Information Loss in Quantum Dynamics

Er'el Granot

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Abstract

The way data is lost from the wavefunction in quantum dynamics is analyzed. The main results are (A) Quantum dynamics is a dispersive process in which any data initially encoded in the wavefunction is gradually lost. The ratio between the distortion's variance and the mean probability density increases in a simple form. (B) For any given amount of information encoded in the wavefunction, there is a time period, beyond which it is impossible to decode the data. (C) The temporal decline of the maximum information density in the wavefunction has an exact analytical expression. (D) For any given time period there is a specific detector resolution, with which the maximum information can be decoded. (E) For this optimal detector size the amount of information is inversely proportional to the square root of the time elapsed.

Keywords: quantum information, quantum encryption, uncertainty principle, quantum decoding

1. Introduction

The field of quantum information received a lot of attention recently due to major development in quantum computing [1–5], quantum cryptography, and quantum communications [6–8].

In most quantum computing, the wavefunction is a superposition of multiple binary states (qubits), which can be in spin states, polarization state, binary energy levels, etc. However, since the wavefucntion is a continuous function, it can carry, in principle, an infinite amount of information. Only the detector dimensions and noises limits the information capacity.

The quantum wavefunction, like any complex signal, carries a large amount of information, which can be decoded in the detection process. Its local amplitude can be detected by measuring the probability density in a direct measurement, while its phase can be retrieved in an interferometric detection, just as in optical coherent detection [9].

The amount of information depends on the detector's capabilities, i.e., it depends on the detector's spatial resolution and its inner noise level. Therefore, the maximum amount of information that can be decoded from the wavefunction is determined by the detector's characteristics. However, unlike the classical wave equation, the quantum Schrödinger dynamics is a dispersive process. During the quantum dynamics, the wavefunction experiences distortions. These distortions increase in time just like the dispersion effects on signals in optical communications [10, 11].

Nevertheless, unlike dispersion compensating modules in optical communications, there is no way to compensate or "undo" the dispersive process in quantum mechanics. Therefore, the amount of information that can be decoded decreases monotonically with time.

The object of this chapter is to investigate the way information is lost during the quantum dynamics.

2. Quantum dynamics of a random sequence

The general idea is to encode the data on the initial wavefunction. In accordance to signals in coherent optical communications, in every point in space the data can be encoded in both the real and imaginary parts of the wavefunction.

The amount of distortion determines the possibility to differentiate between similar values, and therefore, it determines the maximum amount of information that the wavefunction carries.

The detector width Δx determines the highest volume of data that can be stored in a given space, i.e., it determines the data density. All spatial frequencies beyond $1/\Delta x$ cannot be detected and cannot carry information. Moreover, due to this constrain, there is no point in encoding the data with spatial frequency higher than $1/\Delta x$.

A wavefunction, which consists of the infinite random complex sequence $\psi_n = \Re\psi_n + i\Im\psi_n$ for $n = -\infty, \dots -1, 0, 1, 2, \dots \infty$, which occupies the spatial spectral bandwidth $1/\Delta x$ (higher frequencies cannot be detected by the given detector) can be written initially as an infinite sequence of overlapping Nyquist-sinc functions [12, 13] (see **Figure 1**), i.e.,

$$\psi(x, t = 0) = \sum_{n=-\infty}^{\infty} \psi_n \mathrm{sinc}(x/\Delta x - n), \tag{1}$$

where $\mathrm{sinc}(\xi) \equiv \frac{\sin(\pi\xi)}{\pi\xi}$ is the well-known "sinc" function.

After a time period t, in which the wavefunctions obeys the free Schrödinger equation.

