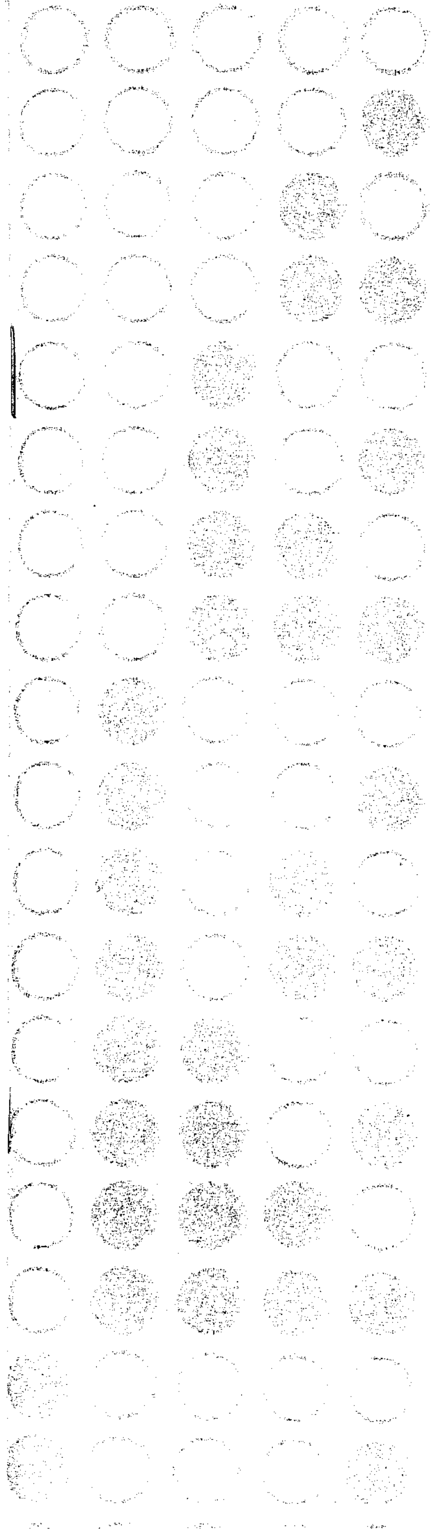

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Wiktor Marek, Zdzisław Pawlak

Information storage and retrieval system– mathematical foundations

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INFORMATION STORAGE AND RETRIEVAL SYSTEM
MATHEMATICAL FOUNDATIONS

Part I

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An approach to information storage and retrieval being an extension of our previous papers is presented. In particular we use methods of formalized languages.

System przechowywania i wyszukiwania informacji
Podstawy matematyczne, cz. I

Przedstawiono podejście do systemów wyszukiwania informacji oparte o matematyczną teorię języków sformalizowanych.

Система хранених и поиска информации
Математические основы, часть I

В работе дается подход к системам поиска информации, основанный на математической теории формальных языков.

In this paper we present new mathematical approach to some problems occurring in information storage and retrieval (i.s.r.) systems. By an i.s.r. system $\mathcal{S}$ we mean a quadruple consisting of set of objects X (like books, documents, etc.) together with the set of descriptors A, the set of attributes I, and the function U which associates a subset of X to each descriptor from A. Attributes are to be understood as sets of elements of A, all of the same "type", e.g. descriptors: green, blue, brown, and black form the attribute colour. Thus each object from X may be described in our system by a vector of descriptors from A exhausting all possible attributes from I. Sometimes "incomplete" descriptions (in the sense that not all possible attributes are specified in the description of an object) are of interest, however we do not consider the case here.

Our first goal is to describe precisely some fundamental facts about i.s.r. systems. To do so we introduce in the first place a formal language tailored to deal with the problem. This language is a sort of intermediate language between propositional and predicate calculi. We further show that the language is adequately chosen for our aims. We show how the language may be used to prove theorems about i.s.r. systems.

Then we introduce the notion of a describable set of objects and find necessary and sufficient condition to determine whether all sets are describable in $\mathcal{S}$ or not. Since in general not all subsets of X are describable in $\mathcal{S}$ we investigate the structure of the family of describable sets.

Since not all the subsets of X are - in general - describable and we may wish to have a more fine description of objects in our systems we sometimes have to add some attributes or/and descriptors. If - on the other hand - our system is "too fine" we may remove some attributes and/or descriptors from the system. The set of objects in the systems may also be varying; it may increase or decrease.
In order to take into account the dynamics of the system (in the above sense) we introduce some algebraic tools i.e. operations on i.s.r. systems, and study properties of the systems thus changed.

Finally a computer implementation algorithm resulting from our considerations is briefly discussed and some other problems rised by our theory are stated at the end of the paper.

Let us finally note that rudimentary versions of this paper were circulated as preprints (Pawlak, Z., 1973, Marek, W., Pawlak, Z., 1973).

Throughout the paper we accept standard mathematical notation and assume the reader to be familiar with it.
In particular $\mathcal{P}(X)$ denotes the power set of X, $f \upharpoonright Y$ is the restriction of f to Y and we distinguish descriptors (elements of A) from their name by underlining the latter.
The end of the proof is denoted by ▮

We express our gratitude to Prof. A. Blikle, Mr. W. Lipski, Jr., and Prof. A. Mazurkiewicz for valuable discussions.

§ 1. Syntax

Let A be a nonempty set and let R be a fixed equivalence on A. We assume that all equivalence classes of R are finite. Since R generates a partition $\{A_i\}_{i \in I}$ of A into family of equivalence classes, $A = \bigcup_{i \in I} A_i$, $i \neq j \Rightarrow A_i \cap A_j = \emptyset$ it is reasonable to call R, R_I.
In the sequel A will be referred to as the set of *descriptors* and I will be called the set of attributes. With each set A we associate the *description language* $\mathcal{L}_A$.
Definition 1.1. (Alphabet of the Language $\mathcal{L}_A$). We define an alphabet of the language $\mathcal{L}_A$ as follows:
1° Constants $\underline{a}$ (for each $a \in A$)
2° Constants $\underline{T}$, $\underline{F}$

3° Constants V, Λ (Truth values, respectively truth and falsity)

4° Symbols $\sim$, $\cdot$, $+$, $\rightarrow$

5° Symbols $\neg$, $\vee$, $\wedge$, $\Rightarrow$

4° Symbol $=$

Definition 1.2. (Terms of the language $\mathcal{L}_A$). The set $\mathcal{T}$ of terms is the least set $\underset{\sim}{T}$ satisfying 1° and 2°

1° $T \in \underset{\sim}{T}$, $F \in \underset{\sim}{T}$, $\underline{a} \in \underset{\sim}{T}$

2° If $t_1, t_2 \in \underset{\sim}{T}$ then $\sim t_1$, $\ulcorner t_1 + t_2 \urcorner$, $\ulcorner t_1 \cdot t_2 \urcorner$, $\ulcorner t_1 \rightarrow t_2 \urcorner \in \underset{\sim}{T}$

As it will turn out later the order of the sum is immaterial and so we shall abbreviate bigger sums as $\sum_{i \in J} t_i$.

Definition 1.3. (Formulas of the language $\mathcal{L}_A$). The set $\mathcal{F}$ of formulas is the least set $\underset{\sim}{F}$ satisfying 1° and 2°.

1° If $t_1, t_2 \in \underset{\sim}{F}$ then $\ulcorner t_1 = t_2 \urcorner \in \underset{\sim}{F}$, $V \in \underset{\sim}{F}$, $\Lambda \in \underset{\sim}{F}$

2° If $\Phi_1, \Phi_2 \in \underset{\sim}{F}$ then $\sim \Phi_1$, $\Phi_1 \wedge \Phi_2$, $\Phi_1 \vee \Phi_2$, $\Phi_1 \Rightarrow \Phi_2 \in \underset{\sim}{F}$

In the sequel the letters s,t (possibly with indices) will denote terms and Φ, Ψ (possibly with indices) formulas.

Definition 1.3. (Axiomatic) We assume as axioms:

1° Substitution of the proposition calculus axioms (see Lyndon, R.1966) for formulas

2° Substitution of the axioms of Boolean Algebra (See Kuratowski, K. Mostowski, A. 1967) for terms (including equality axioms)

3° $$\underline{a} = \sim \sum \{\underline{b} : bR_I a \wedge b \neq a\}$$

(this is sometimes noted as $\underline{a} = \sum_{\substack{bR_I a \\ b \neq a}} \underline{b}$)

As an inference rule we take modus ponens.

Note that the restriction of R_I, namely that all equivalence classes of it are finite is essential in 3°. In case when some A_i is infinite, the expression $\sum \{\underline{b} : b R_I a\}$ may be senseless. We could overcome this obstacle allowing infinite sums operator into the language. This leads to parallel, more general theory. We shall not however pursue the matter in this paper.

