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Infrared Singularities and Small-Distance-Behaviour Analysis

K. Symanzik

Deutsches Elektronen-Synchrotron (DESY), Hamburg, Federal Republic of Germany

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Abstract. The infrared-singularity structure of the vertex functions of massless-particle ϕ^4 theory is studied. This allows to construct the asymptotic forms of the vertex functions of massive-particle ϕ^4 theory in a simpler and more explicit fashion than in a previous paper. With the help of the parquet approximation introduced by Diatlov, Sudakov, and Martirosian we show that the infrared-singularity structure in a theory with besides the massless particles, massive ones is the same as in the theory with massless particles only. All these results in ϕ^4 theory have analoga in other renormalizable theories.

Introduction

In a series of papers [1] a systematic approach to the large-momenta problem for vertex functions[1] (VFs) in renormalizable field theories has been undertaken. A main step hereby was to define the asymptotic forms (AFs) of those functions. At generic momenta, the AFs are the VFs of a corresponding zero-mass theory, and their behaviour under overall scaling of the momenta is described by the renormalization group equations for such a theory, given simplest in the form of homogeneous partial differential equations (PDEs) obeyed by these functions.

The momenta sets at which the zero-mass theory VFs are infrared (UR) singular are called exceptional. The AFs of the massive-theory VFs at Euclidean such momenta are expressible as certain UR finite parts extracted from the zero-mass theory VFs at those momenta, and transform in a fashion, different from case to case, given simplest in terms of certain (in general) inhomogeneous PDEs, the inhomogeneous terms involving other exceptional AFs.

In the formulae derived in Appendix B of SD 2, for the described connection between exceptional finite-mass theory AFs and zero-mass theory VFs, coefficient functions appeared for which definitions only as limites were given. In this paper, we derive equivalent but simpler formulae, with explicit expressions for the coefficient functions. These formulae are developed here for ϕ^4 theory, but have analoga in all renormalizable theories.

[1] The amputated one-particle-irreducible parts of connected Green's functions.

We also make a detailed study of UR singularities of some zero-mass theory VFs near Euclidean exceptional momenta. We then show that the same UR singularity structure holds in theories that have besides the massless scalar particles also (arbitrary) massive ones. The tool hereto is the parquet approximation (PA) introduced by Diatlov, Sudakov, and Ter-Martirosian [2] in a study of the asymptotic behaviour of meson-meson scattering. That the use of the PA is legitimate for UR behaviour problems was shown by Larkin and Khmel'nitskiĭ [3], whose considerations we here sharpen, and generalize as described.

In Section I we recapitulate (from SD 1) the technique of mass vertex insertion, extending it to vertex functions involving also Zimmermann's [5] composite operator $N_2(\phi^2)$. In Section II we recapitulate (from SD $1\frac{1}{2}$) and similarly extend definition and existence proof of AFs at nonexceptional momenta, and (from SD 2) the construction of these forms from directly-defined zero-mass theory VFs. In Section III.1 we recapitulate (from SD 2) definition and PDEs of the AFs to some sets of exceptional momenta, and give an application to a mass-switch-on effect. In Section III.2 we analyze the UR singularities of the corresponding zero-mass theory VFs near those momenta and define, by simpler formulae than in SD 2, the UR finite parts, from which in Section III.3 the AFs of Section III.1 are obtained in explicit fashion. In Section IV we define and analyze the PA, prove that the UR singularities derived from it are actually present in a theory with massless neutral scalar particles even if there are also massive particles, and compute in ϕ^4 theory a correction to the PA to demonstrate the reliability of the reasoning used. We also show why the PA gives no nontrivial result for the σ-model in the Goldstone mode. Appendix A gives a formulary of algebraic deductions from the Bethe-Salpeter (BS) equation, to which also Appendix B gives some technical hints. Section V contains concluding remarks.

I. Mass Vertex Insertion

We here recapitulate the derivation of PDEs for vertex functions in ϕ^4 theory, admitting also $N_2(\phi^2)$ operators [2] in the notation of Zimmermann [5]. These vertex functions are defined by [3]

$$(2\pi)^4\,\delta(\Sigma p+\Sigma q)\,\Gamma(p_1\ldots p_{2n},q_1\ldots q_l;m^2,g)$$
$$=2^{-l}\langle(\tilde{\phi}(p_1)\ldots\tilde{\phi}(p_{2n})\,N_2(\tilde{\phi}^2(q_1))\ldots N_2(\tilde{\phi}^2(q_l)))_+\rangle^{\mathrm{prop}}\,.$$

[2] PDEs for vertex functions involving such operators were already considered by Callan [4].

[3] Momentum conservation is understood and here mostly not expressed in the notation. For Fourier transforms we use the conventions of the last papers of Ref. [5]. Note that $\Gamma(p(-p),)=-G(p)^{-1}$.

The PDEs are obtained simplest, as in SD 1, from the generating functional. For

$$G_{\text{disc}}\{J, K\} = \langle(\exp\{i\int dx[J(x)\,\phi(x) + \tfrac{1}{2}K(x)\,N_2(\phi^2(x))]\})_+\rangle$$

we have, by mass vertex insertion

$$L \to L + \Delta L \equiv L - \tfrac{1}{2}m^2\,\varphi(g)\,N_2(\phi^2)\Delta s$$

with $\varphi(g)$ defined below, a result expressible in two ways:

$$\begin{aligned} &G^{\Delta L}_{\text{disc}}\{J, K\}\,(m^2, g) = \text{const}\,G_{\text{disc}}\{J, K - m^2\,\varphi(g)\Delta s\}\,(m^2, g) \\ &= \exp[\tfrac{1}{2}i\Delta v\int dx\,K(x)^2]\cdot G_{\text{disc}}\{(1+\Delta z)J,\ \ (1+\Delta u)K\}\,(m^2+\Delta m^2, g+\Delta g) \end{aligned} \tag{I.1}$$

where the extra exponential is due to the fact that, while $N_2(\phi^2)$ is like ϕ multiplicatively renormalized, $\langle(N_2(\tilde{\phi}^2(p))\,N_2(\phi^2(0)))_+\rangle$ requires[4] beyond this a subtractive renormalization.

