Information Granularity in Fuzzy Binary GrC Model

Yuhua Qian, *Member, IEEE*, Jiye Liang, Wei-zhi Z. Wu, *Senior Member, IEEE*, and Chuangyin Dang, *Senior Member, IEEE*

***Abstract*—Zadeh's seminal work in theory of fuzzy-information granulation in human reasoning is inspired by the ways in which humans granulate information and reason with it. This has led to an interesting research topic: granular computing (GrC). Although many excellent research contributions have been made, there remains an important issue to be addressed: What is the essence of measuring a fuzzy-information granularity of a fuzzy-granular structure? What is needed to answer this question is an axiomatic constraint with a partial-order relation that is defined in terms of the size of each fuzzy-information granule from a fuzzy-binary granular structure. This viewpoint is demonstrated for fuzzy-binary granular structure, which is called the binary GrC model by Lin. We study this viewpoint from from five aspects in this study, which are fuzzy BINARY-granular-structure operators, partial-order relations, measures for fuzzy-information granularity, an axiomatic approach to fuzzy-information granularity, and fuzzy-information entropies.**

***Index Terms*—Fuzzy-information entropy, fuzzy-information granularity, granular computing (GrC), partial-order relation.**

I. Introduction

GRANULAR computing (GrC), which is a term coined jointly by Zadeh and Lin [50], plays a fundamental role in fuzzy-information granulation of human reasoning. Three basic issues in GrC are information granulation, organization, and causation. As it was pointed out in [48]–[51], the information granulation involves decomposition of whole into parts, the organization involves integration of parts into whole [15], and the causation involves association of causes with effects. This issue has been applied in relevant fields such as interval analysis, rough-set theory, cluster analysis, machine learning, and databases.

A granule is a clump of objects drawn together by indistinguishability, similarity, and proximity of functionality [11], [12], [21], [31], [32], [49]. Granulation of an object leads to a collection of granules. A granular structure is a mathematical structure of the collection of granules, in which the inner structure of each granule is visible (a granule is a white box), and the interactions among granules are detected by the visible structures [15]–[17]. Given these abstract concepts, an important task is to establish a conceptual framework for GrC. For our further development, we will briefly review several focuses of attention in GrC. They include a measure of granularity [11], [13], [14], [38], [39], [42], [45], information processing, including databases [24], [25], a framework of GrC [15], [16], [21], [34], [46], problem solving based on "granulate and conquer" principle and quotient theory [54], and multigranulation view [22], [35], [36] and their applications. It can be seen from the developments that GrC is evolving into a field of cross-disciplinary study.

In GrC, the granulation of objects induced by an equivalent relation is a set of equivalence classes, in which each equivalence class can be regarded as an (Pawlak) information granule [4], [30], [36]; the granulation of objects induced by a tolerance relation generates a set of tolerance classes, in which each tolerance class can also be seen as a tolerance information granule [10], [24], [35]. By using a general binary relation, objects are granulated into a set of information granules, which is called a binary *granular structure*. In GrC, one often needs to measure the granulation degree of objects in a given dataset, which is called *information granularity*.

In the viewpoint of GrC, information granularity of a granular structure is a measure of uncertainty about its actual structure [16], [49]. In general, the information granularity represents discernibility ability of information in a granular structure. The smaller the information granularity, the stronger its discernibility ability [26]. How to calculate the information granularity of a granular structure has always been an important issue. To date, several forms of information granularity have been proposed according to various views and targets [11], [13], [14], [22], [38], [39], [42], [45]. Wierman [42] introduced the concept of granulation measure to measure the uncertainty of information in a knowledge base. This concept has the same form as Shannon's entropy under the axiom definition. Liang *et al.* [13], [14] proposed information granularity in either complete and incomplete datasets, which has been effectively applied in attribute significance measure, feature selection, rule extraction, etc. Qian and Liang [38], [39] presented combination granulation with intuitionistic knowledge-content nature to measure the size of information granulation in a knowledge base. Xu *et al.* [45] gave an improved measure for roughness of a rough set in rough-set theory proposed by Pawlak [29], which is also an information granularity in a broad sense. In the above forms of information granularity, the partial-order relation plays a key role in characterizing the monotonicity of each of them.

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Y. H. Qian and J. Y. Liang are with the Key Laboratory of Computational Intelligence and Chinese Information Processing of Ministry of Education, School of Computer and Information Technology, Shanxi University, Taiyuan 030006, China (e-mail: jinchengqyh@126.com; ljy@sxu.edu.cn).

W. Z. Wu is with the Physic and Information Science, School of Mathematics, Zhejiang Ocean University, Zhejiang 316004, China (e-mail: wzwu@zjou.edu.cn).

C. Y. Dang is with the Department of Manufacture Engineering and Engineering Management, City University of Hong Kong, Kowloon, Hong Kong (e-mail: mecdang@cityu.edu.hk).

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Recently, Qian *et al.* [34] presented an axiomatic definition of information granularity in a knowledge base, in which several existing forms of information granularity become its special cases. From the axiomatic definition, we know that the size of information granularity does not depend on the sizes of equivalence classes (i.e., tolerance classes and maximal consistent blocks) but on some array of these classes.

As we know, by using a fuzzy-binary relation, objects are granulated into a set of fuzzy-information granules [3], [6], [7]–[9], [41], which is called a *fuzzy-binary granular structure*, which is the basis of rough-approximation operations in fuzzy-rough-set theory [1], [2], [23], [43]–[47]. In GrC based on fuzzy-set theory, one often needs to measure the granulation degree of objects in a family of fuzzy-granular structures, which is called *fuzzy-information granularity*. Although many excellent research contributions have been made in the context of fuzzy GrC [52], [53], there remains an important issue to be addressed. What is the essence of measuring a fuzzy-information granularity? As mentioned by Zadeh, in general, information granularity should characterize the granulation degree of objects from the viewpoint of hierarchy [49]. This provides a point of view that an information granularity should characterize hierarchical relationships among fuzzy-binary granular structures. To answer the question, in this investigation, we will develop an axiomatic constraint with a partial-order relation which is defined in terms of the size of each fuzzy-information granule. The viewpoint is systematically demonstrated from five aspects in this study, which include fuzzy-binary granular-structure operators, partial-order relations, measures for fuzzy-information granularity, an axiomatic approach to fuzzy-information granularity, and monotonicity of information entropy.

Information entropy and information granularity are two main approaches to measuring the uncertainty of a granular structure [13], [28], [40]. The entropy of a granular structure, as defined by Shannon [40], gives a measure of the uncertainty about its actual structure. Because that Shannon's entropy is defined by the probability of each equivalence class, it cannot be used to measure the uncertainty of a fuzzy-granular structure. For the consideration, Hu *et al.* extended Shannon's entropy to fuzzy-binary granular structures and used this variant to characterize the uncertainty of fuzzy rough sets and fuzzy probability rough sets [3], [6]. Conveniently, information entropy to measure uncertainty of a fuzzy-binary granular structure is called *fuzzy-information entropy*, which denotes the size of information content of a fuzzy-granular structure. Like the information entropy, as defined by Shannon, the fuzzy-information entropy of a fuzzy-binary granular structure should also possess the performance of measuring the uncertainty about its actual structure.

This paper is organized as follows. Section II reviews several basic concepts, such as Pawlak granular structures, tolerance granular structures, and fuzzy-binary granular structures. Section III presents four fuzzy-binary granular-structure operators to generate new fuzzy-binary granular structures. Section IV introduces three partial-order relations to characterize relationships among fuzzy-granular structures. In Section V, we propose two measures to measure fuzzy-information granularity of fuzzy-binary granular structures. Section VI develops an axiomatic approach to fuzzy-information granularity, under which several fuzzy-information granularity become its special forms. In Section VII, we introduce the concept of fuzzy-information entropy to measure information content of a fuzzy-binary granular structure and characterize its granulation monotonicity by those partial-order relations proposed in Section IV. Finally, Section VIII gives the concluding remarks.

II. Preliminaries

In this section, we will review several basic concepts, such as knowledge bases, information granules, granular structures, and fuzzy-binary granular structures.

In rough-set theory, as Pawlak has defined, a knowledge base is denoted by $(U, \Re) = (U, R_1, R_2, \ldots, R_m)$, where R_i is an equivalence relation [29]. U/R constitutes a partition of U, which is called a granular structure on U, and every equivalence class is called a Pawlak information granule. In a broad sense, an information granularity denotes average measure of Pawlak information granules (equivalence classes) induced by R.

If R_i $(i = 1, 2, \ldots, m)$ is a tolerance relation, then $(U, \Re) = (U, R_1, R_2, \ldots, R_m)$ can be called a tolerance knowledge base. Let similarity classes induced by a similarity relation (SIMR) denote the family sets $\{S_R(u) \mid u \in U\}$, which is the granular structure induced by R. A member $S_R(u)$ from SIMR will be called a tolerance information granule. In fact, $\{S_R(u) : u \in U\}$ is a binary neighborhood system (BNS) [15]–[22]. For a tolerance knowledge base, in a broad sense, an information granularity denotes average measure of tolerance information granules (tolerance classes) induced by R [13], [33].

For the above two types of knowledge bases, one can uniformly represent the granular structure induced by $P \subseteq \Re$ by a vector $K(P) = (S_P(x_1), S_P(x_2), \ldots, S_P(x_n))$ [34], $S_P(x_i)$ is the tolerance class induced by an object $x_i \in U$ with respect to P.

Above modes of information granulation in which the granules are crisp (c-granular) play important roles in a wide variety of methods, approaches, and techniques. Important though it is, crisp-information granulation has a major blind spot. More specifically, it fails to reflect the fact that perhaps most granules of human reasoning and concept formation are fuzzy (f-granular) rather than crisp [49]. Hence, generalization to fuzzy cases is necessary.

Given a universe U, $\widetilde{R}$ a fuzzy-binary relation on U, which is often denoted by the following matrix:

$$M(\widetilde{R}) = \begin{pmatrix} r_{11} & r_{12} & \ldots & r_{1n} \\ r_{21} & r_{22} & \ldots & r_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ r_{n1} & r_{n2} & \ldots & r_{nn} \end{pmatrix} \quad (1)$$

where $r_{ij} \in [0, 1]$ is the similarity between x_i and x_j.

Some operations of relation matrices are defined as

1) $\widetilde{R}_1 = \widetilde{R}_2 \Leftrightarrow \widetilde{R}_1(x, y) = \widetilde{R}_2(x, y)$;
2) $\widetilde{R} = \widetilde{R}_1 \cup \widetilde{R}_2 \Leftrightarrow \widetilde{R} = \max\{\widetilde{R}_1(x, y), \widetilde{R}_2(x, y)\}$;
3) $\widetilde{R} = \widetilde{R}_1 \cap \widetilde{R}_2 \Leftrightarrow \widetilde{R} = \min\{\widetilde{R}_1(x, y), \widetilde{R}_2(x, y)\}$;
4) $\widetilde{R}_1 \subseteq \widetilde{R}_2 \Leftrightarrow \widetilde{R}_1(x, y) \leq \widetilde{R}_2(x, y)$.

A fuzzy-binary relation generates a family of fuzzy-information granules from the universe, which is called a fuzzy-binary granular structure. The fuzzy-binary granular structure of the universe is defined as

$$K(\widetilde{R}) = (S_{\widetilde{R}}(x_1), S_{\widetilde{R}}(x_2), \ldots, S_{\widetilde{R}}(x_n)) \tag{2}$$

where $S_{\widetilde{R}}(x_i) = r_{i1}/x_1 + r_{i2}/x_2 + \cdots + r_{in}/x_n$. $S_{\widetilde{R}}(x_i)$ is the fuzzy-information granule (can be regarded as the fuzzy neighborhood of x_i) induced by x_i, and r_{ij} is the degree of x_i equivalent to x_j. Here, "+" means the union of elements. The cardinality of the fuzzy-information granule $S_{\widetilde{R}}(x_i)$ can be calculated with

$$|S_{\widetilde{R}}(x_i)| = \sum_{j=1}^{n} r_{ij} \tag{3}$$

which appears to be a natural generalization of the cardinality of a crisp set.

Given a family of fuzzy-binary granular structures $(U, \widetilde{\Re})$, for uniform representation in this paper, we also denote the fuzzy-binary granular structure induced by $\widetilde{P} \in \widetilde{\Re}$ by $K(\widetilde{P}) = (S_{\widetilde{P}}(x_1), S_{\widetilde{P}}(x_2), \ldots, S_{\widetilde{P}}(x_n))$, where $S_{\widetilde{P}}(x_i) = p_{i1}/x_i + p_{i2}/x_i + \cdots + p_{in}/x_i$. In this case, the granular structure is a BNS [15]–[22]. In addition, let $\mathbf{K}(U)$ be the collection of all fuzzy-binary granular structures on U.

