when the manufacturer builds new capacity. When these investments are delayed, the manufacturer may face a tighter deadline for building capacity, which would result in a nondecreasing capacity cost structure due to various expediting costs (e.g., second shift premium, premium transportation). Alternatively, if expediting is not possible (i.e., the leadtime is constant), then the manufacturer runs the risk of completing the capacity installment after the start of the sales season, in which case she may lose some potential revenue. This can also be cast as a nondecreasing capacity cost structure.

Advance selling strategy is commonly used in the service sector. The prime example is the airline industry. However, as Xie and Shugan (2001) point out, advance selling does not require industry-specific characteristics; it can be used to enhance profits provided that customers can purchase the service at a time preceding consumption, which is possible for most services. New technology such as electronic tickets, on-line prepayments and smart cards enable more service providers to experiment with advance selling. Customers can buy advance tickets to concerts, sports events, or festivals. They can book hotel rooms, buy railroad tickets, and acquire some other services in advance. The advance sales information is often used to plan and manage capacity. For example, conference organizers offer early registration at discounted prices and use this information in planning the hotel rooms to reserve, meeting rooms to book, as well as catering.

Advance selling is also used in the manufacturing sector, although it is not as prevalent as in the service sector. This may be due to the presence of organizational silos which often decouples demand management and capacity planning. Recent advances in information technology and management practices, however, are enabling firms to coordinate actions across functional areas, such as marketing and operations. Some manufacturers in high technology and apparel industries started to use advance selling strategy to better plan for capacity and production. For example, Ericsson, a telecommunications equipment manufacturer, recently explored this strategy to improve its long-range forecasting for planning the capacity of a new factory for its third-generation (3G) wireless network equipment. Accordingly, the company announced the date for the launch of its 3G stations. Before securing the capacity, Ericsson "presold" 3G wireless base stations to some of its customers such as NTTdocomo, the regional cellular phone operator in Japan.[1] Apparel manufacturers have also been using early sales information to decide on production. To do so, this industry developed several initiatives to reduce the cost of excess inventory and shortage. Fisher and Raman (1996) discuss how apparel manufacturers, such as Sport Obermeyer, commit part of production capacity to certain SKUs after observing some initial demand. Zara uses early market sales information to preposition and decide how much sewing capacity to reserve (Fraiman and Singh 2002). Advance selling is also used in construction projects. Commercial developers sell some units at an advance sales price before construction begins. Revenue from advance sales is used in part to finance the construction. This information can also be used to decide whether to purchase additional land and build more units. Another example is from e-tailers (such as Amazon.com) that collect pre-orders for certain items before their market introduction. As discussed at the outset, the trade-off between delaying a decision and proactively acquiring information (such as demand) versus deciding early (such as building capacity up front) is inherent to many strategic investment decisions. The present paper takes a step in the direction of providing an understanding of this trade-off in capacity planning.

In summary, our primary objective is to determine the effectiveness of advance selling in conjunction with a "let them come and build it later" capacity strategy. In particular, we study a manufacturer who decides on the level of capacity to build for a product that faces price-sensitive stochastic demand. The manufacturer has one opportunity to invest in capacity before the sales season starts. The amount of capacity built defines the upper bound on how much the manufacturer can produce and sell during the sales season. By delaying the capacity decision and offering advance sales, the manufacturer can mitigate the demand uncertainty and obtain additional information about the market potential. We also consider the manufacturer's pricing problem. We study the case in which the manufacturer determines advance sales and sales season prices optimally, as well as the case in which these prices are exogenously specified. For each scenario, we establish the optimality of *control-band* policies that prescribe the optimal time to stop collecting advance sales information. Under this policy, the manufacturer monitors the prevailing advance sales information including the total number of commitments to date, and if this quantity falls within the control band, it is optimal to stop advance selling and to decide on the capacity. Otherwise, the manufacturer continues advance selling. Through an extensive numerical study, we compare the optimal expected profits with and without advance selling, and we show that an advance selling strategy can increase expected profit significantly. We also quantify the value of knowing exactly when to stop advance selling. Our study generates managerial insights on how the value of information acquired through advance selling is influenced by operating and market characteristics, by quantifying the profit-impact of such characteristics. Consequently, we identify the conditions under which advance selling offers the most value (e.g., when demand uncertainty or cost of building capacity is high). Finally, we study the joint benefits of acquiring information for capacity planning through advance selling *and* revenue management of installed capacity through dynamic pricing. Modeling and studying this scenario bridges the revenue and capacity management literatures.

2. Literature Review

There is an extensive body of research dealing with capacity management in different environments. When product lifecycles and sales seasons are long and production leadtimes are short, adjustment of the capacity level over time could be possible. Examples for such products are cement and steel. For such products, Manne (1967) establishes optimal expansion policies for a market with stochastic growth patterns. Luss (1982) provides a comprehensive review of joint capacity expansion and production management problems. This literature assumes steady growth in demand. More recently, Angelus and Porteus (2002) characterize an optimal policy for simultaneous management of capacity and production, whereby stochastic demand can reduce over time. Bradley and Glynn (2002) study a continuous version of a simultaneous capacity and inventory decision problem. Lovejoy and Li (2002) study capacity decisions for hospital operating rooms that consider the objectives of patients, surgical staff and hospital administration. Van Mieghem (2003) provides an extensive review of the recent capacity literature, in which demand is always modeled as an exogenous process. As pointed out by Van Mieghem (2003), the capacity literature has not paid much attention to the more realistic demand models which are partially exogenous and partially endogenous. The present paper takes a step in this direction.

When the capacity cannot be adjusted during the sales horizon and when the capacity is perishable after the sales horizon, the firm can increase expected profit through dynamic pricing and revenue management. Bitran and Caldentey (2003) and Elmaghraby and Keskinocak (2003) provide comprehensive reviews of this topic. The textbook treatment of this literature can be found in Talluri and van Ryzin (2004) and Phillips (2005). This literature takes capacity as given and maximizes revenue by adjusting prices over time based on the level of left-over capacity and price-sensitive customer arrival rates (e.g., Gallego and van Ryzin 1994, Feng and Gallego 1995, Bitran and Mondschein 1997). Carr and Lovejoy (2000) provide an alternative method in which the capacitated firm selects a portfolio of demand distributions from a set of potential customer segments. All these authors also point out that frequent price adjustments could be costly and hence should be exercised with care. In the manufacturing sector, where customer relationships are important, selling capacity and the product at different prices is generally not considered to be a relationship-preserving strategy. In our basic model, a finite number of price adjustments are made prior to the capacity decision, but once advance selling stops and the capacity is set, the sales price remains uniform for all remaining customers. In this way, we retain focus on the capacity planning problem, which constitutes the core of our study. Nevertheless, we also consider an extension that allows installed capacity to be sold via dynamic pricing. This extension bridges capacity planning with revenue management through an advance selling strategy.

A recent line of research in dynamic pricing investigates the effect of demand parameter learning (Braden and Oren 1994; Aviv and Pazgal 2005a, b; Araman and Caldentey 2009 and the references therein). This line of research focuses on the pricing problem for which the perishable capacity at the beginning of a *regular* sales season is given. These authors extend the pricing model of Gallego and van Ryzin (1994) in various interesting directions, in particular, to account for learning demand parameters. They provide methods to compute optimal (and close-to-optimal) prices for left-over inventory until all remaining capacity is sold. Araman and Caldentey (2009) introduce the option of stopping sales before consuming all capacity if the value function is lower than an exogenously specified profit level (which can be interpreted as the reservation profit obtained from selling an alternative product). Unlike these papers, we do not assume that capacity level at the beginning of a regular period is given. Instead, we focus on pricing schedules that collect commitments to purchase earlier, i.e., during *advance* sales periods, and we use this information to decide on the capacity at the beginning of the regular sales season. These studies explicitly examine the role of demand parameter learning in a Bayesian context. In our concluding remarks, we discuss how our proposed model can also account for such a learning process.

Additionally, there is a large body of research that studies how capacity can be used effectively through managing production and inventory. In this literature, capacity is an upper bound on production quantity and it is exogenously specified (see Aviv and Federgruen 1997, Özer and Wei 2004 and references therein). Within this line of research, there is a growing literature that studies advance demand information and its use in capacity constrained production and inventory systems (see Gallego and Özer 2001, Özer and Wei 2004, Hu et al. 2004, Wang and Toktay 2008, Gayon et al. 2009). These papers establish optimal production policies for various environments and show, for example, that advance demand information is a substitute for capacity. This literature takes advance demand information as exogenous to the system. For an inventory system with ample capacity, Weng and Parlar (1999) and Chen (2001) explore the cost and benefit of price incentives to induce time-and-price-sensitive customers to place advance orders. Prasad et al. (2010) consider a newsvendor retailer serving heterogenous customers with uncertain future valuations of a product, and they explore the benefits of advance selling at discounted prices.

Another group of researchers studies how advance purchase or selling affects the allocation of inventory risk within a supply chain (Dana 1998, Cachon 2004, Netessine and Rudi 2006, Taylor 2006, Özer et al. 2007, Dong and Zhu 2007 and references therein). Although advance selling strategy forms the common ground, the primary focus of this stream is different from ours. Modeling inventory decisions within a supply chain under exogenous stochastic demand, these papers focus on allocation of inventory risk, division of profits, and incentives and coordination in a supply chain. In contrast, we investigate how pricing and advance selling can be used to manage demand for the purpose of capacity planning.

The rest of the paper is organized as follows. In §3, we describe the basic elements of our model. In §§4 and 5, we establish the optimality of control-band policies for offering advance sales for exogenous and optimal pricing strategies, respectively. In §6, we present a numerical study and generate managerial insights regarding the value of an advance selling strategy for capacity planning. In §7, we provide an extension where installed capacity can be sold via dynamic pricing. In §8, we conclude and suggest directions for future research. Proofs of all propositions are deferred to the appendix. Some extensions are also discussed in the electronic companion, which is available as part of the online version at http://or.journal.informs.org/.

3. The Model

3.1. Preliminaries

We consider a planning horizon with T time units that consists of the regular sales season and the prior capacity planning period. We do not make any assumptions about the length of each period. The *regular sales season* is indexed as T. Periods $1, \ldots, T-1$ represent possible *advance sales periods*, during which the firm can collect advance commitments from customers before the capacity decision is made and before the start of the regular season, hence the term *advance selling*. The manufacturer faces random, price-sensitive stochastic demand in each period.

The manufacturer has one opportunity to invest in capacity before the regular sales season starts. When the manufacturer stops offering advance sales in period t, she builds capacity at unit cost c_t, based on the acquired advance sales information. The remaining demand is served during the regular sales season. The installed capacity defines the upper bound on how much the manufacturer can produce and sell during the sales season. If the manufacturer under-invests in capacity, she loses potential sales revenue. If she over-invests in capacity, she incurs a unit cost c_u for unused capacity at the end of the sales season. The unit production cost is denoted as c_p.

Let $\mathscr{L}$ denote the nominal leadtime for capacity construction. By expediting the building process, the manufacturer can reduce this leadtime to $\underline{\mathscr{L}}$. Consequently, the latest time for the manufacturer to decide on the capacity level to build is the beginning of period $T - \underline{\mathscr{L}}$. For expositional clarity, we assume $\underline{\mathscr{L}} = 0$, which means that the manufacturer can postpone the capacity decision until the beginning of the sales season T. A positive $\underline{\mathscr{L}}$ can easily be incorporated into the model without changing the nature of the results. As noted earlier, expediting the building of capacity typically results in additional costs, implying that the capacity costs $\{c_t\}$ should be nondecreasing. Although this is plausible, we do *not* posit such an assumption as it is not required in our analysis.

Note that by delaying the capacity investment decision, the manufacturer can acquire demand information through advance selling and use this information for better capacity planning. She also collects revenue earlier. Hence, she may earn interest on advance sales. However, the manufacturer may incur additional costs if this delay results in higher construction costs. The revenue collected later is also discounted. Furthermore, depending on the advance sales prices, she also runs the risk of selling capacity at a lower profit margin. In the presence of such multiple tradeoffs, the manufacturer needs to address two key questions: (i) how much advance sales information is sufficient to decide on the capacity? and (ii) given this information, what is the optimal capacity level? To answer these questions, we develop a dynamic programming formulation of the manufacturer's problem and determine the optimal time for ending advance selling and building capacity.

We also consider the manufacturer's pricing decision. In particular, we study two fundamental pricing strategies that differ in the degree of pricing control the manufacturer can exert during the advance sales and the regular sales periods. First, we study the *exogenous pricing* scenario in which the manufacturer has a predetermined sequence of prices for the advance sales periods as well as the regular selling season. These prices may represent a *mark-up* or a *markdown* structure (see, for example, Feng and Gallego 1995 or Bitran and Mondschein 1997). This prevalent case is modeled by taking an arbitrary sequence of prices $(p_1, \ldots, p_T)$ as given. Next, we study the *optimal pricing* scenario in which the manufacturer determines the advance and regular sales prices optimally in addition to the capacity decision.

For each of the pricing strategies, the manufacturer's decision process is as follows: At the beginning of each period $t < T$, the manufacturer first observes the prevailing advance sales information and the total revenue from advance sales. Based on this information, she decides to either (i) stop advance selling and build capacity, or (ii) delay the capacity investment for one period and continue advance selling at an optimally set price (resp., or an exogenously set price depending on the pricing policy we address). If she stops, she determines the optimal level of capacity based on the advance sales information and the remaining uncertain demand in the market, and she sets the regular sales price (resp., or takes this price as given). At the beginning of regular sales season, i.e., $t = T$, if the capacity investment decision has not already been taken, the manufacturer builds the capacity. As before, she determines the optimal level of capacity and sets the regular sales price (resp., or takes it as given). Next, we describe the demand model, the updating mechanism, and the manufacturer's expected profit.

3.2. Demand Model

We model market demand using an iso-elastic, price-sensitive aggregate demand model with a multiplicative form of uncertainty. Demand in each period depends on the uncertain potential market size ξ_t, the price p_t charged in that period, the cumulative commitments q_t, and relevant historical information[2] $\bar{\mu}_t$ available at the beginning of period t. The pair $(q_t, \bar{\mu}_t)$ defines the *advance sales information* available at the beginning of period t. The actual demand in period t has the form

$$D_t(p_t \mid q_t, \bar{\mu}_t) = f_t(q_t, \bar{\mu}_t)\xi_t p_t^{-b}.$$

The price elasticity b is assumed to be the same across periods for ease of exposition. The potential market size ξ_t in each period is uncertain. They are independent random variables that have increasing failure rates (IFR) with support on $[0, \infty)$.

The nonstationary function $f_t(q_t, \bar{\mu}_t) \geqslant 0$ models the information that the commitments q_t, in relation to the history $\bar{\mu}_t$, provide about future demand. It captures the predictive value of commitments. We call this the *market signal* imparted by the advance sales information regarding the demand potential of the product. Based on the level of this signal, future demand is scaled up or down. For $t = 1$, $q_1 \equiv 0$, $\bar{\mu}_1 \equiv \varnothing$ and hence $f_1(q_1, \bar{\mu}_1) \equiv 1$. We assume that $f_t(q_t, \bar{\mu}_t)$ is linear increasing in q_t for any given $\bar{\mu}_t$ and for all $t > 1$.[3] Based on the information available at time zero, $\mathrm{E}[f_t(q_t, \bar{\mu}_t)] = 1$ for any selection of past prices $\bar{p}_t = (p_1, \ldots, p_{t-1})$ and for all $t > 1$.[4] This property implies that at time zero, before the manufacturer starts to gather advance sales information, she expects demand in any period t to be $\mathrm{E}[\xi_t]p_t^{-b}$, which is independent of prices charged in periods $s < t$. This property ensures that the manufacturer cannot use prices to artificially increase the potential market size for the product. When the manufacturer engages in advance selling, however, the acquired information can have predictive value regarding current and future demand. Depending on the demand realization, the next period's market signal $f_{t+1}(q_{t+1}, \bar{\mu}_{t+1})$ can be lower or higher than or equal to $f_t(q_t, \bar{\mu}_t)$. We do not impose any assumption on its evolution.[5] Our framework and structural results are robust to various forms of market signal function and evolution. This flexibility enables us to model a variety of scenarios as discussed later. In §§6 and 8, we provide specific market signal functions, including those arising from Bayesian learning models, and we show that they satisfy the above assumptions. Hence, in the rest of the paper, we do not restrict ourselves to a specific form but instead study the problem given the above general functional form.