From the equations (cp. SD 1)

$$\begin{aligned} -i\,G_{x,}\{J, K\} &= \mathscr{A}(x) \\ J(x) &= i\,\Gamma_{x,}\{\mathscr{A}, K\} \\ \Gamma\{\mathscr{A}, K\} &= G\{J, K\} - i\int dx\; J(x)\,\mathscr{A}(x) \\ \Gamma_{,x}\{\mathscr{A}, K\} &= G_{,x}\{J, K\} \\ \Delta\Gamma\{\mathscr{A}, K\}_{\mathscr{A}, K\,\text{fixed}} &= \Delta G\{J, K\}_{J, K\,\text{fixed}} \end{aligned}$$

follows similarly as in SD 1, by differentiation of (I.1)

$$\begin{aligned} &\{m^2[\partial/\partial m^2] + \beta(g)\,[\partial/\partial g] - 2n\gamma(g) \\ &\quad + l(2\gamma(g) + \eta(g))\}\;\Gamma(p_1\,\ldots\,p_{2n}, q_1\,\ldots\,q_l; m^2, g) \equiv \mathcal{O}\!\!\not{p}_{2n,l}\,\Gamma(\ldots) \\ &= -im^2\,\varphi(g)\,\Gamma(p_1\ldots p_{2n}, q_1\ldots q_l 0; m^2, g) - i\,\delta_{n0}\,\delta_{l2}\,\kappa(g) \equiv \Delta\Gamma(\ldots) \end{aligned} \tag{I.2}$$

whereby the functions $\beta(g)$, $\gamma(g)$, $\eta(g)$ and $\kappa(g)$ are obtained from consistency of (I.2) with the renormalization conditions

$$\Gamma(p(-p), ; m^2, g)|_{p^2=m^2} = 0\,, \tag{I.3a}$$

$$[\partial/\partial p^2]\;\Gamma(p(-p), ; m^2, g)|_{p^2=m^2} = i\,, \tag{I.3b}$$

$$\Gamma(p_1\,\ldots\,p_4, ; m^2, g)|_{\text{s.pt.}^5\,\text{to}\,m^2} = -ig\,, \tag{I.3c}$$

$$\Gamma(0\,0, 0; m^2, g) = 1\,, \tag{I.3d}$$

$$\Gamma(, 0\,0; m^2, g) = 0\,, \tag{I.3e}$$

[4] A formula for this computation is (A.9). The effect of subtractive renormalization on PDEs was first observed by Coleman and Jackiw [6].

[5] $p_1 p_2 p_3 p_4|_{\text{s.pt. to}\,a^2}$ means $p_i p_j = \frac{1}{3}a^2(4\delta_{ij} - 1)$.

and $\varphi(g)$ is determined by the normalization convention

$$\Delta\Gamma(p(-p),;m^2,g)|_{p^2=m^2}=-im^2 \tag{I.3f}$$

implicit in (I.2).

One finds[6] (SD 1)

$$\beta(g)=b_0g^2+b_1g^3+\cdots \qquad b_0=3(32\pi^2)^{-1}, \tag{I.4a}$$

$$\gamma(g)=c_0g^2+c_1g^3+\cdots \qquad c_0=(2^{11}3\pi^4)^{-1}, \tag{I.4b}$$

and (SD 2)

$$\eta(g)=h_0g+h_1g^2+\cdots \qquad h_0=\tfrac{1}{3}b_0, \tag{I.4c}$$

$$\kappa(g)=k_0+k_1g+\cdots \qquad k_0=\tfrac{1}{3}b_0, \tag{I.4d}$$

$$\varphi(g)=1+f_2g^2+\cdots. \tag{I.4e}$$

For later comparison, we note that (I.3) implies

$$\Gamma(p(-p),;m^2,g)=i(p^2-m^2)+O(g^2), \tag{I.5a}$$

$$\Gamma(p_1p_2p_3p_4,;m^2,g)=-ig+O(g^2), \tag{I.5b}$$

$$\Gamma(p_1p_2,q_1;m^2,g)=1+O(g), \tag{I.5c}$$

$$\Gamma(,q(-q);m^2,g)=O(1). \tag{I.5d}$$

Concerning extensions of mass vertex insertion we remark: PDEs for vertex functions involving N_δ-operators [5] with $\delta>2$ were derived by Christ, Hasslacher, and Mueller [7], Mason [8], and more generally by Mitter [9]. In the case of conserved currents, the corresponding γ-term in PDEs is computable[7] from consistency of the PDEs with the Ward identities, as shown for quantum electrodynamics in SD 1. Insertion into the Lagrangian density of N_δ-operators [5] with $\delta=4$ appearing already in Zimmermann's effective Lagrangian [5] was studied, and called generalized vertex operation, by Lowenstein [10].

II. Asymptotic Forms at Nonexceptional Momenta

II.1. Existence of Asymptotic Forms

To analyze (I.2) we introduce[8]

$$\varrho(g)=\int^g dg'\,\beta(g')^{-1}=-b_0^{-1}g^{-1}-b_0^{-2}b_1\ln g+\text{const}+\cdots g+\cdots \tag{II.1}$$

[6] In Section II.3 we shall prove that the coefficients b_0, b_1, c_0, h_0, and k_0 are independent of the precise forms of (I.3b–d) as long as (I.5a–d) hold.

[7] An equivalent result was obtained by Coleman and Jackiw [6] using a different method.