In particular, for a fuzzy-binary granular structure $K(\widetilde{P}) = (S_{\widetilde{P}}(x_1), S_{\widetilde{P}}(x_2), \ldots, S_{\widetilde{P}}(x_n))$, if $p_{ii} = 1$ and $p_{ij} = 0, j \neq i, i, j \leq n$, then $|S_{\widetilde{P}}(x_i)| = 1, i \leq n$, and $\widetilde{P}$ is called a fuzzy identity relation, and we write as $\widetilde{P} = \omega$; if $p_{ij} = 1, i, j \leq n$, then $|S_{\widetilde{P}}(x_i)| = |U|, i \leq n$, and $\widetilde{P}$ is called a fuzzy universal relation, which is written as $\widetilde{P} = \delta$.

III. Fuzzy-Granular Structures

In the viewpoint of knowledge engineering, the operators on fuzzy granular structures to generate new fuzzy-granular structures are very desirable. These granular structures generated provide indispensable knowledge and find a basis in human reasoning based on a family of fuzzy-binary granular structures, in which there is an underlying algebra structure. In fact, to solve the problem in the context of (crisp) granular structures, Qian *et al.* [34] proposed four operators among granular structures and revealed the algebra structure of these granular structures (lattice structure), which can be used to generate new granular structures. In this section, we extend the four operators to fuzzy-binary granular structures to reveal this underlying algebra structure and data mining from them.

In what follows, for this purpose, we construct four operators in a family of fuzzy-binary granular structures.

Definition 1: Let $\mathbf{K}(U)$ be the collection of all fuzzy-binary granular structures on two $U, K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$ fuzzy-binary granular structures. Four operators $\cap, \cup, -$, and $\wr$ on $\mathbf{K}(U)$ are defined as

$$K(\widetilde{P}) \cap K(\widetilde{Q}) = \{S_{\widetilde{P}\cap\widetilde{Q}}(x_i) \mid S_{\widetilde{P}\cap\widetilde{Q}}(x_i) = S_{\widetilde{P}}(x_i) \cap S_{\widetilde{Q}}(x_i)\} \tag{4}$$

$$K(\widetilde{P}) \cup K(\widetilde{Q}) = \{S_{\widetilde{P}\cup\widetilde{Q}}(x_i) \mid S_{\widetilde{P}\cup\widetilde{Q}}(x_i) = S_{\widetilde{P}}(x_i) \cup S_{\widetilde{Q}}(x_i)\} \tag{5}$$

$$K(\widetilde{P}) - K(\widetilde{Q}) = \{S_{\widetilde{P}-\widetilde{Q}}(x_i) \mid S_{\widetilde{P}-\widetilde{Q}}(x_i) = S_{\widetilde{P}}(x_i) \cap \sim S_{\widetilde{Q}}(x_i)\} \tag{6}$$

$$\wr K(\widetilde{P}) = \{\wr S_{\widetilde{P}}(x_i) \mid \wr S_{\widetilde{P}}(x_i) = \sim S_{\widetilde{P}}(x_i)\} \tag{7}$$

where $x_i \in U, i \leq n$, and $\sim S_{\widetilde{P}}(x_i) = (1 - p_{i1})/x_i + (1 - p_{i2})/x_i + \cdots + (1 - p_{in})/x_i$.

Note: In (5), $K(\widetilde{P}) \cup K(\widetilde{Q})$ is not the union of two attributes but the union of two fuzzy-binary granular structures. The objective of the operator $\cup$ is to obtain a much coarser fuzzy-granular structure than $K(\widetilde{P})$ and $K(\widetilde{Q})$. The proposed four operators can be seen as intersection operation, union operation, subtraction operation and complement operation in-between fuzzy-binary granular structures, which are used to fine, coarsen, decompose fuzzy-binary granular structures and calculate complement of a fuzzy-binary granular structure, respectively. We would like to point out that these operators proposed are natural generalizations of the four operators in a neighborhood system [34].

Here, we regard $\cap, \cup, -$, and $\wr$ as four atomic formulas and finite connections on them are all formulas. Through using these operators, one can obtain new fuzzy-binary granular structures via some known fuzzy-binary granular structures on U. From Definition 1, one easily knows that these four operators $\cap, \cup, -$ and $\wr$ on $\mathbf{K}(U)$ are close.

In GrC, transformation among granular structures is an important issue which involves composition, decomposition and transformation. In Definition 1, $\cap$ and $\cup$ operators can be used to compose two fuzzy-binary granular structures to a new fuzzy-binary granular structure, where one can get a much finer fuzzy-granular space using $\cap$ operator, and one can form a much coarser one using $\cup$ operator. The operator can be understood as a decomposition mechanism and be used to generate much finer fuzzy-binary granular structures. The operator $\wr$ may be viewed as one of mappings between fuzzy-binary granular structures, which can transform one fuzzy-binary granular structure into another fuzzy-binary granular structure.

In what follows, we investigate several fundamental algebra properties of these four operators.

Theorem 1: Letting $\cap, \cup$ be two operators on $\mathbf{K}(U)$, we then have the following.

1) $K(\widetilde{P}) \cap K(\widetilde{P}) = K(\widetilde{P})$
   $K(\widetilde{P}) \cup K(\widetilde{P}) = K(\widetilde{P})$.
2) $K(\widetilde{P}) \cap K(\widetilde{Q}) = K(\widetilde{Q}) \cap K(\widetilde{P})$
   $K(\widetilde{P}) \cup K(\widetilde{Q}) = K(\widetilde{Q}) \cup K(\widetilde{P})$.
3) $K(\widetilde{P}) \cap (K(\widetilde{P}) \cup K(\widetilde{Q})) = K(\widetilde{P})$
   $K(\widetilde{P}) \cup (K(\widetilde{P}) \cap K(\widetilde{Q})) = K(\widetilde{P})$.
4) $(K(\widetilde{P}) \cap K(\widetilde{Q})) \cap K(\widetilde{R}) = K(\widetilde{P}) \cap (K(\widetilde{Q}) \cap K(\widetilde{R}))$
   $(K(\widetilde{P}) \cup K(\widetilde{Q})) \cup K(\widetilde{R}) = K(\widetilde{P}) \cup (K(\widetilde{Q}) \cup K(\widetilde{R}))$.

Theorem 2: Letting $\cap, \cup$, and $\wr$ be three operators on $\mathbf{K}(U)$, then

$$1)\quad \wr\,(K(\widetilde{P}) \cap K(\widetilde{Q})) = \wr K(\widetilde{P}) \cup \wr K(\widetilde{Q}),\ \text{and}$$

$$2)\quad \wr\,(K(\widetilde{P}) \cup K(\widetilde{Q})) = \wr K(\widetilde{P}) \cap \wr K(\widetilde{Q}).$$

Proof: For any $x_i \in U, K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U), S_{\widetilde{P}}(x_i)$ and $S_{\widetilde{Q}}(x_i)$ are the fuzzy-information granules induced by x_i in $K(\widetilde{P})$ and $K(\widetilde{Q})$, respectively.

1) According to Definition 1, for $\forall x_i \in U$, it follows that $\wr(S_{\widetilde{P}}(x_i) \cap S_{\widetilde{Q}}(x_i)) = S_\omega(x_i) \cup \sim(S_{\widetilde{P}}(x_i) \cap S_{\widetilde{Q}}(x_i))$. Hence, for $\forall p_{ij}$ from $S_\omega(x_i)$, if $j = i$, then $p_{ij} = 1$; if $j \neq i$, then $p_{ij} = 0$. Thus, if $j = i$, then $\max\{1, 1 - \min\{p_{ii}, q_{ii}\}\} = 1$; if $j \neq i$, then $\max\{0, 1 - \min\{p_{ij}, q_{ij}\}\} = 1 - \min\{p_{ij}, q_{ij}\} = \max\{1 - p_{ij}, 1 - q_{ij}\} = \max\{\max\{0, 1 - p_{ij}\}, \max\{0, 1 - q_{ij}\}\}$. Therefore, $\wr(K(\widetilde{P}) \cap K(\widetilde{Q})) = \wr K(\widetilde{P}) \cup \wr K(\widetilde{Q})$ holds.

2) From Definition 1, for $\forall x_i \in U$, one has that $\wr(S_{\widetilde{P}}(x_i) \cup S_{\widetilde{Q}}(x_i)) = S_\omega(x_i) \cup \sim(S_{\widetilde{P}}(x_i) \cup S_{\widetilde{Q}}(x_i))$. Therefore, for $\forall p_{ij}$ from $S_\omega(x_i)$, if $j = i$, then $p_{ij} = 1$; if $j \neq i$, then $p_{ij} = 0$. Hence, if $j = i$, then $\max\{1, 1 - \max\{p_{ii}, q_{ii}\}\} = 1$; if $j \neq i$, then $\max\{0, 1 - \max\{p_{ij}, q_{ij}\}\} = 1 - \max\{p_{ij}, q_{ij}\} = \min\{1 - p_{ij}, 1 - q_{ij}\} = \min\{\max\{0, 1 - p_{ij}\}, \max\{0, 1 - q_{ij}\}\}$. Therefore, $\wr(K(\widetilde{P}) \cup K(\widetilde{Q})) = \wr K(\widetilde{P}) \cap \wr K(\widetilde{Q})$ holds. ■

Let $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$, where $K(\widetilde{P}) = (S_{\widetilde{P}}(x_1), S_{\widetilde{P}}(x_2), \ldots, S_{\widetilde{P}}(x_n)), S_{\widetilde{P}}(x_i) = p_{i1}/x_i + p_{i2}/x_i + \cdots + p_{in}/x_i, K(\widetilde{Q}) = (S_{\widetilde{Q}}(x_1), S_{\widetilde{Q}}(x_2), \ldots, S_{\widetilde{Q}}(x_n))$ and $S_{\widetilde{Q}}(x_i) = q_{i1}/x_i + q_{i2}/x_i + \cdots + q_{in}/x_i$. A partial-order relation $\preceq_1$ is defined as

$$K(\widetilde{P}) \preceq_1 K(\widetilde{Q}) \Leftrightarrow S_{\widetilde{P}}(x_i) \subseteq S_{\widetilde{Q}}(x_i), i \leq n \Leftrightarrow p_{ij} \leq q_{ij}, i, j \leq n,$$

just $\widetilde{P} \preceq_1 \widetilde{Q}$.

Furthermore, $K(\widetilde{P}) = K(\widetilde{Q}) \Leftrightarrow S_{\widetilde{P}}(x_i) = S_{\widetilde{Q}}(x_i), i \leq n \Leftrightarrow p_{ij} = q_{ij}, i, j \leq n$ can be written as $\widetilde{P} = \widetilde{Q}$. $K(\widetilde{P}) \prec_1 K(\widetilde{Q}) \Leftrightarrow K(\widetilde{P}) \preceq_1 K(\widetilde{Q})$, and $K(\widetilde{P}) \neq K(\widetilde{Q})$, which is denoted by $\widetilde{P} \prec_1 \widetilde{Q}$. Clearly, $(\mathbf{K}(U), \preceq_1)$ is a poset.

Theorem 3: Let $\cap, \cup$, and $\wr$ be three operators on $\mathbf{K}(U)$, the following properties hold.

1) If $K(\widetilde{P}) \preceq_1 K(\widetilde{Q})$, then $\wr K(\widetilde{Q}) \preceq_1 \wr K(\widetilde{P})$.

2) $K(\widetilde{P}) \cap K(\widetilde{Q}) \preceq_1 K(\widetilde{P}), K(\widetilde{P}) \cap K(\widetilde{Q}) \preceq_1 K(\widetilde{Q})$.

3) $K(\widetilde{P}) \preceq_1 K(\widetilde{P}) \cup K(\widetilde{Q}), K(\widetilde{Q}) \preceq_1 K(\widetilde{P}) \cup K(\widetilde{Q})$.

Proof: The terms (2) and (3) can be easily proved from (4) and (5) in Definition 1, respectively.

From Definition 1, one can obtain that

$K(\widetilde{P}) \preceq_1 K(\widetilde{Q})$

$\Longrightarrow$ for $\forall x_i \in U, S_{\widetilde{P}}(x_i) \subseteq S_{\widetilde{Q}}(x_i), i \leq n$.