$$i\hbar \frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}, \tag{2}$$

the wavefunction can be written as a convolution

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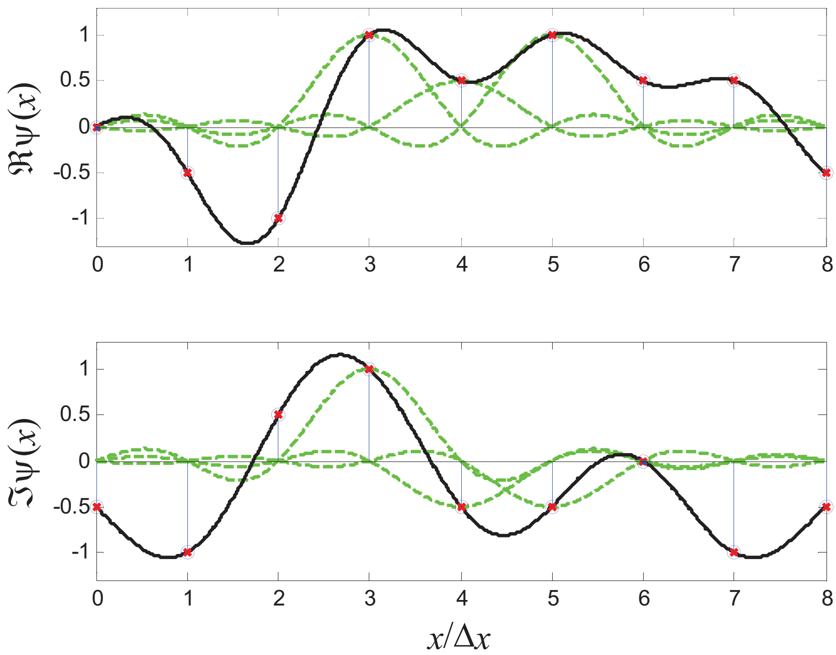

Figure 1. Illustration of the way the data is encoded in the wavefucntion. In every Δx, there is a single complex number $\psi_n = \Re\psi_n + i\Im\psi_n$ (the circles), while the continuous wavefunction is a superposition of these numbers multiplied by sinc's functions (three of which are presented by the dashed curves). The values in the y-axis should be multiplied by the normalization constant of the wavefucntion.

$$\psi(x,t) = \int_{-\infty}^{\infty} K(x-x',t)\psi(x',0)dx' \tag{3}$$

with the Schrödinger Kernel [14].

$$K(x-x',t) = \sqrt{\frac{m}{2\pi i\hbar t}}\ \exp\left[\frac{im}{2\hbar}\frac{(x-x')^2}{t}\right]. \tag{4}$$

Due to the linear nature of the problem, Eq. (3) can be solved directly

$$\psi(x,t>0) = \sum_{n=-\infty}^{\infty} \psi_n \mathrm{dsinc}\left(x/\Delta x - n, (\hbar/m)t/\Delta x^2\right) \tag{5}$$

where "dsinc" is the dynamic-sync function

$$\mathrm{dsinc}(\xi,\tau) \equiv \frac{1}{2}\sqrt{\frac{i}{2\pi\tau}}\exp\left(-i\frac{\xi^2}{2\tau}\right)\left[\mathrm{erf}\left(-\frac{\xi-\pi\tau}{\sqrt{i2\tau}}\right) - \mathrm{erf}\left(-\frac{\xi+\pi\tau}{\sqrt{i2\tau}}\right)\right]. \tag{6}$$

Equation (6) is the "sinc" equivalent of the "srect" function, that describes the dynamics of rectangular pulses (see Ref. [15]).

Note that $\lim_{\tau \to 0} [\text{dsinc}(\xi, \tau)] = \text{sinc}(\xi)$.

Some of the properties of the dsinc function are illustrated in **Figures 2** and **3**. As can be seen, the distortions form dsinc(n, 0) = $\delta(n)$ gradually increase with time.

Hereinafter, we adopt the dimensionless variables

$$\tau \equiv (\hbar/m)t/\Delta x^2 \text{and } \xi \equiv x/\Delta x. \tag{7}$$

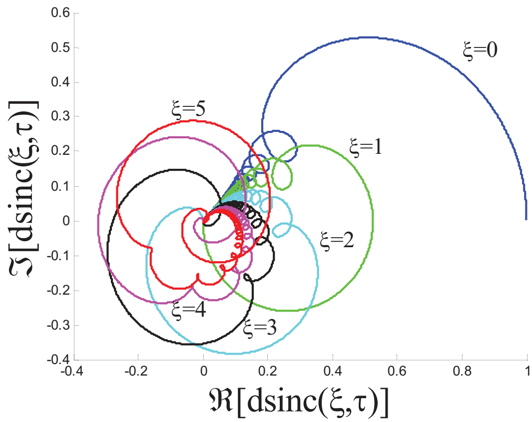

Figure 2. Several plots of the real and imaginary parts of the dsinc function for different discrete values of ξ = 0, 1, 2, … 5.