The evolution of demand is as follows. At the beginning of period t, if the manufacturer decides to offer advance sales, the uncertainty ξ_t is realized as ϵ_t, and accordingly the actual demand $d_t = f_t(q_t, \bar{\mu}_t)\epsilon_t p_t^{-b}$ is observed. The cumulative commitments and the history are updated as $q_{t+1} = q_t + d_t$, and $\bar{\mu}_{t+1} = \phi(\bar{\mu}_t, p_t, d_t)$ for some function $\phi(\cdot)$.[6] If the manufacturer decides to stop offering advance sales, the remaining customers are served during the regular sales season. This remaining demand is a function of the current market signal $f_t(q_t, \bar{\mu}_t)$, the remaining potential market size $\chi_t \equiv \sum_{j=t}^{T} \xi_j$, the price p charged in the selling season, and is given by $X_t(p \mid q_t, \bar{\mu}_t) \equiv f_t(q_t, \bar{\mu}_t)\chi_t p^{-b}$. Since IFR property is closed under convolutions (Barlow and Proschan 1975), χ_t is also IFR. Notice also that χ_t is stochastically decreasing in t. Note that the model accounts for a reduction of future demand due to a longer advance sales period. (In other words, advance sales cannibalizes some portion of future demand.)

3.3. The Manufacturer's Expected Profit

When the manufacturer continues advance selling in period t, she collects the revenue $p_t d_t$. Consider an arbitrary period $t \in \{1, \ldots, T\}$, and suppose that at the beginning of period t the manufacturer has decided to stop advance selling and invest in capacity. The manufacturer already has q_t committed customers at some past prices. At the beginning of period t, the total revenue obtained from advance selling, i.e., $\sum_{k=1}^{t-1} p_k d_k$ is deterministic. The manufacturer is required to serve the committed customers, so she would set the capacity level Q_t above q_t to also meet the remaining demand $X_t(p \mid q_t, \bar{\mu}_t)$ she will face during the selling season at price p. For this reason, it is convenient to write $Q_t = q_t + S_t$, where $S_t \geqslant 0$ denotes the *surplus capacity*. The manufacturer's expected (undiscounted) profit at the time when she stops advance selling and invests in capacity for a given q_t and $\bar{\mu}_t$ is

$$\Pi_t(p, S_t \mid q_t, \bar{\mu}_t) = \sum_{k=1}^{t-1} p_k d_k + \pi_t(p, S_t \mid q_t, \bar{\mu}_t) - (c_p + c_t)q_t, \quad \text{where} \tag{1}$$

$$\pi_t(p, S_t \mid q_t, \bar{\mu}_t) = (p - c_p)\mathrm{E}[\min\{X_t(p \mid q_t, \bar{\mu}_t), S_t\}] - c_t S_t - c_u \mathrm{E}[S_t - X_t(p \mid q_t, \bar{\mu}_t)]^+. \tag{2}$$

Note that $x^+ \equiv \max\{0, x\}$, and the expectation is taken at time t with respect to the remaining uncertain market demand $X_t(p \mid q_t, \bar{\mu}_t)$. Note that maximizing (1) is equivalent to maximizing (2) when the optimization is over the surplus capacity. Next, we analyze the manufacturer's problem of acquiring information through advance selling and pricing for capacity planning. We start with the exogenous pricing case followed by the case where prices are determined optimally.

4. Exogenous Prices

Consider an arbitrary sequence of prices $\mathcal{P} = (p_1, p_2, \ldots, p_T)$, where prices p_1 through p_{T-1} are for advance selling periods and p_T is the price for the regular sales season. At the beginning of period t, the manufacturer observes the advance sales information $(q_t, \bar{\mu}_t)$ and decides whether or not to continue advance sales. If the manufacturer decides to stop advance sales, she determines the optimal surplus capacity S_t^* to build. We assume $p_T \geqslant c_p + c_t$ for $t \leqslant T$. This assumption ensures that the manufacturer makes positive profit from customers who buy during the regular sales season. Otherwise, there is no reason to build capacity more than the total commitments q_t.[7] The manufacturer determines S_t^* by maximizing the profit function in Equation (2), which yields

$$S_t^* \equiv \min\left\{S \mid P(X_t(p_T \mid q_t, \bar{\mu}_t) > S) = \frac{c_t + c_u}{p_T - c_p + c_u}\right\}.$$

The resulting optimal expected (undiscounted) net profit from the remaining customers is then

$$\pi_t^*(q_t, \bar{\mu}_t) = \pi_t(p_T, S_t^* \mid q_t, \bar{\mu}_t).$$

By letting $s_t^* = S_t^*/f_t(q_t, \bar{\mu}_t)$, we can rewrite $\pi_t^*(q_t, \bar{\mu}_t)$ as $\pi_t^*(q_t, \bar{\mu}_t) = f_t(q_t, \bar{\mu}_t)\Gamma_t^*$, where

$$\Gamma_t^* = (p_T - c_p)\,\mathrm{E}[\min\{\chi_t p_T^{-b}, s_t^*\}] - c_t s_t^* - c_u\,\mathrm{E}[s_t^* - \chi_t p_T^{-b}]^+. \quad (3)$$

The manufacturer's optimal capacity level is $Q_t^* = q_t + S_t^*$ and optimal total expected profit is

$$\Pi_t^*(q_t, \bar{\mu}_t) = \Pi_t(p_T, S_t^* \mid q_t, \bar{\mu}_t) = \sum_{k=1}^{t-1} p_k d_k + \pi_t^*(q_t, \bar{\mu}_t) - (c_p + c_t)q_t.$$

Next we formulate a dynamic program to determine the optimal time for the manufacturer to stop acquiring advance sales information. The state space is given by the cumulative commitments q_t and the history $\bar{\mu}_t$. We introduce an auxiliary state (N) in the commitment space to indicate that the capacity decision has already been taken; i.e., $q_t = N$ if the capacity decision has been made, and $q_t \neq N$ otherwise. Let $u_t(q_t, \bar{\mu}_t)$ denote the manufacturer's action in period t:

$$u_t(q_t, \bar{\mu}_t) = \begin{cases} u^c, & \text{continue advance sales at price } p_t, \\ u^s, & \text{stop advance sales, set price to } p_T \text{ and capacity level to } Q_t^*. \end{cases} \quad (4)$$

At the end of period t, the cumulative commitments and the history are updated as

$$q_{t+1} = \begin{cases} q_t + d_t, & \text{if } q_t \neq N \text{ and } u_t(q_t, \bar{\mu}_t) = u^c, \\ N, & \text{if } q_t \neq N \text{ and } u_t(q_t, \bar{\mu}_t) = u^s, \text{ or } q_t = N, \end{cases}$$

$$\bar{\mu}_{t+1} = \begin{cases} \phi(\bar{\mu}_t, p_t, d_t), & \text{if } q_t \neq N \text{ and } u_t(q_t, \bar{\mu}_t) = u^c, \\ \bar{\mu}_t, & \text{if } q_t \neq N \text{ and } u_t(q_t, \bar{\mu}_t) = u^s, \text{ or } q_t = N. \end{cases}$$

Let us now introduce $\alpha \in (0, 1]$ as the discount factor.[8] Revenue is realized at the end of the period when customers place advance orders and at the end of the regular sales period when remaining customers purchase. The costs are incurred in the sales period. All results remain valid if revenues from advance orders are collected at the time of delivery and/or capacity costs are incurred when the investment decision is taken. The reward function for $t \in \{1, \ldots, T-1\}$ is given by

$$g_t(q_t, \bar{\mu}_t) = \begin{cases} p_t d_t, & \text{if } q_t \neq N \text{ and } u_t(q_t, \bar{\mu}_t) = u^c, \\ \alpha^{T-t}\{\pi_t^*(q_t, \bar{\mu}_t) - (c_p + c_t)q_t\}, & \text{if } q_t \neq N \text{ and } u_t(q_t, \bar{\mu}_t) = u^s, \\ 0, & \text{otherwise,} \end{cases}$$

and for $t = T$, it is given by

$$g_T(q_T, \bar{\mu}_T) = \begin{cases} \pi_T^*(q_T, \bar{\mu}_T) - (c_p + c_T)q_T, & \text{if } q_T \neq N, \\ 0, & \text{otherwise.} \end{cases}$$

The function $g_t(q_t, \bar{\mu}_t)$ records the revenue from two sources. The first source is from customers who purchase in period t when the manufacturer decides to continue advance selling (i.e., $u_t(q_t, \bar{\mu}_t) = u^c$). This revenue source is from advance purchases. The second source is the expected revenue from satisfying the remaining market demand (as much as possible) minus the cost of building capacity and the cost of production (i.e., $u_t(q_t, \bar{\mu}_t) = u^s$). The manufacturer's problem is to maximize the total expected profit discounted to the first period:

$$\max_{u_1, u_2, \ldots, u_T} \mathrm{E}\left[\sum_{t=1}^{T} \alpha^{t-1} g_t(q_t, \bar{\mu}_t)\right],$$

where the expectation is taken at time zero over $D_t(p_t \mid q_t, \bar{\mu}_t)$ for all t. The solution to this problem is obtained by the following functional equation:

$$J_T(q_T, \bar{\mu}_T) = \begin{cases} \pi_T^*(q_T, \bar{\mu}_T) - (c_p + c_T)q_T, & \text{if } q_T \neq N \text{ then } u_T(q_T, \bar{\mu}_T) = u^s \text{ (forced decision)}, \\ 0, & \text{if } q_T = N, \end{cases} \quad (5)$$

and for $t = 1, \ldots, T-1$, we solve

$$J_t(q_t, \bar{\mu}_t) = \begin{cases} \max\{\alpha^{T-t}[\pi_t^*(q_t, \bar{\mu}_t) - (c_p + c_t)q_t], \mathrm{E}[p_t D_t(p_t \mid q_t, \bar{\mu}_t) + \alpha J_{t+1}(q_{t+1}, \bar{\mu}_{t+1})]\}, & \text{if } q_t \neq N, \\ 0, & \text{if } q_t = N, \end{cases} \quad (6)$$

where the expectation is taken at time t with respect to $D_t(p_t \mid q_t, \bar{\mu}_t)$. When the maximum in Equation (6) is attained by $\alpha^{T-t}[\pi_t^*(q_t, \bar{\mu}_t) - (c_p + c_t)q_t]$, it is optimal to *stop* advance selling, set the regular sales price to p_T, and set the capacity level to Q_t^*; otherwise, it is optimal to *continue* advance selling at price p_t.

For a clearer representation of the above optimal stopping problem, we define $V_t(q_t, \bar{\mu}_t) = J_t(q_t, \bar{\mu}_t) - \alpha^{T-t}[\pi_t^*(q_t, \bar{\mu}_t) - (c_p + c_t)q_t]$ and substitute $V_t(q_t, \bar{\mu}_t) + \alpha^{T-t}[\pi_t^*(q_t, \bar{\mu}_t) - (c_p + c_t)q_t]$ for J_t in Equations (5), (6) and subtract $\alpha^{T-t}[\pi_t^*(q_t, \bar{\mu}_t) - (c_p + c_t)q_t]$ from both sides of these equations to obtain an equivalent formulation. The resulting dynamic program for $t = 1, \ldots, T-1$, is given by

$$V_t(q_t, \bar{\mu}_t) = \max\{0, H_t(q_t, \bar{\mu}_t) + \alpha \mathrm{E}[V_{t+1}(q_{t+1}, \bar{\mu}_{t+1})]\}, \quad (7)$$

where

$$H_t(q_t, \bar{\mu}_t) \equiv \mathrm{E}[\{p_t - \alpha^{T-t}(c_p + c_{t+1})\}D_t(p_t \mid q_t, \bar{\mu}_t) + \alpha^{T-t}\{\pi_{t+1}^*(q_{t+1}, \bar{\mu}_{t+1}) - \pi_t^*(q_t, \bar{\mu}_t)\}] - \alpha^{T-t}(c_{t+1} - c_t)q_t, \quad (8)$$

and $V_T(q_T, \bar{\mu}_T) = 0$. Under this formulation, if $H_t(q_t, \bar{\mu}_t) + \alpha \mathrm{E}[V_{t+1}(q_{t+1}, \bar{\mu}_{t+1})] > 0$, it is optimal to continue advance selling. Note that the function $H_t(q_t, \bar{\mu}_t)$ can be interpreted as the myopic expected profit that the manufacturer can make by delaying the capacity decision one more period and collecting advance sales without considering the possible benefit of continuing advance selling beyond period $t+1$. The function $\alpha E V_{t+1}(q_{t+1}, \bar{\mu}_{t+1})$ is the additional expected profit due to the impact of the "continue" decision, i.e., advance selling during future profits.

To characterize an optimal policy, we define $\tilde{H}_t(q_t, \bar{\mu}_t) \equiv H_t(q_t, \bar{\mu}_t) + \alpha \mathrm{E}[V_{t+1}(q_{t+1}, \bar{\mu}_{t+1})]$ and identify if and when $\tilde{H}_t(q_t, \bar{\mu}_t)$ crosses the zero line. For $t = 1$, $q_1 \equiv 0$, and $\bar{\mu}_1 = \varnothing$, and hence the decision is based on $V_1(0, \varnothing)$. For $t = 2, \ldots, T-1$, we define

$$L_t(\bar{\mu}_t) = \min\{q_t \mid q_t \geqslant 0\colon \tilde{H}_t(q_t, \bar{\mu}_t) \leqslant 0\}, \quad (9)$$

and we set $L_t(\bar{\mu}_t) = -\infty$ if $\min\{q_t \mid q_t \geqslant 0\colon \tilde{H}_t(q_t, \bar{\mu}_t) \leqslant 0\} = 0$ and $L_t(\bar{\mu}_t) = \infty$ if $\min\{q_t \mid q_t \geqslant 0\colon \tilde{H}_t(q_t, \bar{\mu}_t) \leqslant 0\} = \varnothing$. Similarly, we define

$$U_t(\bar{\mu}_t) = \max\{q_t \mid q_t \geqslant 0\colon \tilde{H}_t(q_t, \bar{\mu}_t) \leqslant 0\}, \quad (10)$$

and we set $U_t(\bar{\mu}_t) = \infty$ if $\max\{q_t \mid q_t \geqslant 0\colon \tilde{H}_t(q_t, \bar{\mu}_t) \leqslant 0\} = \varnothing$.

Theorem 1. *The following statements hold for all* $t \in (1, T)$:

1. *The function* $\tilde{H}_t(q_t, \bar{\mu}_t)$ *is convex in* q_t *for any* $\bar{\mu}_t$.
2. *A state-dependent control-band policy is optimal*; *the optimal decision is*

$$u_t^*(q_t, \bar{\mu}_t) = \begin{cases} u^s, & \text{if } L_t(\bar{\mu}_t) \leqslant q_t \leqslant U_t(\bar{\mu}_t), \\ u^c, & \text{if } q_t < L_t(\bar{\mu}_t) \text{ or } q_t > U_t(\bar{\mu}_t). \end{cases}$$

3. *The function* $V_t(q_t, \bar{\mu}_t)$ *is convex in* q_t *for any* $\bar{\mu}_t$.

Theorem 1 shows that a *state-dependent control-band* policy is optimal. Under this policy, given the history $\bar{\mu}_t$, the manufacturer optimally stops advance selling when the cumulative commitments fall between the control bands, i.e., $q_t \in [L_t(\bar{\mu}_t), U_t(\bar{\mu}_t)]$. When the cumulative commitments are lower than $L_t(\bar{\mu}_t)$, it is optimal to continue acquiring information about future market potential through advance sales. In this case, the benefits of delaying a capacity decision (e.g., acquiring demand information, resolving part of market uncertainty, and collecting revenue) outweighs the costs (e.g., higher cost of building capacity, risk of selling at a lower profit margin later, and earning a discounted revenue). When the cumulative commitments are higher than $U_t(\bar{\mu}_t)$, the commitments signal a strong, significantly more than expected, expanding future market potential. Such a strong market potential allows the manufacturer to have one more opportunity to sell at a different price point by postponing the capacity investment decision for another period when doing so is not too costly. We note that the proposition does not rule out the cases when the upper threshold is very large or infinite. Intuitively, when commitments have no predictive value or delaying the capacity decision is too costly, one may expect the upper threshold to be infinity. The next theorem formalizes this observation and identifies the conditions under which a threshold policy is indeed optimal.