§ 2. Semantics, interpretation of terms and formulas

Definition 2.1. (Basic definition). An information storage and retrieval system (i.s.r. system) is a quadruple

$$\mathcal{S} = \langle X, A, R_I, U \rangle \qquad \text{where}$$

X is some set called carrier of $\mathcal{S}$ and elements of X are referred to as objects of $\mathcal{S}$. A is the set of descriptors in $\mathcal{S}$ and R_I is an equivalence on A. U maps A into $\mathcal{P}(X)$ $(U : A \longrightarrow \mathcal{P}(X))$ and satisfies the following two conditions:

1° If $a\, R_I\, b \wedge a \neq b$ then $U(a) \cap U(b) = \emptyset$

2° $\bigcup \{U(b) : b\, R_I\, a\} = X$ (for each $a \in A$)

The conditions 1° and 2° may be expressed equivalently as $1^{\prime\circ}$, $2^{\prime\circ}$

$1^{\prime\circ}$ If $i \in I$, $a \in A_i$, $b \in A_i$, $a \neq b$ then $U(a) \cap U(b) = \emptyset$

$2^{\prime\circ}$ $\bigcup_{a \in A_i} U(a) = X$ (for each $i \in I$)

Definition 2.2. (Valuation of terms) Let $\mathfrak{J} = \langle X, A, R_I, U \rangle$ be an i.s.r. system. We define inductively the value of a term t in $\mathfrak{J}$, $\| t \|_{\mathfrak{J}}$ as follows;

(a) $\| \underline{a} \|_{\mathfrak{J}} = U(a)$

(b) $\| \sim t \|_{\mathfrak{J}} = X - \| t \|_{\mathfrak{J}}$

(c) $\| t_1 \cdot t_2 \|_{\mathfrak{J}} = \| t_1 \|_{\mathfrak{J}} \cap \| t_2 \|_{\mathfrak{J}}$

(d) $\| t_1 + t_2 \|_{\mathfrak{J}} = \| t_1 \|_{\mathfrak{J}} \cup \| t_2 \|_{\mathfrak{J}}$

(e) $\| t_1 \longrightarrow t_2 \|_{\mathfrak{J}} = (X - \| t_1 \|_{\mathfrak{J}}) \cup \| t_2 \|_{\mathfrak{J}}$

(f) $\| F \|_{\mathfrak{J}} = \emptyset$

(g) $\| T \|_{\mathfrak{J}} = X$

Definition 2.3. (Valuation of formulas). Unlikely to the terms, formulas will take as values truth values V and Λ, we define inductively $\| \Phi \|_{\mathfrak{J}}$ (we assume that $\| t \|_{\mathfrak{J}}$ is already defined).

(a) $\| V \|_{\mathfrak{J}} = V$, $\| \Lambda \|_{\mathfrak{J}} = \Lambda$

(b) $$\| t_1 = t_2 \|_{\mathfrak{J}} = \begin{cases} V & \text{if } \| t_1 \|_{\mathfrak{J}} = \| t_2 \|_{\mathfrak{J}} \\ \Lambda & \text{otherwise} \end{cases}$$

(c) $$\| \neg \Phi \|_{\mathfrak{J}} = \begin{cases} V & \text{if } \| \Phi \|_{\mathfrak{J}} = \Lambda \\ \Lambda & \text{if } \| \Phi \|_{\mathfrak{J}} = V \end{cases}$$

(d) For other connectives we extend our definition in natural way.

Theorem 2.4. (Adequacy of axiomatic) If Φ is an axiom, then $\| \Phi \|_{\mathfrak{J}} = V$.

Proof: As our valuation was defined in a way to make first two groups of axioms true, it is enough to check the axiom three. Therefore we need to show that $\| \underline{a} = \sim \sum_{\substack{b R_I a \\ b \neq a}} \underline{b} \|_{\mathfrak{J}} = V$ i.e. according to 2.3 (b) that

$$\| \underline{a} \|_{\mathfrak{J}} = \| \sim \sum_{\substack{b R_I a \\ b \neq a}} \underline{b} \|_{\mathfrak{J}}.$$

easy transformation, according to 2.2(b) and (c) reduces the problem to checking $\| \underline{a} \|_{\mathfrak{J}} = X - \bigcup_{\substack{b R_I a \\ b \neq a}} \| \underline{b} \|_{\mathfrak{J}}$

This however is easily seen to be equivalent to 2.1. 1° and 2°. ∎

There is nothing strange in the fact that we used in our proof only conditions 2.2.(a), (b) and (c), since other connectives may be expressed with help of $\sim$ and $+$.

Definition 2.5. Let $\mathfrak{S} = \langle X, A, R_I, U \rangle$ be an i.s.r. system. Let $x \in X$.

(a) An <u>information</u> on x in $\mathfrak{S}$ is a function $f_x : I \to A$ such that for all $i \in I$ $f_x(i) \in A_i$ and $x \in U(f_x(i))$

(b) A <u>description</u> of x in $\mathfrak{S}$ is a term $t_x = \bigsqcap_{i \in I} \underline{f_x(i)}$

An information on x in $\mathfrak{S}$ determines several terms, all of them provably equivalent (They differ only in that, that order of constants occurring in them may be different), this explains our usage of one symbol, t_x.

This leads to natural:

Definition 2.6. An i.s.r. system $\mathfrak{S}$ is <u>selective</u> iff for all $x \in X$, $\|t_x\|_{\mathfrak{S}} = \{x\}$.

Thus selective system is one in which different elements have necessarily different descriptions i.e. are distinguishable.

§ 3. Completeness properties of i.s.r. systemes

Since we introduced in the § 1 an axioms system and the rule of inference we are able to prove formulas. We denote by $\vdash \Phi$ the fact that Φ has the proof. It is immediate (from the

theorem 2.4 and the fact that the rule modus ponens is preserved under $\|\cdot\|_{\mathfrak{S}}$ (cf. 2.2 for definition of $\|\cdot\|_{\mathfrak{S}}$) (i.e. $\|\Phi\|_{\mathfrak{S}} = V$ and $\|\Phi \Rightarrow \Psi\|_{\mathfrak{S}} = V$ implies $\|\Psi\|_{\mathfrak{S}} = V$)) that the following lemma holds:

Lemma 3.1. (Adequacy of inference) If $\vdash \Phi$ then, for all i.s.r. systems $\mathfrak{S}$, $\|\Phi\|_{\mathfrak{S}} = V$.

We shall also obtain converse result soon.

Definition 3.2. We define relations $\leqslant$ and $\approx$ on $\mathcal{T}$ as follows: $t_1 \leqslant t_2$ iff there is a term t_3 such that $\vdash t_2 = t_1 + t_3$

$t_1 \approx t_2$ iff $\vdash t_1 = t_2$.

Lemma 3.3. (a) $\leqslant$ is reflexive and transitive

(b) $\approx$ is an equivalence

(c) $\leqslant$ has antysymmetry property with respect to $\approx$ i.e.

$$t_1 \leqslant t_2 \wedge t_2 \leqslant t_1 \Rightarrow t_1 \approx t_2$$

Proof: (a) is obvious

(b) follows immediately from the fact that we accepted equality axioms

(c) Assume $t_1 \leqslant t_2$ and $t_2 \leqslant t_1$ i.e. assume that there are terms t_3 and t_4 such that $\vdash t_2 = t_1 + t_3$ and $\vdash t_1 = t_2 + t_4$. Substituting we get

$\vdash t_2 = t_2 + t_3 + t_4$. Thus $\vdash t_2 + t_4 = t_2 + t_3 + t_4 + t_4$. Using the axiom $t+t = t$ we get $\vdash t_2 + t_4 = t_2 + t_3 + t_4$. But $\vdash t_2 + t_4 = t_1$ and $\vdash t_2 + t_3 + t_4 = t_2$. So we get $\vdash t_1 = t_2$ i.e. $t_1 \simeq t_2$. ▮

Each i.s.r. system $\mathfrak{S}$ generates relations $\leqslant_{\mathfrak{S}}$ and $\approx_{\mathfrak{S}}$ as follows:

Definition 3.4. (a) $t_1 \leqslant_{\mathfrak{S}} t_2 \Leftrightarrow \| t_1 \|_{\mathfrak{S}} \subseteq \| t_2 \|_{\mathfrak{S}}$

(b) $t_1 \approx_{\mathfrak{S}} t_2 \Leftrightarrow \| t_1 \|_{\mathfrak{S}} = \| t_2 \|_{\mathfrak{S}}$

Lemma 3.5. $t_1 \leqslant t_2 \Rightarrow (\forall \mathfrak{S})\ t_1 \leqslant_{\mathfrak{S}} t_2$

$t_1 \approx t_2 \Rightarrow (\forall \mathfrak{S})\ t_1 \approx_{\mathfrak{S}} t_2$.

Proof: As in 3.1. ▮

This leads to the following:

Definition 3.6. $t_1 \leqslant^* t_2 \Leftrightarrow (\forall \mathfrak{S})\ t_1 \leqslant_{\mathfrak{S}} t_2$

$t_1 \approx^* t_2 \Leftrightarrow (\forall \mathfrak{S})\ t_1 \approx_{\mathfrak{S}} t_2$.

Notice that $t_1 \leqslant^* t_2$ means that in every interpretation, t_1 determines smaller set then t_2 does. Similarly $t_1 \approx^* t_2$ means that in every interpretation both terms determine the same set.

Thus 3.5 says that $\leqslant \subseteq \leqslant^*$ and $\approx \subseteq \approx^*$

In the sequel we shall prove converse inclusions.

Definition 3.7. (a) We define $\underline{a}^0 = \underline{a}$ $\underline{a}^1 = \sim \underline{a}$

(b) A term t is called <u>primitive</u> iff

$t = \prod_{j \in J} \underline{a}_j^{\varepsilon_j}$ where each ε_j is 0 or 1.

(c) A term t is in <u>normal additive</u> form iff $t = \sum_{j\ J} t_j$ where each t_j is primitive

(d) A term t is in positive form if $\sim$, $\rightarrow$ does not occur in t.

Theorem 3.8. (Normal form I) (a) If t is a term then there is term t_1 in normal additive form such that $\vdash t = t_1$

(b) If t is a term then there is term t_2 in positive normal additive form such that $\vdash t = t_2$.

Proof: (a) A reasoning used in this case is a standard one; we refer reader to Lyndon R. 1966.

(b) By (a) we may assume that t is already in normal additive form. Using the axioms $x = y \Rightarrow \sim x = \sim y$, $\sim\sim x = x$ and 1.3. 3° we get $\sim \underline{a} = \sum_{\substack{b R_I a \\ b \neq a}} \underline{b}$. Substituting the right hand side in every place where left hand side occurs we eliminate negation from t_1. Consecutive application of distributive law makes the rest.

Definition 3.9 (a) A primitive term t is called <u>complete</u> iff for every $i \in I$ there is exactly one $a \in A_i$ such that $\underline{a}$ occurs in t.