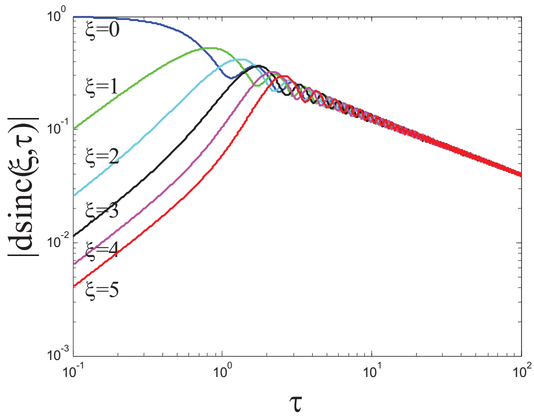

Figure 3. The dependence of the absolute value of the dsinc function on τ for different discrete values of ξ = 0, 1, 2, … 5.

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Thus, Eq. (2) can be rewritten

$$i\frac{\partial\psi(\xi,\tau)}{\partial\tau} = -\frac{1}{2}\frac{\partial^2\psi(\xi,\tau)}{\partial\xi^2} \tag{8}$$

and Eq. (5) simply reads

$$\psi(\xi,\tau>0) = \sum_{n=-\infty}^{\infty}\psi_n \mathrm{dsinc}(\xi-n,\tau). \tag{9}$$

Therefore, the wavefunction at the detection point of the mth symbol (center of the symbol at $\xi = m$) is a simple convolution

$$\psi(m,\tau) = \sum_n \psi_n h(m-n) = \psi_m + \sum_n \psi_n \delta h(m-n) \tag{10}$$

where

$$h(n) \equiv \mathrm{dsinc}(n,\tau) \text{ and } \delta h(n) \equiv \mathrm{dsinc}(n,\tau) - \delta(n). \tag{11}$$

Since

$$\left.\frac{\partial^2 \mathrm{sinc}(\xi)}{\partial\xi^2}\right|_{\tau=n\neq 0} = \frac{2}{n^2}(-1)^{n+1} \quad \text{and} \quad \left.\frac{\partial^2 \mathrm{sinc}(\xi)}{\partial\xi^2}\right|_{\tau=0} = -\frac{\pi^2}{3}, \tag{12}$$

then Eq. (9) can be written as a linear set of differential equations

$$\frac{d\psi(m,\tau)}{d\tau} = i\sum_n w(m-n)\psi(n,\tau) \equiv iw(m) * \psi(m,\tau) \tag{13}$$

with the dimensionless

$$w(m) \equiv \left[\cdots \quad \frac{1}{3^2} \quad -\frac{1}{2^2} \quad 1 \quad -\frac{\pi^2}{6} \quad 1 \quad -\frac{1}{2^2} \quad \frac{1}{3^2} \quad \cdots\right] = \begin{cases} (-1)^{m+1}/m^2 & m \neq 0 \\ -\pi^2/6 & m = 0 \end{cases}. \tag{14}$$

It should be noted that the fact that Eq. (14) is a universal sequence, i.e. it is independent of time, is not a trivial one. It is a consequence of the properties of the sinc function. Unlike rectangular pulses, which due to their singularity has short time dynamics is mostly nonlocal (and therefore, time-dependent) [15, 16], sinc pulses are smooth and therefore, their dynamics is local and consequently $w(m)$ is time-independent.

3. Quantum distortion noise

After a short period of time, the error (distortion) in the wavefunction (i.e., the wavefunction deformation)

$$\Delta\psi(\xi,\tau) \equiv \psi(\xi,\tau) - \psi(\xi,0) \tag{15}$$

can be approximated by

$$\Delta\psi(\xi,\tau) \equiv \psi(\xi,\tau) - \psi(\xi,0) \cong \tau \frac{\partial\psi(\xi,\tau)}{\partial\tau}\bigg|_{\tau=0}. \tag{16}$$

Then we can define the Quantum Noise as the variance of the error

$$N = \left\langle |\Delta\psi(\xi,\tau)|^2 \right\rangle \cong \tau^2 \left\langle \left| \frac{\partial\psi(\xi,\tau)}{\partial\tau}\bigg|_{\tau=0} \right|^2 \right\rangle \tag{17}$$

where the triangular brackets stand for spatial averaging, i.e., $\langle f(x)\rangle \equiv \frac{1}{X}\int_{-X/2}^{X/2} f(x')dx'$.