Theorem 2. *The following statements hold for all for* $t \in (1, T)$:

1. *Suppose that the advance commitments have no predictive value (i.e.,* $f_t(q_t, \bar{\mu}_t) = 1$ *for all* q_t, $\bar{\mu}_t$, *and* t*). If* $c_{t+1} > c_t \; \forall\, t$, *then a threshold policy is optimal, i.e.,*

$$u_t^*(q_t) = \begin{cases} u^s, & \text{if } q_t \geqslant L_t, \\ u^c, & \text{if } q_t < L_t. \end{cases}$$

If $c_{t+1} = c_t \; \forall\, t$, *then the optimal policy does not depend on the advance sales information, and the functions* $\tilde{H}_t(\cdot, \cdot)$ *equal constants* $\tilde{H}_t$ *for each* t. *In this case, the optimal stopping time is the first* $t \in [1, T]$ *such that* $\tilde{H}_t = 0$.

2. *Suppose that* c_{t+1} *is sufficiently larger than* c_t *for all* t. *Then a state-dependent threshold policy is optimal.*

Theorem 2, shows that even when commitments carry no predictive value;[9] i.e., $f_t(\cdot) = 1$ for all t, it can be optimal to engage in advance selling. Recall that the manufacturer collects revenue earlier by advance selling. Hence, she earns interest on advance sales. She also reduces demand uncertainty by inducing customers to place early orders, reducing her risk of excess capacity and shortage. Yet, delaying the capacity decision is costly when the construction cost is increasing, i.e., $c_{t+1} > c_t$. The theorem characterizes when it is optimal for the manufacturer to stop advance selling. It also shows that if it is optimal to stop advance selling when the commitments exceed the threshold L_t, then it is never optimal to continue advance selling for total commitments above this threshold. So, part 1 of Theorem 2 shows that in the absence of the predictive value of commitments, there is no reason for the manufacturer to reverse the stopping decision. In this case the manufacturer also does not need to take the history $\bar{\mu}_t$ into account. She only needs to know the cumulative commitments because the cost of building capacity depends on t and hence q_t. Hence, an optimal stopping policy for advance sales is a *threshold* policy and is *state independent*. In addition, if the cost of capacity does not depend on t, then the manufacturer does not need to track cumulative commitments either. Part 2 of Theorem 2 shows that a state-dependent threshold policy is optimal when the capacity cost in period $t+1$ is sufficiently larger than that of period t for all periods. In short, the optimal policy is provably a threshold one when *commitments have no predictive value* or when capacity cost increases excessively over time.

5. Optimal Prices for Advance Sales and Regular Season

We study a manufacturer who sets the advance sales prices and the price for the regular sales season. We first characterize the manufacturer's optimal pricing and capacity decisions and the resulting profits when she stops advance selling. We conclude by characterizing the optimal advance sales prices and the optimal stopping policy.

THEOREM 3. *Suppose that the manufacturer decides to stop advance selling and to build capacity in period t with an existing level of cumulative commitments q_t and history $\bar{\mu}_t$, and that $c_p \geqslant c_u$. Let $z_t = S_t/(f_t(q_t, \bar{\mu}_t)p_t^{-b})$. Then, there exists a unique optimal regular sales price p_t^s given as*

$$p_t^s = p_t(z_t^*) = \left(\frac{b}{b-1}\right)\left(c_p + \frac{c_t z_t^* + c_u \mathrm{E}[z_t^* - \chi_t]^+}{z_t^* - \mathrm{E}[z_t^* - \chi_t]^+}\right), \qquad (11)$$

where z_t^* is the unique solution of $P(\chi_t > z_t) = (c_t + c_u)/(p_t(z_t) - c_p + c_u)$. The resulting optimal surplus capacity is $S_t^* = f_t(q_t, \bar{\mu}_t) z_t^* (p_t^s)^{-b}$ and optimal expected profit from remaining customers is $\pi_t^*(q_t, \bar{\mu}_t) = f_t(q_t, \mu)\Gamma_t^*$, where

$$\Gamma_t^* = \frac{1}{b}(p_t^s)^{-(b-1)}(z_t^* - \mathrm{E}[z_t^* - \chi_t]^+). \qquad (12)$$

The optimal capacity level is $Q_t^* = q_t + S_t^*$ and optimal expected undiscounted profit $\Pi_t^*(q_t, \bar{\mu}_t)$ is

$$\Pi_t^*(q_t, \bar{\mu}_t) = \Pi_t(p_t^s, S_t^* \mid q_t, \bar{\mu}_t) = \sum_{k=1}^{t-1} p_k d_k + \pi_t^*(q_t, \bar{\mu}_t) - (c_p + c_t) q_t. \qquad (13)$$

Theorem 3 provides closed-form solutions for both optimal surplus capacity and sales price during the regular season. Note that the manufacturer uses the market signal, hence the advance sales information, to set the optimal capacity level. Intuitively, the manufacturer is prompted to build optimal surplus capacity to account for the market signal accordingly. As a result, the manufacturer's expected profit from the remaining customers already takes into consideration the market signal. Hence, the optimal regular sales prices p_t^s do not need to depend on the advance sales information. To elaborate more on the structure of the optimal prices for the regular selling season, we establish the following basic properties:

THEOREM 4. *Suppose that the manufacturer has decided to stop advance selling in period t. The optimal regular sales season price satisfies $p_t^s > c_p + c_t$. Furthermore, p_t^s is increasing in c_t, c_p, and c_u. Consequently, the optimal capacity level Q_t^* is decreasing in c_t, c_p, and c_u.*

This result shows that when setting the selling season prices, the manufacturer guarantees herself a positive margin from sales. This margin ensures that the optimal surplus capacity S_t^* is positive. Intuitively, when the manufacturer has the ability to set the regular sales price, she would set the price such that building surplus capacity is profitable. Furthermore, the higher the costs, the higher the prices charged to customers. Higher prices reduce demand, and prompt the manufacturer to build less capacity.

To derive the manufacturer's optimal stopping policy and the optimal advance sales prices, we modify the dynamic programming in §4. For $t < T$, the functional equation is given by

$$\begin{aligned} V_t(q_t, \bar{\mu}_t) &= \max\Big\{0, \max_{p_t \in \mathcal{R}_t} \mathrm{E}[(p_t - \alpha^{T-t}(c_p + c_{t+1}))D_t(p_t \mid q_t, \bar{\mu}_t) \\ &\quad + \alpha^{T-t}(\pi_{t+1}^*(q_{t+1}, \bar{\mu}_{t+1}) - \pi_t^*(q_t, \bar{\mu}_t))] \\ &\quad - \alpha^{T-t}(c_{t+1} - c_t)q_t + \alpha \mathrm{E}[V_{t+1}(q_{t+1}, \bar{\mu}_{t+1})]\Big\} \qquad (14) \\ &\equiv \max\Big\{0, \max_{p_t \in \mathcal{R}_t}\{H_t(p_t, q_t, \bar{\mu}_t) + \alpha \mathrm{E}[V_{t+1}(q_{t+1}, \bar{\mu}_{t+1})]\}\Big\} \qquad (15) \\ &\equiv \max\Big\{0, \max_{p_t \in \mathcal{R}_t} R_t(p_t, q_t, \bar{\mu}_t)\Big\} \qquad (16) \\ &\equiv \max\{0, \tilde{H}_t(q_t, \bar{\mu}_t)\}, \qquad (17) \end{aligned}$$

and $V_T(\cdot, \cdot) \equiv 0$. The expectations are taken at period t with respect to ξ_t. $\mathcal{R}_t$ is a convex set of possible advance sales prices for each period t. Let p_t^c denote the optimal advance sales price in period t. To state the optimal stopping result, we define if and when $\tilde{H}_t(\cdot, \cdot)$ crosses the zero line. As before, these points are $L_t(\bar{\mu}_t)$ and $U_t(\bar{\mu}_t)$, which are defined as in (9) and (10), respectively.

THEOREM 5. *The following statements hold for all $t \in (1, T)$:*

1. *The function $R_t(p_t, q_t, \bar{\mu}_t)$ is convex in q_t for any p_t and $\bar{\mu}_t$.*
2. *The function $\tilde{H}_t(q_t, \bar{\mu}_t)$ is convex in q_t for any $\bar{\mu}_t$.*

3. *A state-dependent control-band policy is optimal, i.e.,*

$$u_t^*(q_t, \bar{\mu}_t) = \begin{cases} u^s, & \text{if } L_t(\bar{\mu}_t) \leqslant q_t \leqslant U_t(\bar{\mu}_t), \\ u^c, & \text{if } q_t < L_t(\bar{\mu}_t) \text{ or } q_t > U_t(\bar{\mu}_t). \end{cases}$$

4. *The function $V_t(q_t, \bar{\mu}_t)$ is convex in q_t for any $\bar{\mu}_t$.*

This result shows that the *structure* of the optimal stopping policy for acquiring advance sales information remains the same when the manufacturer determines advance and regular sales prices optimally. The actual threshold values, however, depend on sales prices. Optimal advance sales prices and thresholds can be determined numerically, for example, by a backward induction algorithm.

6. Numerical Study

We conduct a numerical study to illustrate the impact of different operating and market/demand factors on the advance selling policy and on the manufacturer's profit using a specific a market signal function. We base this analysis on the scenario in which the manufacturer sets advance and regular sales prices optimally in addition to the capacity.

6.1. A Specific Market Signal Function

So far, we have characterized the optimal advance selling policy under a generic market signal function. The manufacturer keeps track of the total commitments q_t and the history $\bar{\mu}_t$ in order to specify and update the market signal and demand. Hence, the dimension of the state space depends on the functional form of $f_t(q_t, \bar{\mu}_t)$. In certain cases, it may increase as time progresses, for example, when one needs to keep track of all past prices. Nevertheless, when $\bar{\mu}_t$ is a scalar representing summary statistics of the advance sales information, then the state space is given by the two-tuple $(q_t, \bar{\mu}_t)$, achieving state-space reduction. Consider the following market signal function, which we use in our numerical experiments:

$$f_t(q_t, \bar{\mu}_t) = 1 + \theta\left(\frac{q_t - \bar{\mu}_t}{\bar{\mu}_t}\right) = (1-\theta) + \theta\frac{q_t}{\bar{\mu}_t} \quad \text{for } t = 2, \ldots, T, \tag{18}$$

where $\bar{\mu}_t = \sum_{j=1}^{t-1} \mathrm{E}[\xi_j] p_j{}^{-b}$ and $\theta \in [0, 1)$ is a constant. Note that given the past prices $(p_1, \ldots, p_{t-1})$, the manufacturer's estimate of the demand for each period t, based on the information available at time zero, is $\mathrm{E}[\xi_t] p_t{}^{-b}$. Hence, before acquiring any advance sales information, i.e., at time zero, the manufacturer *expects* to collect $\bar{\mu}_t$ units of commitments by period t. Note that when the expectation is taken at time zero, $\mathrm{E}[f_t(q_t, \bar{\mu}_t)] = 1$ for all t and any price path. Depending on the demand realizations, however, the *actual* commitment level q_t can exceed or fall below the expected amount $\bar{\mu}_t$ and the market signal fluctuates around 1. If q_t exceeds $\bar{\mu}_t$, the manufacturer has collected more advance sales than she initially expected, and vice versa. In consequence, next period's expected market signal can be higher or lower than $f_t(q_t, \bar{\mu}_t)$. The parameter $\theta \in [0, 1)$ is akin to a smoothing constant in forecasting and defines the *extent of correlation* between the market signal provided by advance sales and future demand. Note that as the manufacturer continues advance selling, the cumulative commitments and the summary statistics for advance sales information are updated as $q_{t+1} = q_t + d_t = q_t + f_t(q_t, \bar{\mu}_t)\epsilon_t p_t^{-b}$ and $\bar{\mu}_{t+1} = \bar{\mu}_t + \mathrm{E}[\xi_t] p_t{}^{-b}$, respectively.[10]

6.2. Numerical Study Setup

Market Signal and the Predictive Value of Commitments. We use the market signal function $f_t(\cdot, \cdot)$ specified in Equation (18). The extent of correlation between advance purchase commitments and future demand is measured through the smoothing constant θ. When $\theta = 0$, demand in each period is independent of prior commitments, and as $\theta \to 1$, the signal provided by the commitments is a very strong indicator of future demand. A strong dependence between past and future demand could be observed, for example, in fashion products and the apparel industry. In contrast a low level of θ would likely apply more to mature consumer products.

Customer Time Preferences. We model ξ_{t+1} as an independent random variable with a *distribution* identical to the distribution of $(1+k)\xi_t$ for $t > 1$ and $k \in (-1, 1)$. Note that k $(k > -1)$ is a measure of customers' time preference for purchasing decisions. When $k > 0$ (resp., $k < 0$), more customers prefer to purchase later (resp., earlier), indicating a higher (resp., lower) anticipation in the market for potential shortages in capacity. In other words, when $k > 0$ the distribution of the future period's market potential ξ_{t+1} is stochastically larger than the previous period's ξ_t. When $k = 0$, there is no clear time preference. Hence, we refer to k as the "late purchase tendency." We also model ξ_t as normally distributed random variables with mean $[(1+k)^{t-1}/\sum_{j=1}^{T}(1+k)^{j-1}]\mu$ and standard deviation $[(1+k)^{t-1}/\sqrt{\sum_{j=1}^{T}(1+k)^{2(j-1)}}]\sigma$. Hence, the total market potential $\sum_{t=1}^{T}\xi_t$ is normally distributed with mean μ and σ, and it is *independent* of k.

Capacity Cost. The unit cost of capacity c_t is of the form $c_t = c_0 + \delta t$, where c_0 is the base cost of capacity and δ $(\delta \geqslant 0)$ is the measure of how this cost increases as the sales season is approached. By varying c_0 we investigate the effects of overall cost of capacity, whereas different δ values indicate the importance of time in building capacity.

Price Sensitivity and Set of Advance Sales Prices $\mathscr{R}_t$. The price sensitivity of customers in the model is measured by the parameter b. The set $\mathscr{R}_t$ has $n > 0$ finite number of prices that are uniformly distributed in the region

$[(1-a)p_t^s, (1+a)p_t^s]$, where $a > 0$ and p_t^s is defined in Equation (11), i.e., $\mathcal{R}_t = \{(1-a)p_t^s, \ldots, p_t^s, \ldots, (1+a)p_t^s\}$. We include the optimal price for the regular selling season p_t^s among the possible prices to be offered during the advance selling period. For practical motivations of considering discrete prices, see Gallego and van Ryzin (1994).

6.3. Measures of Interest

In addition to the optimal policy, the manufacturer can follow two advance selling strategies. One extreme is to set the regular sales price and capacity in period 1 without advance selling. We refer to this scenario as the *no advance selling scenario*, for which the corresponding expected profit G_{no} is given by $\alpha^{T-1}\Pi_1^*(0, \varnothing)$, where $\Pi_1^*(0, \varnothing)$ is defined in Equation (13). The other extreme is to continue advance selling until the last advance sales period, $T-1$. We refer to this one as the *full advance selling scenario*. The corresponding expected profit G_f is obtained by policy evaluation in which the decision $u_t(q_t, \bar{\mu}_t)$ is forced to be u^c instead of choosing the maximum in Equation (17). The expected optimal profit G^* is given by $V_1(0, \varnothing)$ in Equation (14). The difference between the optimal strategy and the first extreme is the *expected value of advance selling* or *value of information acquisition*. To quantify this value, we report $I_{no} = [(G^* - G_{no})/G_{no}] \times 100\%$. The profit difference between the optimal strategy and the second extreme is the *expected value of knowing when to stop* advance selling or the *value of optimal advance selling*. To quantify this measure, we report $I_f = [(G^* - G_f)/G_f] \times 100\%$. Figure 1 illustrates the resulting expected profits under optimal advance selling, no advance selling, and full advance selling. The resulting percentages are $I_{no} = 10.93\%$ and $I_f = 1.02\%$. For this example, the expected profit to advance sell and build capacity later is 50.85, which is larger than the profit of stopping and building capacity of 45.84 in the first period. The figure also illustrates the optimal lower threshold $L_t(\bar{\mu}_t)$ as a function of *expected* commitments $\bar{\mu}_t$ when $t = 2$. For example, if the manufacturer expects more commitments at the beginning of period two (such as $\bar{\mu}_2 = 3$) and the quantity committed so far is low (such as $q_2 = 2$), then it is optimal to continue advance selling. Note also that the threshold increases with $\bar{\mu}_2$. Intuitively, the manufacturer is more likely to continue advance selling and acquiring information when the *expected* commitments are high. We note that the upper threshold U_t was large relative to the L_t and the mean demand, or it was infinite in our structured numerical study. However, it is possible to construct cases where both L_t and U_t are finite, and are relatively close to each other. For example, when $T = 5$, $c_p = 3$, $c_u = 2$, $c_0 = 1.2$, $\delta = -0.1$, $b = 2$, $\mu = 1{,}000$, $\sigma = 80$, $\theta = 0.3$, $\alpha = 1$, $k = 0$, for the pricing policy $\mathcal{P} = (4.2, 4.1, 4.0, 3.9, 4.65)$, the optimal control band for the second period is $L_2 = 11.33$ and $U_2 = 24.66$, resulting in optimal expected profit of 18.79, and $I_{no} = 4.06\%$ and $I_f = 4.78\%$.