(b) A term t is in complete positive normal additive form iff $t = \overline{\underline{\sum}}_{k \in K} t_k$ and each t_k is complete positive primitive term.

Theorem 3.10. (Normal form II) If I is finite then for each term t there is term t_3 in complete positive normal additive form such that $\vdash t = t_3$.

Proof: It is clear that it is enough to find such a term for positive primitive term (by 3.8 (b)). Since $\vdash \sim \underline{a} = \overline{\underline{\sum}}_{\substack{bR_I a \\ b \neq a}} \underline{b}$ therefore we have $\vdash \overline{\underline{\sum}}_{bR_I a} \underline{b} = T$. Using in turn $\vdash t \cdot T = t$ we get $t \cdot \overline{\underline{\sum}}_{b \in A_i} = t$. Assume no $\underline{b}$ (for $b \in A_i$) occurs in t then $t = \sum_{b \in A_i} t \cdot \underline{b}$. Thus we diminished in t a number of i such that no $\underline{b}$ (for $b \in A_i$) occurs in t. Since I is finite this gives an inductive procedure to make our task.∎

Notice that t_3 is unique up to possible order of primitive terms and possible order within the terms.

Theorem 3.11. (Completeness property for terms) Assume I finite. Then (a) $t_1 \leq t_2$ iff $t_1 \leq^* t_2$

(b) $t_1 \approx t_2$ iff $t_1 \approx^* t_2$

Proof: Clearly (b) follows from (a).

(a) $\Rightarrow$ was already proved in 3.5.

(a) $\Leftarrow$ Assume $t_1 \leqslant^* t_2$. We may assume that both t_1 and t_2 are in complete normal positive additive form.

It is clear that if every primitive term occurring in t_1 occurs also in t_2 then $t_1 \leqslant t_2$. Thus it is enough to show the this first property holds. Assume it is not true. Let t_o be primitive term occurring in t_1 but not in t_2. We construct a i.s.r. system $\mathfrak{S}'$ in which $\| t_o \|_{\mathfrak{S}'} \neq \emptyset$. Using the fact (which we leave to the reader) that different primitive complete positive terms have disjoint values we find out that $\| t_1 \|_{\mathfrak{S}'}$ is not included in $\| t_2 \|_{\mathfrak{S}'}$ which contradicts $t_1 \leqslant^* t_2$. ∎

Apart of additive normal form one may - as usually introduce multiplicative normal form.

A construction from the proof of the theorem 3.11 suggests the following question: Is there on i.s.r. system $\mathfrak{S}$ such that $\approx_{\mathfrak{S}}$ is identical with $\approx$?

In fact there is one. A construction of it strongly resembles construction of the family $\{ A_n \}$ such that all components corresponding to it are nonempty (cf. Kuratowski, K., Mostowski A., 1967).

Construction: Let each A_i be in the form $\{ a^i_1 \ldots a^i_{k_i} \}$

We produce Cartesian product $\mathbb{P}_{i\,I} A_i$ and define $\mathfrak{S}_{\max}$ as follows

$$\mathfrak{S}_{\max} = \left\langle \mathbb{P}_{i \in I} A_i,\ A,\ R_I,\ U \right\rangle$$

where $U(a) = \{ f \in \mathbb{P}_{i \in I} A_i : f(i) = a,$ for unique i such that $a \in A_i \}$.

We leave to the reader checking that the system $\mathfrak{S}_{\max}$ has the property that each complete primitive positive term has in $\mathfrak{S}_{\max}$ non void value. ∎

Before we prove completeness theorem for formulas we need some terminology and facts.

1) By a similar procedure to that applied in case of transformation of a term into normal form, we are able to transform every formula into the following form: $\Phi := \Phi_1 \wedge \ldots \wedge \Phi_k$ where each Φ_j is of the form $\Psi_{i_1} \vee \ldots \vee \Psi_{i_j}$ and each Ψ_l is of the form $t_l = t_m$ or of the form $t_k \neq t_r$ for some terms t being in normal, positive, additive, complete form. We describe this fact symbolically as $\Phi = \bigwedge \bigvee \Psi_j$

2) Another fact needed in proof is the following:
$t = s$ is equivalent to conjunction of equations of the form $t_1' = F \ \ldots \ t_k' = F$.

Indeed assume both t and s are in the positive normal additive complete form. Then $t = t_1 + \dots\ t_n$ and $s = s_1 + \dots + s_1$. There are possibly some primitive terms which appear in both expansions. Let $t'_1, \dots, t'_k$ be primitive terms which appear in either t or s but <u>not</u> in both. We leave to the reader that $\vdash (t=s) \Leftrightarrow (t'_1 = F \wedge \dots \wedge t'_k = F)$.

Similarly $t \neq s \Leftrightarrow (t'_1 \neq F \vee \dots \vee t'_k \neq F)$

3) Finally let us note that if $\Phi := \Phi_1 \wedge \dots \wedge \Phi_k$ then $\vdash \Phi$ iff for all $1 \leq j \leq k$ $\vdash \Phi_j$.

Theorem 3.12. (Completeness theorem for formulas).

$$\vdash \Phi \text{ iff for all } \triangle, \quad \|\Phi\|_\triangle = V$$

Proof: $\Rightarrow$ was already proved in 2.4 (adequacy theorem)

$\Leftarrow$ By our remark 1) we may assume that Φ is $\bigwedge\bigvee \Psi_j$ where each Ψ_j is of the form $t_m = t_s$ or of the form $t_n \neq t_k$. We want to prove that $\vdash \Phi$. By the remark 3) it is sufficient to show that $\vdash \bigvee \Psi_j$. We shall transform $\bigvee \Psi_j$ to certain form which finally allows to find a proof for it. Indeed, using remark 2 we may substitute for Ψ_j either a conjunction $(t_{l_1} = F \wedge \dots \wedge t_{l_k} = F)$ or alternative $(t_{j_1} \neq F \vee \dots \vee t_{j_m} \neq F)$ depending whether Ψ_j is $t = s$ or else $t \neq s$.

Thus instead of the proof of $\bigvee \Psi_j$ we need a proof of certain formula Ξ built from the primitive formulas of the form $t_r = F$ and $t_u \neq F$ where each t_v is in primitive, positive, complete form.

Using our remark 1) once more we find a formula Ξ_1 in conjunctive normal form equivalent to Ξ. Since in the process of building of conjunctive normal form ne new atomic formula is used therefore we find that our formula Ξ_1 has the form $\bigwedge \bigvee \Theta_m$ where each Θ_m is of the form $t_1 = F$ or $t_j \neq F$ and t_k are in primitive, positive, complete, normal form.

Since $\bigwedge \bigvee \Psi_j$ was valid in every i.s.r. system therefore so is $\bigvee \Psi_j$. Thus also Ξ and Ξ_1 are valid in all i.s.r. systems since transformation used in above reasoning preserves validity. Thus $\bigwedge \bigvee \Theta_m$ is valid in every i.s.r. system. This in turn is equivalent to the fact that $\bigvee \Theta_m$ is valid in every i.s.r. system. But $\bigvee \Theta_m$ is of the form:

$$t_1' = F \vee \dots \vee t_k' = F \vee t''_{k+1} \neq F \vee \dots \vee t''_{k+m} \neq F$$

If we show a proof for $\bigvee \Theta_m$ then we are done.

Here is main step of the proof: We claim that if $\bigvee \Theta_m$ is valid in every i.s.r. system then there must be primitive term t such that both $t = F$ and $t \neq F$ occur among Θ_m.

Assume this is not true. We construct a i.s.r. system $\mathfrak{S}$ in which $\| \bigvee \Theta_m \|_{\mathfrak{S}} = F$.

Indeed such a system is produced from the previously constructed system $\mathfrak{S}_{\max}$ by throwing out generalized components corresponding to $t''_{k+1}, \dots, t''_{k+m}$. Then since none t_j' is t''_r, in this particular system $\mathfrak{S}$ $t_j' \neq F$ for all $1 \leqslant j \leqslant k$, and in the same time $t''_r = F$ for all $k+1 \leqslant r \leqslant k+m$. thus $\| t_1' = F \vee \dots \vee t_k' = F \vee t''_{k+1} \neq F \vee \dots \vee t''_{k+m} \neq F \|_{\mathfrak{S}} = \Lambda$, contradicting validity of $\bigvee \Theta_m$. Therefore there must be primitive term t such that both $t = F$ and $t \neq F$ occur among Θ_m. Since however formula $t = F \vee t \neq F$ is provable therefore so is formula $\bigvee \Theta_m$. Since all alternatives $\bigvee \Theta_m$ are provable so is $\bigwedge \bigvee \Theta_m$. Thus $\bigvee \Psi_j$ is provable, being equivalent to provable formula. So finally $\bigwedge \bigvee \Psi_j$ is provable and thus $\vdash \Phi$. ∎

Let us notice, that as in any formalized systems we may consider theories based on some additional axioms. (We shall encounter this situation in the sequel). In such enriched system we may as well prove theorems. Let us denote the fact that the proof, using eventually additional axioms from T exists, by $T \vdash \Phi$.

By the reasonings virtually identical with that of 3.11 and 3.12 we prove

Theorem 3.11′ (Generalized completeness property for terms)

$T \vdash t_1 = t_2$ iff for every i.s.r. system $\mathfrak{S}$ such that

$(\Phi)(\Phi \in T \Rightarrow \|\Phi\|_{\mathfrak{S}} = V)$ $\|t_1 = t_2\|_{\mathfrak{S}} = V$

Theorem 3.12' (Generalized completeness property for formulas)

$T \vdash \Psi$ iff for every i.s.r. system $\mathfrak{S}$ such that $(\forall \Phi)(\Phi \in T \Rightarrow \| \Phi \|_{\mathfrak{S}} = V) \Rightarrow \| \Psi \|_{\mathfrak{S}} = V$

As a corollary we get the following .