Using the Schrödinger equation, Eq. (17) can be rewritten as follows:

$$N = \left\langle |\Delta\psi(\xi,\tau)|^2 \right\rangle \cong \frac{\tau^2}{4} \left\langle \left| \frac{\partial^2\psi(\xi,0)}{\partial^2\xi} \right|^2 \right\rangle. \tag{18}$$

Similarly, we can define the average density as

$$\rho = \left\langle |\psi(\xi,\tau)|^2 \right\rangle. \tag{19}$$

Now, from the Parseval theorem [12], the spatial integral (average) can be replaced by a spatial frequency integral over the Fourier transform, i.e.,

$$N = \frac{1}{2\pi} \left\langle |\Delta\psi(\kappa,\tau)|^2 \right\rangle \tag{20}$$

and

$$\rho = \frac{1}{2\pi} \left\langle |\psi(\kappa,\tau)|^2 \right\rangle \tag{21}$$

where

$$\psi(\kappa,\tau) \equiv (2\pi)^{-1} \int d\xi \exp(-i\kappa\xi)\psi(\xi,\tau) \text{ and } \Delta\psi(\kappa,\tau) \equiv (2\pi)^{-1} \int d\xi \exp(-i\kappa\xi)\Delta\psi(\xi,\tau). \tag{22}$$

Therefore, the ratio between the noise and the density (i.e., the reciprocal of the Signal-to-Noise Ratio, SNR) satisfies the surprisingly simple expression

$$\frac{N}{\rho} = \frac{\left\langle |\Delta\psi(\xi,\tau)|^2 \right\rangle}{\left\langle |\psi(\xi,0)|^2 \right\rangle} \cong \tau^2 \frac{\frac{1}{2\pi}\int d\kappa \frac{\kappa^4}{4} |\psi(\kappa,0)|^2}{\frac{1}{2\pi}\int d\kappa |\psi(\kappa,0)|^2} = \tau^2 \frac{\pi^4}{20} \tag{23}$$

and with physical dimensions

$$\frac{N}{\rho} = \frac{t^2}{\Delta x^4}\left(\frac{\hbar}{m}\right)^2\frac{\pi^4}{20}. \tag{24}$$

We, therefore, find a universal relation: the relative noise (the ratio between the noise and the density) depends only on a single dimensionless parameter $\tau \equiv (\hbar/m)t/\Delta x^2$.

It should be stressed that this is a universal property, which emerges from the quantum dynamics. This relation is valid regardless of the specific data encoded in the wavefunction provided the data's spectral density is approximately homogenous in the spectral bandwidth $[-1/\Delta x, 1/\Delta x]$.

Clearly, since the noise increases gradually, it will becomes more difficult to decode the data from the wavefucntion. In fact, as is well known from Shannon celebrated equation [17], the amount of noise determines the data capacity that can be decoded. Therefore, the amount of information must decrease gradually.

4. The rate of information loss

We assume that at every Δx interval the wavefunction can have one of M different complex values. In this case, both the real and imaginary parts can have $\sqrt{M}$ different values (this form is equivalent to the Quadrature Amplitude Modulation, QAM, in electrical and optical modulation scheme [18]), i.e., any complex $\psi_n = \psi(n) = \Re\psi_n + i\Im\psi_n = \tilde{N}v_{p,q}$ can have one of the values

$$v_{p,q} = \frac{2p-\sqrt{M}-1}{\sqrt{M}-1} + i\frac{2q-\sqrt{M}-1}{\sqrt{M}-1} \text{ for } p,q = 1,2,\ldots\sqrt{M} \tag{25}$$

where $\tilde{N}$ is the normalization constant.

Since $b = \log_2 M$ is the number of bits encapsulated in each one of the complex symbol, then the difference between adjacent symbol

$$\Delta v = \Re v_{p,q} - \Re v_{p-1,q} = \Im v_{p,q} - \Im v_{p,q-1} \tag{26}$$

decreases exponentially with the number of bits, i.e.,.

$$\Delta v = \frac{2}{\sqrt{M}-1} = \frac{2}{2^{b/2}-1} \cong 2^{1-b/2} = 2\exp[-b(\ln 2/2)] \tag{27}$$

Therefore, as the number of bits per symbol increases, it becomes more difficult to distinguish between the symbols.