$\Longrightarrow$ for $\forall x_i \in U, p_{ij} \leq q_{ij}, i, j \leq n$.

$\Longrightarrow$ for $\forall x_i \in U$, if $j \neq i$, then $1 - q_{ij} \leq 1 - p_{ij}, i, j \leq n$; if $j = i$, then $\max\{1, 1 - q_{ij}\} = 1 = \max\{1, 1 - p_{ij}\}$.

$\Longrightarrow$ for $\forall x_i \in U, S_\omega(x_i) \cup S_{\widetilde{P}}(x_i) \subseteq S_\omega(x_i) \cup S_{\widetilde{P}}(x_i)$.

$\Longrightarrow$ for $\forall x_i \in U, \wr S_{\widetilde{P}}(x_i) \subseteq \wr S_{\widetilde{P}}(x_i)$.

$\Longrightarrow \wr K(Q) \preceq_1 \wr K(P)$.

Hence, the term (1) in this theorem holds. ■

Definition 2: Let $(L, \leq)$ be a poset, if there exist two operators $\wedge, \vee$ on $L : L^2 \longrightarrow L$ such that

1) $a \wedge b = b \wedge a, a \vee b = b \vee a$;

2) $(a \wedge b) \wedge c = a \wedge (b \wedge c), (a \vee b) \vee c = a \vee (b \vee c)$;

3) $a \wedge b = b \Longleftrightarrow b \leq a, a \vee b = b \Longleftrightarrow a \leq b$.

Then, we call L a lattice.

Furthermore, we call L a complemented lattice, if for any $a \in L$, there exists a' such that $(a')' = a$ and $a \leq b \Longleftrightarrow b' \leq a'$. If there exist $0, 1 \in L$ such that $0 \leq a \leq 1$ for any $a \in L$, then we call 0 and 1 its minimal element and maximal element, respectively.

Theorem 4: Letting $\mathbf{K}(U)$ be the collection of all fuzzy-binary granular structures on U, then $(\mathbf{K}(U), \cup, \cap)$ is a lattice.

Proof: At first, we prove that $(\mathbf{K}(U), \preceq_1)$ is a lattice.

From 2) and 4) in Theorem 1, the terms 1) and 2) in Definition 2 are obvious.

Let $K(\widetilde{P}), K(\widetilde{Q}), K(\widetilde{R}) \in \mathbf{K}(U)$, where $K(\widetilde{P}) = (S_{\widetilde{P}}(x_1), S_{\widetilde{P}}(x_2), \ldots, S_{\widetilde{P}}(x_n))$, $K(\widetilde{Q}) = (S_{\widetilde{Q}}(x_1), S_{\widetilde{Q}}(x_2), \ldots, S_{\widetilde{Q}}(x_n))$, and $K(\widetilde{R}) = (S_{\widetilde{R}}(x_1), S_{\widetilde{R}}(x_2), \ldots, S_{\widetilde{R}}(x_n))$. One can obtain that

$K(\widetilde{P}) \cap K(\widetilde{Q}) = K(\widetilde{P})$

$\Longleftrightarrow$ for $\forall x_i \in U, [x_i]_{\widetilde{P} \cap \widetilde{Q}} = S_{\widetilde{P}}(x_i)$

$\Longleftrightarrow$ for $\forall x_i \in U, S_{\widetilde{P}}(x_i) \cap S_{\widetilde{Q}}(x_i) = S_{\widetilde{P}}(x_i)$

$\Longleftrightarrow S_{\widetilde{P}}(x_i) \subseteq S_{\widetilde{Q}}(x_i)$, for $\forall x_i \in U$

$\Longleftrightarrow K(\widetilde{P}) \preceq_1 K(\widetilde{Q})$.

According to the dual principle in a lattice, one can easily get that $K(\widetilde{P}) \cup K(\widetilde{Q}) = K(\widetilde{P}) \Longleftrightarrow K(\widetilde{Q}) \preceq_1 K(\widetilde{P})$. Thus, the term (3) in Definition 2 holds.

Therefore, $(K, \cup, \cap)$ is a lattice. ■

Theorem 5: Letting $\mathbf{K}(U)$ be the collection of all fuzzy-binary granular structures on U, then $(\mathbf{K}(U), \cup, \cap, \wr)$ is a complemented lattice.

Proof: From Theorem 4, it is obvious that $(\mathbf{K}(U), \cup, \cap)$ is a lattice. Furthermore, from 1) in Theorem 2, one can get that $\wr(\wr K(\widetilde{P})) = K(\widetilde{P})$. In addition, from 3) in Definition 2, one has that

$K(\widetilde{P}) \preceq_1 K(\widetilde{Q})$

$\Longleftrightarrow$ for $\forall x_i \in U, S_{\widetilde{P}}(x_i) \subseteq S_{\widetilde{Q}}(x_i)$

$\Longleftrightarrow$ for $\forall x_i \in U, \sim S_{\widetilde{P}}(x_i) \supseteq \sim S_{\widetilde{Q}}(x_i)$

$\Longleftrightarrow$ for $\forall x_i \in U, S_\omega(x_i) \cup \sim S_{\widetilde{P}}(x_i) \supseteq S_\omega(x_i) \cup \sim S_{\widetilde{Q}}(x_i)$

$\Longleftrightarrow$ for $\forall x_i \in U, \wr S_{\widetilde{P}}(x_i) \supseteq \wr S_{\widetilde{Q}}(x_i)$

$\Longleftrightarrow \wr K(\widetilde{Q}) \preceq_1 \wr K(\widetilde{P})$.

Hence, $(\mathbf{K}(U), \cup, \cap, \wr)$ is a complemented lattice. ■

In the complemented lattice, $(\mathbf{K}(U), \cup, \cap, \wr), K(\omega)$, and $K(\delta)$ are two special fuzzy-granular spaces. For any $K(\widetilde{P}) \in \mathbf{K}(U)$, one has that $K(\omega) \preceq_1 K(\widetilde{P}) \preceq_1 K(\delta)$. Then, we call $K(\omega)$ and $K(\delta)$ the minimal element and the maximal element on the lattice $(\mathbf{K}(U), \cup, \cap, \wr)$, respectively.

From the above analysis, it is shown that these four operators ($\cup, \cap, \wr$, and $-$) can be applied to generate new fuzzy-binary granular structures from given fuzzy-binary granular structures. Therefore, this mechanism may be used for data mining and knowledge discovery from a family of fuzzy granular structures.

IV. Partial-Order Relations on Fuzzy-Granular Structures

To characterize the uncertainty of a granular structure, a partial relation plays a very important role. In this section, through revealing the drawback of partial-order relations $\preceq_1$, we will introduce two new partial-order relations in fuzzy-binary granular structures and establish the relationships among them.

In GrC, measures of information granularity usually take the size of each of information granules into account [13], [32], [34], which are used to calculate the degree of granulation of fuzzy information (or fuzzy knowledge). Clearly, the partial-order relation $\preceq_1$ cannot characterize the size nature of information granules. Based on this consideration, we introduce the following binary relation.

Definition 3: Let $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$, where $K(\widetilde{P}) = (S_{\widetilde{P}}(x_1), S_{\widetilde{P}}(x_2), \ldots, S_{\widetilde{P}}(x_n)), S_{\widetilde{P}}(x_i) = p_{i1}/x_i + p_{i2}/x_i + \cdots + p_{in}/x_i, K(\widetilde{Q}) = (S_{\widetilde{Q}}(x_1), S_{\widetilde{Q}}(x_2), \ldots, S_{\widetilde{Q}}(x_n))$, and $S_{\widetilde{Q}}(x_i) = q_{i1}/x_i + q_{i2}/x_i + \cdots + q_{in}/x_i$. One defines a binary relation $\preceq_2$ as $K(\widetilde{P}) \preceq_2 K(\widetilde{Q}) \Leftrightarrow |S_{\widetilde{P}}(x_i)| \leq |S_{\widetilde{Q}}(x_i)|$, $i \leq n$, where $|S_{\widetilde{P}}(x_i)| = \sum_{j=1}^{n} p_{ij}, |S_{\widetilde{Q}}(x_i)| = \sum_{j=1}^{n} q_{ij}$, and $\widetilde{P} \preceq_2 \widetilde{Q}$.

Furthermore, $K(\widetilde{P}) \simeq K(\widetilde{Q}) \Leftrightarrow |S_{\widetilde{P}}(x_i)| = |S_{\widetilde{Q}}(x_i)|, i \leq n$, and $\widetilde{P} \simeq \widetilde{Q}$. $K(\widetilde{P}) \prec_2 K(\widetilde{Q}) \Leftrightarrow K(\widetilde{P}) \preceq_2 K(\widetilde{Q})$ and $K(\widetilde{P}) \not\simeq K(\widetilde{Q})$ should be written as $\widetilde{P} \prec_2 \widetilde{Q}$.

Theorem 6: Letting $\mathbf{K}(U)$ be the collection of all fuzzy-binary granular structures on U, then $(\mathbf{K}(U), \preceq_2)$ is a poset.

Proof: Let $K(\widetilde{P}), K(\widetilde{Q}), K(\widetilde{R}) \in \mathbf{K}(U), K(\widetilde{P}) = (S_{\widetilde{P}}(x_1), S_{\widetilde{P}}(x_2), \ldots, S_{\widetilde{P}}(x_n)), K(\widetilde{Q}) = (S_{\widetilde{Q}}(x_1), S_{\widetilde{Q}}(x_2), \ldots, S_{\widetilde{Q}}(x_n))$, and $K(\widetilde{R}) = (S_{\widetilde{R}}(x_1), S_{\widetilde{R}}(x_2), \ldots, S_{\widetilde{R}}(x_n))$.

1) For arbitrary $x \in U$, we have $|S_{\widetilde{P}}(x_i)| = |S_{\widetilde{P}}(x_i)|$, and hence, $\widetilde{P} \preceq_2 \widetilde{P}$.
2) Suppose that $\widetilde{P} \preceq_2 \widetilde{Q}$ and $\widetilde{Q} \preceq_2 \widetilde{P}$. From Definition 3, it follows that
$$\widetilde{P} \preceq_2 \widetilde{Q} \Leftrightarrow |S_{\widetilde{P}}(x_i)| \leq |S_{\widetilde{Q}}(x_i)|, \qquad i \leq n$$
and
$$\widetilde{Q} \preceq_2 \widetilde{P} \Leftrightarrow |S_{\widetilde{Q}}(x_i)| \leq |S_{\widetilde{P}}(x_i)|, \qquad i \leq n.$$
Therefore, we have that $|S_{\widetilde{P}}(x_i)| \leq |S_{\widetilde{Q}}(x_i)| \leq |S_{\widetilde{P}}(x_i)|$, i.e., $|S_{\widetilde{P}}(x_i)| = |S_{\widetilde{Q}}(x_i)|$. Thus, for every $i \leq n$, one has $|S_{\widetilde{P}}(x_i)| = |S_{\widetilde{Q}}(x_i)|$, i.e., $\widetilde{P} \simeq \widetilde{Q}$.
3) Suppose that $\widetilde{P} \preceq_2 \widetilde{Q}$ and $\widetilde{Q} \preceq_2 \widetilde{R}$. It follows from Definition 3 that
$$\widetilde{P} \preceq_2 \widetilde{Q} \Leftrightarrow |S_{\widetilde{P}}(x_i)| \leq |S_{\widetilde{Q}}(x_i)|, \qquad i \leq n$$
and
$$\widetilde{Q} \preceq_2 \widetilde{R} \Leftrightarrow |S_{\widetilde{Q}}(x_i)| \leq |S_{\widetilde{R}}(x_i)|, \qquad i \leq n.$$

Therefore, we obtain that $|S_{\widetilde{P}}(x_i)| \leq |S_{\widetilde{Q}}(x_i)| \leq |S_{\widetilde{R}}(x_i)|$, $i \leq n$, i.e., $|S_{\widetilde{P}}(x_i)| \leq |S_{\widetilde{R}}(x_i)|$. Hence, $\widetilde{P} \preceq_2 \widetilde{R}$.

Summarizing (1)–(3), $(\mathbf{K}(U), \preceq_2)$ is a poset. ■

Theorem 7: The partial-order relation $\preceq_1$ is a special instance of partial relation $\preceq_2$.