6.4. Effect of the Environment

We quantify the effect of various operating and market/demand factors on the optimal advance selling strategy. In particular, we investigate how these factors affect the optimal expected profit, the value of advance selling, and the value of knowing when to stop. We use the following parameters in the base scenario: $T = 5$, $\mu = 1{,}000$, $\sigma = 100$, $c_u = 2$, $c_p = 3$, $c_0 = 1.2$, $\delta = 0.3$, $k = 0$, $b = 2$, $\alpha = 0.95$, $\theta = 0.3$, $a = 0.1$, and $n = 7$. For this base case, the expected optimal profit is $G^* = 48.02$, the value of advance selling is $I_{no} = 4.75\%$, and the value of knowing when to stop is $I_f = 4.73\%$. To have a balanced view, we chose the base parameter set such that these two measures of value are equal. We change one parameter at a time while keeping the others constant.

Impact of Overall Market Uncertainty. We test the effect of the coefficient of variation of the total market demand potential by varying $\sigma \in [40, 110]$. The results are illustrated in Figure 2. The expected profit decreases

Figure 1. Expected profits and thresholds for $T = 5$, $\mu = 1{,}000$, $\sigma = 80$, $c_u = 2$, $c_p = 3$, $c_0 = 1.2$, $\delta = 0.18$, $k = 0$, $b = 2$, $\alpha = 0.95$, $\theta = 0.3$, $a = 0.1$, and $n = 7$.

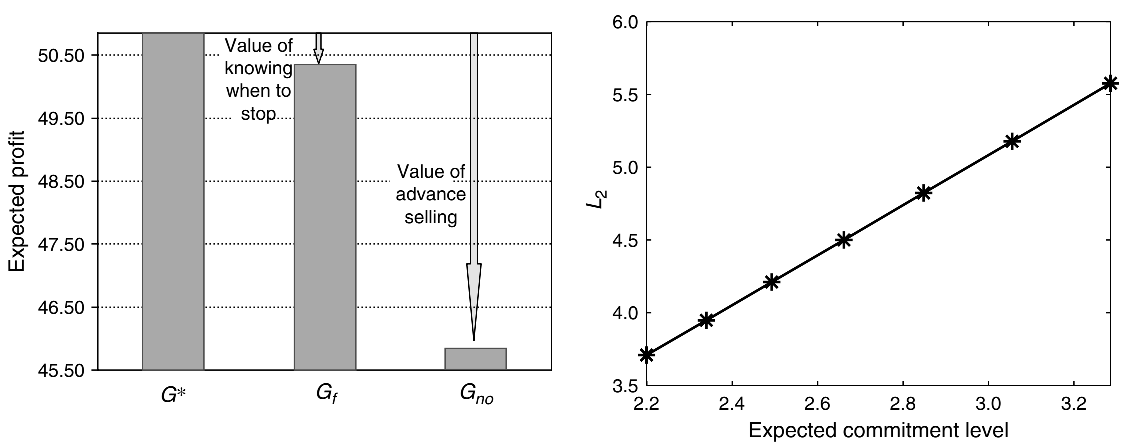
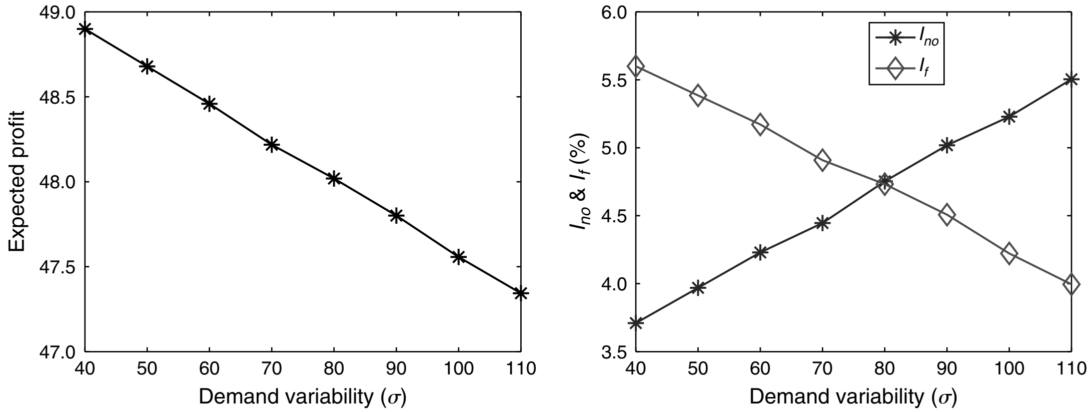


Figure 2. The impact of overall market demand variability (σ).



with higher demand variability. This is consistent with known results for the traditional newsvendor problem. We also observe that the value of information acquisition (or advance selling) increases with market uncertainty while the value of knowing when to stop decreases. Intuitively, when the market uncertainty is high, the value, of acquiring information through advance selling is high. Hence, stopping closer to the last period is more likely to be an optimal policy. In most cases, the two measures I_f and I_{no} would be complements. While one is increasing, the other will be decreasing. These observations suggest that advance selling mitigates the adverse effect of demand uncertainty.

Impact of Capacity Cost Structure. Figure 3 illustrates the effects of the *incremental* cost of capacity as measured by δ (i.e., how rapidly cost of building capacity increases as the regular sales season nears). In general, increasing capacity construction costs reduces profit. Note also that the value of information acquisition through advance selling is also decreasing with δ. Essentially, large δ penalizes late construction. If the late construction is too expensive, the optimal solution is to build capacity sooner rather than later and not to acquire information. From the above results we conclude that the value of information acquisition is greater when time is not a major constraint for the manufacturer in building capacity.

Figure 4 illustrates the effects of *base* capacity cost c_0 (i.e., how expensive capacity cost is in general). Note that the manufacturer's profit naturally decreases as capacity becomes more costly, but the reduction in profits is even higher when the manufacturer does not offer advance sales. Consequently, the benefit of advance selling actually increases as base capacity cost increases. These results show that the value of information acquisition is higher when capacity is more expensive relative to the penalty cost of late construction.

Impact of Predictive Value of Commitments. Figure 5 shows the impact of the predictive value of commitments, as measured by the smoothing parameter θ. Observe that as θ increases, the value of advance selling first increases. However, when θ is very high, the commitments send strong signals of future demand, indicating a potential for high mean and variance. High uncertainty reduces the expected profit, the value of information acquisition, as well as knowing when to stop. Consequently, advance selling is most beneficial when the predictive value

Figure 3. The impact of incremental cost of capacity (δ).

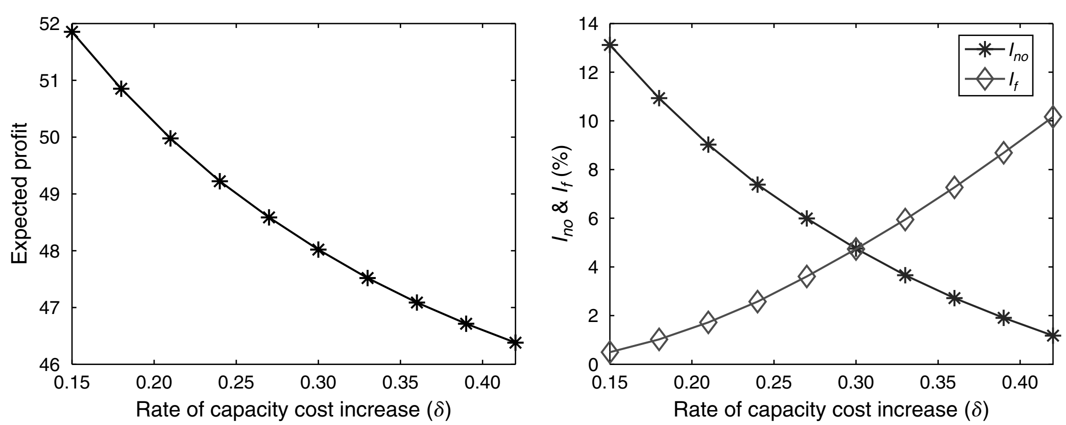


Figure 4. The impact of base capacity cost (c_0).

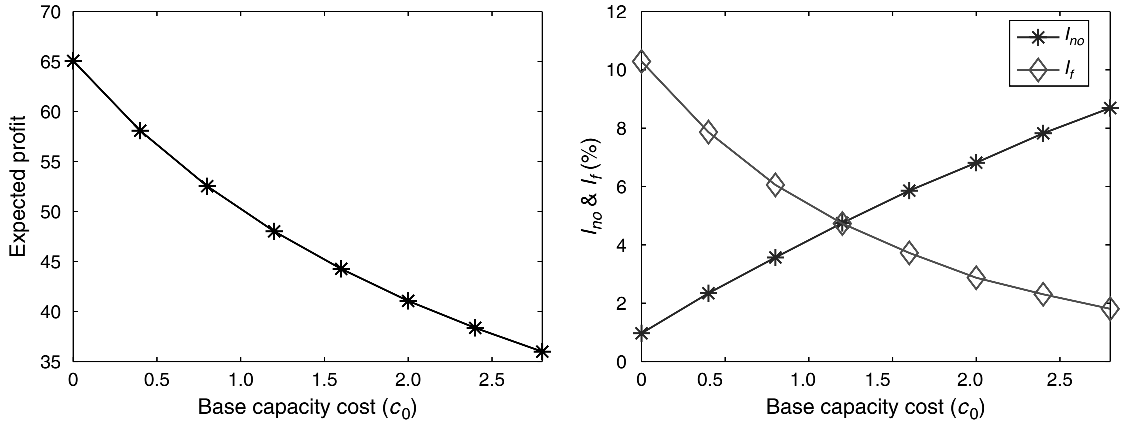


Figure 5. The impact of predictive value of commitments (θ).

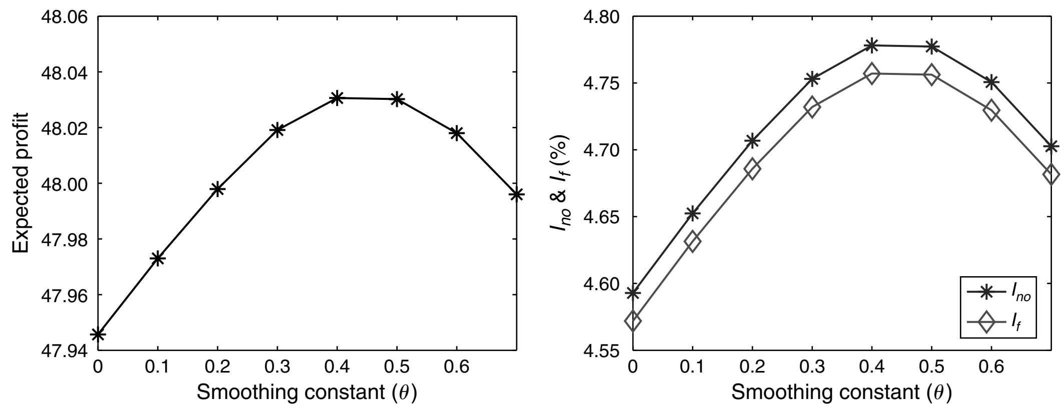
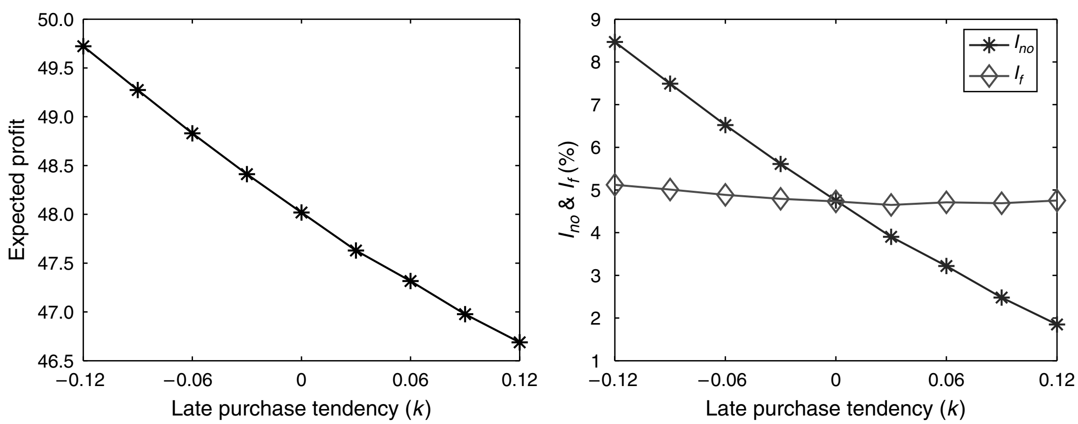


Figure 6. The impact of customer time preferences (k).

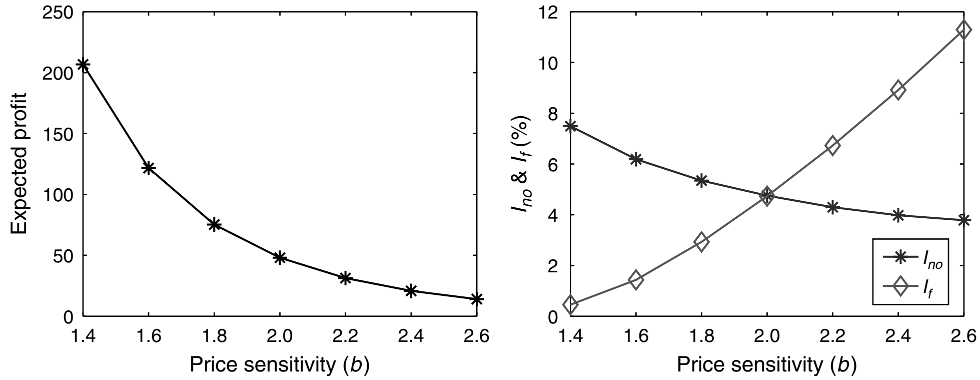


of commitments is moderate. Note, however, that the absolute scale of changes is quite small.

Impact of Customer Time Preferences. The value of information acquisition is also related to whether customers anticipate shortages or not. In the event of potential shortages in supply, customers would be more tempted to commit to advance purchases and also to commit earlier in time. Figure 6 demonstrates the influence of customers' time preferences through the late purchase tendency parameter k. As more customers tend to commit earlier (smaller k), the profit and the value of information acquisition increases. Figure 6 supports the claim that advance selling yields a higher benefit when there is more anticipation of capacity shortages in the market.

Impact of Price Sensitivity. Figure 7 illustrates the effects of customer price sensitivity. When customers are

Figure 7. The impact of customer price sensitivity (b).



more price concerned, the manufacturer's profitability is reduced. This results in a lower value of information acquisition. Note also that when the customers are not price sensitive, acquiring information until the last period becomes more likely. Hence the expected value of knowing when to stop advance selling decreases. These observations suggest that low customer price sensitivity is another condition for maximizing the gains from advance selling.

These numerical results help identify that advance selling coupled with a *let-them-come-and-build-it-later* approach is a profitable strategy, in particular, when (i) demand uncertainty is high, (ii) more customers anticipate capacity shortages in the market, (iii) building capacity is expensive but timing is not a major concern, (iv) commitments have moderate predictive value about market potential, and (v) customer price sensitivity is relatively low. We note that, with these conditions, one can construct cases where the value of advance selling I_{no} is arbitrarily high.