[8] Throughout this paper we take $0<g<g_\infty$ where g_∞ is the first positive zero of $\beta(g)$.

and define

$$g(\lambda) = \varrho^{-1}(\ln \lambda^2 + \varrho(g)) \tag{II.2a}$$

such that

$$F(g(\lambda)) = \sum_{l=0}^{\infty} (l!)^{-1} (\ln \lambda^2)^l [\beta(g)\, \partial/\partial g]^l F(g) \tag{II.2b}$$

and, as $\lambda \to +0$

$$g(\lambda) = b_0^{-1}(\ln \lambda^{-2})^{-1} - b_0^{-3} b_1 (\ln \lambda^{-2})^{-2} \ln \ln \lambda^{-2} + O((\ln \lambda)^{-2}). \tag{II.3}$$

We also introduce

$$a(g) = \exp\left[2 \int_0^g dg'\, \beta(g')^{-1} \gamma(g')\right] = 1 + 2b_0^{-1} c_0 g + \cdots, \tag{II.4a}$$

$$h(g) = \exp\left[\int^g dg'\, \beta(g')^{-1} \eta(g')\right] = g^{\frac{1}{3}}(1 + \cdots g + \cdots), \tag{II.4b}$$

$$k(g) \doteq \int^g dg'\, \beta(g')^{-1} a(g')^2 h(g')^2 \kappa(g') = -g^{-\frac{1}{3}}(1 + \cdots g + \cdots) \tag{II.4c}$$

where in (II.4b–c) we have made a convenient choice of the integration constants.

(I.2) becomes

$$\begin{aligned}
&\Gamma(p_1 \ldots p_{2n}, q_1 \ldots q_l; m^2, g) \\
&= a(g)^{n-l} a(g(\lambda))^{-n+l} h(g)^{-l} h(g(\lambda))^l \Gamma(p_1 \ldots p_{2n}, q_1 \ldots q_l; m^2 \lambda^2, g(\lambda)) \\
&\quad - i\delta_{n0}\delta_{l2}\, a(g)^{-2} h(g)^{-2} [k(g) - k(g(\lambda))] \\
&\quad - im^2 \int_{\lambda^2}^1 d\lambda'^2\, a(g)^{n-l} a(g(\lambda'))^{-n+l} h(g)^{-l} h(g(\lambda'))^l \\
&\quad \cdot \varphi(g(\lambda'))\, \Gamma(p_1 \ldots p_{2n}, q_1 \ldots q_l 0; m^2 \lambda'^2, g(\lambda')).
\end{aligned} \tag{II.5}$$

Now for any momenta

$$\begin{aligned}
&\Gamma(\lambda^{-1} p_1 \ldots \lambda^{-1} p_{2n}, \lambda^{-1} q_1 \ldots \lambda^{-1} q_l; m^2, g) \\
&\qquad = \lambda^{-4+2n+2l}\, \Gamma(p_1 \ldots p_{2n}, q_1 \ldots q_l; m^2 \lambda^2, g).
\end{aligned} \tag{II.6}$$

Nonexceptional momenta are defined (SD 1 $\frac{1}{2}$) by

$$\Gamma(p_1 \ldots p_{2n}, q_1 \ldots q_l 0; m^2 \lambda^2, g) = O((\ln \lambda)^c) \tag{II.7}$$

for $\lambda \to 0$ to hold in all orders of renormalized perturbation theory, with c depending on the order. Weinberg [11] power counting (see also Fink [12] and Westwater [13], and SD 2) applied to (II.6) with $\lambda \to 0$ yields: Euclidean momenta sets are nonexceptional if no (in the present model, even[9]) partial sum of momenta vanishes. Other considerations (SD 2, and unpublished) yield: Minkowskian momenta sets are non-

[9] Hereby the q-type momenta count as even.

exceptional if no (in the present model, even) partial sum of momenta is lightlike and certain other sets of momenta more complicated to describe, where at least one (in the present model, even) partial sum of momenta is timelike, are also excluded. For nonexceptional momenta, as $\lambda \to 0$ *afortiori*

$$\Gamma(p_1 \ldots p_{2n}, q_1 \ldots q_l; m^2 \lambda^2, g) = O((\ln \lambda)^c) \tag{II.8}$$

holds and so, by comparison with (II.7), also for momenta sets that are not too exceptional.

For nonexceptional momenta, (II.7) and (II.2b) show that the integral in (II.5) allows $\lambda \to 0$ in all orders of perturbation theory. Thus (SD 1 $\frac{1}{2}$), in this sense, except for $n=0$, $l=2$

$$\begin{aligned} &\lim_{\lambda \to 0} [a(g)^{n-l} a(g(\lambda))^{-n+l} h(g)^{-l} h(g(\lambda))^l \\ &\quad \cdot \Gamma(p_1 \ldots p_{2n}, q_1 \ldots q_l; m^2 \lambda^2, g(\lambda))] \\ &\quad \equiv \Gamma_{as}(p_1 \ldots p_{2n}, q_1 \ldots q_l; m^2, g) \end{aligned} \tag{II.9a}$$

exists, while for $n=0$, $l=2$

$$\begin{aligned} &\lim_{\lambda \to 0} \{a(g)^{-2} a(g(\lambda))^2 h(g)^{-2} h(g(\lambda))^2 \, \Gamma(\,, q(-q); m^2 \lambda^2, g(\lambda)) \\ &\quad - i a(g)^{-2} h(g)^{-2} [k(g) - k(g(\lambda))]\} \equiv \Gamma_{as}(\,, q(-q); m^2, g)\,. \end{aligned} \tag{II.9b}$$

These are the AFs of these VFs.

II.2. Properties of Asymptotic Forms

The formulae (II.9) show that the Γ_{as} are the VFs of a massless (see below) ϕ^4 theory, which (for $0 < g < g_\infty$) we call the prae-asymptotic theory: performing the limit in the renormalized Euclidean integral equations (involving e.g. skeleton expansions in Dyson's sense) written with subtractions[10] at nonexceptional Euclidean momenta one can interchange the integrations with the limit $\lambda \to 0$ since the power behaviour of the integrands does, in any case in renormalized perturbation theory, not change during the limiting process, the integrals remaining absolutely convergent (UV, and also UR if the external momenta are nonexceptional). It follows that one could construct the Γ_{as} directly as those of a massless theory from renormalization conditions obtained by using (II.9) merely for the renormalization functions at suitable (nonexceptional) normalization momenta. This inconvenient method is avoided by the technique of Section II.3.