Proof: Suppose that $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$ with $\widetilde{P} \preceq_1 \widetilde{Q}, K(\widetilde{P}) = (S_{\widetilde{P}}(x_1), S_{\widetilde{P}}(x_2), \ldots, S_{\widetilde{P}}(x_n)), K(\widetilde{Q}) = (S_{\widetilde{Q}}(x_1), S_{\widetilde{Q}}(x_2), \ldots, S_{\widetilde{Q}}(x_n))$; $S_{\widetilde{P}}(x_i) = p_{i1}/x_i + p_{i2}/x_i + \cdots + p_{in}/x_i$, and $S_{\widetilde{Q}}(x_i) = q_{i1}/x_i + q_{i2}/x_i + \cdots + q_{in}/x_i$.

Since $\widetilde{P} \preceq_1 \widetilde{Q}$, one knows $S_{\widetilde{P}}(x_i) \subseteq S_{\widetilde{Q}}(x_i), i \leq n \Leftrightarrow p_{ij} \leq q_{ij}, i, j \leq n$. Thus, for arbitrary $i \leq n$, one has $|S_{\widetilde{P}}(x_i)| \leq |S_{\widetilde{Q}}(x_i)|$, where $|S_{\widetilde{P}}(x_i)| = \sum_{j=1}^{n} p_{ij}, |S_{\widetilde{Q}}(x_i)| = \sum_{j=1}^{n} q_{ij}$. It is clear that $\widetilde{P} \preceq_2 \widetilde{Q}$.

Therefore, partial-order relation $\preceq_1$ is a special instance of partial-order relation $\preceq_2$. ■

Example 1: Let $K(\widetilde{P}) = (S_{\widetilde{P}}(x_1), S_{\widetilde{P}}(x_2), S_{\widetilde{P}}(x_3), S_{\widetilde{P}}(x_4))$, $K(\widetilde{Q}) = (S_{\widetilde{Q}}(x_1), S_{\widetilde{Q}}(x_2), S_{\widetilde{Q}}(x_3), S_{\widetilde{Q}}(x_4))$, $|S_{\widetilde{P}}(x_1)| = 3, |S_{\widetilde{P}}(x_2)| = 4, |S_{\widetilde{P}}(x_3)| = 1, |S_{\widetilde{P}}(x_4)| = 3$, and $|S_{\widetilde{Q}}(x_1)| = 2, |S_{\widetilde{Q}}(x_2)| = 3, |S_{\widetilde{Q}}(x_3)| = 4, |S_{\widetilde{Q}}(x_4)| = 3$.

From the values of these fuzzy neighborhoods and the definition of $\preceq_2$, we know that $K(\widetilde{P}) \npreceq_2 K(\widetilde{Q})$. However, it is obvious that $K(\widetilde{Q})$ has a much coarser granularity than $K(\widetilde{P})$. Hence, it is desirable to develop a new partial order relation.

If we rearrange the rank of those information granules in $K(\widetilde{Q})$, one can obtain such a rank: $K'(\widetilde{Q}) = (S_{\widetilde{Q}}(x_2), S_{\widetilde{Q}}(x_3), S_{\widetilde{Q}}(x_1), S_{\widetilde{Q}}(x_4))$. From this point of view, we can differentiate the fuzzy-information granularity of these two fuzzy granular structures.

Definition 4: Let $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$, where $K(\widetilde{P}) = (S_{\widetilde{P}}(x_1), S_{\widetilde{P}}(x_2), \ldots, S_{\widetilde{P}}(x_n)), S_{\widetilde{P}}(x_i) = p_{i1}/x_i + p_{i2}/x_i + \cdots + p_{in}/x_i, K(\widetilde{Q}) = (S_{\widetilde{Q}}(x_1), S_{\widetilde{Q}}(x_2), \ldots, S_{\widetilde{Q}}(x_n))$, and $S_{\widetilde{Q}}(x_i) = q_{i1}/x_i + q_{i2}/x_i + \cdots + q_{in}/x_i$. One defines a binary relation $\preceq_3$ as $K(\widetilde{P}) \preceq_3 K(\widetilde{Q}) \Leftrightarrow$ for $K(\widetilde{P})$, where there exists a sequence $K'(\widetilde{Q})$ of $K(\widetilde{Q})$ such that $|S_{\widetilde{P}}(x_i)| \leq |S_{\widetilde{Q}}(x_i)|, i \leq n$, and $\widetilde{P} \preceq_3 \widetilde{Q}$, where $K'(\widetilde{Q}) = (S_{\widetilde{Q}}(x'_1), S_{\widetilde{Q}}(x'_2), \ldots, S_{\widetilde{Q}}(x'_n))$.

In particular, $K(\widetilde{P}) \approx K(\widetilde{Q}) \Leftrightarrow |S_{\widetilde{P}}(x_i)| = |S_{\widetilde{Q}}(x'_i)|, i \leq n$, which is simply denoted by $\widetilde{P} \approx \widetilde{Q}$. $K(\widetilde{P}) \prec_3 K(\widetilde{Q}) \Leftrightarrow K(\widetilde{P}) \preceq_3 K(\widetilde{Q})$ and $K(\widetilde{P}) \not\approx K(\widetilde{Q})$ and written $\widetilde{P} \prec_3 \widetilde{Q}$.

Theorem 8: Let $\mathbf{K}(U)$ be the collection of all fuzzy-binary granular structures on U, then $(\mathbf{K}(U), \preceq_3)$ is a poset.

Proof: Letting $K(\widetilde{P}), K(\widetilde{Q}), K(\widetilde{R}) \in \mathbf{K}(U), K(\widetilde{P}) = (S_{\widetilde{P}}(x_1), S_{\widetilde{P}}(x_2), \ldots, S_{\widetilde{P}}(x_n)), K(\widetilde{Q}) = (S_{\widetilde{Q}}(x_1), S_{\widetilde{Q}}(x_2), \ldots, S_{\widetilde{Q}}(x_n))$, and $K(\widetilde{R}) = (S_{\widetilde{R}}(x_1), S_{\widetilde{R}}(x_2), \ldots, S_{\widetilde{R}}(x_n))$.

1) For arbitrary $x \in U, |S_{\widetilde{P}}(x_i)| = |S_{\widetilde{P}}(x_i)|$ holds, and hence, $\widetilde{P} \preceq_3 \widetilde{P}$.

2) Suppose that $\widetilde{P} \preceq_3 \widetilde{Q}$ and $\widetilde{Q} \preceq_3 \widetilde{P}$. It follows from Definition 4 that $\widetilde{P} \preceq_3 \widetilde{Q} \Leftrightarrow$ for $K(\widetilde{P})$ and that there exists a sequence $K'(\widetilde{Q})$ of $K(\widetilde{Q})$, where $K'(\widetilde{Q}) = (S_{\widetilde{Q}}(x'_1), S_{\widetilde{Q}}(x'_2), \ldots, S_{\widetilde{Q}}(x'_n))$, such that $|S_{\widetilde{P}}(x_i)| \leq |S_{\widetilde{Q}}(x'_i)|, i \leq n$.

$\widetilde{Q} \preceq_3 \widetilde{P} \Leftrightarrow$ for $K(\widetilde{Q})$, and there exists a sequence $K'(\widetilde{P})$ of $K(\widetilde{P})$ such that $|S_{\widetilde{Q}}(x_i)| \leq |S_{\widetilde{P}}(x'_i)|, i \leq n$, where $K'(\widetilde{P}) = (S_{\widetilde{P}}(x'_1), S_{\widetilde{P}}(x'_2), \ldots, S_{\widetilde{P}}(x'_n))$.

Therefore, we have that

$$\sum_{i=1}^{n} |S_{\widetilde{P}}(x_i)| \le \sum_{i=1}^{n} |S_{\widetilde{Q}}(x'_i)| = \sum_{i=1}^{n} |S_{\widetilde{Q}}(x_i)| \le \sum_{i=1}^{n} |S_{\widetilde{P}}(x'_i)| .$$

In addition, from $\sum_{i=1}^{n} |S_{\widetilde{P}}(x_i)| = \sum_{i=1}^{n} |S_{\widetilde{P}}(x'_i)|$, one knows $\sum_{i=1}^{n} |S_{\widetilde{P}}(x_i)| = \sum_{i=1}^{n} |S_{\widetilde{Q}}(x'_i)|$. Considering $|S_{\widetilde{P}}(x_i)| \le |S_{\widetilde{Q}}(x'_i)|$, we have $|S_{\widetilde{P}}(x_i)| = |S_{\widetilde{Q}}(x'_i)|$. Hence, for arbitrary $i \le n$, it follows that $|S_{\widetilde{P}}(x_i)| = |S_{\widetilde{Q}}(x'_i)|$, i.e., $\widetilde{P} \approx \widetilde{Q}$.

3) Suppose that $\widetilde{P} \preceq_3 \widetilde{Q}$ and $\widetilde{Q} \preceq_3 \widetilde{R}$. From Definition 4, it follows that $\widetilde{P} \preceq_3 \widetilde{Q} \Leftrightarrow$ for $K(\widetilde{P})$, and there exists a sequence $K'(\widetilde{Q})$ of $K(\widetilde{Q})$, where $K'(\widetilde{Q}) = (S_{\widetilde{Q}}(x'_1), S_{\widetilde{Q}}(x'_2), \ldots, S_{\widetilde{Q}}(x'_n))$, such that $|S_{\widetilde{P}}(x_i)| \le |S_{\widetilde{Q}}(x'_i)|, i \le n$.

$\widetilde{Q} \preceq_3 \widetilde{R} \Leftrightarrow$ for $K(\widetilde{Q})$, and there exists a sequence $K'(\widetilde{R})$ of $K(\widetilde{R})$, where $K'(\widetilde{R}) = (S_{\widetilde{R}}(x'_1), S_{\widetilde{R}}(x'_2), \ldots, S_{\widetilde{R}}(x'_n))$, such that $|S_{\widetilde{Q}}(x_i)| \le |S_{\widetilde{R}}(x'_i)|, i \le n$.

Hence, for the sequence $K'(\widetilde{Q})$, there must exist a sequence $K''(\widetilde{R})$ of $K(\widetilde{R})$ such that $|S_{\widetilde{Q}}(x'_i)| \le |S_{\widetilde{R}}(x''_i)|$, where $K''(\widetilde{R}) = (S_{\widetilde{R}}(x''_1), S_{\widetilde{R}}(x''_2), \ldots, S_{\widetilde{R}}(x''_n))$.

Therefore, for $K(\widetilde{P})$, there exists the sequence $K''(\widetilde{R})$ of $K(\widetilde{R})$ such that $|S_{\widetilde{P}}(x_i)| \le |S_{\widetilde{R}}(x''_i)|$, i.e., $\widetilde{P} \preceq_3 \widetilde{R}$.

Summarizing the above, $(\mathbf{K}(U), \preceq_3)$ is a poset. ■

Theorem 9: Partial-order relation $\preceq_2$ is a special instance of partial relation $\preceq_3$.

Proof: Suppose that $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$ with $\widetilde{P} \preceq_2 \widetilde{Q}$, $K(\widetilde{P}) = (S_{\widetilde{P}}(x_1), S_{\widetilde{P}}(x_2), \ldots, S_{\widetilde{P}}(x_n))$, and $K(\widetilde{Q}) = (S_{\widetilde{Q}}(x_1), S_{\widetilde{Q}}(x_2), \ldots, S_{\widetilde{Q}}(x_n))$.

Since $\widetilde{P} \preceq_2 \widetilde{Q}$, one knows that $|S_{\widetilde{P}}(x_i)| \le |S_{\widetilde{Q}}(x_i)|, i \le n$. That is to say, there exists a sequence such that $|S_{\widetilde{P}}(x_i)| \le |S_{\widetilde{Q}}(x_i)|, i \le n$. Hence, $\widetilde{P} \preceq_3 \widetilde{Q}$.

Therefore, partial-order relation $\preceq_2$ is a special instance of partial-order relation $\preceq_3$. ■

Corollary 1: Partial-order relation $\preceq_1$ is a special instance of partial relation $\preceq_3$.

From Theorems 7 and 9 and Corollary 1, one can draw the conclusion that the partial-order relation $\preceq_3$ is the best to characterize the nature of granulation of fuzzy information (or fuzzy knowledge).

V. Fuzzy-Information Granularity

As we know, information granularity, in a broad sense, is the average measure of information (knowledge) granules of a granular structure [13], [34]. It can be used to characterize the classification ability of a granular structure. In fuzzy-binary granular structures, a fuzzy-information granularity plays the same role, which should also be used to depict the classification ability of a fuzzy-binary granular structure. In this section, we develop two measures to evaluate the degree of granulation of a fuzzy-binary granular structure.