Theorem 3.13. Let $\mathfrak{S}$ be an i.s.r. system. Then there is a single formula $\Phi_{\mathfrak{S}}$ such that, for all formulas Ψ

$$\| \Psi \|_{\mathfrak{S}} = V \Leftrightarrow \Phi_{\mathfrak{S}} \vdash \Psi$$

Theorem 3.13 was also proved - using different reasoning by Mr. W. Lipski.

Definition 3.14. Let $\mathfrak{S}$ and $\mathfrak{S}'$ be i.s.r. systems.

We say that $\mathfrak{S}$ is equivalent to $\mathfrak{S}'$ $(\mathfrak{S} \equiv \mathfrak{S}')$ iff for every Φ

$$\| \Phi \|_{\mathfrak{S}} = \| \Phi \|_{\mathfrak{S}'}$$

Obviously $\equiv$ is an equivalence relation.

The equivalence classes of $\equiv$ are determined (according to 3.13) by some special formulas. In fact the formula Ψ determining equivalence class of $\equiv$ is of the form $(\bigwedge t_i = F) \wedge (\bigwedge t_j' \neq F)$ where t_i are those primitive complete positive terms whose value in $\mathfrak{S}$ is empty, whereas t_j' are those whose value in $\mathfrak{S}$ is nonempty. Using this remark we have

Theorem 3.15. In every $\equiv$ - equivalence class there is exactly one (up to isomorphism) selective system. Thus for every system $\mathfrak{S}$ there is a selective system $\mathfrak{S}$ such that $\mathfrak{S} \equiv \mathfrak{S}$

We also get - as a corollary - the following fact:

We are not able to express within the formal language of i.s.r. the power of the i.s.r. system.

§ 4. Algebraic properties of i.s.r. systems

Definition 4.1. Let $\mathfrak{S}_X = \langle X, A, R_I, U_X \rangle$ and $\mathfrak{S}_Y = \langle Y, B, R_J, U_Y \rangle$ be two i.s.r. systems. We say that $\mathfrak{S}_X \subseteq \mathfrak{S}_Y$ iff (a) $X \subseteq Y$

(b) $A \subseteq B$

(c) $R_J \cap A^2 = R_I$

(d) $\forall_{a \in A} \; U_Y(a) \cap X = U_X(a)$

Let us note that whenever $A \subseteq B$ then $\mathcal{L}_A \subseteq \mathcal{L}_B$. Thus the terms of $\mathcal{L}_A$ are, in particular, terms of $\mathcal{L}_B$.

The adequacy of our definition is shown by the following

Lemma 4.2. If $\mathfrak{S}_X \subseteq \mathfrak{S}_Y$ and t is a term of the language $\mathcal{L}_A$ then $\| t \|_{\mathfrak{S}_X} = \| t \|_{\mathfrak{S}_Y} \cap X$.

Proof: By induction on complexity of t. If t is $\underline{a}$ then the desired equality is nothing else but 4.1 (d). In case of F and T the condition is seen immediately. Assume now that t is $\sim t_1$.

We have $\| t \|_{\mathfrak{S}_X} = \| \sim t_1 \|_{\mathfrak{S}_X} = X - \| t_1 \|_{\mathfrak{S}_X}$

by inductive assumption we have $\| t \|_{\mathfrak{S}_X} = X - (X \cap \| t_1 \|_{\mathfrak{S}_Y})$.

But $X - (X \cap \| t_1 \|_{\mathfrak{S}_Y}) = X \cap Y - (X \cap \| t_1 \|_{\mathfrak{S}_Y}) =$

$= (Y - \| t_1 \|_{\mathfrak{S}_Y}) \cap X = X \cap \| \sim t_1 \|_{\mathfrak{S}_Y}$.

If $t = t_1 \cdot t_2$ then we have:

$\| t \|_{\mathfrak{S}_X} = \| t_1 \|_{\mathfrak{S}_X} \cap \| t_2 \|_{\mathfrak{S}_X} = \| t_1 \|_{\mathfrak{S}_Y} \cap X \cap \| t_2 \|_{\mathfrak{S}_Y} \cap X$ (here inductive assumption is used) thus $\| t \|_{\mathfrak{S}_X} = \| t_1 \|_{\mathfrak{S}_Y} \cap \| t_2 \|_{\mathfrak{S}_Y} \cap X =$

$\| t_1 \cdot t_2 \|_{\mathfrak{S}_Y} \cap X = \| t \|_{\mathfrak{S}_Y} \cap X$.

The case when $t = t_1 + t_2$ is similar. Finally when $t = t_1 \longrightarrow t_2$ we eliminate the case using equality $t_1 \to t_2 = (\sim t_1) + t_2$ and then applying inductive assumption. ∎

Definition 4.3. (a) $\mathfrak{S}_X \overset{*}{\subseteq} \mathfrak{S}_Y$ iff $\mathfrak{S}_X \subseteq \mathfrak{S}_Y$ and $X = Y$

(b) $\mathfrak{S}_X \underset{*}{\subseteq} \mathfrak{S}_Y$ iff $\mathfrak{S}_X \subseteq \mathfrak{S}_Y$ and $A = B$.

Fact 4.4. If $\mathfrak{S}_X \underset{*}{\subseteq} \mathfrak{S}_Y$ then $R_I = R_J$

Proof: Immediate by 4.1 (c) ∎

Theorem 4.5. (Interpolation property). If $\mathfrak{S}_X \subseteq \mathfrak{S}_Y$ then there are systems $\mathfrak{S}'$ and $\mathfrak{S}''$ such that the following holds:

1° $\mathfrak{S}_X \underset{*}{\subseteq} \mathfrak{S}' \overset{*}{\subseteq} \mathfrak{S}_Y$

2° $\mathfrak{S}_X \overset{*}{\subseteq} \mathfrak{S}'' \underset{*}{\subseteq} \mathfrak{S}_Y$

Proof: Define $\mathfrak{S}$ as follows: $\mathfrak{S} = \langle Y, A, R_I, U' \rangle$ where $U' = U_Y \upharpoonright A$ and $\mathfrak{S}''$ as follows: $\mathfrak{S} = \langle X, B, R_J, U'' \rangle$ where $U''(b) = U_Y(b) \cap X$.

It is enough to check that $\mathfrak{S}_X \leq \mathfrak{S}'$, $\mathfrak{S}' \leq \mathfrak{S}_Y$ $\mathfrak{S}_X \leq \mathfrak{S}''$, $\mathfrak{S}'' \leq \mathfrak{S}_Y$ because all remaining conditions hold by our construction. This however is only direct computation. ∎

Definition 4.6. Let $\mathfrak{S} = \langle X, A, R_I, U \rangle$ be an i.s.r. system. Let $\{ I_j \}_{j \in J}$ be a **partition** of the set I. An induced family $\{\mathfrak{S}_j\}_{j \in J}$ of i.s.r. systems is formed as follows:

$\mathfrak{S}_j = \langle X, A^j, R_{I_j}, U_j \rangle$ where

(a) $A^j = \bigcup_{i \in I_j} A_i$

(b) $R_{I_j} = R_I \cap (A^j \times A^j)$

(c) $U_j = U \upharpoonright A^j$

Lemma 4.7. Under the assumptions of 4.6, for each $j \in J$, $\mathfrak{S}_j \overset{*}{\subseteq} \mathfrak{S}$

Proof: Since universe of $\mathfrak{S}_j$ is X if is enough to prove that $\mathfrak{S}_j \leq \mathfrak{S}$ But all the conditions 4.1(b), (c) and (d) are seen easily to be satisfied. ∎

Since each subset $I' \subseteq I$ induces partition $I = I' \cup (I - I')$ therefore we naturally get restriction of $\mathfrak{S}_I$ of $\mathfrak{S}$ to $I' \leq I$ and complementary system $\mathfrak{S}_{I-I'}$

Definition 4.8. Let $\{\mathfrak{S}_j\}_{j\in J}$ be a family of i.s.r. systems with the same carrier $(\mathfrak{S}_j = \langle X, A^j, R_{I_j}, U_j \rangle)$ and suppose moreover that $i \neq j \Rightarrow A^i \cap A^j = \emptyset$

Define $\bigoplus_{j\in J} \mathfrak{S}_j$ as follows:

$$\bigoplus_{j\in J} \mathfrak{S}_j = \langle X, A, R_I, U \rangle$$

where $A = \bigcup_{j\in J} A^j$, $R_I = \bigcup_{j\in J} R_{I_j}$, $U = \bigcup_{j\in J} U_j$

Lemma 4.9. Under the assumptions of 2.8 $\mathfrak{S}_j \subseteq \bigoplus_{j\in J} \mathfrak{S}_j$ moreover if the family $\{\mathfrak{S}_j\}_{j\in J}$ is obtained as in 2.6. then $\mathfrak{S} = \bigoplus_{j\in J} \mathfrak{S}_j$

We leave the proof to the reader. ▮

Definition 4.10. Let R, S be equivalences on a set Z we say that $S \prec R$ iff $S \subseteq R$ i.e.

$$(\forall x)(\forall y)\ (xSy \Rightarrow xRy)$$

It is clear that $\prec$ is a partial ordering (i.e. that it is reflexive, antysymmetric and transitive).

Definition 4.11. (a) Let R be an equivalence on Z. $Z/_R$ consists of all equivalence classes of R in Z.

(b) Let S be an equivalence on Z and R an equivalence on $Z/_S$. We define a relation $R * S$ on Z as follows

$$x\ R * S y \iff (x/_S) R (y/_S)$$

Lemma 4.12. (a) Under assumptions of 4.11 $R * S$ is an equivalence relation on Z.

(b) Moreover $S \prec R * S$

Proof: (a) is a simple computation.