[10] The prescription (II.9b) is due to this VF involving a subtractive renormalization, and the convention (I.3e) cannot be upheld in a massless theory.

The limites in (II.9) were shown to exist in all orders of renormalized perturbation theory. In view of (II.3) it is reasonable to assume that the Γ_{as} exist also outside of perturbation theory if $0 \leqq g < g_\infty$. (On the other hand, for $g<0$ the behaviour of $g(\lambda)$ as $\lambda \to 0$ is unknown and also the existence of the Γ themselves doubtful [14].) In this sense, we make no distinction in the following between validity in and/or outside perturbation theory unless there is reason to.

Replacing in (II.9) λ by $\lambda\lambda'$ and letting $\lambda' \to 0$ yields, using (II.2a), the transformation laws (except $n=0, l=2$)

$$\begin{aligned} &a(g)^{n-l}\, a(g(\lambda))^{-n+l}\, h(g)^{-l}\, h(g(\lambda))^{l}\, \Gamma_{as}(p_1 \ldots p_{2n}, q_1 \ldots q_l; m^2\lambda^2, g(\lambda)) \\ &\qquad = \Gamma_{as}(p_1 \ldots p_{2n}, q_1 \ldots q_l; m^2, g) \end{aligned} \tag{II.10a}$$

and

$$\begin{aligned} &a(g)^{-2}\, a(g(\lambda))^2\, h(g)^{-2}\, h(g(\lambda))^2\, \Gamma_{as}(\,, q(-q); m^2\lambda^2, g(\lambda)) \\ &- i\,a(g)^{-2}\, h(g)^{-2}[k(g) - k(g(\lambda))] = \Gamma_{as}(\,, q(-q); m^2, g) \end{aligned} \tag{II.10b}$$

which are equivalent to

$$\mathcal{O}\!\!\not{\!\mathscr{P}}_{2n,l}\Gamma_{as}(p_1 \ldots p_{2n}, q_1 \ldots q_l; m^2, g) = -i\delta_{n0}\delta_{l2}\kappa(g)\,. \tag{II.11}$$

(II.10a) gives, for $p^2 \to 0$

$$\begin{aligned} &-\Gamma_{as}(p(-p),\,; m^2, g)^{-1} \equiv G_{as}(p) \\ &= a(g)^{-1}\, i(p^2+i\varepsilon)^{-1}\,[1 + 2b_0^{-2}c_0(\ln[m^2(-p^2-i\varepsilon)^{-1}])^{-1} \\ &\qquad + O((\ln p^2)^{-2}\ln\ln p^2)] \end{aligned} \tag{II.12}$$

showing that the Γ_{as} are VFs of a theory with discrete massless particles (SD 1 $\frac{1}{2}$), a property, like (II.3) it is derived from, not visible in perturbation theory.

The justification of the nomenclature "asymptotic form" lies in the consequence of (II.5) and (II.9)

$$\begin{aligned} &\Gamma(\lambda p_1 \ldots \lambda p_{2n}, \lambda q_1 \ldots \lambda q_l; m^2, g) \\ &\quad - \Gamma_{as}(\lambda p_1 \ldots \lambda p_{2n}, \lambda q_1 \ldots \lambda q_l; m^2, g) \\ &= -im^2\lambda^{2-2n-2l}\int_0^1 d\lambda'^2\, a(g)^{n-l}\, a(g(\lambda'))^{-n+l} \\ &\quad \cdot h(g)^{-l}\, h(g(\lambda'))^{l}\, \Gamma(p_1 \ldots p_{2n}, q_1 \ldots q_l 0; m^2\lambda^{-2}\lambda'^2, g(\lambda'))\,. \end{aligned} \tag{II.13}$$

Thus from (II.6–8) follows that Γ_{as} describes all the logarithmic terms of Γ in the leading λ order as $\lambda \to \infty$, the RHS of (II.13) being $O(\lambda^{2-2n-2l}(\ln\lambda)^c)$, while (II.10) with (II.2b) shows that Γ_{as} comprises these logarithmic terms only. In Section III.3 of SD 2 we showed how to obtain for $n=1$, $l=0$ a refined AF by evaluating the RHS of (II.13) to its logarithmic accuracy;

in Appendix C of Ref. [15] this was generalized to all n. With formula (III.10) below the extension to all n, l is obvious.

Formulae (II.6, 8, 13) can be complemented (cp. SD 2 (II.15)) with

$$\begin{aligned}&\Gamma((\lambda p_1 + r_1) \dots (\lambda p_{2n} + r_{2n}), (\lambda q_1 + s_1) \dots (\lambda q_l + s_l); m^2, g)\\&= \Gamma(\lambda p_1 \dots \lambda p_{2n}, \lambda q_1 \dots \lambda q_l; m^2, g) + O(\lambda^{3-2n-2l}(\ln \lambda)^c)\end{aligned} \tag{II.14}$$

as $\lambda \to \infty$ for nonexceptional p, q.

From (II.2b) and the considerations on (II.13) follows: The evaluation of the limites (II.9) in perturbation theory amounts to keep in the expansion, as $\lambda \to 0$, of $\Gamma(p_1 \dots p_{2n}, q_1 \dots q_l; m^2 \lambda^2, g)$ as double power series[11] in $\ln \lambda^2$ and λ^2 merely the λ-independent terms. This procedure we call "elementary recipe" (SD 2). It yields from (I.5) immediately

$$\Gamma_{as}(p(-p), ; m^2, g) = ip^2 + O(g^2), \tag{II.15a}$$

$$\Gamma_{as}(p_1 p_2 p_3 p_4, ; m^2, g) = -ig + O(g^2), \tag{II.15b}$$

$$\Gamma_{as}(p_1 p_2, q_1 ; m^2, g) = 1 + O(g), \tag{II.15c}$$

$$\Gamma_{as}(, q(-q); m^2, g) = O(1), \tag{II.15d}$$

to be used below. Otherwise, however, this (in renormalization group applications, traditional) procedure is highly uneconomical, and avoided by the technique (Appendix B of SD 2, abbreviated SD 2 B) reviewed in the following section.