From the viewpoint of sizes of information granules, in the following, we introduce a definition of fuzzy-information granularity of a fuzzy-binary granular structure.

Definition 5: [3] Let $K(\widetilde{R}) = (S_{\widetilde{R}}(x_1), S_{\widetilde{R}}(x_2), \ldots, S_{\widetilde{R}}(x_n))$. Then, fuzzy-information granularity of $\widetilde{R}$ is defined as

$$\mathrm{GK}(\widetilde{R}) = \frac{1}{n}\sum_{i=1}^{n} \frac{|S_{\widetilde{R}}(x_i)|}{n} \quad (8)$$

where $|S_{\widetilde{R}}(x_i)|$ is the cardinality of the fuzzy information granule $S_{\widetilde{R}}(x_i)$.

Theorem 10: Let $U/R = \{X_1, X_2, \ldots, X_m\}$ be a Pawlak granular structure. Then, the fuzzy-information granularity of R degenerates to the information granularity

$$\mathrm{GK}(R) = \frac{1}{n^2}\sum_{k=1}^{m} |X_k|^2 \quad (9)$$

where $\sum_{i=1}^{m} |X_i|^2$ is the number of objects in the equivalence relation induced by $\cup_{i=1}^{m}(X_i \times X_i)$.

Proof: Let us denote $K(\widetilde{R}) = (S_{\widetilde{A}}(x_1), S_{\widetilde{R}}(x_2), \ldots, S_{\widetilde{A}}(x_n))$, and $S_{\widetilde{R}}(x_i) = a_{i1}/x_i + a_{i2}/x_i + \cdots + a_{in}/x_i$. For an equivalence relation R, as we know, if $R(x, y) = 1$ and $R(y, z) = 1$, then $R(x, z) = 1$. That is to say $a_{ij} = a_{ji} = 1$ or $0, j \le n$. Let $X_k = \{x_{k1}, x_{k2}, \ldots, x_{ks_k}\}, k \le m$, where $|X_k| = |S_R(x_{kl})| = s_k, l \le s_k$, and $\sum_{k=1}^{m} s_k = n$. Hence

$$\begin{aligned}
&\frac{1}{n^2}\sum_{k=1}^{m} |X_k|^2 \\
&= \frac{1}{n^2}\sum_{k=1}^{m}(|S_R(x_{k1})| + |S_R(x_{k2})| + \cdots + |S_R(x_{ks_k})|) \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} = \frac{1}{n}\sum_{i=1}^{n}\frac{\sum_{j=1}^{n} a_{ij}}{n} \\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{|S_{\widetilde{R}}(x_i)|}{n} = \mathrm{GK}(\widetilde{R}).
\end{aligned}$$

This completes the proof. ■

From Theorem 10, it follows that the information granularity in a Pawlak knowledge base is a special instance of fuzzy-information granularity in fuzzy-binary granular structures.

Theorem 11: Letting $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$. If $K(\widetilde{P}) \prec_1 K(\widetilde{Q})$, then $\mathrm{GK}(\widetilde{P}) < \mathrm{GK}(\widetilde{Q})$.

Theorem 12: Letting $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$. If $K(\widetilde{P}) \prec_2 K(\widetilde{Q})$, then $\mathrm{GK}(\widetilde{P}) < \mathrm{GK}(\widetilde{Q})$.

Theorem 13: Letting $K(\widetilde{P}) \in \mathbf{K}(U)$, then $\mathrm{GK}(\widetilde{P}) + \mathrm{GK}(\wr\widetilde{P}) = 1 + \frac{1}{n}$.

Proof: From Definition 1, one can easily see that $\wr S_{\widetilde{P}}(x_i) = S_\omega(x_i) \cup \sim S_{\widetilde{P}}(x_i)$. Letting $S_{\widetilde{P}}(x_i) = p_{i1}/x_i + p_{i2}/x_i + \cdots + p_{in}/x_i$, one has that $\sim S_{\widetilde{P}}(x_i) = (1 - p_{i1})/x_i + (1 - p_{i2})/x_i + \cdots + (1 - p_{in})/x_i$. Since $S_\omega(x_i) = p_{i1}/x_i + p_{i2}/x_i + \cdots + p_{i(i-1)}/x_i + 1/x_i + p_{i(i+1)}/x_i + \cdots + p_{in}/x_i$, where $p_{ij} = 0, i \ne j$, we have that for $\forall p_{ij}$, if $j = i$, then $\max\{1, 1 - \max\{1, 1 - 1\}\} = 1$; if $j \ne i$, then $\max\{0, 1 - \max\{0, 1 - p_{ij}\}\} = p_{ij}$. Hence, $\wr(\wr S_{\widetilde{P}}(x_i)) = S_\omega(x_i) \cup \sim (S_\omega(x_i) \cup \sim S_{\widetilde{P}}(x_i)) = S_{\widetilde{P}}(x_i)$, i.e., $\wr(\wr K(\widetilde{P})) = K(\widetilde{P})$. ■

In rough-set theory, there is a kind of especial uncertainty, i.e., roughness [13], [29]. For a given granular structure, we need to assess its roughness for a target concept or a target decision. An uncertainty measure, which is called rough entropy, is always employed to calculate roughness degree of a granular structure. Liang *et al.* [13] introduced the concept of rough entropy to measure the roughness degree of a granular structure. Like the opinion proposed by Qian *et al.* [34], each of the rough entropies also can be induced as an information granularity. The following definition gives the depiction of the rough entropy.

Definition 6: [13] Let $K(R) = (S_R(x_1), S_R(x_2), \ldots, S_R(x_n))$ be a tolerance granular structure. Then, the rough entropy of R is defined as

$$E_r(R) = -\sum_{i=1}^{n} \frac{1}{n} \log_2 \frac{1}{|S_R(x_i)|}. \tag{10}$$

Due to the property of the above rough entropy that can be used to measure information granularity, we can construct the definition of fuzzy rough entropy of a family of fuzzy-binary granular structures, which is used to characterize the fuzzy-information granularity of a fuzzy-binary granular structure.

Definition 7: Let $K(\widetilde{R}) = (S_{\widetilde{R}}(x_1), S_{\widetilde{R}}(x_2), \ldots, S_{\widetilde{R}}(x_n))$. Then, fuzzy-information granularity of $\widetilde{R}$ is defined as

$$E_r(\widetilde{R}) = -\sum_{i=1}^{n} \frac{1}{n} \log_2 \frac{1}{|S_{\widetilde{R}}(x_i)|}. \tag{11}$$

Theorem 14: Let $U/R = \{X_1, X_2, \ldots, X_m\}$ be a Pawlak granular structure. Then, the fuzzy-information granularity of R degenerates to the rough entropy

$$E_r(R) = -\sum_{k=1}^{m} \frac{|X_k|}{n} \log_2 \frac{1}{|X_k|}. \tag{12}$$

Proof: Let $K(\widetilde{R}) = (S_{\widetilde{R}}(x_1), S_{\widetilde{R}}(x_2), \ldots, S_{\widetilde{R}}(x_n)), S_{\widetilde{R}}(x_i) = a_{i1}/x_i + a_{i2}/x_i + \cdots + a_{in}/x_i$. For an equivalence relation R, we know that if $R(x,y) = 1$ and $R(y,z) = 1$, then $R(x,z) = 1$. In other words, $a_{ij} = a_{ji} = 1$ or $0, j \leq n$. We denote $X_k = \{x_{k1}, x_{k2}, \ldots, x_{ks_k}\}, k \leq m$, where $|X_k| = |[x_{kl}]_R| = s_k, l \leq s_k$, and $\sum_{k=1}^{m} s_k = n$. Therefore

$$\begin{aligned}
&-\sum_{k=1}^{m} \frac{|X_k|}{n} \log_2 \frac{1}{|X_k|} \\
&= -\sum_{k=1}^{m} \left(\frac{1}{n} \log_2 \frac{1}{|S_R(x_{k1})|} + \frac{1}{n} \log_2 \frac{1}{|S_R(x_{k2})|} + \cdots \right. \\
&\quad \left. + \frac{1}{n} \log_2 \frac{1}{|S_R(x_{ks_k})|} \right) \\
&= \sum_{k=1}^{m} \left(\frac{1}{n} \log_2 |S_R(x_{k1})| + \frac{1}{n} \log_2 |S_R(x_{k2})| + \cdots \right. \\
&\quad \left. + \frac{1}{n} \log_2 |S_R(x_{ks_k})| \right) \\
&= \frac{1}{n} \log_2 |S_R(x_1)| + \frac{1}{n} \log_2 |S_R(x_2)| + \cdots + \frac{1}{n} \log_2 |S_R(x_n)| \\
&= -\left(\frac{1}{n} \log_2 \frac{1}{|S_R(x_1)|} + \frac{1}{n} \log_2 \frac{1}{|S_R(x_2)|} + \cdots \right. \\
&\quad \left. + \frac{1}{n} \log_2 \frac{1}{|S_R(x_n)|} \right) \\
&= -\sum_{i=1}^{n} \frac{1}{n} \log_2 \frac{1}{|S_{\widetilde{R}}(x_i)|} = E_r(\widetilde{R}).
\end{aligned}$$

This completes the proof. ■

Theorem 13 shows that the rough entropy in Pawlak knowledge bases is a special case of the fuzzy-information granularity in a family of fuzzy-binary granular structures.

Theorem 15: Letting $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$. If $K(\widetilde{P}) \prec_1 K(\widetilde{Q})$, then $E_r(\widetilde{P}) < E_r(\widetilde{Q})$.

Theorem 16: Letting $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$. If $K(\widetilde{P}) \prec_2 K(\widetilde{Q})$, then $E_r(\widetilde{P}) < E_r(\widetilde{Q})$.

VI. Axiomatic Approach to Fuzzy-Information Granularity

In recent years, some researchers have already started to pay attention to such the problem of what is the essence of information granularity in (crisp) granular structures. Liang and Qian [11] attempted to unify the definitions by some axiomatic approaches in granular structures. Qian *et al.* [34] developed a more reasonable and comprehensive form of axiomatic definition of information granularity. In this section, we will propose an axiomatic definition to fuzzy information granularity.

By employing the partial-order relation $\preceq_3$, we give the following axiomatic constraint to define a fuzzy-information granularity in the context of fuzzy-binary granular structures.

Definition 8: Letting $\mathbf{K}(U)$ be the collection of all fuzzy-binary granular structures on U if, for $\forall K(\widetilde{P}) \in \mathbf{K}(U)$, there is a real number $G(\widetilde{P})$ with the following properties:

1) $G(\widetilde{P}) \geq 0$ (nonnegative);
2) for $\forall K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$, if $K(\widetilde{P}) \approx K(\widetilde{Q})$, then $G(\widetilde{P}) = G(\widetilde{Q})$ (invariability);
3) for $\forall K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$, if $K(\widetilde{P}) \prec_3 K(\widetilde{Q})$, then $G(\widetilde{P}) < G(\widetilde{Q})$ (monotonicity);

then G is called a fuzzy-information granularity.

As a result of the above discussions, we come to the following four theorems.

Theorem 17: (Extremum) Letting $\mathbf{K}(U)$ be the collection of all fuzzy-binary granular structures on $U, K(\widetilde{P}) \in \mathbf{K}(U)$, then $G(\widetilde{P})$ achieves its minimum value if $K(\widetilde{P}) = \omega$, and $G(\widetilde{P})$ achieves its maximum value if $K(\widetilde{P}) = \delta$.

Proof: Given arbitrary $\widetilde{P} \in \widetilde{R}$, one get its fuzzy-relation matrix

$$M(\widetilde{P}) = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}$$

From the definition of $K(\omega)$, it is obvious that $|S_\omega(x_i)| = 1 \leq |S_{\widetilde{P}}(x_i)|$, i.e., $\omega \preceq_3 \widetilde{P}$. Similarly, it follows from the definition of $K(\delta)$ that $|S_{\widetilde{P}}(x_i)| \leq |S_\delta(x_i)| = |U|$, that is, $\widetilde{P} \preceq_3 \delta$. Therefore, from 2) and 3) in Definition 8, one knows that $G(\omega) \leq G(\widetilde{P}) \leq G(\delta)$. This completes the proof. ■

From Definition 8 and Theorem 17, it is easy to see that the size of $G(\widetilde{P})$ only depends on the cardinality of every fuzzy information granule in the fuzzy-binary granular structure $K(\widetilde{P})$.