(b) assume xRy, then $x/_R$, then $x/_R = y/_R$ and so, by reflexivity of S we get $x/_R\ S\ y/_R$ i.e. $x\ S * Ry$ ▮

Lemma 4.13. Assume $S * (T * R)$ is defined. Then $(S * T) * R$ is defined and $S * (T * R) = (S * T) * R$.

Proof: Assume T is defined on $A/_R$ and S defined on $(A/_R)/T$. Then $S * T$ is defined on $A/_R$ and so $(S * T) * R$ is defined.

Let $x\ S*(T * R)y$. Then $(x/_T)/_R\ S\,(y/_T)/_R$. Having in mind that $(x/_T)/_R$ consists of all $y/_R$ which are (with $x/_R$) in relation T we find that

$$(x/_R)\ S*T(y/_R)$$

which is the desired result. ▮

Lemma 4.14. If $S \prec R$ then there is unique T such that:

$$R = T * S$$

Proof: We define T as follows:

$$(x/_S)\,T\,(y/_S) \text{ iff } x\,R\,y.$$

It is enough to prove that T is an equivalence. Clearly T is both reflexive and symmetric. If $(x/_S)T(y/_S)$ and $y/_S\ T\ z/_S$ then $x\,R\,y$ and $y\,R\,z$ i.e. $(x/_S)T(z/_S)$.

The uniqueness of T is easily proved e.g. by contraposition.

Definition 4.15. Let S be an equivalence on A, $S < R_I$, $\mathfrak{S} = \langle X, A, I, U\rangle$ be an i.s.r. system. We define quotient system $\mathfrak{S}/_S$ as follows

$$\mathfrak{S}/_S = \langle X, A/_S, T_I, U/_S\rangle$$

Where (a) T_I is the unique relation such that $R_I = T_I \cdot S$

(b) $U/_S(a/_S) = \bigcup\{U(b) : bSa\}$

Note that the equivalence classes of T may be indexed by I which makes our 4.15 (a) reasonable.

Clearly $\mathfrak{S}/_S$ determines its language $\mathcal{L}(A/_S)$.

Let us form now $\mathfrak{S} \oplus \mathfrak{S}/_S$. The language corresponding to this system consists of constants $\underline{a}$ for $a \in A$ and $a/_S$ for $a/_S \in A/_S$.

The system obtained is such a way is denoted by $\mathfrak{S}_S$.

Lemma 4.16. If S satisfies the assumptions of 4.15, $\mathfrak{S}_S$ is the resulting system then $\| a/_S = \sum_{bSa} \underline{b} \|_{\mathfrak{S}_S} = V$

Proof: We need only to show that $\| \underline{a}/_S \|_{\mathfrak{S}_S} = \bigcup_{bSa} \| \underline{b} \|_{\mathfrak{S}_S}$

but since $\| \underline{b} \|_{\mathfrak{S}_S} = \| \underline{b} \|_{\mathfrak{S}}$ therefore the right hand side is $\bigcup_{bSa} \| \underline{b} \|_{\mathfrak{S}}$ i.e. $\bigcup\{U(b) : bSa\}$ On the other hand

$\| \underline{a}/_S \|_{\mathfrak{S}_S} = \| \underline{a}/_S \|_{\mathfrak{S}/_S}$ i.e. $U/_S(a/_S)$ which is by definition $\{U(b) : bSa\}$. ∎

The full power of the operation $\oplus$ and in the same time the generality of our approach allowing including of hierarchical approach is seen after theorem 4.18.

Definition 4.17. Let $S_1 < \ldots < S_n < R_I$ be an increasing sequence of equivalence relations on A, we define $\mathfrak{S}_{S_1 \ldots S_n}$ as follows:

$$\mathfrak{S}_{S_1 \ldots S_n} = \mathfrak{S} \oplus \left(\bigoplus_{i=1}^{n} \mathfrak{S}/_{S_i} \right).$$

Let $T_1, \ldots, T_{n-1}$ be equivalences such that $S_{i+1} = T_i \cdot S_i$.

Theorem 4.18

$$\| \underline{a}/_{S_{i+1}} = \sum_{b/_{S_i} T_i a/_{S_i}} \underline{b}/_{S_i} \|_{\mathfrak{S}_{S_1 \ldots S_n}} = V$$

Proof: It is clear that it is enough to give the proof for the case $S_1 \prec S_2 \prec R_I$, $S_2 = T * S_1$.

Indeed, for $a \in A$ $\| \underline{a}/_{S_2} = \overline{\sum}_{bS_2a} \underline{b} \|_{\mathfrak{S}_{S_2}} = V$ and so,

$$\| \underline{a}/_{S_2} = \overline{\sum}_{bS_2a} \underline{b} \|_{\mathfrak{S}_{S_1S_2}} = V.$$

Similarly $\| \underline{a}/_{S_1} = \overline{\sum}_{bS_1a} \underline{b} \|_{\mathfrak{S}_{S_1S_2}} = V$

Since however $S_1 \prec S_2$ therefore we have, for $a \in A$

$$\| \underline{a}/_{S_1} \|_{\mathfrak{S}_{S_1S_2}} \subseteq \| a/_{S_2} \|_{\mathfrak{S}_{S_1S_2}}$$

Using the idempotence laws we get

$$\| \underline{a}/_{S_2} = \overline{\sum}_{bS_2a} \underline{b}/_{S_1} \|_{\mathfrak{S}_{S_1S_2}} = V$$

There are indentical terms on the right hand side and grouping them together we find that they correspond exactly to the equivalence classes of the relation T, which gives the desired equation. ▌

The hierarchical construction $\mathfrak{S}/_S$ is used when our system is "too fine". Similar construction works when the system is "to crude".

As introduced, for each $i \in I$ $\{U(a) : a \in A_i\}$ is a decomposition of X. Let T_i be an equivalence relation (on X) corresponding to this decomposition.

Assume now, that for each $i \in I$ there is an equivalence W_i on X such that $W_i \prec T_i$.

The family $\{W_i\}_{i \in I}$ generates an i.s.r. system $\mathfrak{S}^w = \langle X, B, R_j, V \rangle$ as follows:

$$B = \bigcup_{i \in I} \{ x/_{W_i} : i \in I \}$$

$$R_J = \{ \langle x/_{W_i}, y/_{W_j} \rangle : i = j \}$$

$$V(x/_{W_i}) = \{ y : y\, W_i\, x \}$$

Theorem 4.19. $\mathfrak{S}$ **is isomorphic to certain quotient system of** $\mathfrak{S}^w$.

Proof: It is enough to give the relation S **such that** $\mathfrak{S}$ is isomorphic to $\mathfrak{S}^w/_S$.

Since for each $i \in I$ $W_i \prec T_i$ therefore there is unique S_i such that $T_i = S_i * W_i$. Put $S = \bigcup S_i$.

We leave to the reader details of the proof that $\mathfrak{S}^w/_S$ is isomorphic to $\mathfrak{S}$.

Similarly we have:

Theorem 4.20. If $S \prec R_I$ then there is W such that $(\mathcal{S}/_S)^W$ is isomorphic to $\mathcal{S}$.

§ 5. Describable sets

Definition 5.1. Let $\mathcal{S} = \langle X, A, R_I, U \rangle$ be an i.s.r. system. Let $Y \subseteq X$.

(a) Y is said to be describable within $\mathcal{S}$ iff there is a term t such that $\|t\|_{\mathcal{S}} = Y$

(b) $\mathcal{B}(\mathcal{S})$ is the family of all subsets of X describable within $\mathcal{S}$.

Lemma 5.2. (a) Describable subsets of X form a Boolean algebra.

(b) Moreover if Y is describable subset of X then describable in X subsets of Y also form a Boolean algebra.

Proof: (a) follows directly from the choice of the axioms for our system.

(b) follows from the fact that if t is a description of Y in $\mathcal{S}$ (i.e. $\|t\|_{\mathcal{S}} = Y$) then the values of terms of the form $t \cdot s$ (s ranging over $\mathcal{T}$) form a Boolean algebra. ∎

Lemma 5.3. If $\mathcal{S}$ is a finite selective i.s.r. system, then $\mathcal{B}(\mathcal{S}) = 2^X$ (Recall that 2^X is Boolean algebra of all subsets of X).

Proof: Assume t_x is a description of x in $\mathcal{S}$ (i.e. $\|t_x\|_{\mathcal{S}} = \{x\}$) then $t_Y = \sum_{y \in Y} t_y$ is a description of Y in $\mathcal{S}$. ∎

Remark: Here is a point in which a difference between finite and infinite i.s.r. systems occurs. Indeed assuming the language $\mathcal{L}_A$ finitary (i.e. allowing only finitary conjunctions and disjunctions) with A infinite it is easy to produce infinite selective system with underscribable subset (by cardinality argument).

Theorem 5.4. If $\mathcal{S}$ is a finite i.s.r. system then $\mathcal{S}$ is selective iff $\mathcal{B}(\mathcal{S}) = 2^X$

Proof: $\Rightarrow$ was proved in 5.3.

$\Leftarrow$ Since $\mathcal{B}(\mathcal{S}) = 2^X$ then in particular $\{x\} \in \mathcal{B}(\mathcal{S})$. We need to show that $\|t_x\|_{\mathcal{S}} = \{x\}$, (where t_x was introduced in 2.5). Let $\|t\| = \{x\}$. We may assume that t is in complete positive additive normal form. Thus $t = \sum \underset{\sim}{t}$ where each t is primitive term. if, for each $\underset{\sim}{t}$ occurring in t, $\|\underset{\sim}{t}\|_{\mathcal{S}} \neq \{x\}$ then, since $\|\underset{\sim}{t}\|_{\mathcal{S}} \leq \|t\|_{\mathcal{S}}$ we have $\|\underset{\sim}{t}\| = \emptyset$ and so $\|t\| = \emptyset$. But this is not the case and so for some $\underset{\sim}{t}$, $\|\underset{\sim}{t}\|_{\mathcal{S}} = \{x\}$. Thus $\underset{\sim}{t}$ is description of x.