6.5. Optimal Prices

To quantify the value of advance selling and the value of knowing when to stop in the previous section, we compute, for each period, the optimal advance selling price p_t^c and the optimal regular season price p_t^s when the manufacturer decides to stop advance selling.

Figure 8 plots the optimal prices as a function of commitments q_t for a given expected commitment level $\bar{\mu}_t = 2.2$ when $t = 2$. The base data set is the same as in Figure 1. Note that the optimal market price p_2^s does not depend on the commitments (Theorem 3). The optimal advance selling price p_2^c, however, is increasing in the number of commitments. This is because having more commitments suggests a stronger market and allows the manufacturer to charge higher advance selling prices. Figure 8 also depicts the evolution of the prices over time for different commitment levels, $q_t = 0.7\bar{\mu}_t$, $q_t = \bar{\mu}_t$, $q_t = 1.3\bar{\mu}_t$ for each t, corresponding to low, medium, and high levels of commitment. For $t = T = 5$, the manufacturer is forced to build capacity. As before, for a given t, p_t^c is nondecreasing in q_t (high commitments induce high advance sales prices). Furthermore, both the advance sales prices and the regular season prices are increasing over time. Finally, note that the advance sales prices are always lower than the regular season prices, implying that the customers committing to buy earlier are garnering discounts.

Figure 8. Optimal advance selling and regular season prices.

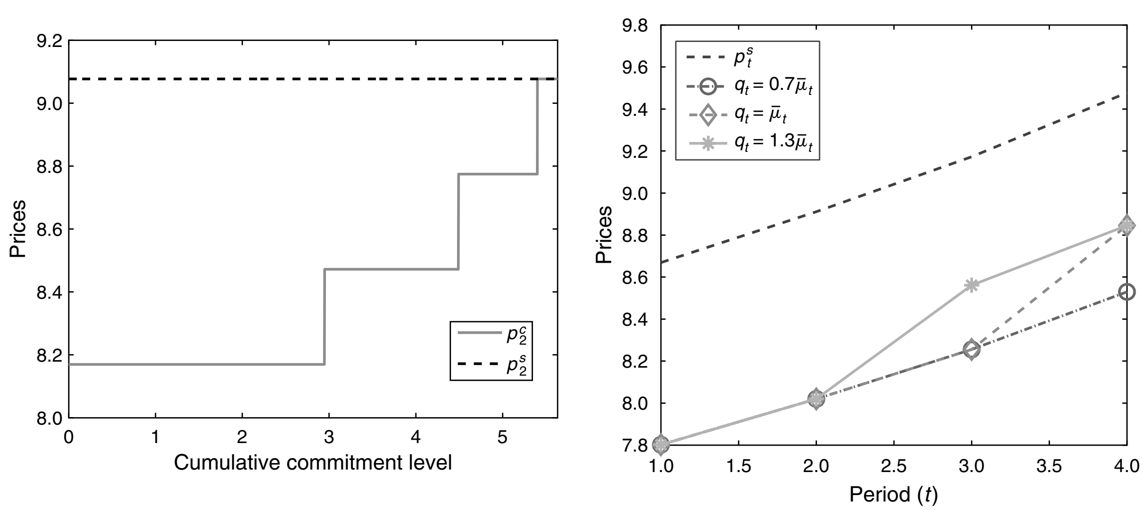


Figure 9. Optimal advance sales prices as δ changes for $q_t = 0.7\bar{\mu}_t$, $q_t = \bar{\mu}_t$, and $q_t = 1.3\bar{\mu}_t$.

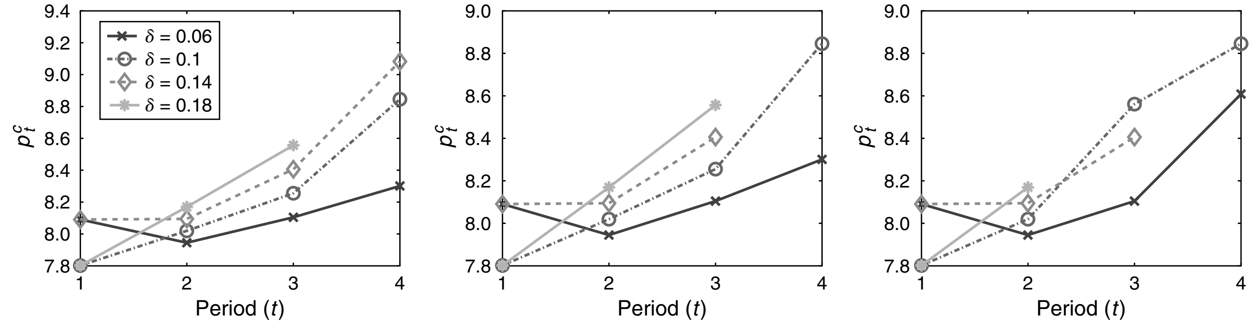
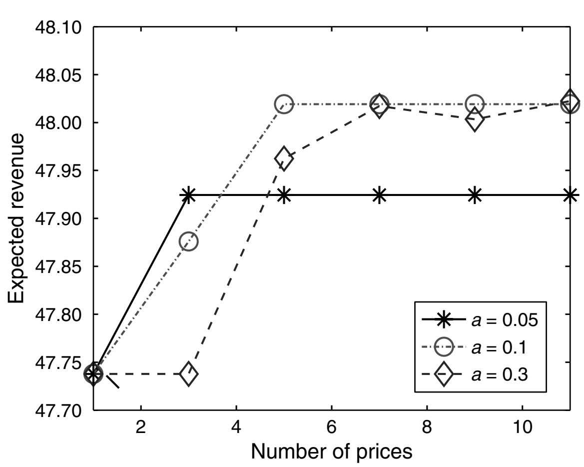


Figure 9 depicts the optimal advance sales prices as the incremental cost of building capacity δ changes for different commitment levels $q_t = 0.7\bar{\mu}_t$, $q_t = \bar{\mu}_t$, and $q_t = 1.3\bar{\mu}_t$. Note that for any given t where p_t^c is not reported, it is optimal for the manufacturer to stop advance selling in that period. It is possible to infer from Figure 9 that for any capacity cost structure, higher levels of commitment, induce the manufacturer to stop advance selling earlier. Generally speaking, increasing the incremental cost of capacity δ results in higher advance selling prices. The exceptions are due to discretization of the advance selling price set $\mathcal{R}_t$, which are set based on the optimal regular season price p_t^s. When δ changes, so does the set of advance prices considered during that period, which can result in nonmonotonic results. Consequently, the optimal advance sales prices can display nonmonotonic behavior over time as well (although within a fairly restricted range). We note that the changes in other cost or market parameters have rather predictable effects on optimal prices, and hence are omitted for brevity.

6.6. Computational Aspects, Algorithmical Complexity, and a Heuristic

In determining the optimal advance selling prices, we search n uniformly distributed prices in the range $[(1-a)p_t^s, (1+a)p_t^s]$ for each period t. Figure 10 illustrates the optimal expected profit as a function of the number of prices n and the price range a. We highlight two observations. First, the expected profit increases with either factor. Essentially, increasing n and a is equivalent to relaxing the constraint set, which in turn increases the optimal expected profit. Second, note that the marginal return on profit is decreasing with these factors. For example, when $a = 0.1$ considering seven or more prices does not change the profit significantly. Hence, there is a decreasing return to considering larger price sets, which require significantly more computational effort.

Figure 10. The impact of price set $\mathcal{R}$.



The algorithm to solve the general case is of complexity $O(n^T)$, which implies that finding optimal advance and regular season prices is a computationally expensive task. For this reason we examined heuristic pricing policies. Recall that p_t^s is the optimal price for the regular sales season when the manufacturer stops collecting commitments at period t. Consider a heuristic advance sales pricing policy under which the manufacturer charges the same price p_t^s even when she continues to collect commitments in period t. The computational effort required to numerically solve this heuristic is the same as that of the exogenous pricing case (i.e., $O(T)$). We have tested the performance of this heuristic for different values of δ, c_0, and σ and used regression to compare the heuristic profit to the optimal profit (for details refer to the electronic companion to this paper). The resulting that R^2 was close to 1 for all factors, suggesting that the heuristic can safely be used to investigate the impact of parameter changes. The average optimality gap across all experiments was also very small (0.535%).

7. Dynamic Pricing to Sell Capacity

Our objective here is to investigate the tradeoff between two strategies that mitigate the adverse effect of demand uncertainty: (1) information acquisition for capacity planning through advance selling, and (2) revenue management of installed capacity through pricing during the regular sales season. To do so, we study a manufacturer who, in addition to employing advance selling and pricing to determine the capacity level, sets prices dynamically to sell the installed capacity during the regular sales season. The manufacturer first acquires information through pricing and advance selling. When it is optimal to do so, she stops collecting advance sales information and uses this information to build capacity optimally. During the regular sales season, the manufacturer sells the installed capacity through dynamically setting prices. Note that this selling process combines a strategic pricing and capacity decision with an operational pricing decision. Hence, the time scale and periods for a strategic versus tactical level decision could be different. All actual sales and deliveries take place in the regular sales season during which the manufacturer is allowed to adjust prices M times, $M \geqslant 1$.[11]

In order to solve this problem, we have to embed a second-stage dynamic program that prescribes the optimal dynamic pricing policy and the resulting profit during the regular sales season for a given level of capacity and then determine the optimal capacity level. Since the manufacturer has M opportunities to adjust prices over the regular sales season, let $A_{t,m}$, $m = 1, \ldots, M$ denote the random market demand potential she will face in the mth pricing epoch (or subperiod). We assume that $A_{t,m}$s are independently distributed IFR random variables with support on $[0, \infty)$. Recalling that χ_t is the remaining demand potential the manufacturer serves after stopping advance selling, for logical consistency it is desirable that the distribution of $\sum_{m=1}^{M} A_{t,m}$ is the same as the distribution of χ_t. Denoting the price charged in the mth epoch as $p_{T,m}$, the random demand during the mth pricing epoch is $D_{T,m}(p_{T,m} \mid q_t, \bar{\mu}_t) = f_t(q_t, \bar{\mu}_t) A_{t,m} p_{T,m}^{-b}$. Note that our original model corresponds to the case when $M = 1$. For a given pricing policy $\mathcal{P} = (p_{T,1}, \ldots, p_{T,M})$, the manufacturer's expected profit from remaining customers during the regular sales season is

$$\begin{aligned}
&\Gamma_t(S_t, q_t, \bar{\mu}_t \mid \mathcal{P}) \\
&= \sum_{m=1}^{M} (p_{T,m} - c_p) \,\mathrm{E}[\min\{S_m, D_{T,m}(p_{T,m} \mid q_t, \bar{\mu}_t)\}] \\
&\quad - c_u \,\mathrm{E}[S_M - D_{T,m}(p_{T,m} \mid q_t, \bar{\mu}_t)]^+ \\
&= \sum_{m=1}^{M} p_{T,m} \,\mathrm{E}[\min\{S_m, D_{T,m}(p_{T,m} \mid q_t, \bar{\mu}_t)\}] \\
&\quad + (c_p - c_u) \,\mathrm{E}[S_M - D_{T,m}(p_{T,m} \mid q_t, \bar{\mu}_t),]^+ - c_p S_1 \\
&= f_t(q_t, \bar{\mu}_t) \Bigg(\sum_{m=1}^{M} p_{T,m} \,\mathrm{E}[\min\{s_m, A_{t,m} p_{T,m}^{-b}\}] \\
&\qquad + (c_p - c_u) \,\mathrm{E}[s_M - A_{t,m} p_{T,M}^{-b}]^+ - c_p s_1 \Bigg) \\
&\equiv f_t(q_t, \bar{\mu}_t) \tilde{\Gamma}_t(s_t \mid \mathcal{P}),
\end{aligned}$$

where we define $S_m \equiv S_t$ for $m = 1$, and $S_{m+1} = [S_m - A_{t,m} p_{T,m}^{-b}]^+$ for $m > 1$, and $s_m \equiv S_m / f_t(q_t, \bar{\mu}_t)$. The second equality holds because total sales during the regular season plus the remaining capacity equals the total capacity at the beginning of regular sales season, i.e., $S_1 = \sum_{m=1}^{M} \min\{S_m, D_{T,m}\} + [S_M - D_{T,M}]^+$. The objective is to solve $\tilde{\Gamma}_t(s_t) = \min_{\mathcal{P} \in \mathbf{P}} \tilde{\Gamma}_t(s_t \mid \mathcal{P})$, where $\mathbf{P}$ denotes the set of all policies. Finding $\tilde{\Gamma}_t(s_t)$ involves solving the following dynamic program:

$$\begin{aligned}
\tilde{\Gamma}_t^m(s_m) &= \max_{p_{T,m}} \{ p_{T,m} \,\mathrm{E}[\min\{s_m, A_{t,m} p_{T,m}^{-b}\}] \\
&\qquad + \mathrm{E}[\tilde{\Gamma}_t^{m+1}([s_m - A_{t,m} p_{T,m}^{-b}]^+)] \} \\
&\qquad\qquad \text{for } 1 \leqslant m < M, \qquad (19) \\
\tilde{\Gamma}_t^M(s_M) &= \max_{p_{T,M}} \{ p_{T,M} \,\mathrm{E}[\min\{s_M, A_{t,m} p_{T,M}^{-b}\}] \\
&\qquad + (c_p - c_u) \,\mathrm{E}[s_M - A_{t,m} p_{T,M}^{-b}]^+ \}, \qquad (20)
\end{aligned}$$

where the expectation is with respect to the random variable $A_{t,m}$ in each period m. Notice that by definition $\tilde{\Gamma}_t(s_t) \equiv \tilde{\Gamma}_t^1(s_1)$.

The manufacturer's optimal net profit from remaining customers when she stops advance selling in period t and sells the surplus capacity S_t via dynamic pricing is given by

$$\begin{aligned}
&f_t(q_t, \bar{\mu}_t) \left[\tilde{\Gamma}_t\left(\frac{S_t}{f_t(q_t, \bar{\mu}_t)} \right) - (c_t + c_p) \frac{S_t}{f_t(q_t, \bar{\mu}_t)} \right] \\
&\quad = f_t(q_t, \bar{\mu}_t)[\tilde{\Gamma}_t(s_t) - (c_t + c_p) s_t] \\
&\quad = f_t(q_t, \bar{\mu}_t)[\tilde{\Gamma}_t^1(s_1) - (c_t + c_p) s_1].
\end{aligned}$$

Optimizing over the capacity level also, the manufacturer's optimal net profit from remaining customers becomes

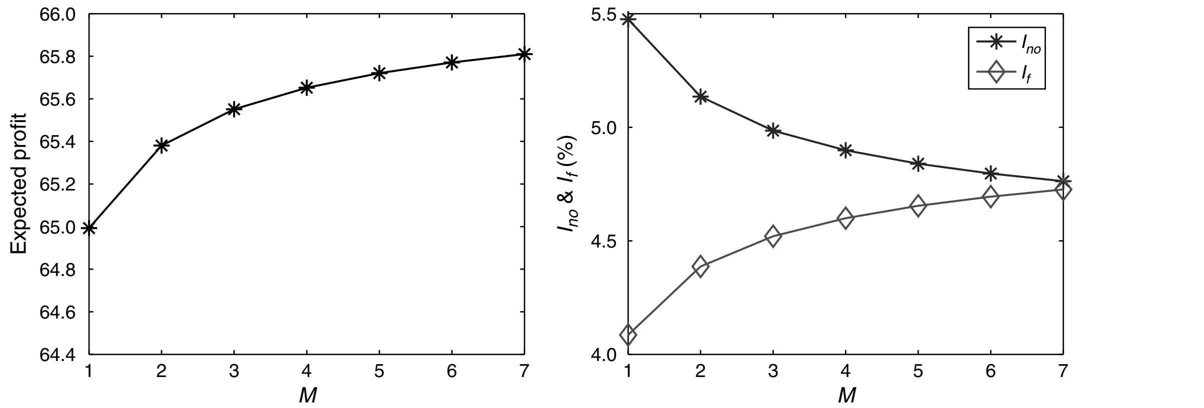
$$\begin{aligned}
\pi_t^*(q_t, \bar{\mu}_t) &= f_t(q_t, \bar{\mu}_t)[\tilde{\Gamma}_t(s_t^*) - (c_t + c_p) s_t^*] \\
&= f_t(q_t, \bar{\mu}_t)[\tilde{\Gamma}_t^1(s_1^*) - (c_t + c_p) s_1^*]. \qquad (21)
\end{aligned}$$

Defining $\Gamma_t^* \equiv \tilde{\Gamma}_t(s_t^*) - (c_t + c_p) s_t^*$, we have a similar structure as in Equation (3), i.e., $\pi_t^*(q_t, \bar{\mu}_t) = f_t(q_t, \bar{\mu}_t) \Gamma_t^*$. This implies that all of the preceding results regarding the optimal stopping policy for acquiring advance sales information hold with the profit $\pi_t^*(q_t, \bar{\mu}_t)$ replaced by its new definition above. We formalize this in the next theorem, which we state without proof.