Theorem 18: Letting $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$, then $G(\widetilde{P}) \leq G(\widetilde{Q})$ if $K(\widetilde{P}) \preceq_1 K(\widetilde{Q})$.

Theorem 19: Letting $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$, then $G(\widetilde{P}) \leq G(\widetilde{Q})$ if $K(\widetilde{P}) \preceq_2 K(\widetilde{Q})$.

The following theorem reveals several important properties of the fuzzy-information granularity G.

Theorem 20: The following properties hold:

1) $G(\widetilde{P}) = G(\wr\wr\widetilde{P})$;
2) $G(\widetilde{P} \cap \widetilde{Q}) \leq G(\widetilde{P}), G(\widetilde{P} \cap \widetilde{Q}) \leq G(\widetilde{Q})$; and
3) $G(\widetilde{P}) \leq G(\widetilde{P} \cup \widetilde{Q}), G(\widetilde{Q}) \leq G(\widetilde{P} \cup \widetilde{Q})$.

Proof: They are straightforward. ■

In what follows, we observe whether GK in Definition 5 and E_r in Definition 7 satisfy the proposed axiomatic definition of fuzzy information granularity or not.

Theorem 21: GK in Definition 5 is a fuzzy-information granularity under Definition 8.

Proof: 1) Obviously, it is nonnegative.

2) Letting $K(\widetilde{P})$, $K(\widetilde{Q}) \in \mathbf{K}(U)$, $K(\widetilde{P}) = (S_{\widetilde{P}}(x_1), S_{\widetilde{P}}(x_2), \ldots, S_{\widetilde{P}}(x_n))$, $K(\widetilde{Q}) = (S_{\widetilde{Q}}(x_1), S_{\widetilde{Q}}(x_2), \ldots, S_{\widetilde{Q}}(x_n))$. If $\widetilde{P} \approx \widetilde{Q}$, then there exists a sequence $K'(\widetilde{Q})$ of $K(\widetilde{Q})$, where $K'(\widetilde{Q}) = (S_{\widetilde{Q}}(x'_1), S_{\widetilde{Q}}(x'_2), \ldots, S_{\widetilde{Q}}(x'_n))$, such that $|S_{\widetilde{P}}(x_i)| = |S_{\widetilde{Q}}(x'_i)|, i \leq n$. Therefore

$$\begin{aligned} \mathrm{GK}(\widetilde{P}) &= \frac{1}{n}\sum_{i=1}^{n}\frac{|S_{\widetilde{P}}(x_i)|}{n} = \frac{1}{n}\sum_{i=1}^{n}\frac{|S_{\widetilde{Q}}(x'_i)|}{n} \\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{|S_{\widetilde{Q}}(x_i)|}{n} = \mathrm{GK}(\widetilde{Q}). \end{aligned}$$

3) Letting $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$ with $\widetilde{P} \prec_3 \widetilde{Q}$, then there exists a sequence $K'(\widetilde{Q})$ of $K(\widetilde{Q})$, where $K'(\widetilde{Q}) = (S_{\widetilde{Q}}(x'_1), S_{\widetilde{Q}}(x'_2), \ldots, S_{\widetilde{Q}}(x'_n))$, such that $|S_{\widetilde{P}}(x_i)| \leq |S_{\widetilde{Q}}(x'_i)|, i \leq n$, and there exists $x_0 \in U$ such that $|S_{\widetilde{P}}(x_0)| < |S_{\widetilde{Q}}(x'_0)|$. Hence

$$\begin{aligned} \mathrm{GK}(\widetilde{P}) &= \frac{1}{n}\sum_{i=1}^{n}\frac{|S_{\widetilde{P}}(x_i)|}{n} = \frac{1}{n}\Bigg(\sum_{i=1, x_i \neq x_0}^{n}\frac{|S_{\widetilde{P}}(x_i)|}{n} \\ &\quad + \frac{|S_{\widetilde{P}}(x_0)|}{n}\Bigg) < \frac{1}{n}\Bigg(\sum_{i=1, x_i \neq x_0}^{n}\frac{|S_{\widetilde{Q}}(x'_i)|}{n} + \frac{|S_{\widetilde{Q}}(x'_0)|}{n}\Bigg) \\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{|S_{\widetilde{Q}}(x_i)|}{n} = \mathrm{GK}(\widetilde{Q}) \end{aligned}$$

i.e., $\mathrm{GK}(\widetilde{P}) < \mathrm{GK}(\widetilde{Q})$.

Summarizing the above, GK in Definition 5 is a fuzzy-information granularity under Definition 8. ■

Theorem 22: E_r in Definition 7 is a fuzzy-information granularity under Definition 8.

Proof: 1) Obviously, it is nonnegative.

2) Let $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U), K(\widetilde{P}) = (S_{\widetilde{P}}(x_1), S_{\widetilde{P}}(x_2), \ldots, S_{\widetilde{P}}(x_n)), K(\widetilde{Q}) = (S_{\widetilde{Q}}(x_1), S_{\widetilde{Q}}(x_2), \ldots, S_{\widetilde{Q}}(x_n))$. If $\widetilde{P} \approx \widetilde{Q}$, then there exists a sequence $K'(\widetilde{Q})$ of $K(\widetilde{Q})$, where $K'(\widetilde{Q}) = (S_{\widetilde{Q}}(x'_1), S_{\widetilde{Q}}(x'_2), \ldots, S_{\widetilde{Q}}(x'_n))$, such that $|S_{\widetilde{P}}(x_i)| = |S_{\widetilde{Q}}(x'_i)|, i \leq n$. Therefore

$$\begin{aligned} E_r(\widetilde{P}) &= -\sum_{i=1}^{n}\frac{1}{n}\log_2\frac{1}{|S_{\widetilde{P}}(x_i)|} = -\sum_{i=1}^{n}\frac{1}{n}\log_2\frac{1}{|S_{\widetilde{Q}}(x'_i)|} \\ &= -\sum_{i=1}^{n}\frac{1}{n}\log_2\frac{1}{|S_{\widetilde{Q}}(x_i)|} = E_r(\widetilde{Q}). \end{aligned}$$

3) Letting $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$, with $\widetilde{P} \prec_3 \widetilde{Q}$, then there exists a sequence $K'(\widetilde{Q})$ of $K(\widetilde{Q})$, where $K'(\widetilde{Q}) = (S_{\widetilde{Q}}(x'_1), S_{\widetilde{Q}}(x'_2), \ldots, S_{\widetilde{Q}}(x'_n))$, such that $|S_{\widetilde{P}}(x_i)| \leq |S_{\widetilde{Q}}(x'_i)|, i \leq n$, and there exists $x_0 \in U$ such that $|S_{\widetilde{P}}(x_0)| < |S_{\widetilde{Q}}(x'_0)|$. Hence

$$\begin{aligned} E_r(\widetilde{P}) &= -\sum_{i=1}^{n}\frac{1}{n}\log_2\frac{1}{|S_{\widetilde{P}}(x_i)|} = \sum_{i=1}^{n}\frac{1}{n}\log_2|S_{\widetilde{P}}(x_i)| \\ &= \frac{1}{n}\sum_{i=1 x_i \neq x_0}^{n}\log_2|S_{\widetilde{P}}(x_i)| + \frac{1}{n}\log_2|S_{\widetilde{P}}(x_0)| \\ &< \frac{1}{n}\sum_{i=1 x_i \neq x_0}^{n}\log_2|S_{\widetilde{Q}}(x_i)| + \frac{1}{n}\log_2|S_{\widetilde{Q}}(x_0)| \\ &= -\sum_{i=1}^{n}\frac{1}{n}\log_2\frac{1}{|S_{\widetilde{P}}(x_i)|} = E_r(\widetilde{Q}) \end{aligned}$$

i.e., $E_r(\widetilde{P}) < E_r(\widetilde{Q})$.

From the above, we conclude that E_r in Definition 7 is a fuzzy-information granularity under Definition 8. ■

VII. Fuzzy-Information Entropy and Its Granulation Monotonicity

In physics, entropy is often used to measure out-of-order degree of a system. The bigger entropy value is, the higher out of order of a system will be. Shannon introduced the concept of entropy in physics to information theory to measure uncertainty of the structure of a system [40]. The entropy is called information entropy, which can be used to measure information content of an information system.

In the framework of granular structures, Liang *et al.* [13] established the relationship between information entropy, rough entropy, and information granulation in Pawlak/tolerance granular structures. Qian *et al.* [37] proved that the existing information entropies all satisfy the granulation monotonicity of an information granularity. Hu *et al.* [3] extended Shannon's entropy to a fuzzy-granular structure and used this variant to characterize the uncertainty of fuzzy rough sets and fuzzy-probability rough sets. Hence, in this section, we will discuss how to measure the uncertainty of a fuzzy-granular structure using entropy theory and establish the relationship between fuzzy-information entropy and fuzzy-information granularity. It is noted that the entropy in this paper represents information entropy but not fuzzy entropy to measure fuzziness of a fuzzy set.

Definition 9: [40] Letting $U/R = \{X_1, X_2, \ldots, X_m\}$ with the probability distribution $p_i = |X_i|/n$, one then calls

$$H(R) = -\sum_{i=1}^{m} p_i \log_2 p_i \tag{13}$$

the information entropy of the Pawlak granular structure. When $p_i = 0, 0 \cdot \log_2 0 = 0$.

Because that Shannon's entropy is defined by the probability of each equivalence class, it cannot be used to measure the uncertainty of a fuzzy-granular structure. For the consideration, Hu *et al.* [3] extended Shannon's entropy to fuzzy-binary granular structures, and used this variant to characterize the uncertainty of fuzzy rough sets and fuzzy probability rough sets. Conveniently, information entropy to measure uncertainty of a fuzzy-binary granular structure is called *fuzzy-information entropy*, which denotes the size of information content of a fuzzy-granular structure.

Definition 10: [3] Letting $K(\widetilde{R}) = (S_{\widetilde{R}}(x_1), S_{\widetilde{R}}(x_2), \ldots, S_{\widetilde{R}}(x_n))$, then fuzzy-information entropy of $\widetilde{R}$ is defined as

$$H(\widetilde{R}) = -\frac{1}{n}\sum_{i=1}^{n} \log_2 \frac{|S_{\widetilde{R}}(x_i)|}{n}. \tag{14}$$

In fact, for a Pawlak granular structure, the fuzzy-information entropy will degenerate to the form of Shannon's entropy. That is to say, the definition with a uniform configuration also can be used to measure the uncertainty of a Pawlak granular structure. In what follows, we examine whether the fuzzy-information entropy $H(\widetilde{A})$ satisfies monotonicity or not.

Theorem 23: Let $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$. If $K(\widetilde{P}) \prec_3 K(\widetilde{Q})$, then $H(\widetilde{Q}) < H(\widetilde{P})$.

Proof: Let us denote $K(\widetilde{P}) = (S_{\widetilde{P}}(x_1), S_{\widetilde{P}}(x_2), \ldots, S_{\widetilde{P}}(x_n))$, and $K(\widetilde{Q}) = (S_{\widetilde{Q}}(x_1), S_{\widetilde{Q}}(x_2), \ldots, S_{\widetilde{Q}}(x_n))$.

If $\widetilde{P} \prec_3 \widetilde{Q}$, then there exists a sequence $K'(\widetilde{Q})$ of $K(\widetilde{Q})$, where $K'(\widetilde{Q}) = (S_{\widetilde{Q}}(x'_1), S_{\widetilde{Q}}(x'_2), \ldots, S_{\widetilde{Q}}(x'_n))$, such that $|S_{\widetilde{P}}(x_i)| \le |S_{\widetilde{Q}}(x'_i)|, i \le n$, and there exists $x_0 \in U$ such that $|S_{\widetilde{P}}(x_0)| < |S_{\widetilde{Q}}(x'_0)|$. Therefore

$$\begin{aligned} H(\widetilde{P}) &= -\frac{1}{n}\sum_{i=1}^{n} \log_2 \frac{|S_{\widetilde{P}}(x_i)|}{n} \\ &= -\frac{1}{n}\sum_{i=1, x_i \ne x_0}^{n} \log_2 \frac{|S_{\widetilde{P}}(x_i)|}{n} - \frac{1}{n}\log_2 \frac{|S_{\widetilde{P}}(x_0)|}{n} \\ &> -\frac{1}{n}\sum_{i=1, x_i \ne x_0}^{n} \log_2 \frac{|S_{\widetilde{Q}}(x_i)|}{n} - \frac{1}{n}\log_2 \frac{|S_{\widetilde{Q}}(x_0)|}{n} \\ &= -\frac{1}{n}\sum_{i=1}^{n} \log_2 \frac{|S_{\widetilde{Q}}(x_i)|}{n} = H(\widetilde{Q}). \end{aligned}$$

It is clear that $H(\widetilde{Q}) < H(\widetilde{P})$. ■

For convenience, we call the monotonicity of entropy induced by partial-order relation $\preceq_3$ *fuzzy-granulation monotonicity*.