Theorem 5.5. (a) If $\mathcal{S}$ is an i.s.r. system and $Y \subseteq X$ then there is $\mathcal{S}'$ such that $\mathcal{S} \leq \mathcal{S}'$ and $Y \in \mathcal{B}(\mathcal{S}')$

(b) If $\mathcal{S}$ is an i.s.r. system and $\mathcal{B}$ a Boolean algebra such that $\mathcal{B}(\mathcal{S}) \subseteq \mathcal{B} \subseteq 2^X$ then there is $\mathcal{S}'$ such that $\mathcal{S} \leq \mathcal{S}'$ and $\mathcal{B}(\mathcal{S}') = \mathcal{B}$

Proof: (a) If Y is describable in $\mathcal{S}$ put $\mathcal{S}' = \mathcal{S}$. Assume Y not describable within $\mathcal{S}$. Add two new elements a and a' (both not in $A \cup I$) to the set A. Define R' on $A \cup \{a, a'\}$ as follows

$$R' = R \cup \{\langle a, a'\rangle, \langle a', a\rangle, \langle a, a\rangle, \langle a', a'\rangle\}$$

form $\mathcal{S}'$ as follows

$$\mathcal{S}' = \langle X, A \cup \{a, a'\}, R', U'\rangle$$

where $U'(b) = U(b)$ whenever $b \in A$

$U'(a) = Y$

$U'(a') = X - Y.$

(b) Let us notice the following easy fact from the theory of Boolean algebras.

If $Y \in \mathcal{B}$, $\mathcal{A} \subseteq \mathcal{B}$ ($\mathcal{A}$, $\mathcal{B}$ Boolean algebras of sets) then the smallest Boolean algebra containing $\mathcal{A}$ and Y, $[\mathcal{A}, Y]$ is included in $\mathcal{B}$.

Now we proceed as follows. We order the elements of $\mathcal{B} - \mathcal{B}(\mathcal{S})$ into (possibly transfinite) sequence $\{Y_\alpha\}_{\alpha<\beta}$ and form an increasing family of i.s.r. systems $\{\mathcal{S}_\alpha\}_{\alpha<\beta}$ as follows:

$\mathcal{S}_{\alpha+1}$ is $\mathcal{S}_\alpha'$ (Operation $'$ was described in the proof of 5.5) if $Y_\alpha \notin \mathcal{B}(\mathcal{S}_\alpha)$ or $\mathcal{S}_\alpha$ if $Y_\alpha \in \mathcal{B}(\mathcal{S}_\alpha)$.

In the limit step λ we take a limit of $\mathcal{S}_\beta$, $\beta \in \lambda$

Using the fact mentioned at the beginning of the proof we find that for all $\alpha < \beta$, $\mathcal{B}(\mathcal{S}_\beta) \subseteq \mathcal{B}$ and since $Y_\alpha \in \mathcal{B}(\mathcal{S}_{\alpha+1})$, and $\mathcal{B}(\mathcal{S}) \subseteq \mathcal{B}(\mathcal{S}_\alpha)$ for all $\alpha < \beta$ we get

$$\mathcal{B} = \mathcal{B}(\mathcal{S}) \cup \bigcup_{\alpha<\beta} \{Y_\alpha\} \subseteq \bigcup_{\alpha<\beta} \mathcal{B}(\mathcal{S}_\alpha) \subseteq \mathcal{B}$$

$$\text{thus} \quad \bigcup_{\beta<\alpha} \mathcal{B}(\mathcal{S}_\beta) \subseteq \mathcal{B}$$

But, by construction, the left hand side is $\mathcal{B}(\mathcal{S}_\beta)$ and

$$\mathcal{S} \leq \mathcal{S}_\beta$$

The construction given in § 3, as we mentioned resembles that of components (cf. Kuratowski K., Mostowski A., (1967)). The selectiveness of the i.s.r. system means that each generalized component (i.e. value of primitive complete positive term) is at most one element set. If each of the components is nonempty then the system is isomorphic with the universal system, constructed in § 3. It is clear that every nonempty set of the set of components determines selective system and conversly. In this way we are able calculate power of the family of all selective systems (up to isomorphism) over A and I.

Indeed let $I = \{0 \ldots k\}$, $\bar{\bar{A}}_i = n_i$. Then we have

Theorem 5.6. There is exactly $2^{\prod_{i=0}^{k} n_i} - 1$ of nonempty selective systems over A and R_I.

Producing an isomorphic copy of each $\mathfrak{S}_j$, $0 \leq j < (\prod_{i=0}^{k} n_i)-1$ we are able to produce a finite system (in extended language) such that each $\mathfrak{S}_j$ is isomorphic with certain subsystem of $\mathfrak{S}$. One may even produce an infinite system $\mathfrak{S}'$ universal in the above sense for all finite (even non selective) systems over A and I.

Let us remark that $\mathcal{B}(\mathfrak{S})$ is a Boolean algebra of subsets of X generated by $\|t\|_{\mathfrak{S}}$ where t is primitive complete, positive normal term. This fact has deep implementational consequences. While performing $\leq^{*}$ operation generalized components change; but by our previous remark they are unions of new generalized components.

While performing $\leq_{*}$ operation generalized components do not change in the sense that the trace of a generalized component in new system on the carrier of old system is again a generalized component (in old system).

In the hierarchical operations $\mathfrak{S}/_S$ and $\mathfrak{S}^W$ the components are glued together (in the first case) and split in parts (in the second).

§ 6. Implementation, combinatorial problems

The syntactical approach as we did suggests the following implementational proposal: We store in the memory documents as follows; documents belonging to the some generalized component are stored "together". Then, any query is transformed into the alternative of the description of generalized components; thus we need only to find the generalized components.

A quasi-practical suggestion is the following: In the linearly ordered memory, the documents are stored such that the generalized components form segments in the ordering. Then, each component is determined by the address of its beginning and the end. Thus, while the query is received we transform it in to the normal, positive complete form and find the addresses corresponding to the primitive components of the term obtained. Similarly the question in the form of statement about our system is reduced as in the proof of completeness theorem to the conjunction of alternatives of terms of the form $t_i = F$ or $t_j \neq F$ where t_i and t_j are primitive complete positive terms and thus checked.

Definition 6.1. Let $\langle T, \leqslant \rangle$ be a linearly ordered set and let $\mathfrak{S} = \langle X, A, R_I, U \rangle$ be an i.s.r. system.

(a) A function $\varphi : T \xrightarrow{onto} X$ is called enumeration of $\mathfrak{S}$.

(b) A function $\varphi : T \xrightarrow[1-1]{onto} X$ is called one-one enumeration of $\mathfrak{S}$.

Roughly speaking enumeration is a listing of element of X in certain order possibly with repetition.

Definition 6.2. A term t is called segmential in the enumeration φ iff there is a segment $W \subseteq T$ such that the image of W, $\varphi * W$ is $\|t\|_{\mathfrak{S}}$.

It is obvious that segmential terms are particularly useful in the i.s.r. processes. We need therefore some criterious to find whether we may find an enumeration in which given term is segmential.

Lemma 6.3. There always is a linearly ordered set $\langle T, \leqslant \rangle$ and enumeration φ such that all terms $t \in \mathcal{T}$ are segmential.

Proof: List all terms t $(t \in \mathcal{T})$ and consequtively order $\|t\|_{\mathfrak{S}}$. ∎

However this enumeration can be useful only in case of very simple i.s.r. systems. In fact there will be a lot of repetitions, and so the memory will be used completely uneconomically. The most important case is when the enumeration used is one-one.

Definition 6.4. A family of terms H is admissible over $\mathfrak{S}$ if there is a set $\langle T, \leqslant \rangle$ and one-one enumeration φ of $\mathfrak{S}$ in T such that for all $t \in H$, t is segmential in φ.

We have important:

Theorem 6.5. If for all $t_1, t_2 \in H$, $t_1 \neq t_2 \longrightarrow \|t_1\|_{\mathfrak{S}} \cap \|t_2\|_{\mathfrak{S}} = \emptyset$ then H is admissible over $\mathfrak{S}$.

Proof: Let us list all elements of H and order them consecutively, the elements of $X - \bigcup_{t \in H} \|t\|_{\mathfrak{S}}$ are listed at the end. ∎

Corollary 6.6. The family of primitive complete, positive normal terms is admissible over every i.s.r. system $\mathfrak{S}$.

Proof: They satisfy assumptions of 6.5. ∎

Definition 6.7. Let H be a family of terms. $Sub_{\mathfrak{S}}(H)$ is a family of all primitive, normal, complete possitive terms which are implicants of elements of H i.e. $t_1 \in Sub_{\mathfrak{S}}(H)$ iff t_1 is primitive normal, complete positive and there is $t \in H$ such that $\|t_1\|_{\mathfrak{S}} \subseteq \|t\|_{\mathfrak{S}}$.

Theorem 6.8. If H is admissible then also $H \cup \mathrm{Sub}_{\Diamond}(H)$ is admissible.

Proof: Let φ be one-one enumeration of X in which all elements of H were segmential. We show how to change φ in such a way that all generalized components of terms occurring in H become segments. Indeed the component may be split into the segments; then we fix one of them and transform all other into it;

It is immediate that all generalized components - when such an operation is consecutively applied become segments. ∎

Therefore we may conclude that in order to know whether or not given family H of terms is admissible over $\Diamond$ it is enough to check whether or not this family admissible over unique (up to isomorphism) selective system with the same theory.

Question whether or not given family H of terms over is admissible or not may be reduced to the problem of so called incidence graphs (cf. Boland J. Ch., Lekkerkerker C. C. 1962).

In Boland J. Ch. Lekkerkerker 1962 there is a condition under which a graph is an isomorphic to the incidence graph on the real line.