Clearly, maximum distortion occurs, when all the *other* symbols oscillates with maximum amplitude, i.e.,

$$\psi_n = \psi(n,0) = \begin{cases} \psi(m,0) & n = m \\ (-1)^{n-m} & n \neq m \end{cases}, \tag{28}$$

in which case the differential Eq. (13) can be written (for short periods)

$$\frac{d\psi_{\max/\min}(m,\tau)}{d\tau} = -iw(0)\psi_{\max/\min}(m,\tau) \mp i\sum_{n\neq 0} w(m-n)(-1)^n = i\frac{\pi^2}{6}\psi_{\max/\min}(m,\tau) \mp i\pi^2/3. \tag{29}$$

The solution of Eq. (29) is

$$\psi_{\max/\min}(m,\tau) = \psi(m,0)\exp\left(i\pi^2\tau/6\right) \pm 2\left(1 - \exp\left(i\pi^2\tau/6\right)\right). \tag{30}$$

Therefore, each cluster is bounded by a circle whose center is

$$\psi(m,0)\exp\left(i\pi^2\tau/6\right) \tag{31}$$

and its radius is

$$R = 2\left|1 - \exp\left(i\pi^2\tau/6\right)\right| = 4\left|\sin\left(\pi^2\tau/12\right)\right|. \tag{32}$$

Since this result applies only for short periods, then the entire cluster is bounded by the radius

$$R = \pi^2\tau/3, \tag{33}$$

which is clearly larger than the cluster's standard deviation $\sigma = \pi^2\tau/\sqrt{20} < R$.

A simulation based on Eq. (1) with $2^{11} - 1$ symbols, which were randomly selected from the pool (25) for $M = 16$ was taken. That is, the probability that ψ_n is equal to $v_{p,q}$ is $1/M$ for all ns, or mathematically

$$P\left(\psi_n = v_{p,q}\right) = 1/M, \quad \text{for} \;\; n = 1, 2, 3, \ldots, 2^{11} - 1, \;\; \text{and} \; p, q = 1, 2, \ldots\sqrt{M}. \tag{34}$$

The temporal dependence of the calculated SNR is presented in **Figure 4**. As can be seen, Eq. (23) is indeed an excellent approximation for short τ.

Since the symbols were selected randomly (with uniform distribution), then when all the symbols $\psi(n, 0) = \psi_n$ are plotted on the complex plain, an ideal constellation image is shown (see the upper left subfigure of **Figure 5**).

In **Figure 5**, a numerical simulation for a QAM 16 scenario is presented initially and after a time period, $\tau = 0.1$. Moreover, the dashed circles represents the standard deviation, i.e., the noise level (radius $\pi^2\tau/\sqrt{20}$), and the bounding circles (radius $R = \pi^2\tau/3 > \pi^2\tau/\sqrt{20}$).

Since the initial distance between centers of adjacent clusters is $\frac{2}{\sqrt{M}-1}$, then decoding is impossible for $\frac{1}{\sqrt{M}-1} = \frac{\pi^2\tau_{\max}}{3}$, i.e., we finally have an expression for the maximum time $\tau_{\max}$, beyond

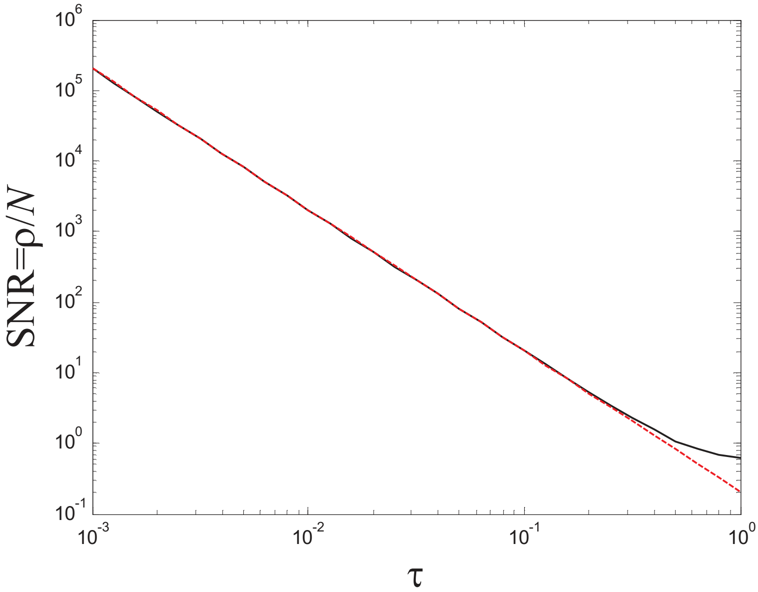

Figure 4. Plot of the SNR as a function of τ. The solid curve represents the simulation result, and the dashed line represents the approximation for short τ (the reciprocal of Eq. (23)).