II.3. Relation to Zero-Mass Theory

The vertex functions Γ_0 of a zero-mass ϕ^4 theory are defined by imposing the renormalization conditions

$$\Gamma_0(0\,0, ; U^2, V) = 0, \tag{II.16a}$$

$$\Gamma_0(p(-p), ; U^2, V)|_{p^2=-U^2} = -iU^2, \tag{II.16b}$$

$$\Gamma_0(p_1 \dots p_4, ; U^2, V)|_{\text{s.pt.to}\,-U^2} = -iV, \tag{II.16c}$$

$$\Gamma_0(\tfrac{1}{2}q\,\tfrac{1}{2}q, -q; U^2, V)|_{q^2=-U^2} = 1, \tag{II.16d}$$

$$\Gamma_0(, q(-q); U^2, V)|_{q^2=-U^2} = 0. \tag{II.16e}$$

[11] Our remarks on (II.13), and extension of the procedure discussed, to also the corrections to the AF of the Γ in the integrand prove that such double power series expansion exists.

A possible construction of the Γ_0 without q-arguments was described in SD 2 B and is easily extended to all Γ_0 functions[12]. We have (cp. (I.9))

$$\Gamma_0(p(-p),\,;U^2,V)=ip^2+O(V^2)\,, \tag{II.17a}$$

$$\Gamma_0(p_1\dots p_4;U^2,V)=-iV+O(V^2)\,, \tag{II.17b}$$

$$\Gamma_0(p_1p_2,q_1;U^2,V)=1+O(V)\,, \tag{II.17c}$$

$$\Gamma_0(\,,q(-q);U^2,V)=O(1)\,. \tag{II.17d}$$

The normalization momenta square $-U^2$ in (II.16) is not intrinsic, such that its change can be compensated by V- and normalization change, plus an additive term if $n=0$, $l=2$. The functional equations expressing this fact are the well-known renormalization group equations[13] of a massless theory [17]. Their differentiated forms are the PDEs

$$\begin{aligned}&\{U^2[\partial/\partial U^2]+\hat\beta(V)[\partial/\partial V]-2n\hat\gamma(V)\\&\quad+l(2\hat\gamma(V)+\hat\eta(V))\}\,\Gamma_0(p_1\dots p_{2n},q_1\dots q_l;U^2,V)\\&\quad\equiv\hat{\mathcal{O}}_{\not p_{2n,l}}\Gamma_0(\dots)=-i\delta_{n0}\delta_{l2}\hat\kappa(V)\end{aligned} \tag{II.18}$$

with coefficient functions obtained from (II.16) as

$$\hat\beta(V)=\hat b_0V^2+\hat b_1V^3+\cdots\,, \tag{II.19a}$$

$$\hat\gamma(V)=\hat c_0V^2+\hat c_1V^3+\cdots\,, \tag{II.19b}$$

$$\hat\eta(V)=\hat h_0V+\hat h_1V^2+\cdots\,, \tag{II.19c}$$

$$\hat\kappa(V)=\hat k_0+\hat k_1V+\cdots\,. \tag{II.19d}$$

From the observation at the beginning of Section II.2 it follows that we necessarily have

$$\begin{aligned}&\Gamma_{as}(p_1\dots p_{2n},q_1\dots q_l;m^2,g)\\&=Z_1(g)^{-n}Z_2(g)^{-l}\Gamma_0(p_1\dots p_{2n},q_1\dots q_l;m^2,V(g))+i\delta_{n0}\delta_{l2}f(g)\end{aligned} \tag{II.20}$$

and comparison of (II.17) with (II.15) yields

$$V(g)=g+O(g^2)\,, \tag{II.21a}$$

$$Z_1(g)=1+O(g^2)\,, \tag{II.21b}$$

$$Z_2(g)=1+O(g)\,, \tag{II.21c}$$

$$f(g)=O(1)\,. \tag{II.21d}$$

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[12] The results of Blanchard and Sénéor [16] imply that the Γ_0 functions are tempered distributions in Minkowski space to all finite orders of perturbation theory.

[13] These equations are described before (II.27) below.

Applying $\mathcal{O}\not{p}_{2n,l}$ of (I.2) on (II.20) and using (II.18) yields, due to linear independence (verified by special choices of n, l, and momenta)

$$\beta(g)^{-1}\,dg=\hat{\beta}(V)^{-1}\,dV\,, \tag{II.22a}$$

$$\beta(g)\,[d/dg]\ln Z_1(g)=2\hat{\gamma}(V)-2\gamma(g)\,, \tag{II.22b}$$

$$\beta(g)\,[d/dg]\ln Z_2(g)=-2\hat{\gamma}(V)-\hat{\eta}(V)+2\gamma(g)+\eta(g)\,, \tag{II.22c}$$

$$\begin{aligned}&\beta(g)\,[d/dg]\,f(g)\\&=-2(2\gamma(g)+\eta(g))\,f(g)-\kappa(g)+Z_2(g)^{-2}\,\hat{\kappa}(V)\,.\end{aligned} \tag{II.22d}$$

Integration yields

$$V(g)=\hat{\varrho}^{-1}(\varrho(g))\,, \tag{II.23a}$$

$$Z_1(g)=\hat{a}(V(g))\,a(g)^{-1}\,, \tag{II.23b}$$

$$Z_2(g)=\hat{a}(V(g))^{-1}\,\hat{h}(V(g))^{-1}\,a(g)\,h(g)\,, \tag{II.23c}$$

$$f(g)=a(g)^{-2}\,h(g)^{-2}[-k(g)+\hat{k}(V(g))]\,, \tag{II.23d}$$

where we have introduced, analogous to (II.1) and (II.4)

$$\begin{aligned}\hat{\varrho}(V)&=\int^{V}dV'\,\hat{\beta}(V')^{-1}\\&=-\hat{b}_0^{-1}\,V^{-1}-\hat{b}_0^{-2}\,\hat{b}_1\ln V+\text{const}+\cdots V+\cdots\,,\end{aligned} \tag{II.24}$$