Thus considering a family $H \cup \mathrm{Sub}_{\Diamond}(H)$ we are able to find whether it is admissible or not. The method given there, together with our theorem 6.8. allows to check admissibility of H. We shall not pursue the matter in this paper.

If however H is not admissible, the problem of chosing of an enumeration (which is not one-one then) optimal (for instance with respect to the power of T) occurs.

§ 7. Dynamical treatment

One can remark that our approach allows to see an i.s.r. system in "microscopic" i.e. static situation. Yet in a "real" situation, we have to modify our system according to requirements which may consist of:

(a) Changing of the set of documents - increasing or decreasing

(b) Adding or delating of attributes (and so descripters too)

(c) Changing a descriptors within the attributes.

Let us note that relations $\overset{x}{\subseteq}$, $\overset{x}{\supseteq}$, $\underset{x}{\subsetneq}$, $\underset{x}{\supsetneq}$ serve to enable us to speak about first two problems; the third one is treated as follows; With the help of hierarchical relationship we are able to make the attributes "more crude" and with the help of division relationship "more fine".

Thus we want to express the following "meta-theorem".
Relations: $\overset{*}{\subseteq}$, $\overset{*}{\supseteq}$, $\underset{*}{\subseteq}$, $\underset{*}{\supseteq}$, $\curvearrowleft/_S$, $\curvearrowleft^W$ are sufficient to describe what happens in real time while the i.s.r. system is subjected to accommodation changes.

§ 8. Problems

The following general question seems to be of great importance:

Q1 How should be the memory of a computer organized to simplify the implementation of i.s.r. systems?

The important results in this direction were recently obtained by Mr. W. Lipski Jr. and will be published soon.

Another problem which seems to be of great practical importance is the following:

In the axioms of i.s.r. systems we assume that the classification is complete i.e. every element of our i.s.r. has full description. Thus:

Q2 What properties of our theory are preserved if we admit that some elements are not fully classified?

Again some results were obtained in this direction by Mr. W. Lipski Jr. and the first author.

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Appendix 1

While the i.s.r. system is implemented, it is necessary to enumerate the generalized components. If $\overline{\overline{A}}_i = n_i$ $(i \in I)$ then there is exactly $\prod_{i \in I} n_i$ of generalized components.

We may assume that $I = \{1 \dots k\}$ (i.e. $\overline{\overline{I}} = k$).

Thus generalized components may be viewed as sequences $\langle b_1 \dots b_k \rangle$, where $0 \leqslant b_1 \leqslant n_1 - 1 \dots 0 \leqslant b_k \leqslant n_k - 1$

We know that the set of all these sequences has power $\prod_{i=1}^{k} n_i$ and so is equipollent with the set $\{0, \dots, n_1 \cdot \dots \cdot n_k - 1\}$

However we should be able to decode in some simple way, from the number $0 \leqslant a \leqslant n_1 \cdot n_2 \dots n_k - 1$ the sequence $\langle b_1, \dots, b_k \rangle$ it codes.

We may assume that each $n_i \neq 1$ since if $n_i = 1$ then in the representing sequence there will be always 0 at i-th position.

We define

$$u_0 = n_1 \cdot \dots \cdot n_k$$

$$u_1 = n_2 \cdot \dots \cdot n_k$$

$$\dots\dots\dots\dots\dots$$

$$u_{k-1} = . n_k$$

$$u_k = 1$$

By our assumption $u_0 > u_1 > u_2 > \dots > u_k$

Theorem A1. For every integer $0 \leqslant a \leqslant n_1 \cdot \dots \cdot n_k - 1$ there is exactly one sequence $\langle b_1, \dots, b_k \rangle$ such that

(a) $0 \leqslant b_i \leqslant n_i - 1$

(b) $a = \sum_{i=1}^{k} b_i u_i$ (note the sum is taken from i=1)

Proof: Existence Define b_i as follows

$$b_1 = E\left(\frac{a}{u_i}\right)$$

$$b_{n+1} = E\left(\frac{a - \sum_{j=1}^{n} b_j u_j}{u_{n+1}}\right)$$

(where E is "entier" function).

We prove first $0 \leqslant b_i \leqslant n_i - 1$

This we show by simultaneous induction together with

$$(*) \qquad 0 \leqslant a - \sum_{j=1}^{m-1} b_j u_j \leqslant n_m \cdot \dots \cdot n_k - 1 \qquad (\text{i.e. } u_{m-1} - 1)$$

Indeed, for $n = 1$

$$0 \leqslant a \leqslant n_1 \dots n_k - 1$$

Then $0 \leqslant E\left(\frac{a}{u_1}\right) = E\left(\frac{a}{n_1 \cdot \dots \cdot n_k}\right) \leqslant E\left(\frac{n_1 \cdot \dots \cdot n_k - 1}{n_2 \cdot \dots \cdot n_k}\right) = n_1 - 1$

i.e. $0 \leqslant b_1 \leqslant n_1 - 1$

Let us assume now

$$0 \leqslant b_{r-1} \leqslant n_{r-1} - 1$$

$$0 \leqslant a - \sum_{j=1}^{r-1} b_j u_j \leqslant u_{r-1} - 1$$

Then

$$0 \leqslant b_r = E\left(\frac{a - \sum_{j=1}^{r-1} b_j u_j}{u_r}\right) \leqslant E\left(\frac{u_{r-1} - 1}{u_r}\right) =$$

$$= E\left(\frac{n_r \cdots n_k - 1}{n_{r+1} \cdots n_k}\right) = n_r - 1$$

thus

$$0 \leqslant b_r \leqslant n_r - 1$$

On the other hand

$$b_r = E\left(\frac{a - \sum_{j=1}^{r-1} b_j u_j}{u_r}\right) \qquad \text{means}$$

$$0 \leqslant \frac{a - \sum_{j=1}^{r-1} b_j u_j}{u_r} - b_r < 1$$

thus $\quad 0 \leqslant a - \sum_{j=1}^{r-1} b_j u_j - b_r u_r < u_r$

so

$$0 \leqslant a - \sum_{j=1}^{r} b_j u_j \leqslant u_r - 1$$

Now we prove $\quad a = \sum_{j=1}^{k} b_j u_j$

Notice that according to the definition

$$b_k = E\left(\frac{a - \sum_{j=1}^{k-1} b_j u_j}{u_k}\right)$$

since however $u_k = 1$ we have

$$b_k = a - \sum_{j=1}^{k-1} b_j u_j$$

thus $\quad a = \sum_{j=1}^{k-1} b_j u_j + b_k = \sum_{j=1}^{k-1} b_j u_j + b_k u_k = \sum_{j=1}^{k} b_j u_j .$

<u>Uniqueness</u>. Instead of showing this directly which is also possible (by the method we employ later) we notice that denoting $B = \{0, \ldots, n_1 \cdots n_k - 1\}$ $A = \{0, \ldots, n_1 - 1\} x \ldots x \{0, \ldots, n_k - 1\}$.

We have $\bar{A} = \bar{\bar{B}}$.

Our proof of existence exhibits 1-1 function of B into A. Since they have the same power it has to be onto, which shows uniqueness.

Let us notice that in the proof of existence we exhibited effective, iterative procedure which for every $0 \leqslant a \leqslant n_1 \cdots n_k - 1$ gives the sequence $\langle b_1, \ldots, b_k \rangle$.

Let us write $\nu^{-1}(a) = \langle b_1, \ldots, b_k \rangle$ and

$$\nu(b_1 \ldots b_k) = a$$

when $a = \sum_{j=1}^{k} b_j u_j$.

Question: what is the relation $\preccurlyeq$ defined as follows

$$\langle b_1, \ldots, b_k \rangle \preccurlyeq \langle b_1', \ldots, b_k' \rangle \longleftrightarrow \nu(b_1, \ldots, b_k) \leqslant \nu(b_1', \ldots, b_k')$$

<u>Theorem A2</u> $\preccurlyeq$ is lexicographic ordering of A.

Proof: Since $\preccurlyeq$ is connective therefore there is enough to show that $\langle b_1, \ldots, b_k \rangle \preccurlyeq_{lex} \langle b_1', \ldots, b_k \rangle \rightarrow \langle b_1, \ldots, b_k \rangle \preccurlyeq \langle b_1', \ldots, b_k' \rangle$ (where $\preccurlyeq_{lex}$ is lexicographic ordering on A).

I.e. $\langle b_1, \ldots, b_k \rangle \prec_{lex} \langle b_1', \ldots, b_k' \rangle \rightarrow \nu(b_1, \ldots, b_k) < \nu(b_1', \ldots, b_k')$

Clearly it is enough to show

$$\langle b_1, \ldots, b_k \rangle \prec_{lex} \langle b_1', \ldots, b_k' \rangle \rightarrow \nu(b_1, \ldots, b_k) < \nu(b_1', \ldots, b_k')$$

Thus we need to show that under our assumption

$\sum_{j=1}^{k} b_j u_j < \sum_{j=1}^{k} b_j' u_j$ holds.

We have $b_1 = b_1', \ldots, b_{r-1} = b_{r-1}'$, $b_r < b_r'$

Thus we have to show $\sum_{j=r}^{k} b_j u_j < \sum_{j=r}^{k} b_j' u_j$

Let us consider $\sum_{j=r+1}^{k} b_j u_j$

$$\sum_{j=r+1}^{k} b_j u_j = \nu(b_1, \ldots, b_k) - \sum_{j=1}^{r} b_j u_j \leqslant u_r - 1$$

(last inequality holds by (*))

Thus $\sum_{j=r}^{k} b_j u_j = b_r u_r + \sum_{j=r+1}^{k} b_j u_j \leqslant b_r u_r + u_r - 1$

$$= (b_r + 1) u_r - 1$$

but $(b_r + 1) u_r - 1 \leqslant b_r' u_r - 1 < b_r' u_r$

therefore we have

$$\sum_{j=r}^{k} b_j u_j < b_r' u_r \leqslant \sum_{j=r}^{k} b_j' u_j$$

thus

$$\sum_{j=r}^{k} b_j u_j < \sum_{j=1}^{k} b'_j u_j$$

▮

As is clear from our construction, $\nu(b_1,\dots,b_k) = \sum_{j=1}^{k} b_j u_j$

Since $b_j \leqslant n_j - 1$ therefore

$$\nu(b_1 \dots b_k) \leqslant \nu(n_1-1,\dots,n_k-1) = \sum_{j=1}^{k} (n_j-1)u_j = n_1 \dots n_k - 1$$

the last equality is easily provable by induction and we leave it to reader.