Theorem 6. *When the manufacturer sells installed capacity by dynamically adjusting prices, the optimal stopping policy for advance selling is a state-dependent control-band policy.*

The above result establishes the structure of the optimal policy. Yet, computing policy parameters, such as the optimal prices and thresholds, remains a difficult task. Essentially, the problem is a two-stage, nested, stochastic dynamic program with multiple decision epochs and continuous, multidimensional state spaces. The first stage is the optimal stopping problem whose solution depends on the second-stage dynamic program specified in Equations (19)–(20). Even this second-stage problem is a challenging one to solve numerically. Monahan et al. (2004) study a

Figure 11. Multiple pricing opportunities to sell capacity during the regular sales season.



similar problem as the second-stage DP and report that efficient results can only be obtained when $c_p = c_u$. For this case, they show that $\tilde{\Gamma}_t^m(s_m) = r_m^*(s_m)^n$, where

$$r_m^* = \max_z \frac{z - \mathrm{E}[z - A_{t,m}]^+ + r_{m+1}^* E([z - A_{t,m}]^+)^n}{z^n},$$

$$s_1^* = \left(\frac{nr_1^*}{c_t + c_p}\right)^b,$$

$$\tilde{\Gamma}_t(s_t^*) - (c_t + c_p)s_t^* = \frac{1-n}{n}(c_t + c_p)\left(\frac{nr_1^*}{c_t + c_p}\right)^b,$$

where $n = 1 - 1/b$ and $r_{M+1}^* = 0$. Hence, $\pi_t^*(q_t, \bar{\mu}_t) = f_t(q_t, \bar{\mu}_t)\Gamma_t^*$, where $\Gamma_t^* = (1-n)/n(nr_1^*/c_t + c_p)^b$.

We use this result to numerically solve the second-stage dynamic program and embed its solution to the optimal stopping problem and solve for the optimal advance sales prices and stopping thresholds. Figure 11 illustrates the results of a numerical study in which $A_{t,m}$ is normally distributed with same mean and variance across all m. The parameters of this example are the same as the base set, except $\sigma = 40$ and $\delta = 0.2$.

Three observations are worth noting. First note that the expected profit is increasing with the number of pricing opportunities. The percentage increase in expected profit between having $M = 7$ pricing epochs to having one is $1.25\% = (65.83 - 65.02)/65.02$. Second, the marginal increase in profit is decreasing. Third, the expected value of advance selling is decreasing, but the marginal decrease is also decreasing. This observation suggests that dynamic pricing during the regular sales season is only a partial substitute for dynamic pricing during advance sales periods. Altogether, these observations suggest that using a small number of price adjustments or even a single price during the regular sales season is reasonably close to optimal, considering also the fact that such price adjustments are costly due to transaction costs (e.g., due to advertising new prices). Similar observations for this second-stage problem are also reported in Gallego and van Ryzin (1994).

8. Summary and Discussion

In this paper, we study the strategy of obtaining information about market potential through advance sales for a better capacity decision. In particular, we consider a manufacturer who collects revenue and information through advance selling prior to building capacity. Using advance sales information, the manufacturer sets the capacity and satisfies the remaining demand during the regular season as much as possible, subject to the available capacity. We establish the optimal pricing strategy both for the advance and regular selling seasons and the optimal capacity to build. We also establish the optimality of a control-band policy that prescribes when to stop collecting advance sales information. This policy is also optimal when prices are set exogenously (e.g., as mark-up or mark-down schedules). Through a numerical study, we quantify the expected value of this capacity planning strategy under different market and operating conditions. We show that advance selling coupled with a *let-them-come-and-build-it-later* approach is a profitable strategy, in particular, when (i) demand uncertainty is high, (ii) more customers anticipate capacity shortages in the market, (iii) building capacity is expensive but timing is not a major concern, (iv) commitments have moderate predictive value about market potential, and (v) customer price sensitivity is relatively low. We also show that the extreme strategy of collecting full advance selling information or not collecting any information leads to inferior solutions in comparison to the optimal pricing strategy. These results suggest that the practice of advance selling is of most value for, e.g., high technology, apparel, and pharmaceutical industries. For example, telecommunication and semiconductor industries face high capacity building costs. They often introduce new products for which the market uncertainty is also high relative to commodity type products. Finally, we study a scenario in which the manufacturer continues to sell installed capacity through dynamic pricing. Modeling this scenario bridges the revenue and capacity management literatures. We show that selling capacity by dynamically adjusting regular sales prices increases the expected profit but only to a limited extent. Next, we revisit a few aspects of our framework.

Other Market Signal Functions. In our numerical examples we have utilized a specific form of the market signal function, which smoothed the predictive value of cumulative commitments over time. As noted earlier, it is possible to envision a case where next period's expected market signal is the same as the current level (i.e., $\mathrm{E}[f_{t+1}(q_{t+1}, \bar{\mu}_{t+1})] = f_t(q_t, \bar{\mu}_t)$) for any selected price p_t. In other words, the evolution of the market signal is a martingale. This would correspond to an extreme case where the current market signal is a firm indicator of future demand, such that the manufacturer expects the current signal to sustain regardless of the price charged. As an example, consider the following scenario where the manufacturer tracks the *value* f_t of the market signal itself and recursively updates it:

$$\begin{aligned} f_t &= (1-\theta)f_{t-1} + \theta \frac{d_{t-1}}{\mathrm{E}[\xi_{t-1}]p_{t-1}^{-b}} \\ &= (1-\theta)f_{t-1} + \theta f_{t-1} \frac{\epsilon_{t-1}}{\mathrm{E}[\xi_{t-1}]}. \end{aligned} \tag{22}$$

In this case, the manufacturer needs to know only f_t to update the market signal and the cumulative commitments q_t, to determine its profit and optimal course of action. This does not suggest, however, that f_t is independent of the q_t. As a matter of fact, substituting $q_t - q_{t-1}$ for d_t and accordingly taking $\bar{\mu}_t = (f_{t-1}, q_{t-1}, p_{t-1})$, Equation (22) can be stated equivalently as $f_t(q_t, \bar{\mu}_t) = (1-\theta)f_{t-1} + \theta(q_t - q_{t-1})/(\mathrm{E}[\xi_{t-1}]p_{t-1}{}^{-b})$, which is a linear increasing function of cumulative commitments q_t. Also, at time zero, $\mathrm{E}[f_t(q_t, \bar{\mu}_t)] = 1$ for any given t and price path. Hence, our structural results on the form of the optimal policy for acquiring advance sales information apply to this martingale evolution model as well.

It is also possible to envision a case where the market signal depends only on the cumulative commitments q_t and not on past prices or other historical information. In this case, the linear increasing signal $f_t(q_t)$ would model strictly the "word-of-mouth effect" created by the cumulative number of early purchasers. This type of dependency of future demand on past sales is a common feature in new product diffusion models (as in Bass 1969). This case may occur when the manufacturer announces the prices privately to each potential customer without revealing historical information. Residential real estate developers sometimes use such a selling strategy. Alternatively, such a market signal function could approximately model a case where early customers are relatively insensitive to prices. Our structural results in §§4 and 5 would apply to this case as well, with the added simplification that the optimal control-band policies would be state independent.

Connection with Demand Learning Models. Although there is no formal learning of demand parameters in our demand model, the functional form of the market signal makes it applicable to a class of Bayesian models. Bayesian models of demand learning involve a multiple period horizon whereby the demand in each period follows a known distribution with an unknown parameter or vector of parameters (say, ω). There is a known prior distribution of ω, which is updated on the basis of a sufficient statistic $\mathcal{S}_t$ as time progresses and demand realizes. For a certain class of a conjugate family of distributions, $\mathcal{S}_t$ is cumulative past sales q_t or a function of it, and its effect on the demand distribution can be factored out as a scaling function in the same spirit as our market signal function $f_t(\cdot, \cdot)$. This approach was first used by Scarf (1960) to solve the dynamic inventory management problem efficiently, and it was later extended by Azoury (1985). Next we provide some specific examples to illustrate the applicability of our framework and results in this setting. We refer the reader to Azoury (1985) for a full account of the required conditions, details, and other examples.

Suppose that the market demand potential in each period ξ_t are independent with distribution $\xi_t = k_t \xi$, where k_ts ($\sum_{t=1}^{T} k_t = 1$) are known scalars. Consider the case $k_t = 1/T$ and an exogenous, fixed pricing scheme $p_t = p$ for $t = 1, \ldots, T$. This implies the demands across periods are independent and distributed identically. When the distribution belongs to the Gamma family with unknown scale parameter ω that also has a Gamma prior distribution, cumulative sales q_t is the sufficient statistic for updating demand (hence $\bar{\mu}_t = \varnothing$ for all t). Furthermore, the Bayes estimate of demand in period t can be written as $f_t(q_t)D_t$, where $f_t(q_t) = a + q_t$ and $a > 0$ is a known constant, and the distribution of D_t only depends on t (Scarf 1960, Azoury 1985). Notice that $f_t(q_t)$ is increasing linear in q_t. Consequently, our optimal policy results in §4 apply to this scenario. When k_t and p_t are nonidentical (but still exogenous), defining $k'_t = k_t p_t^{-b}$, we have $\xi_t = k'_t \xi$, which are independent but no longer identically distributed. In this case, q_t is no longer a sufficient statistic; it is necessary to know the history of past demand realizations (hence q_ts), causing a significant increase in the dimensionality of the state space.[12] When prices are decision variables, it becomes necessary to track past prices that define the scalars k'_t. Nevertheless, given the history of past prices and commitments, the scaling function can be expressed as a linear function of q_t, which implies that our state-dependent optimal policy results apply. The details are deferred to the electronic companion to this paper.

Other Price Functions. In a multiplicative demand environment, there are alternatives to the iso-elastic price function $d(p) = p^{-b}$. This form facilitates the derivation of unique optimal regular sales prices p_t^s in Theorem 3. As shown in Song et al. (2008), the uniqueness is guaranteed when the curvature of $d(p)$, given as $d(p)(d''(p)/d'(p)^2)$, is increasing in p and is not too large (see their Assumption 2 for details). A large class of price functions, including the iso-elastic one, fits into this category. Some other examples include $d(p) = (a - bp)^\gamma (a > 0, b > 0, \gamma > 0)$, $d(p) = ae^{-bp} (a > 0, b > 0)$, $d(p) = a - p^b (a > 0, b > 1)$, and $a - e^{-bp} (a > 0, b > 0)$. Hence all the results remain valid for more general price functions. The only complication would arise when the installed capacity is sold via dynamic pricing because the iso-elastic function facilitates solving the second-stage DP numerically.

Other Pricing Strategies. The flexibility to set advance and regular sales prices optimally would generate the highest profits for the manufacturer. However, as also noted earlier, there are some practical scenarios in which these prices are predetermined by the manufacturer or the market. For example, Bitran and Mondschein (1997) discuss a pricing strategy with announced premiums or discounts. Under this pricing strategy only the initial price p_1 is a decision; the remaining prices are $p_t = \beta p_{t-1}$, where $\beta > 1$ for the announced premium (mark-up pricing) strategy and $\beta < 1$ for the announced discount (mark-down pricing) strategy. Note that for any p_1 and β, the resulting problem is equivalent to the exogenous pricing case. Hence the optimal initial p_1 and the resulting profit can be obtained by taking one extra step and searching over $p_1 \in \mathcal{R}_1$ to maximize expected profit.

The area of information acquisition for capacity planning offers a fertile avenue for future research. This paper takes a first step toward addressing pricing strategies to acquire demand information used for capacity planning. There are other possible research directions. One possibility is to explore the impact of advance selling when multiple products can be produced given a flexible capacity or when product substitution is a possibility. For example, Netessine et al. (2002) show that one can gain significant benefits if the capacity decision incorporates the possibility of upward substitution, i.e., satisfying customer demand by a better product. We consider the impact of pricing and advance selling strategy implemented prior to the capacity decision, whereas they consider the impact of a substitution strategy implemented after capacity is set. An interesting research avenue is to investigate the joint effect of both. We leave these for future research.

9. Appendix. Proofs

We use the notation f_{χ_t} and F_{χ_t} to denote the pdf and cdf of distribution χ_t.

Proof of Theorem 1. The proof is based on an induction argument. Before the inductive proof we first show that $H_t(q_t, \bar{\mu}_t)$ is linear in q_t for all t. To do so, we substitute $\pi_t^*(q_t, \bar{\mu}_t)$ defined in (3) to Equation (8) and rearrange terms to derive

$$H_t(q_t,\bar{\mu}_t) = f_t(q_t,\bar{\mu}_t)\{(p_t-\alpha^{T-t}(c_p+c_{t+1}))\mathrm{E}[\xi_t]p_t^{-b}-\alpha^{T-t}\Gamma_t^*\} + \alpha^{T-t}E[f_{t+1}(q_{t+1},\bar{\mu}_{t+1})]\Gamma_{t+1}^* - \alpha^{T-t}(c_{t+1}-c_t)q_t. \quad (23)$$

The expectation of $f_{t+1}(\cdot\,,\cdot)$ is with respect to $D_t(p_t \mid q_t, \bar{\mu}_t)$. The third term is linear. As $f_t(q_t, \bar{\mu}_t)$ is linear, the first term is also linear in q_t. Since $q_{t+1} = q_t + f_t(q_t, \bar{\mu}_t)\xi_t p_t^{-b}$, by same reason, $\mathrm{E}[f_{t+1}(q_{t+1}, \bar{\mu}_{t+1})]$ is also linear in q_t. This proves the linearity of $H_t(q_t, \bar{\mu}_t)$.

To initiate the inductive argument, note for $t = T-1$ that $\tilde{H}_{T-1}(q_{T-1}, \bar{\mu}_{T-1}) = H_{T-1}(q_{T-1}, \bar{\mu}_{T-1})$, which is linear (and hence convex) in q_{T-1}, proving part 1 for $T-1$. If $\tilde{H}_{T-1}(q_{T-1}, \bar{\mu}_{T-1})$ is increasing in q_{T-1}, then it can cross zero either once or not at all. In the former case, $L_{T-1}(\bar{\mu}_{T-1}) = -\infty$ and $U_{T-1}(\bar{\mu}_{T-1}) < \infty$, and it is optimal to continue advance selling if $q_{T-1} > U_{T-1}(\bar{\mu}_{T-1})$. In the latter case, $L_{T-1}(\bar{\mu}_{T-1}) = U_{T-1}(\bar{\mu}_{T-1}) = \infty$ and it is optimal to continue advance selling. If $\tilde{H}_{T-1}(q_{T-1}, \bar{\mu}_{T-1})$ is decreasing in q_{T-1}, it can again hit zero either once or not at all. In either case, we have $L_{T-1}(\bar{\mu}_{T-1}) < \infty$ and $U_{T-1}(\bar{\mu}_{T-1}) = \infty$, and it is optimal to continue advance selling if $q_{T-1} \leqslant L_{T-1}(\bar{\mu}_{T-1})$ and stop otherwise. Noticing that the function $\max\{0, x\}$ is increasing convex, and recognizing that increasing convex transformation of a convex function is still convex, $V_{T-1}(q_{T-1}, \bar{\mu}_{T-1}) = \max\{0, \tilde{H}_{T-1}(q_{T-1}, \bar{\mu}_{T-1})\}$ is convex, proving part 3 for $t = T-1$.