Theorem 24: Let $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$. If $K(\widetilde{P}) \prec_1 K(\widetilde{Q})$, then $H(\widetilde{P}) > H(\widetilde{Q})$.

Theorem 25: Let $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$. If $K(\widetilde{P}) \prec_2 K(\widetilde{Q})$, then $H(\widetilde{P}) > H(\widetilde{Q})$.

Liang *et al.* [13] developed a new measure in a tolerance granular structure through generalizing the information entropy, which has been successfully used to depict uncertainty of a tolerance granular structure.

Definition 11: [13] Letting $K(R) = (S_R(x_1), S_R(x_2), \ldots, S_R(x_n))$ be a tolerance granular structure, then information entropy of R is defined as

$$E(R) = \sum_{i=1}^{n} \frac{1}{n}\left(1 - \frac{|S_R(x_i)|}{n}\right). \tag{15}$$

Similar to Shannon's entropy, Liang's information entropy E also encounters the same challenge for dealing with fuzzy-granular structures. In the following definition, we define another form of fuzzy-information entropy.

Definition 12: Letting $K(\widetilde{R}) = (S_{\widetilde{R}}(x_1), S_{\widetilde{R}}(x_2), \ldots, S_{\widetilde{R}}(x_n))$, then fuzzy-information entropy of $\widetilde{R}$ is constructed by

$$E(\widetilde{R}) = \sum_{i=1}^{n} \frac{1}{n}\left(1 - \frac{|S_{\widetilde{R}}(x_i)|}{n}\right). \tag{16}$$

The proposed fuzzy-information entropy of a fuzzy-binary granular structure has several nice properties. From the above point of view, one can give the definition of joint entropy between two fuzzy-granular structures. Letting $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$, the joint entropy of $\widetilde{P}$ and $\widetilde{Q}$ is formalized as

$$E(\widetilde{P}; \widetilde{Q}) = \sum_{i=1}^{n} \frac{1}{n}\left(1 - \frac{|S_{\widetilde{P}}(x_i) \cap S_{\widetilde{Q}}(x_i)|}{n}\right) \tag{17}$$

and, its condition entropy can be defined as

$$E(\widetilde{P} \mid \widetilde{Q}) = \sum_{i=1}^{n} \frac{1}{n}\left(\frac{|S_{\widetilde{Q}}(x_i)|}{n} - \frac{|S_{\widetilde{P}}(x_i) \cap S_{\widetilde{Q}}(x_i)|}{n}\right). \tag{18}$$

Similar to the property of Shannon entropy, the relationship among above three concepts can be formalized as

$$E(\widetilde{P} \mid \widetilde{Q}) = E(\widetilde{P}; \widetilde{Q}) - E(\widetilde{P}). \tag{19}$$

Theorem 26: Let $U/R = \{X_1, X_2, \ldots, X_m\}$ be a Pawlak granular structure. Then, the fuzzy-information entropy R generates to the information entropy

$$E(R) = \sum_{k=1}^{m} \frac{|X_k|}{n} \left(1 - \frac{|X_k|}{n}\right). \tag{20}$$

Proof: Let $K(\widetilde{R}) = (S_{\widetilde{R}}(x_1), S_{\widetilde{R}}(x_2), \ldots, S_{\widetilde{R}}(x_n))$, with $S_{\widetilde{R}}(x_i) = a_{i1}/x_i + a_{i2}/x_i + \cdots + a_{in}/x_i$. For an equivalence relation R, if $R(x,y) = 1$ and $R(y,z) = 1$, then $R(x,z) = 1$. That is to say, $a_{ij} = a_{ji} = 1$ or $0, j \leq n$. Let us denote $X_k = \{x_{k1}, x_{k2}, \ldots, x_{ks_k}\}, k \leq m$, where $|X_k| = |S_R(x_{kl})| = s_k, l \leq s_k$, and $\sum_{k=1}^{m} s_k = n$. Hence

$$\begin{aligned}
&\sum_{k=1}^{m} \frac{|X_k|}{n} \left(1 - \frac{|X_k|}{n}\right) \\
&= \sum_{k=1}^{m} \left(\frac{1}{n}(1 - \frac{|S_R(x_{k1})|}{n}\right) + \frac{1}{n}\left(1 - \frac{|S_R(x_{k2})|}{n}\right) \\
&\quad + \cdots + \frac{1}{n}\left(1 - \frac{|S_R(x_{ks_k})|}{n}\right)\Bigg) \\
&= \frac{1}{n}\left(1 - \frac{|S_R(x_1)|}{n}\right) + \frac{1}{n}\left(1 - \frac{|S_R(x_2)|}{n}\right) \\
&\quad + \cdots + \frac{1}{n}\left(1 - \frac{|S_R(x_n)|}{n}\right) \\
&= \sum_{i=1}^{n} \frac{1}{n}\left(1 - \frac{|S_R(x_i)|}{n}\right) \\
&= E(\widetilde{R}).
\end{aligned}$$

This completes the proof. ■

Theorem 23 states that the information entropy $E(R)$ is a special case of the fuzzy-information entropy $E(\widetilde{R})$ in fuzzy granular structures.

Using the following theorem, we verify the fuzzy granulation monotonicity of fuzzy-information entropy $E(\widetilde{A})$.

Theorem 27: Let $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$. If $K(\widetilde{P}) \prec_3 K(\widetilde{Q})$, then $E(\widetilde{Q}) < E(\widetilde{P})$.

Proof: Let $K(\widetilde{P}) = (S_{\widetilde{P}}(x_1), S_{\widetilde{P}}(x_2), \ldots, S_{\widetilde{P}}(x_n))$, $K(\widetilde{Q}) = (S_{\widetilde{Q}}(x_1), S_{\widetilde{Q}}(x_2), \ldots, S_{\widetilde{Q}}(x_n))$.

If $\widetilde{P} \prec_3 \widetilde{Q}$, then there exists a sequence $K'(\widetilde{Q})$ of $K(\widetilde{Q})$, where $K'(\widetilde{Q}) = (S_{\widetilde{Q}}(x'_1), S_{\widetilde{Q}}(x'_2), \ldots, S_{\widetilde{Q}}(x'_n))$, such that $|S_{\widetilde{P}}(x_i)| \leq |S_{\widetilde{Q}}(x'_i)|, i \leq n$, and there exists $x_0 \in U$ such that $|S_{\widetilde{P}}(x_0)| < |S_{\widetilde{Q}}(x'_0)|$. Therefore

$$\begin{aligned}
E(\widetilde{P}) &= \sum_{i=1}^{n} \frac{1}{n}\left(1 - \frac{|S_{\widetilde{P}}(x_i)|}{n}\right) \\
&= \sum_{i=1, x_i \neq x_0}^{n} \frac{1}{n}\left(1 - \frac{|S_{\widetilde{P}}(x_i)|}{n}\right) + \frac{1}{n}\left(1 - \frac{|S_{\widetilde{P}}(x_0)|}{n}\right) \\
&> \sum_{i=1, x_i \neq x_0}^{n} \frac{1}{n}\left(1 - \frac{|S_{\widetilde{Q}}(x_i)|}{n}\right) + \frac{1}{n}\left(1 - \frac{|S_{\widetilde{Q}}(x_0)|}{n}\right) \\
&= \sum_{i=1}^{n} \frac{1}{n}\left(1 - \frac{|S_{\widetilde{Q}}(x_i)|}{n}\right) = E(\widetilde{Q}).
\end{aligned}$$

Obviously, $E(\widetilde{Q}) < E(\widetilde{P})$. This completes the proof. ■

Theorem 28: Letting $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$. If $K(\widetilde{P}) \prec_1 K(\widetilde{Q})$, then $E(\widetilde{P}) > E(\widetilde{Q})$.

Theorem 29: Let $K(\widetilde{P}), K(\widetilde{Q}) \in \mathbf{K}(U)$. If $K(\widetilde{P}) \prec_2 K(\widetilde{Q})$, then $E(\widetilde{P}) > E(\widetilde{Q})$.

From the above analysis, one can draw a conclusion that each of fuzzy-information entropies satisfies the fuzzy granulation monotonicity induced by the partial-order relation $\preceq_3$. From the viewpoint of fuzzy GrC, it might be better the partial-order relation $\preceq_3$ to reveal the essence of fuzzy-information granularity than partial-order relations $\preceq_1$ and $\preceq_2$.

In what follows, we establish the relationship between fuzzy information entropy and fuzzy-information granularity in the context of fuzzy-knowledge bases.

Theorem 30: Letting $K(\widetilde{R}) = (S_{\widetilde{R}}(x_1), S_{\widetilde{R}}(x_2), \ldots, S_{\widetilde{R}}(x_n))$, then the relationship between the fuzzy-information entropy $H(\widetilde{R})$ and the fuzzy rough entropy $E_r(\widetilde{R})$ is depicted by

$$H(\widetilde{R}) + E_r(\widetilde{R}) = \log_2 n. \tag{21}$$

Theorem 31: Letting $K(\widetilde{R}) = (S_{\widetilde{R}}(x_1), S_{\widetilde{R}}(x_2), \ldots, S_{\widetilde{R}}(x_n))$, then the relationship between the fuzzy-information entropy $E(\widetilde{R})$ and the fuzzy-information granularity $\mathrm{GK}(\widetilde{R})$ is

$$E(\widetilde{R}) + \mathrm{GK}(\widetilde{R}) = 1. \tag{22}$$

These two theorems above show that the relationship between fuzzy-information entropy and fuzzy-information granularity, in some sense, may be a complement relationship, i.e., they possess the same capability on depicting the uncertainty of a fuzzy-granular structure in the context of fuzzy-binary granular structures.

VIII. Conclusion

Zadeh's seminal work in theory of fuzzy-information granulation in human reasoning is inspired by the ways in which humans granulate information and reason with it. Although many excellent research contributions have been made, there remains an important issue to be addressed: What is the essence of measuring a fuzzy-information granularity of a fuzzy-binary granular structure?

In this research, to reveal the essence of measuring a fuzzy information granularity, a partial-order relation with set-size character has first been introduced, and the relationship between the proposed partial-order relation and the other two partial-order relations have been also established. Then, based on this partial order relation, an axiomatic definition to measure the uncertainty of a fuzzy-binary granular structure has been proposed in terms of the size of each fuzzy-information granule. The proposed two forms of fuzzy-information granularity are all special instances under the axiomatic definition. Furthermore, we also have investigated the theory of fuzzy-information entropy and proved their granulation monotonicity induced by the proposed partial-order relation. Note that the relationship between fuzzy-information entropy and fuzzy information granularity, in some sense, may be a complement relationship. They possess the same capability on depicting the uncertainty of a fuzzy-binary granular structure. These results show that the partial-order relation proposed in this paper may be better for characterizing the essence of fuzzy-information granularity for measuring uncertainty of fuzzy-binary granular structures, which will be very helpful for establishing a uniform framework for GrC.