$$\hat{a}(V)=\exp\left[2\int_0^V dV'\,\hat{\beta}(V')^{-1}\,\hat{\gamma}(V')\right]=1+2\hat{b}_0^{-1}\,\hat{c}_0\,V+\cdots\,, \tag{II.25a}$$

$$\hat{h}(V)=\exp\left[\int^V dV'\,\hat{\beta}(V')^{-1}\,\hat{\eta}(V')\right]=V^{\hat{b}_0^{-1}\hat{h}_0}(1+\cdots V+\cdots)\,, \tag{II.25b}$$

$$\begin{aligned}\hat{k}(V)&=\int^{V}dV'\,\hat{\beta}(V)^{-1}\,\hat{h}(V')^2\,\hat{a}(V')^2\,\hat{\kappa}(V')\\&=-(1-2\hat{b}_0^{-1}\,\hat{h}_0)^{-1}\,\hat{k}_0\,V^{-1+2\hat{b}_0^{-1}\hat{h}_0}(1+\cdots V+\cdots)\,.\end{aligned} \tag{II.25c}$$

Consistency of (II.23)[14] with (II.21) requires

$$\hat{b}_0=b_0,\ \hat{b}_1=b_1,\ \hat{c}_0=c_0,\ \hat{h}_0=h_0,\ \hat{k}_0=k_0 \tag{II.26}$$

which relations (except the second, which involves some computation) are easily verified directly, and yield $\hat{b}_0^{-1}\,\hat{h}_0=\frac{1}{3}$. The integrated form of (II.18) is analogous to (II.10), with functions (II.25) if we define

$$V(\lambda)=\hat{\varrho}^{-1}(\ln\lambda^2+\hat{\varrho}(V))\,. \tag{II.27}$$

We now consider the massive-theory vertex functions $\bar{\Gamma}$ defined by replacing the renormalization conditions (I.3b–e) by others that also

[14] In the calculation of $V(g)$ from (II.23a) an integration constant needs to be computed (most easily, with the help of the "elementary recipe") from the second order four-point graphs.

lead to relations (I.5) for $\bar{\Gamma}$ with g replaced by $\bar{g}$, e.g. by merely replacing the renormalization momenta in (I.3b–e) by others. The relation of the $\bar{\Gamma}$ to the Γ is then analogous to the one between Γ_{as} and Γ_0 in (II.20), by familiar renormalization group considerations. Going then in this relation to the asymptotic forms by, e.g., the "elementary recipe" yields the same relation between $\bar{\Gamma}_{as}$ and Γ_{as} as just described for $\bar{\Gamma}$ and Γ. The same steps that led from (II.20) to (II.26) now lead to

$$\bar{b}_0 = b_0, \ \bar{b}_1 = b_1, \ \bar{c}_0 = c_0, \ \bar{h}_0 = h_0, \ \bar{k}_0 = k_0 \tag{II.28}$$

where $\bar{b}_0$ etc. are the coefficients in the functions $\bar{\beta}(\bar{g})$ etc. of the $\bar{\Gamma}$ theory. Likewise, if we change (II.16b–e) such that (II.17) holds for new $\bar{\Gamma}_0, \cdot\bar{V}$, then $\bar{\bar{b}}_0 = \bar{b}_0 = b_0$ etc. Thus, the coefficients in (II.28) are universal, e.g. independent of the renormalization momenta chosen, and the same in the massive and massless theory. A consequence of this result, that the imaginary part of the four-point vertex function should not change sign in Euclidean and Minkowskian momentum space, was elaborated in Ref. [14].

III. Exceptional Momenta and Infrared Singularities

III.1. Asymptotic Forms at Exceptional Momenta

At exceptional momenta, by definition the estimate (II.7) does not hold, which requires to remove from the integrand in (II.5) the offending part with the help of the appropriate Wilson expansion [18]. For the four-point vertex, the momenta sets $p(-p)\,0\,0$ and $p(-p)\,q(-q)$ are exceptional. In Section III.1 of SD 2 we derived the PDEs

$$[\mathcal{O}\!\!/\!p_{4,0} - \eta(g)]\, \Gamma_{\underline{as}}(p(-p)\,0\,0, ; m^2, g) = 0 \tag{III.1}$$

and

$$\begin{aligned}\mathcal{O}\!\!/\!p_{4,0}\, &\Gamma_{\underline{as}}(p(-p)\,q(-q), ; m^2, g)\\ &= i\kappa(g)\, \Gamma_{\underline{as}}(p(-p)\,0\,0, ; m^2, g)\, \Gamma_{\underline{as}}(q(-q)\,0\,0, ; m^2, g)\end{aligned} \tag{III.2}$$

with $\eta(g)$ and $\kappa(g)$ of (I.4c,d), where immediate consequences of the formula (A.6) had been used. (The underlined subscript indicates that the asymptotics of that function is exceptional.) The analoga of (II.9a) are SD 2 (III.8) and SD 2 (III.15), and the analoga of (II.10) the formulae SD 2 (III.7) and the analogous one derived from SD 2 (III.15). For the vertex, $p(-p), 0$ are the exceptional momenta sets, and we obtained, again as a consequence of (A.6), Eq. SD 2 (III.19)

$$\mathcal{O}\!\!/\!p_{2,1}\, \Gamma_{\underline{as}}(p(-p), 0; m^2, g) = i\kappa(g)\, \Gamma_{\underline{as}}(p(-p)\,0\,0, ; m^2, g) \tag{III.3}$$

with limit formula SD 2 (III.21) and corresponding transformation formula [cp. (III.9) below].

In all these cases, the correct limit formulae show that the corresponding Γ_{as} would not exist (or, in the $\Gamma(p(-p)0\,0,)$ case, would vanish), the reason being an infrared (UR) singularity of the Γ_{as} at the exceptional momenta, to be investigated in Section III.2. The mentioned correct limit formulae, however, allow to read off the $\lambda \to 0$, i.e. small-mass, behaviour of the vertex function (s) in the limitand, from the $\lambda \to 0$ behaviour of the factors that had to be inserted to render the limit finite. The reader is referred to SD 2 Section III for details, and also for the proof that the "elementary recipe" of Section II.3 is applicable also to the functions considered here.