which it is impossible to encode the data (i.e., to differentiate between symbols). This maximum time is

$$\tau_{\max} = \frac{3}{\left(\sqrt{M} - 1\right)\pi^2} \tag{35}$$

It should be noted that this result coincides with the On-Off-Keying (OOK) dispersion limit, for which case $\sqrt{M} = 2$, and then $\tau_{\max} = 1/\pi \cong 3/\pi^2$ (see Ref. [19]).

Similarly, Eq. (35) can be rewritten to find the maximum M for a given distance, i.e.,

$$\sqrt{M_{\max}} = 1 + \frac{3}{\pi^2 \tau}. \tag{36}$$

However, it is clear that this formulae for $\sqrt{M_{\max}}$ is meaningful only under the constraint that $\sqrt{M_{\max}}$ is an integer.

Since the number of bits per symbol is $\log_2 M$, then the maximum data density (bit/distance) is

$$S_{\max} = \frac{2}{\Delta x} \log_2 \sqrt{M_{\max}} = \frac{2}{\Delta x} \log_2\left(1 + 3/\pi^2 \tau\right). \tag{37}$$

Using $\Delta x = \sqrt{\frac{(\hbar/m)t}{\tau}}$, we finally have

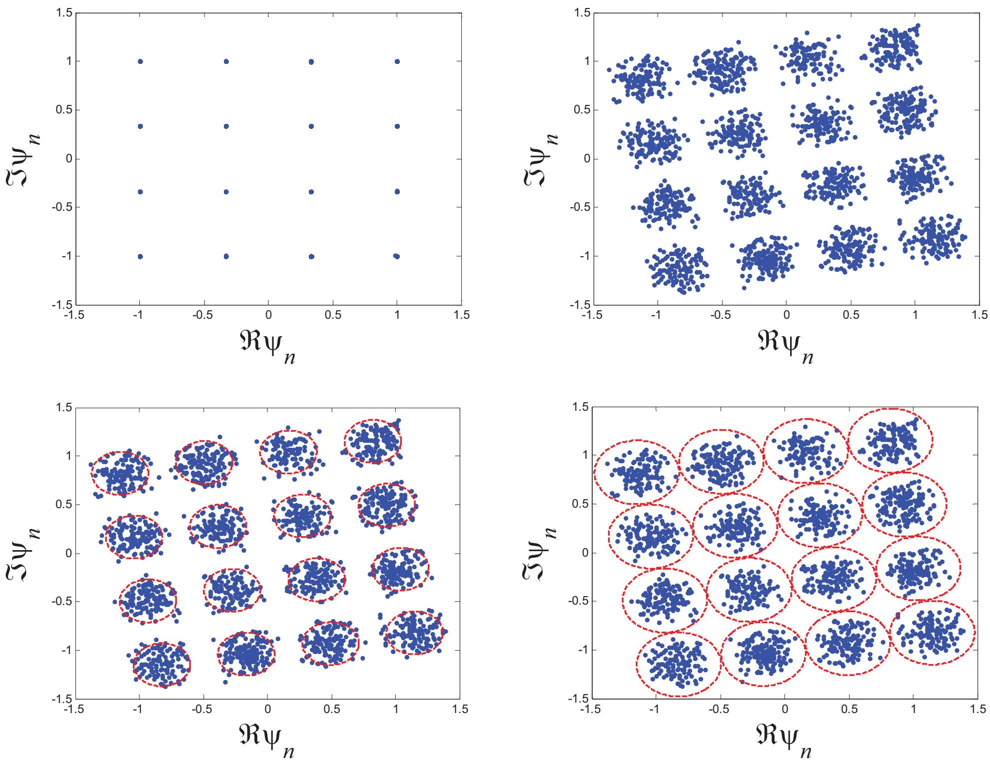

Figure 5. Upper left: the initial constellation of the data in the wavefunction. Upper right: the data constellation after τ = 0.1. Bottom left: the constellation with the circles that stands for the standard deviation $\sigma = \pi^2\tau/\sqrt{20}$. Bottom right: The constellation with the circles that represents the bounding circles $R = \pi^2\tau/3$.