Let us finally note, that lexicographical ordering is specially convenient when, while extending the language we increase the number of attributes.

After finishing the paper we found that elementary considerations of similar kind were already performed by Wong, E. and Chiang T. C. 1971.

Appendix B (by W. Lipski Jr.)

An i.s.r. system can be given also a purely algebraic description in terms of algebras (with three binary operations $+, \cdot, \rightarrow$, one unary operation $\sim$ and two constants T, F) and homomorphisms between them. We shall restrict ourselves to the part of the theory concerning terms and their valuation. The considerations will be illustrated by the (commutative)diagram depicted in Fig. 1.
$\mathcal{T}$ is the algebra of terms, i.e. absolutely free $\langle +,\cdot,\rightarrow,\sim,T,F\rangle$-algebra generated by A. Each other algebra $\mathcal{C}$ in the diagram is its image by some homomorphism $h:\mathcal{T}\rightarrow\mathcal{C}$ or, in other words, is (up to isomorphism) of the form $\mathcal{T}/\approx$ where $\approx$ is a congruence relation on $\mathcal{T}$. All homomorphisms in the diagram are epimorphisms.
$\mathcal{B} = \mathcal{T}/\approx$ is the free Boolean algebra (generated by A), i.e. the algebra of equivalence classes of terms provably equal in the theory of Boolean algebras. $\mathcal{L} = \mathcal{T}/\approx$ is the algebra of equivalence classes of terms provably equal in the theory of i.s.r. systems (from the algebraic point of view it can be called the free i.s.r. system).
$\mathcal{L}_{\mathcal{S}} = \mathcal{T}/\approx_{\mathcal{S}}$ is the algebra of equivalence classes of terms equal in an i.s.r. system $\mathcal{S}$. p_1, p_2 and p_3 are canonical projections.
$v: t \longmapsto \|t\|_{\mathcal{S}}$ is the valuation of terms in $\mathcal{S}$. Since $\approx_{\mathcal{B}} \subseteq \approx \subseteq \approx_{\mathcal{S}}$ there exist unique homomorphisms p_{12}, p_{23} and v_1, v_2. $\mathcal{D}$ is the Boolean algebra of describable sets in $\mathcal{S}$. Since $\mathcal{L}_{\mathcal{S}} = \mathcal{T}/\approx_{\mathcal{S}}$ and $\approx_{\mathcal{S}} = \ker v$, $\mathcal{D}$ is canonically isomorphic to $\mathcal{L}_{\mathcal{S}}$. The completeness theorem for terms and the construction of $\mathcal{S}_{max}$ (see § imply that $\bigcap_{\mathcal{S}} \approx_{\mathcal{S}} = \approx = \approx_{\mathcal{S}_{max}}$. Notice that $\mathcal{S}_{max}, \mathcal{L}, \mathcal{P}(\prod_{i\in I} \tilde{A}_i)$ and $\mathcal{P}(T)$ are isomorphic. The set $\approx_{\mathcal{S}} \setminus \approx$ of equalities (treated as ordered pairs of terms) can be viewed as a set of "specific features" of $\mathcal{S}$. Clearly $\approx_{\mathcal{S}_{max}} - \approx = \emptyset$. If in an i.s.r. system $\mathcal{S}$ we replace v by $\acute{s}\, v$ we get an i.s.r. system $\mathcal{S}'$ (with the carrier

$\mathcal{S}(\mathcal{A})$ which is selective and equivalent to δ. The algebras have isomorphic copies $\mathcal{P}(2^{\widetilde{A}})$, $\mathcal{P}(\prod_{i\in I} \widetilde{A}_i)(\mathcal{P}(T))$ and $\mathcal{P}(R)(\mathcal{P}(Q))$ respectively, where

$A = \{a_1, a_2, \dots, a_N\}$

$\widetilde{A} = \{1, 2, \dots, N\}$

$\widetilde{A}_i = \{k : a_k \in A_i\}$

$R = \{\langle i_1, i_2, \dots, i_n\rangle : a_{i_j} \in A_j \wedge v(a_{i_1}\, a_{i_2} \dots a_{i_n}) \neq \emptyset\}$, $n = \overline{\overline{I}}$

$$T = \{\langle \varepsilon_1, \varepsilon_2, \dots, \varepsilon_N\rangle : \varepsilon_j \in \{0,1\} \wedge (\forall i,k,l)(\varepsilon_i = \varepsilon_k = 1 \wedge i,$$
$$k \in \widetilde{A}_l \Rightarrow i = k)\}$$

$Q = \{\langle \varepsilon_1, \varepsilon_2, \dots, \varepsilon_N\rangle \in T : v(a_1^{\varepsilon_1}\, a_2^{\varepsilon_2} \dots a_N^{\varepsilon_N}) \neq \emptyset\}$

Notice that the elements of the above algebras can easily be coded in the memory of a computer. The set Q or R can serve as a complete characterization of an i.s.r. system (up to equivalence). Here are additional explanations concerning the diagram:

$\mathcal{A} = \{v(a) : a \in A\}$,

$$\mathcal{S}(\mathcal{A}) = \{v(a_1^{\varepsilon_1}\, a_2^{\varepsilon_2} \dots a_N^{\varepsilon_N}) : \varepsilon_1, \varepsilon_2, \dots, \varepsilon_N \in \{0,1\} \wedge v(a_1^{\varepsilon_1}\, a_2^{\varepsilon_2} \dots$$
$$\dots a_N^{\varepsilon_N}) \neq \emptyset\},$$

$\sigma(D) = \{S \in \mathcal{S}(\mathcal{A}) : S \subseteq D\}$ for each $D \in \mathcal{D}$,

$\rho(t/\approx_\delta) = \{S \in \mathcal{S}(\mathcal{A}) : S \subseteq \|t\|_\delta\}$,

$r_2(M) = M \cap R$ for each $M \in \mathcal{P}(\prod_{i\in I} \widetilde{A}_i)$,

$r_3(N) = N \cap Q$ for each $N \in \mathcal{P}(T)$,

$r_4(P) = P \cap T$ for each $P \in \mathcal{P}(2^{\widetilde{A}})$,

$$I_1\Big(\sum_{\gamma\in\Gamma} a_1^{\varepsilon_1^{(\gamma)}}\, a_2^{\varepsilon_2^{(\gamma)}} \dots a_N^{\varepsilon_N^{(\gamma)}} / \approx_B\Big) = \{\langle \varepsilon_1^{(\gamma)}, \dots, \varepsilon_N^{(\gamma)}\rangle : \gamma \in \Gamma\},$$

$$I_2\Big(\sum_{\gamma\in\Gamma} a_{i_1^{(\gamma)}}\, a_{i_2^{(\gamma)}} \dots a_{i_n^{(\gamma)}}\Big) = \{\langle i_1^{(\gamma)}, i_2^{(\gamma)}, \dots, i_n^{(\gamma)}\rangle : \gamma \in \Gamma\}$$

$$I_3\Big(\sum_{\gamma\in\Gamma} a_{i_1^{(\gamma)}}\, a_{i_2^{(\gamma)}} \dots a_{i_n^{(\gamma)}}\Big) = \{\langle i_1^{(\gamma)}, i_2^{(\gamma)}, \dots, i_n^{(\gamma)}\rangle : \gamma \in \Gamma\} \cap R.$$

Other homomorphisms are determined by the above ones. The arrowe $\rightarrowtail$ and $\twoheadrightarrow$ denotes epimorphisms and isomorphisms respectively. For the systematic treatment of all algebraic notions used here see for instance Cohn (1965).

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Fig. 1. Valuation of terms in an i.s.r. system.

$$\begin{tikzcd}
 & & \mathcal{P}(T) \arrow[r, tail, "r_3"'] & \mathcal{P}(Q) \\
 & \mathcal{P}(2^{\tilde{A}}) \arrow[r, tail, "r_1"'] \arrow[ur, tail, "r_4"'] & \mathcal{P}(\prod_{i \in I} \tilde{A}_i) \arrow[r, tail, "r_2"'] \arrow[u, tail, two heads, "I_5"'] & \mathcal{P}(R) \arrow[u, tail, two heads, "I_4"'] \arrow[dd, tail, two heads, bend left, "\mu"] \\
\mathcal{T} \arrow[ur, tail, "I_0"] \arrow[r, tail, "p_1"'] \arrow[rr, tail, bend left, "p_2"] \arrow[rrr, tail, bend left, "p_3"] \arrow[drr, tail, "v"'] & \mathcal{B} \arrow[u, tail, two heads, "I_1"'] \arrow[r, tail, "p_{12}"'] \arrow[dr, tail, "v_1"'] & \mathcal{L} \arrow[u, tail, two heads, "I_2"'] \arrow[r, tail, "p_{23}"'] \arrow[d, tail, "v_2"'] & \mathcal{L}_{\mathcal{S}} \arrow[u, tail, two heads, "I_3"'] \arrow[dl, tail, "v_3"'] \arrow[d, tail, "\varrho"'] \\
 & & \mathcal{D} \arrow[r, tail, "\sigma"'] & \mathcal{P}(\mathcal{S}(\mathcal{A}))
\end{tikzcd}$$