There are analoga of (II.14) which show the relation between the asymptotics near, and the one at, exceptional momenta. They are obtained from (A.6), (A.11), and (A.8), respectively, as

$$\begin{aligned}&\Gamma((\lambda p + r_1)(-\lambda p + r_2)\, r_3 r_4,)\\ &= \Gamma(\lambda p(-\lambda p)\,0\,0,)\,\Gamma(r_3 r_4, (-r_3 - r_4)) + O(\lambda^{-1}(\ln\lambda)^c)\,,\end{aligned} \tag{III.4}$$

$$\begin{aligned}&\Gamma((\lambda p + r_1)(-\lambda p + r_2)(\lambda q + r_3)(-\lambda q + r_4),)\\ &= \Gamma(\lambda p(-\lambda p)\,\lambda q(-\lambda q),)\\ &\quad + \Gamma(\lambda p(-\lambda p)\,0\,0,)\,\Gamma(,(r_1 + r_2)(r_3 + r_4))\,\Gamma(\lambda q(-\lambda q)\,0\,0,)\\ &\quad + O(\lambda^{-1}(\ln\lambda)^c)\,,\end{aligned} \tag{III.5}$$

and

$$\begin{aligned}&\Gamma((\lambda p + r_1)(-\lambda p + r_2), r_3) = \Gamma(\lambda p(-\lambda p), 0)\\ &\quad + \Gamma(\,,(r_1 + r_2)\, r_3)\,\Gamma(\lambda p(-\lambda p)\,0\,0,) + O(\lambda^{-1}(\ln\lambda)^c)\end{aligned} \tag{III.6}$$

to be contrasted with (II.14). It is the $\lambda \to \infty$ asymptotics of the functions on the RHSs that the formulae (III.4–6) and the cited ones of SD 2 apply to.

There is an analog of (III.3) for many arguments:

$$\begin{aligned}&\mathcal{O}\!\!\not{p}_{2n,l+1}\,\Gamma_{\underline{as}}(p_1 \ldots p_{2n}, q_1 \ldots q_l 0; m^2, g)\\ &\quad = i\kappa(g)\,\Gamma'_{\underline{as}}(p_1 \ldots p_{2n}\,0\,0, q_1 \ldots q_l; m^2, g)\end{aligned} \tag{III.7}$$

with, analogous to (III.1),

$$[\mathcal{O}\!\!\not{p}_{2n+2,l} - \eta(g)]\,\Gamma'_{\underline{as}}(p_1 \ldots p_{2n}\,0\,0, q_1 \ldots q_l; m^2, g) = 0 \tag{III.8}$$

whereby the set $p_1 \ldots p_{2n}, q_1 \ldots q_l$ must not be exceptional. In Eqs. (III.7–10), $\Gamma'_{\underline{as}}$ is the (one-particle-irreducible) VF plus the sum, over all partitions of arguments, of multilinear combinations of VFs times connecting propagators as correspond to contributions to the amputated Green's function forming a chain, with partition of arguments such that in each of the end VFs there is one of the two zero-momentum arguments. (III.7, 8) with $l=0$ were used in Appendic C of Ref. [15] to evaluate the mass correction term to the Gell-Mann-Low limit

$$\Gamma_{as}(\lambda p_1 \dots \lambda p_{2n}; m^2, g_\infty) \quad \text{of} \quad \Gamma(\lambda p_1 \dots \lambda p_{2n}; m^2, g).$$

(III.7, 8) also yield a precise estimate of the $\lambda \to 0$ behaviour of the limitand in (II.9). Namely, (II.5) can be rewritten

$$\begin{aligned}
&a(g)^{n-l}\, a(g(\lambda))^{-n+l}\, h(g)^{-l}\, h(g(\lambda))^l \\
&\quad \cdot \Gamma(p_1 \dots p_{2n}, q_1 \dots q_l; m^2 \lambda^2, g(\lambda)) \\
&\quad - i\delta_{n0}\delta_{l2}\, a(g)^{-2}\, h(g)^{-2}\, [k(g) - k(g(\lambda))] \\
&\quad - \Gamma_{as}(p_1 \dots p_{2n}, q_1 \dots q_l; m^2, g) \\
&= -im^2 \int_0^{\lambda^2} d\lambda'^2\, a(g)^{n-l} a(g(\lambda'))^{-n+l}\, h(g)^{-l}\, h(g(\lambda'))^l \\
&\qquad \cdot \varphi(g(\lambda'))\, \Gamma(p_1 \dots p_{2n}, q_1 \dots q_l 0; m^2 \lambda'^2, g(\lambda')) \\
&\approx -im^2 \int_0^{\lambda^2} d\lambda'^2\, a(g)^{n-l} a(g(\lambda'))^{-n+l}\, h(g)^{-l}\, h(g(\lambda'))^l \qquad \text{(III.9)} \\
&\qquad \cdot \varphi(g(\lambda'))\, \Gamma_{\underline{as}}(p_1 \dots p_{2n}, q_1 \dots q_l\, 0; m^2 \lambda'^2, g(\lambda')) \\
&\approx m^2\, a(g)^{-1}\, h(g)^{-1}\, \Gamma'_{\underline{as}}(p_1 \dots p_{2n}\, 0\, 0, q_1 \dots q_l; m^2, g) \\
&\qquad \cdot \int_0^{\lambda^2} d\lambda'^2\, h(g(\lambda'))^{-1}\, k(g(\lambda')) \\
&\approx -\lambda^2 (\ln \lambda^{-2})^{\frac{2}{3}}\, m^2\, a(g)^{-1}\, h(g)^{-1}\, b_0^{\frac{2}{3}} \\
&\qquad \cdot \Gamma'_{\underline{as}}(p_1 \dots p_{2n}\, 0\, 0, q_1 \dots q_l; m^2, g)
\end{aligned}$$