Suppose for an induction argument that part 1 is true for some $t+1 < T-1$. This implies $\tilde{H}_{t+1}(\cdot, \bar{\mu}_{t+1})$ can cross zero at most twice and those points are precisely defined by $L_{t+1}(\bar{\mu}_{t+1})$ and $U_{t+1}(\bar{\mu}_{t+1})$. Hence, for $q \in [L_{t+1}(\bar{\mu}_{t+1}), U_{t+1}(\bar{\mu}_{t+1})]$, we have $\tilde{H}_{t+1}(q, \bar{\mu}_{t+1}) < 0$, which implies it is optimal to stop advance selling. Otherwise, it is optimal to continue advance selling, proving part 2 for $t+1$. Since $\max\{0, x\}$ is increasing convex and increasing convex transformation of a convex function is still convex, $V_{t+1}(q, \bar{\mu}_{t+1})$ is convex in q, proving part 4 for $t+1$. To conclude the induction argument, we show that part 4 for $t+1$ implies part 1 for t. Note that $\alpha E_{\xi_t}[V_{t+1}(q_{t+1}, \bar{\mu}_{t+1})]$ is convex because (i) the update $q_{t+1} = q_t + f_t(q_t, \bar{\mu}_t)\xi_t p_t^{-b}$ is increasing convex in q_t, (ii) the composition of increasing convex function is convex, and (iii) convexity is preserved under expectation. Since $H_t(q_t, \bar{\mu}_t)$ is linear, the sum $\tilde{H}_t(q_t, \bar{\mu}_t)$ is also convex, proving part 1 for t. This concludes the induction argument. □

Proof of Theorem 2. To prove part 1, note that when $f_t(q_t, \bar{\mu}_t) \equiv 1 \;\; \forall\, t, q_t$, and $\bar{\mu}_t$, $H_t(q_t, \bar{\mu}_t)$ in Equation (8) is independent of $\bar{\mu}_t$ and is given as

$$H_t(q_t) = (p_t - \alpha^{T-t}(c_p + c_{t+1}))\,\mathrm{E}[\xi_t]p_t^{-b} - \alpha^{T-t}[\Gamma_t^* - \Gamma_{t+1}^*] - \alpha^{T-t}(c_{t+1} - c_t)q_t.$$

Hence, if $c_{t+1} > c_t$ for all t, then $H_t(q_t)$ is strictly decreasing and linear in q_t. Then it is easy to establish inductively that $\tilde{H}_t(q_t)$ and $V_t(q_t)$ are also independent of $\bar{\mu}_t$ and strictly decreasing convex functions of q_t. Hence $U_t = \infty$ and $L_t < \infty$ for all t, proving the optimality of state-independent threshold policy.

When $c_{t+1} = c_t \;\; \forall\, t$, the last term in $H_t(q_t)$ also drops. Consequently, $H_t(q_t)$, $\tilde{H}_t(q_t)$ and $V_t(q_t)$ are all independent of q_t and equal constants H_t, $\tilde{H}_t$, and V_t, respectively. Then it is easy to verify that the optimal stopping time for acquiring advance sales information is the first $t \in [1, T]$ such that $\tilde{H}_t = 0$.

To prove part 2, recall from Theorem 1 and its proof that the structure of the policy is driven by $H_t(q_t, \bar{\mu}_t)$ given in Equation (23), which is linear in q_t. For any t, $\bar{\mu}_t$, and c_t, for a sufficiently large c_{t+1}, the term in $\{\cdot\}$ is decreasing because Γ_t^*, defined in (3), does not depend on c_{t+1}. The second term is also decreasing in q_t because Γ_{t+1}^* is negative for large c_{t+1} (the higher the cost of building capacity, the lower will be optimal profit from remaining customers), while $E[f_{t+1}(q_{t+1}, \bar{\mu}_{t+1})]$ does not depend on c_{t+1}. The last terms are clearly decreasing. As a result, $H_t(q_t, \bar{\mu}_t)$ is decreasing in q_t. This means that when c_{t+1} is sufficiently larger than c_t,, $H_t(q_t, \bar{\mu}_t)$ would be decreasing in q_t. Next, one can establish inductively that $\tilde{H}_t(q_t, \bar{\mu}_t)$ and $V_t(q_t, \bar{\mu}_t)$ are decreasing convex functions of q_t because this property is preserved under $\max\{0, f(x)\}$ operator. Hence, $U_t(\bar{\mu}_t) = \infty$ for all t, proving the optimality of state-dependent threshold policy. $\square$

Proof of Theorem 3. Let p_t denote the selling season price set in period t, and p_t^s is optimal value. Clearly, p_t^s maximizes $\pi_t(p_t, S_t \mid q_t, \bar{\mu}_t)$ defined in (2) over p_t. Substituting $y_t \equiv S_t/(f_t(q_t, \bar{\mu}_t))$ in (2), we get

$$\begin{aligned} &\pi_t(p_t, S_t \mid q_t, \bar{\mu}_t) = f_t(q_t, \bar{\mu}_t)\hat{\pi}_t(p_t, y_t), \qquad \text{where} \\ &\hat{\pi}_t(p_t, y_t) = \{(p_t - c_p - c_t)y_t \\ &\qquad\qquad - (p_t - c_p + c_u)\,\mathrm{E}[y_t - \chi_t p_t^{-b}]^+\}. \end{aligned} \tag{24}$$

Hence, maximizing (2) for a given commitment q_t and history $\bar{\mu}_t$ boils down to maximizing (24). Recall that since IFR property is closed under convolutions (Barlow and Proschan 1975), χ_t is also IFR, meaning its failure rate $h_t(x) = f_{\chi_t}(x)/(1 - F_{\chi_t}(x))$ is increasing. Since χ_t is IFR, it also has increasing generalized failure rate (IGFR), i.e., $xh_t(x)$ is also increasing (Lariviere and Porteus 2001).

Song et al. (2008) study the optimal ordering and pricing problem for a newsvendor with order-up-to level y, retail price p, purchase cost w, and salvage value b ($p > w > b \geqslant 0$). They establish (in Theorem 1) the existence of a unique optimal (y^*, p^*) pair under multiplicative demand for a large class of demand functions that includes the iso-elastic function when the distribution of the underlying uncertainty is IGFR. They derive the optimal pair (y^*, p^*) sequentially, i.e., they first determine the unique optimal price $p(y)$ for a given y and then derive the unique optimal y^* and resulting p^*. Observe that (24) is equivalent to the standard newsvendor function with $p = p_t$, $w = c_p + c_t$, and $b = c_p - c_u$. Hence, their result applies as long as $p_t > c_p + c_t > c_p - c_u \geqslant 0$. Note that the second inequality is immediately satisfied. Consequently, when $c_p \geqslant c_u$ and the first inequality holds, (24) has a unique optimal (y_t^*, p_t^s). Next we show that the first inequality is true for a candidate optimal stocking level and price pair and hence the pair is also the unique maximizer.

To do so, it is convenient to conduct the *stocking factor* transformation (Petruzzi and Dada 1999) $z_t = y_t/p_t^{-b}$ and hence write (24) equivalently as

$$\begin{aligned} \hat{\pi}_t(p_t, z_t) = p_t^{-b}\{&(p_t - c_p - c_t)z_t \\ &- (p_t - c_p + c_u)\,\mathrm{E}[z_t - \chi_t]^+\}. \end{aligned} \tag{25}$$

For a fixed z_t, taking the derivative of (25), we obtain after some manipulation

$$\begin{aligned} &\frac{\partial \hat{\pi}_t(p_t, z_t)}{\partial p_t} \\ &\quad = (b-1)p_t^{-(b+1)}(z_t - \mathrm{E}[z_t - \chi_t]^+) \\ &\qquad \cdot \left\{ \frac{b}{b-1}\left(c_p + \frac{c_t z_t + c_u \,\mathrm{E}[z_t - \chi_t]^+}{z_t - \mathrm{E}[z_t - \chi_t]^+} \right) - p_t \right\}. \end{aligned} \tag{26}$$

Setting the derivative to zero and solving for p_t yields the unique optimal price

$$p_t(z_t) = \left(\frac{b}{b-1}\right)\left(c_p + \frac{c_t z_t + c_u \,\mathrm{E}[z_t - \chi_t]^+}{z_t - \mathrm{E}[z_t - \chi_t]^+} \right)$$

as in (11) because $p_t(z_t) > c_p + c_t$ for all $z_t \geqslant 0$. Substituting $p_t(z_t)$ in (25) results in

$$\begin{aligned} \hat{\pi}_t(z_t) &= \hat{\pi}_t(p_t(z_t), z_t) \\ &= \frac{1}{b} p_t(z_t)^{-(b-1)}(z_t - \mathrm{E}[z_t - \chi_t]^+). \end{aligned} \tag{27}$$

Taking the derivative of (27) we get

$$\begin{aligned} &\frac{\partial \hat{\pi}_t(z_t)}{\partial z_t} \\ &\quad = p_t(z_t)^{-b}(1 - F_{\chi_t}(z_t)) \left\{ (p_t(z_t) - c_p + c_u) - \frac{c_t + c_u}{1 - F_{\chi_t}(z_t)} \right\}. \end{aligned}$$

Hence z_t^* is the unique solution of $(p_t(z_t) - c_p + c_u) - (c_t + c_u)/(1 - F_{\chi_t}(z_t)) = 0$ as stated in the theorem, and $p_t^s = p_t(z_t^*)$.

By definition $S_t^* = f_t(q_t, \bar{\mu}_t)z_t^*(p_t^s)^{-b}$, so that $Q_t^* = q_t + S_t^*$. Substituting z_t^* in (27) results in Γ_t^* given by (12) and $\pi_t^*(q_t, \bar{\mu}_t) = f_t(q_t, \bar{\mu}_t)\Gamma_t^*$. Hence, $\Pi_t^*(q_t, \bar{\mu}_t) = \Pi_t(p_t^s, S_t^* \mid q_t, \bar{\mu}_t) = \sum_{k=1}^{t-1} p_k d_k - (c_p + c_t)q_t + f_t(q_t, \bar{\mu}_t)\Gamma_t^*$. $\square$

Proof of Theorem 4. Note that $p_t^s > c_p + c_t$ follows directly from (11) in Theorem 3, since $b/(b-1) > 1$ and $z_t/(z_t - \mathrm{E}[z_t - \chi_t]^+) > 1$. In order to prove the remaining results, it is convenient to change the order of optimization of the profit function $\hat{\pi}_t(p_t, z_t)$ given by (25). Suppose that p_t is fixed. Let $z_t(p_t)$ denote the corresponding optimal stocking factor, which is given uniquely by the equation

$$P(\chi_t > z_t) = \frac{c_t + c_u}{p_t - c_p + c_u}. \tag{28}$$

Substituting $z_t(p_t)$ into $\hat{\pi}_t(p_t, z_t)$, we can write

$$\frac{\partial\hat{\pi}_t(p_t, z_t(p_t))}{\partial p_t} = \frac{\partial\hat{\pi}_t(p_t, z_t)}{\partial p_t} + \frac{\partial\hat{\pi}_t(p_t, z_t)}{\partial z_t}\frac{\partial z_t}{\partial p_t}$$

at $z_t = z_t(p_t)$. Noting that the second term in $(\partial\hat{\pi}_t(p_t, z_t(p_t)))/\partial p_t$ equals zero at $z_t = z_t(p_t)$, it follows from (26) that the optimal price p_t^s is given as the unique solution to

$$\left(\frac{b}{b-1}\right)\left\{c_p + \frac{c_t z_t(p_t) + c_u \mathrm{E}[z_t(p_t)-\chi_t]^+}{z_t(p_t) - \mathrm{E}[z_t(p_t)-\chi_t]^+}\right\} - p_t = 0. \quad (29)$$

Furthermore, for $p_t < p_t^s$, the left-hand side of (29) is positive and for $p_t > p_t^s$, it is negative.

We first characterize the change in the optimal stocking factor (28) and then the resulting change in (29). This enables us to prove the impact on optimal price p_t^s. From the joint effect, we can then establish the effect on optimal capacity level $Q_t^* = q_t + f_t(q_t, \bar{\mu}_t) z_t^*(p_t^s)^{-b}$. The proof for each cost element (c_p, c_t, c_u) follows identical logic and intermediate steps. For this reason, in order to avoid repetition, we provide a detailed proof for only one of them, namely the production cost c_p. Taking derivatives with respect to c_p, for a fixed p_t, we have $\partial z_t(p_t)/\partial c_p = -(c_t + c_u/((p_t - c_p + c_u)^2 f_{\chi_t}(z_t(p_t))) < 0$. Let

$$\tilde{O}(p_t) = c_p + \frac{c_t z_t(p_t) + c_u E[z_t(p_t) - \chi_t]^+}{z_t(p_t) - E[z_t(p_t) - \chi_t]^+}.$$

Note from (29) that p_t^s is increasing in c_p if $\partial\tilde{O}(p_t)/\partial c_p > 0$.

$$\begin{aligned}
&\frac{\partial\tilde{O}(p_t)}{\partial c_p}\\
&= 1 + (c_t + c_u)\frac{z_t(p_t)F_{\chi_t}(z_t(p_t)) - E[z_t(p_t)-\chi_t]^+}{(z_t(p_t) - E[z_t(p_t)-\chi_t]^+)^2}\frac{\partial z_t(p_t)}{\partial c_p},\\
&= 1 - \left(\frac{c_t + c_u}{p_t - c_p + c_u}\right)^2 \frac{z_t(p_t)F_{\chi_t}(z_t(p_t)) - E[z_t(p_t)-\chi_t]^+}{f_{\chi_t}(z_t(p_t))(z_t(p_t) - E[z_t(p_t)-\chi_t]^+)^2},\\
&= 1 - \frac{(1-F_{\chi_t}(z_t(p_t)))^2}{f_{\chi_t}(z_t(p_t))}\frac{z_t(p_t)F_{\chi_t}(z_t(p_t)) - E[z_t(p_t)-\chi_t]^+}{(z_t(p_t) - E[z_t(p_t)-\chi_t]^+)^2}.
\end{aligned}$$

The last equality is from (28). Notice that

$$\begin{aligned}
&\frac{(1-F_{\chi_t}(z_t(p_t)))^2}{f_{\chi_t}(z_t(p_t))}\frac{z_t(p_t)F_{\chi_t}(z_t(p_t)) - E[z_t(p_t)-\chi_t]^+}{(z_t(p_t) - E[z_t(p_t)-\chi_t]^+)^2}\\
&< \frac{(1-F_{\chi_t}(z_t(p_t)))^2}{f_{\chi_t}(z_t(p_t))}\frac{F_{\chi_t}(z_t(p_t))}{z_t(p_t) - E[z_t(p_t)-\chi_t]^+}\\
&< 1 - F_{\chi_t}(z_t(p_t)) < 1.
\end{aligned}$$

The first inequality is evident. The second one is obtained by bounding the function $z_t - E[z_t - \chi_t]^+$ using the fact that χ_t is IFR. In particular,

$$z_t - E[z_t - \chi_t]^+ = \int_0^{z_t}(1 - F_{\chi_t}(u))\,du \geqslant \frac{(1-F_{\chi_t}(z_t))}{f_{\chi_t}(z_t)}F_{\chi_t}(z_t).$$

Consequently, $\partial\tilde{O}(p_t)/\partial c_p > 0$, and hence p_t^s is increasing in c_p. With this result, using implicit differentiation on (29), it is easy to verify that $\partial p_t^s/\partial c_p > 1$, which immediately implies (from (28)) that $\partial z_t^*/\partial c_p < 0$. Since p_t^s is increasing and z_t^* is decreasing, Q_t^* is decreasing in c_p. □

Proof of Theorem 5. Similar to the proof of Theorem 1, first note that $H_t(p_t, q_t, \bar{\mu}_t)$ in Equation (15) is linear in q_t for a given p_t and $\bar{\mu}_t$. The rest of the proof is based on an induction argument. For $t = T-1$, note that $R_{T-1}(p_{T-1}, q, \bar{\mu}_{T-1}) = H_{T-1}(p_{T-1}, q, \bar{\mu}_{T-1})$ which is linear in q, proving part 1 for $T-1$. This implies that if $H_{T-1}(p_{T-1}, q, \bar{\mu}_{T-1})$ is increasing (resp., decreasing) in q then $R_{T-1}(p_{T-1}, q, \bar{\mu}_{T-1})$ is also increasing (resp., decreasing) in q. Next we show that this implies $\tilde{H}_{T-1}(q, \bar{\mu}_{T-1})$ is also increasing (resp., decreasing) in q. Define $q_1 < q_2$ and let $p_1 \equiv \arg\max_p R_{T-1}(p, q_1, \bar{\mu}_{T-1})$ and $p_2 \equiv \arg\max_p R_{T-1}(p, q_2, \bar{\mu}_{T-1})$. When $R_{T-1}(p, q, \bar{\mu}_{T-1})$ is decreasing in q, we have $\tilde{H}_{T-1}(q_2, \bar{\mu}_{T-1}) = R_{T-1}(p_2, q_2, \bar{\mu}_{T-1}) < R_{T-1}(p_2, q_1, \bar{\mu}_{T-1}) < R_{T-1}(p_1, q_1, \bar{\mu}_{T-1}) = \tilde{H}_{T-1}(q_1, \bar{\mu}_{T-1})$. Hence, $\tilde{H}_{T-1}(q, \bar{\mu}_{T-1})$ is also decreasing in q. When $R_{T-1}(p, q, \bar{\mu}_{T-1})$ is increasing in q, we have $\tilde{H}_{T-1}(q_1, \bar{\mu}_{T-1}) = R_{T-1}(p_1, q_1, \bar{\mu}_{T-1}) < R_{T-1}(p_1, q_2, \bar{\mu}_{T-1}) < R_{T-1}(p_2, q_2, \bar{\mu}_{T-1}) = \tilde{H}_{T-1}(q_2, \bar{\mu}_{T-1})$. Hence $\tilde{H}_{T-1}(q, \bar{\mu}_{T-1})$ is also increasing in q. This implies parts 3 and 4 for $T-1$ along the same arguments as in the proof of Theorem 1.