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References

[1] M. D. Cock, C. Cornelis, and E. E. Kerre, “Fuzzy rough sets: The forgotten step,” *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 1, pp. 121–130, Feb. 2007.
[2] D. Dübois and H. Prade, “Rough fuzzy sets and fuzzy rough sets,” *Int. J. Gen. Syst.*, vol. 17, pp. 191–209, 1990.
[3] Q. H. Hu, D. R. Yu, Z. X. Xie, and J. F. Liu, “Fuzzy probabilistic approximation spaces and their information measures,” *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 2, pp. 191–201, Apr. 2006.
[4] Q. H. Hu, D. R. Yu, Z. X. Xie, and X. D. Li, “EROS: Ensemble rough subspaces,” *Pattern Recognit.*, vol. 40, pp. 3728–3739, 2007.
[5] Q. H. Hu, Z. X. Xie, and D. R. Yu, “Hybrid attribute reduction based on a novel fuzzy-rough model and information granulation,” *Pattern Recognit.*, vol. 40, pp. 3509–3521, 2007.
[6] Q. H. Hu, Z. X. Xie, and D. R. Yu, “Comment on Fuzzy probabilistic approximation spaces and their information measures,” *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 2, pp. 549–551, Apr. 2008.
[7] R. Jensen and Q. Shen, “Semantics-preserving dimensionality reduction: Rough and fuzzy-rough-based approaches,” *IEEE Trans. Knowl. Data Eng.*, vol. 16, no. 12, pp. 1457–1471, Dec. 2004.
[8] R. Jensen and Q. Shen, “Fuzzy-rough sets assisted attribute reduction,” *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 1, pp. 73–89, Feb. 2007.
[9] R. Jensen and Q. Shen, “New approaches to fuzzy-rough feature selection,” *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 4, pp. 824–838, Aug. 2009.
[10] M. Kryszkiewicz, “Rough set approach to incomplete information systems,” *Inf. Sci.*, vol. 112, pp. 39–49, 1998.
[11] J. Y. Liang and Y. H. Qian, “Information granules and entropy theory,” *Sci. China., Ser. F*, vol. 51, no. 10, pp. 1427–1444, 2008.
[12] J. Y. Liang, C. Y. Dang, K. S. Chin, and C. M. Yam Richard, “A new method for measuring uncertainty and fuzziness in rough set theory,” *Int. J. Gen. Syst.*, vol. 31, no. 4, pp. 331–342, 2002.
[13] J. Y. Liang, Z. Z. Shi, D. Y. Li, and M. J. Wierman, “The information entropy, rough entropy and knowledge granulation in incomplete information system,” *Int. J. Gen. Syst.*, vol. 35, no. 6, pp. 641–654, 2006.
[14] J. Y. Liang and Z. Z. Shi, “The information entropy, rough entropy and knowledge granulation in rough set theory,” *Int. J. Uncertain., Fuzziness Knowl.-Based Syst.*, vol. 12, no. 1, pp. 37–46, 2004.
[15] T. Y. Lin, “Granular computing: Practices, theories, and future directions,” in *Encyclopedia of Complexity and Systems Science*, R. A. Meyers, Ed., Berlin, Heidelberg: Springer, 2009, pp. 4339–4355.
[16] T. Y. Lin, “Granular computing I: The concept of granulation and its formal model,” *Int. J. Granular Comput., Rough Sets Intell. Syst.*, vol. 1, no. 1, pp. 21–42, 2009.
[17] T. Y. Lin, “Deductive data mining using granular computing,” *Encyclopedia Database Syst.*, pp. 772–778, 2009.
[18] T. Y. Lin, “Granular computing: Information hiding and integration,” NAFIPS2008, Tech. Rep.
[19] T. Y. Lin, “Data mining and machine oriented modeling: a granular computing approach,” *Appl. Intell.*, vol. 13, no. 2, pp. 113–124, 2000.
[20] T. Y. Lin, “Granular computing: fuzzy logic and rough sets,” in *Computing with Words in Information/Intelligent Systems*, L. A. Zadeh and J. Kacprzyk, Eds. New York: Physica-Verlag, 1999, pp. 183–200.
[21] T. Y. Lin, “Granular computing on binary relations I: Data mining and neighborhood systems,” in *Rough Sets In Knowledge Discovery*, A. Skowron and L. Polkowski, Eds. New York: Physica-Verlag, 1998, pp. 107–121.
[22] T. Y. Lin, “Granular computing on binary relations II: Rough set representations and belief functions,” in *Rough Sets in Knowledge Discovery*, A. Skowron and L. Polkowski, Eds. New York: Physica-Verlag, 1998, pp. 121–140.
[23] T. Y. Lin, “Topological and fuzzy rough sets,” in *Decision Support by Experience-Application of the Rough Sets Theory*, R. Slowinski, Ed. Dordrecht, The Netherlands: Kluwer, 1992, pp. 287–304.
[24] T. Y. Lin, “Neighborhood systems and approximation in database and knowledge base systems,” in *Proc. 4th Int. Symp. Methodol. Intell. Syst.*, Charlotte, NC, 1989, pp. 75–86.
[25] T. Y. Lin, “Neighborhood systems and relational database,” in *Proc. 16th ACM Annu. Conf. Comput. Sci.*, Altanta, GA, 1988, pp. 725–728.
[26] J. M. Ma, W. X. Zhang, Y. Leung, and X. X. Song, “Granular computing and dual Galois connection,” *Inf. Sci.*, vol. 177, pp. 5365–5377, 2007.
[27] P. Maji and S. K. Pal, “Feature selection using f-information measures in fuzzy approximation spaces,” *IEEE Trans. Knowl. Data Eng.*, vol. 22, no. 6, pp. 854–867, Jun. 2009.
[28] J. S. Mi, Y. Leung, and W. Z. Wu, “A uncertainty measure in partition-based fuzzy rough sets,” *Int. J. Gen. Syst.*, vol. 34, no. 1, pp. 77–90, 2005.
[29] Z. Pawlak, *Rough Sets: Theoretical Aspects of Reasoning about Data*. Dordrecht, The Netherlands: Kluwer, 1991.
[30] Z. Pawlak and A., “Skowron, Rudiments of rough sets,” *Inf. Sci.*, vol. 177, pp. 3–27, 2007.
[31] W. Pedrycz, “Relational and directional aspects in the construction of information granules,” *IEEE Trans. Syst. Man. Cybern. A, Syst., Humans*, vol. 32, no. 5, pp. 605–614, Sep. 2002.
[32] W. Pedrycz and A. Bargiela, “Granular clustering: a granular signature of data,” *IEEE Trans. Syst. Man. Cybern. B, Cybern.*, vol. 32, no. 2, pp. 212–224, Apr. 2002.
[33] Y. H. Qian, J. Y. Liang, and C. Y. Dang, “Consistency measure, inclusion degree and fuzzy measure in decision tables,” *Fuzzy Sets Syst.*, vol. 159, pp. 2353–2377, 2008.
[34] Y. H. Qian, J. Y. Liang, and C. Y. Dang, “Knowledge structure, knowledge granulation and knowledge distance in a knowledge base,” *Int. J. Approx. Reason.*, vol. 50, no. 1, pp. 174–188, 2009.
[35] Y. H. Qian, J. Y. Liang, and C. Y. Dang, “Incomplete multi-granulation rough set,” *IEEE Trans. Syst. Man. Cybern. A, Syst., Humans*, vol. 40, no. 2, pp. 420–431, Mar. 2010.
[36] Y. H. Qian, J. Y. Liang, Y. Y. Yao, and C. Y. Dang, “MGRS: A multi-granulation rough set,” *Inf. Sci.*, vol. 180, pp. 949–970, 2010.
[37] Y. H. Qian, J. Y. Liang, W. Z. Wu, and C. Y. Dang, “Partial orderings of information granulation: A further investigation,” *Expert. Syst.*, DOI: 10.1111/j.1468-0394.2010.00551.x.
[38] Y. H. Qian and J. Y. Liang, “A new method for measuring the uncertainty in incomplete information systems,” *Int. J. Uncertain., Fuzziness Knowl.-Based Syst.*, vol. 17, no. 6, pp. 855–880, 2009.
[39] Y. H. Qian and J. Y. Liang, “Combination entropy and combination granulation in rough set theory,” *Int. J. Uncertain., Fuzziness Knowl.-Based Syst.*, vol. 16, no. 2, pp. 179–193, 2008.
[40] C. E. Shannon, “The mathematical theory of communication,” *Bell Syst. Tech. J.*, vol. 27, no. 3/4, pp. 373–423, 623–656, 1948.
[41] E. C. C. Tsang, D. G. Chen, D. S. Yeung, X. Z. Wang, and J. W. T. Lee, “Attributes reduction using fuzzy rough sets,” *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 5, pp. 1130–1141, Oct. 2008.

[42] M. J. Wierman, “Measuring uncertainty in rough set theory,” *Int. J. Gen. Syst.*, vol. 28, no. 4, pp. 283–297, 1999.

[43] W. Z. Wu, J. S. Mi, and W. X. Zhang, “Generalized fuzzy rough sets,” *Inf. Sci.*, vol. 151, pp. 263–282, 2003.

[44] W. Z. Wu and W. X. Zhang, “Constructive and axiomatic approaches of fuzzy approximation operators,” *Inf. Sci.*, vol. 159, pp. 233–254, 2004.

[45] B. W. Xu, Y. M. Zhou, and H. M. Lu, “An improved accuracy measure for rough sets,” *J. Comput. Syst. Sci.*, vol. 71, pp. 163–173, 2005.

[46] Y. Y. Yao, “Information granulation and rough set approximation,” *Int. J. Intell. Syst.*, vol. 16, no. 1, pp. 87–104, 2001.

[47] D. S. Yeung, D. G. Chen, E. C. C. Tsang, J. W. T. Lee, and X. Z. Wang, “On the generalization of fuzzy rough sets,” *IEEE Trans. Fuzzy Syst.*, vol. 13, no. 3, pp. 343–361, Jun. 2005.

[48] L. Zadeh, “Fuzzy logic equals computing with words,” *IEEE Trans. Fuzzy Syst.*, vol. 4, no. 2, pp. 103–111, May 1996.

[49] L. Zadeh, “Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic,” *Fuzzy Sets Syst.*, vol. 90, no. 2, pp. 111–127, 1997.

[50] L. Zadeh, “Some reflections on soft computing, granular computing and their roles in the conception, design and utilization of information/intelligent systems,” *Soft Comput.*, vol. 2, pp. 23–25, 1998.

[51] L. Zadeh, “A new direction in AI-Toward a computational theory of perceptions,” *AI Mag.*, vol. 22, no. 1, pp. 73–84, 2001.

[52] S. Y. Zhao, E. C. C. Tsang, and D. G. Chen, “The model of fuzzy variable precision rough sets,” *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 2, pp. 451–467, Apr. 2009.

[53] S. Y. Zhao, E. C. C. Tsang, and X. Z. Wang, “Building a rule-based classifier-a fuzzy rough set approach,” *IEEE Trans. Knowl. Data Eng.*, vol. 22, no. 5, pp. 624–638, May. 2009.

[54] L. Zhang and B. Zhang, *Theory and Application of Problem Solving-Theory and Application of Granular Computing in Quotient Space*, 2nd ed. Beijing, China: Tsinghua Univ. Press, 2007.

Yuhua Qian (M’10) received the M.S. degree in computers with applications from Shanxi University, Taiyuan, China, in 2005.

He is an Assistant Professor with the Key Laboratory of Computational Intelligence and Chinese Information Processing of the Ministry of Education, China, and is also a Senior Research Associate with the Department of Manufacturing Engineering and Engineering Management, City University of Hong Kong. He is actively pursuing research in pattern recognition, feature selection, rough set theory, granular computing, and artificial intelligence. He has published more than 20 articles in international journals.

Dr. Qian has also served on the editorial board of the *International Journal of Knowledge-Based Organizations*.

Jiye Liang received the B.S. degree in computational mathematics and the Ph.D degree in information science.from Xi’an Jiaotong University, Xi’an, China.

He a Professor with the School of Computer and Information Technology and Key Laboratory of Computational Intelligence and Chinese Information Processing of the Ministry of Education, Shanxi University, Taiyuan, China. His research interests include artificial intelligence, granular computing, data mining, and knowledge discovery. He has published more than 20 articles in international journals.

Wei-zhi Z. Wu (SM’03) received the B.Sc. degree in mathematics from Zhejiang Normal University, Jinhua, China, in 1986, the M.Sc. degree in mathematics from East China Normal University, Shanghai, China, in 1992, and the Ph.D. degree in applied mathematics from Xi’an Jiaotong University, Xi’an, China, in 2002.

He is currently a Professor of Mathematics with the School of Mathematics, Physics, and Information Science, Zhejiang Ocean University, Zhejiang, China. He has published book chapters, two monographs, and more than 70 articles in international journals. His current research interests include approximate reasoning, rough sets, random sets, formal concept analysis, and granular computing.

Dr. Wu serves in the editorial boards of several international journals.

Chuangyin Dang (SM’03) received the B.S. degree in computational mathematics from Shanxi University, Taiyuan, China, in 1983, the M.S. degree in applied mathematics from Xidian University, Xi’an, China, in 1986, and the Ph.D. degree in operations research/economics from the University of Tilburg, Tilburg, The Netherlands, in 1991.

He is an Associate Professor with the City University of Hong Kong. He is best known for the development of the D1-triangulation of the Euclidean space and the simplified method for integer programming. His current research interests include computational intelligence, optimization theory and techniques, and applied general equilibrium modeling and computation.

Dr. Dang is a member of INFORS and MPS.