$$S_{\max} = \frac{1}{\sqrt{(\hbar/m)t}} F(\tau) \tag{38}$$

where $F(\tau) \equiv 2\sqrt{\tau}\log_2\left(1 + 3/\pi^2\tau\right)$ is a universal dimensionless function, which is plotted in **Figure 6** and receives its maximum value $F(x_{\max}) \cong 1.28$ for $x_{\max} \cong 0.0775$. However, under the restriction that $\sqrt{M_{\max}}$ must be an integer, then as can be shown in **Figure 6**, the maximum bit-rate is reached for

$$M_{\max} = 25, \tag{39}$$

for which case

$$\tau_{\max} = \frac{3}{4\pi^2} \cong 0.076, \tag{40}$$

Which means that for a given time of measurement t, the largest amount of information would survive provided the detector size (i.e., the sampling interval) is equal to

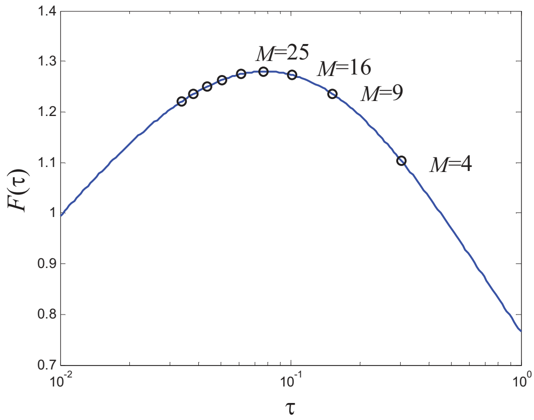

Figure 6. Plot of the function $F(x) \equiv 2\sqrt{x}\log_2\left(1+3/\pi^2 x\right)$. The circles stands for different values of integer $\sqrt{M}$ ($M = 2^2, 3^2, 4^2, \ldots 10^2$). The closest circle to the maximum point is $M = 5^2$.

$$\Delta x_{\max} = 2\pi\sqrt{\hbar t/3m}. \tag{41}$$

For this value $F(\tau_{\max}) = \frac{\sqrt{3}}{\pi}\log_2(5) \cong 1.28$, and therefore, the maximum information density that can last after a time period t is

$$S_{\max} = \sqrt{\frac{3}{(\hbar/m)t}}\frac{\log_2(5)}{\pi} \cong \frac{1.28}{\sqrt{(\hbar/m)t}}. \tag{42}$$

This equation reveals the loss of information from the wave function.

It should be stressed that this expression is universal and the only parameter, which it depends on, is the particle's mass. The higher the mass is, the longer is the distance the information can last.

5. Summary and conclusion

We investigate the decay of information from the wavefunction in the quantum dynamics.

The main conclusions are the following:

A. The signal-to-noise ratio, i.e., the ratio between the mean probability and the variance of the distortion, has a simple analytical expression for short times

$$SNR = \frac{\rho}{N} = \frac{20}{\tau^2\pi^4}$$

where $\tau \equiv (\hbar/m)t/\Delta x^2$ and Δx is the data resolution (the detector size).

B. When there are M possible symbols (as in QAM M), then the maximum time, beyond which the data cannot be decoded is $\tau_{\max} = \frac{3}{(\sqrt{M}-1)\pi^2}$

C. For a given symbol density (Δx) and a given measurement time, the maximum data density (bit/distance) is $S_{\max} = \frac{2}{\Delta x}\log_2\sqrt{M_{\max}} = \frac{2}{\Delta x}\log_2\left(1 + 3/\pi^2\tau\right)$.

D. For a given measurement time, the sampling interval with the highest amount of decoded information is $\Delta x_{\max} = 2\pi\sqrt{\hbar t/3m}$,

E. In which case the highest data density is $S_{\max} = \sqrt{\frac{3}{(\hbar/m)t}}\frac{\log_2(5)}{\pi}$

Author details

Er'el Granot

Address all correspondence to: erel@ariel.ac.il

Department of Electrical and Electronics Engineering, Ariel University, Ariel, Israel

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