Next, suppose for an induction argument that Part 1 is true for t. This implies part 2 because convexity is preserved under maximization (Porteus 2002, p. 226). Hence, given $\bar{\mu}_t$ $\tilde{H}_t(q, \bar{\mu}_t)$ can cross zero at most twice, and those points are given precisely as $L_t(\bar{\mu}_t)$ and $U_t(\bar{\mu}_t)$. From convexity, it follows also that $\tilde{H}_t(q, \bar{\mu}_t) \leqslant 0$ on $L_t(\bar{\mu}_t) \leqslant q \leqslant U_t(\bar{\mu}_t)$, in which case it is optimal to stop advance selling. Otherwise, it is optimal to continue advance selling, proving part 3. Noting that the function $\max\{0, x\}$ is increasing convex, and increasing convex transformation of a convex function is still convex, $V_t(q, \bar{\mu}_t) = \max\{0, \tilde{H}_t(q, \bar{\mu}_t)\}$ is also convex, proving part 4 for t. To complete the proof, we show that part 4 for t implies part 1 for $t-1$. Note that $q_t = q_{t-1} + f_{t-1}(q_{t-1}, \bar{\mu}_{t-1})\xi_{t-1}{p_{t-1}^*}^{-b}$ is linear increasing in q_{t-1}. Hence, $\alpha E[V_t(q_t, \bar{\mu}_t)]$ is also convex in q_{t-1}. Since $H_{t-1}(p_{t-1}, q_{t-1}, \bar{\mu}_{t-1})$ is linear in q_{t-1}, the sum $R_{t-1}(p_{t-1}, q_{t-1}, \bar{\mu}_{t-1})$ is also convex in q_{t-1}, proving part 1 for $t-1$ and concluding the induction argument. □

10. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at http://or.journal.informs.org/.

Endnotes

1. During September 2002, Christer Lundberg from Ericsson presented an advance selling strategy for long range forecasting during the Ericsson Supply Chain Academy Conference in Sweden.

2. This information could be the vector of past prices $\bar{p}_t \equiv (p_1, \ldots, p_{t-1})$ offered up until that period or some function of them. It could also include any other relevant information excluding q_t.
3. Linear increasing means that higher commitments signal the potential for stronger future demand. For example, Carlson (1983) studies apparel sales data of a department store and shows that given the regular and mark-down prices, the post-mark-down sales rate is a linear function of the pre-mark-down sales rate. Some Bayesian learning models satisfy this assumption as well.
4. Note that $\mathrm{E}[f_t(q_t, \bar{\mu}_t)]$ can be equal to any constant. Without loss of generality, we take the constant to be 1.
5. Similarly, its expected value $\mathrm{E}[f_{t+1}(q_{t+1}, \bar{\mu}_{t+1})]$ taken at time t can be higher than, lower than, or equal to $f_t(q_t, \bar{\mu}_t)$. This models a scenario where the current market signal is only a partial determinant of (expected) future demand. In the more extreme case, $\mathrm{E}[f_{t+1}(q_{t+1}, \bar{\mu}_{t+1})] = f_t(q_t, \bar{\mu}_t)$ for any price p_t, which models a scenario where the predictive value of the current market signal is very strong. In this case the manufacturer expects the current signal to sustain at the same level, regardless of the price charged.
6. The specification of $\phi(\cdot)$ depends on the definition of $\bar{\mu}_t$. For example, it may not be necessary to know both p_t and d_t to update $\bar{\mu}_t$.
7. The manufacturer can also ensure positive return from advance purchasers by requiring prices to be such that $p_t \geqslant c_p + \max_{t=1,\ldots,T}\{c_t\}$ for all t. This assumption, however, is not required for our analysis.
8. A constant α implies that the discount factor is stationary over time and the length of the periods are not too different. Otherwise, period-specific discount factors α_t can be used.
9. We remark that the optimal policy remains as a control-band when predictive value is very strong, i.e., when expectation $\mathrm{E}[f_{t+1}(q_{t+1}, \bar{\mu}_{t+1})]$ taken at time period t equals $f_t(q_t, \bar{\mu}_t)$ for any price path.
10. Note that the market signal can also be expressed recursively, i.e.,

$$f_{t+1}(q_{t+1}, \bar{\mu}_{t+1}) = f_t(q_t, \bar{\mu}_t) + \theta\epsilon_t p_t^{-b}(f_t(q_t, \bar{\mu}_t)\bar{\mu}_t - q_t).$$

We observe that when $q_t > \bar{\mu}_t$ (which implies $f_t(q_t, \bar{\mu}_t) > 1$), the next period's expected market signal $\mathrm{E}[f_{t+1}(q_{t+1}, \bar{\mu}_{t+1})]$ is still larger than 1 but is less than $f_t(q_t, \bar{\mu}_t)$. Hence, although the manufacturer has sold more than she initially expected, she does not necessarily expect the future demand to arrive at exactly the same strength. In other words, there is smoothing of the advance sales information provided by the market signal function, the extent of which is determined by θ. The opposite scenario $q_t < \bar{\mu}_t$ can be interpreted similarly.
11. An alternative way to model is to assume that the manufacturer can change prices in periods $t+1, \ldots, T$ where t is the stopping time. Our structural results remain valid under this model as well.
12. The sufficient statistics is $\mathcal{S}_t = \sum_{j=1}^{t-1}(d_j/k'_j)$, where d_j is the realized demand in period j, and the scaling function is given as $k'_t(a + \mathcal{S}_t)$. Since $d_j = q_{j+1} - q_j$, taking $\bar{\mu}_t = (q_1, \ldots, q_{j-1})$, $\mathcal{S}_t$ and hence the scaling function can be stated equivalently as a linear increasing function of q_t.

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An Electronic Companion to
"Information Acquisition for Capacity Planning via Pricing and Advance Selling: When to Stop and Act?"

Tamer Boyaci and Özalp Özer

Operations Research

Performance of the Heuristic Pricing Policy

We test the performance of this heuristic by comparing the resulting expected profit to that of optimal pricing policy. To do so, we test nine settings for each δ, c_0 and σ and report the resulting profits and the optimality gap ϵ in Table 1. Let G_h be the optimal expected profit under the heuristic pricing strategy. We use regression to compare G_h to the optimal profit G^* and report the R^2.

δ	G^*	G_h	$\epsilon(\%)$	c_0	G^*	G_h	$\epsilon(\%)$	σ	G^*	G_h	$\epsilon(\%)$
0.18	50.852	50.490	0.713	0	65.061	64.827	0.360	30	49.120	48.921	0.405
0.21	49.977	49.642	0.670	0.4	58.088	57.807	0.483	40	48.900	48.684	0.440
0.24	49.223	48.896	0.664	0.8	52.527	52.250	0.529	50	48.68	48.477	0.415
0.27	48.587	48.275	0.642	1.2	48.019	47.738	0.586	60	48.459	48.210	0.513
0.30	48.019	47.738	0.586	1.6	44.270	43.988	0.637	70	48.218	47.973	0.508
0.33	47.518	47.265	0.533	2.0	41.068	40.818	0.609	80	48.019	47.738	0.586
0.36	47.087	46.853	0.497	2.4	38.362	38.106	0.664	90	47.801	47.505	0.619
0.39	46.714	46.493	0.472	2.8	35.986	35.762	0.624	100	47.557	47.276	0.592
0.42	46.379	46.171	0.449	3.2	33.938	33.704	0.690	110	47.344	47.050	0.620
$R^2 = 99.99\%$, $c_0 = 2, \sigma = 80$				$R^2 = 100\%$, $\delta = 0.03, \sigma = 80$				$R^2 = 99.95\%$, $\delta = 0.03, c_0 = 2$			

Table 1: Performance of the heuristic pricing policy

Connection with Demand Learning Models

Consider the following demand learning setting which is suitable to our decision problem and framework. The market demand potential in each period is given as $\xi_t \equiv k_t\xi$ for $t = 1, ..., T$, where ξ's in each period are iid, $k_t's$ are known scalars, and hence ξ_t are independent random variables. One practical way to interpret this setup is to think of ξ as the total random market size and k_t the fraction of customers who potentially buy in period t (in this case it makes sense to have $\sum k_t = 1$). Let $d_t(p_t)$ denote the deterministic price function, which captures the effects of prices. Although any function can be used, to be able to make parallels with our framework, suppose that $d_t(p_t) = p_t^{-b}$. As a result, the distribution of demand in each period is given as $d_t(p_t)\xi_t \equiv d_t(p_t)k_t\xi \equiv m_t(p_t)\xi$.

Suppose that the distribution of ξ is unknown but belongs to a gamma distribution with unknown scale parameter ω. The density of ξ for a fixed value of ω is: $\phi(z|\omega) = \frac{\omega^\lambda z^{\lambda-1}e^{-\omega z}}{\Gamma(\lambda)}, \quad \lambda > 0, z \geq 0$. Suppose that the scale parameter ω itself has a gamma prior distribution $g(\omega) = \frac{b^a \omega^{a-1}e^{-b\omega}}{\Gamma(a)}, \quad a, b > 0, \omega \geq 0$, and the firm updates its information in a Bayesian manner over time as demand realizes. Let $\mathrm{d_t}$ denote the realized demand in period t. It is well-known from Scarf (1960) and Azoury (1985) that the sufficient statistic $\mathcal{S}_\mathbf{t}$ for updating the demand distribution is $\mathcal{S}_\mathbf{t} = \sum_{\mathbf{j=1}}^{\mathbf{t-1}} \frac{\mathrm{d_j}}{\mathbf{m_j(p_j)}}$. Furthermore, the Bayesian estimate (i.e., posterior distribution) of demand in period t has density

$$\phi_t(\mathrm{d}|\mathcal{S}_\mathbf{t}) = \frac{\Gamma(a+\lambda t)(b+\mathcal{S}_\mathbf{t})^{a+\lambda(t-1)}(\mathrm{d}/m_t(p_t))^{\lambda-1}}{m_t(p_t)\Gamma(\lambda)\Gamma(a+\lambda(t-1))(b+\mathcal{S}_\mathbf{t}+\mathrm{d}/m_t(p_t))^{a+\lambda t}}.$$

Furthermore, $\phi_t(\mathrm{d}|\mathcal{S}_\mathbf{t}) = \frac{1}{\varphi_t(\mathcal{S}_\mathbf{t})}\psi_t(\mathrm{d}/\varphi_t(\mathcal{S}_\mathbf{t}))$, where $\varphi_t(\mathcal{S}_\mathbf{t}) = m_t(p_t)(b+\mathcal{S}_\mathbf{t})$ and $\psi_t(u) = (\Gamma(a+\lambda t)u^{\lambda-1})/(\Gamma(\lambda)\Gamma(a+\lambda(t-1))(1+u)^{a+\lambda t})$. The above states that $\varphi_t(\mathcal{S}_\mathbf{t})$ is a function that scales random demand. Let $D_t(p_t|\mathcal{S}_\mathbf{t})$ denote the random demand each period given the sufficient statistic $\mathcal{S}_\mathbf{t}$. We have

$$D_t(p_t|\mathcal{S}_\mathbf{t}) = \varphi_t(\mathcal{S}_\mathbf{t})\tilde{D}_t = m_t(p_t)(b+\mathcal{S}_\mathbf{t})\tilde{D}_t, \tag{30}$$

where the distribution of $\tilde{D}_t$ (given by $\psi_t(u)$) only depends on t. If the firm decides to stop advance selling in period t, then there is no more learning, so the demand distribution does not get updated. For any given selling season price p, the remaining demand in the market is $X_t(p|\mathcal{S}_\mathbf{t}) = \sum_{j=t}^T D_j(p|\mathcal{S}_\mathbf{t}) = (b+\mathcal{S}_\mathbf{t})\sum_{j=t}^T m_j(p)\tilde{D}_j$, where $\tilde{D}_j$'s are iid with distribution given by $\psi_t(u)$.

The demand function in (30) is consistent with our market signal based demand framework. In particular, the function $(b + \mathrm{S_t})$ is akin to our market signal function $f_t(q_t, \bar{\mu}_t)$. In fact, since $q_t = \sum_{j=1}^{t-1} \mathrm{d_j}$, we can write $\mathcal{S}_\mathbf{t}$ as a function of q_t and define $f_t(q_t, \bar{\mu}_t)$ accordingly. This would require the knowledge of the entire past sequence of $\mathrm{d_j}$ for $j = 1, .., t-1$. Hence the history $\bar{\mu}_t$ would contain q_j and p_j for $j = 1, ...t-1$, (hence $m_j(p_j)$ will also be known). [14] This follows because $\mathcal{S}_\mathbf{1} = 0$, $\mathcal{S}_\mathbf{2} = \frac{q_2}{m_1(p_1)}, \ldots, \mathcal{S}_\mathbf{t} = \frac{q_t}{m_{t-1}(p_{t-1})} + \sum_{i=1}^{t-2} q_{t-i}\left(\frac{1}{m_{t-i-1}(p_{t-i-1})} - \frac{1}{m_{t-i}(p_{t-i})}\right)$ for $t \geq 2$. Consequently, for $t = 1$, $q_1 \equiv 0$, $\bar{\mu}_1 \equiv \emptyset$, $f_1(q_1, \bar{\mu}_1) \equiv 1$, and for $t \geq 2$, we have $f_t(q_t, \bar{\mu}_t)$

$$f_t(q_t, \bar{\mu}_t) = b + \frac{q_t}{m_{t-1}(p_{t-1})} + \sum_{i=1}^{t-2} q_{t-i}\left(\frac{1}{m_{t-i-1}(p_{t-i-1})} - \frac{1}{m_{t-i}(p_{t-i})}\right) \tag{31}$$

[14]Note that state space reduction is possible if the manufacturer tracks the value f_t of the market signal itself. In this case, the manufacturer would need $(f_t, \mathrm{d_t}, \mathrm{p_t})$ to update the market signal and q_t to determine the profit and the optimal course of action.

Observe that $f_t(q_t, \bar{\mu}_t)$ is linear increasing in q_t given the history $\bar{\mu}_t$. Hence the demand function given by (30) can be equivalently stated as $D_t(p_t|q_t, \bar{\mu}_t) = f_t(q_t, \bar{\mu}_t)p_t^{-b}\left(k_t\tilde{D}_t\right)$, which is has the same form as our demand model. Furthermore, when the firm stops advance selling in period t, the remaining demand can be stated as $X_t(p_t|q_t, \bar{\mu}_t) = f_t(q_t, \bar{\mu}_t)p_t^{-b}\sum_{j=t}^{T}\left(k_j\tilde{D}_j\right)$. Defining $\chi_t \equiv \sum_{j=t}^{T}\left(k_j\tilde{D}_j\right)$, we have, as in our framework $X_t(p_t|q_t, \bar{\mu}_t) = f_t(q_t, \bar{\mu}_t)p_t^{-b}\chi_t$.