where the formula resulting from (III.7) and (III.8)

$$\begin{aligned}
&\Gamma_{\underline{as}}(p_1 \dots p_{2n}, q_1 \dots q_l\, 0; m^2 \lambda^2, g(\lambda)) \\
&\quad = a(g)^{-n+l+1}\, a(g(\lambda))^{n-l-1}\, h(g)^{l+1}\, h(g(\lambda))^{-l-1} \\
&\qquad \cdot \Gamma_{\underline{as}}(p_1 \dots p_{2n}, q_1 \dots q_l\, 0; m^2, g) \qquad \text{(III.10)} \\
&\quad + i[k(g(\lambda)) - k(g)]\, a(g)^{-n-1+l}\, a(g(\lambda))^{n-l-1} \\
&\qquad \cdot h(g)^{l-1}\, h(g(\lambda))^{-l-1}\, \Gamma'_{\underline{as}}(p_1 \dots p_{2n}\, 0\, 0, q_1 \dots q_l; m^2, g)
\end{aligned}$$

has been used and only the terms most singular as $\lambda \to 0$ have been kept. (III.9) shows that a small-mass correction to a zero-mass theory, as the LHS represents according to (II.9), is nonanalytic[15] in the mass, naively as a consequence of the UR divergence of the small-mass-correction integral obtained e.g. by the Schwinger action principle [20]. The logarithmic power two-third in (III.9) can be understood as one, from the massless-propagator pair, minus one-third from the "mass analog", SD 2 (III.8) and following, to (III.18) below.

III.2. Infrared Singularities in Zero-Mass Theory

The vertex functions Γ_0 of the zero-mass theory of Section II.3 are UR singular at exceptional momenta. In this section we analyze this singularity structure in the simplest cases and define certain UR finite parts $\underline{\Gamma}_0$. The formulae we here obtain for these are simpler than those of SD 2 B and will allow us in Section III.3 to calculate the $\Gamma_{\underline{as}}$ therefrom without need of any asymptotic consideration.

For Euclidean momenta, UR divergences arise only from two-particle links through which zero momentum flows, and from three-particle links differentiated at zero momentum as will not occur in our cases. Thus, in the formulae of Appendix A, to which we now must refer, there is UR divergence danger from factors $\overset{0}{G}$ and, related thereto, $\overset{0}{I}$ but not from $\overset{0}{B}$. Thus, W in (A.2b) does not exist. However, a $\overset{0}{G}$ is admissible if the factor to its right or left vanishes at zero relative momentum since (the Euclidean integral)

$$\int dk[(k+p)^2(k-p)^2]^{-1} f(p,k)$$

is UR divergent only if $p=0$, and thus UR convergent if $f(p,k)$ vanishes at $k=0$ of first order. Thus, on the RHS of (A.6) the second term is UR convergent, and so are, for nonzero total momentum (and momenta so chosen that there is nonzero momentum in the other channels also, as we will always assume in the following) all other terms except

$$\overset{0}{I}{}^0 \langle (\mathbf{1}+W) = \overset{0}{I}{}^0 (\overset{0}{\boldsymbol{\Gamma}}{}^0)^{-1} \boldsymbol{\Gamma} \tag{III.11}$$

where (A.7) is used. Since (III.11) must therefore also be UR finite, and we normalize $\boldsymbol{\Gamma}$ to be UR finite at nonexceptional momenta by (II.16d),

$$\underline{\overset{0}{I}}{}^0 \equiv \text{“}\overset{0}{I}{}^0 (\overset{0}{\boldsymbol{\Gamma}}{}^0)^{-1}\text{”} \tag{III.12a}$$

[15] The analytic structure of a Feynman integral as a function of the complex-valued mass of one propagator near zero of that mass has been determined by Speer and Westwater [19]. (III.9) can be looked at as resulting from summing over all insertions in all diagrams, the broken power of the logarithm being due to summing over two-nearly massless-particle intermediate states lying behind each other, which essentially leads to a binomial series. See also the end of Section IV.2.

is UR finite, where here and in the following quotation marks mean, optionally, obtaining the expression either by a limit from the nonzero-mass theory [as in (II.9) for the nonexceptional case] or by a limit within the $\boldsymbol{\Gamma}_0$ theory letting the momentum in the through-channel go to zero. (A.6) allows, however, to give the definition without any limiting process

$$\overset{0}{\underline{I}}{}^{0} = \overset{U}{I}{}^{0} - (\overset{0}{B} - \overset{0}{B}{}^{0})\overset{U}{W}{}^{0} - (\overset{U}{B} - \overset{0}{B})(\mathbf{1} + \overset{U}{G}\overset{U}{I})^{0} \tag{III.12b}$$

where the U on top means that the through-momentum[16] has square $-U^2$ and (II.16d) has been used. In Section III.3 we need a PDE to relate $\overset{0}{\underline{I}}{}^{0}$ to $\boldsymbol{\Gamma}_{\underline{as}}$. This is simplest obtained by considering instead of (III.12b)

$$\overset{0}{\underline{I}}{}^{0} = (\overset{p}{\boldsymbol{\Gamma}}{}^{0})^{-1}\,[\overset{p}{I}{}^{0} - (\overset{0}{B} - \overset{0}{B}{}^{0})\overset{p}{W}{}^{0} - (\overset{p}{B} - \overset{0}{B})(\mathbf{1} + \overset{p}{G}\overset{p}{I})^{0}]$$

with p arbitrary and unrelated to U. Namely, (II.18) gives then immediately

$$[\hat{\mathcal{O}}\!\not{p}_{4,0} - \hat{\eta}(V)]\,\overset{0}{\underline{I}}{}^{0} = 0\,. \tag{III.12c}$$

Turning to $\overset{0}{I}$, (A.11) suggests the definition

$$\overset{0}{\underline{I}} \equiv \text{“}\overset{0}{I} - \overset{0}{\underline{I}}{}^{0}\,\overset{0}{\boldsymbol{\Pi}}\,{}^{0}\overset{0}{\underline{I}}\text{”} \tag{III.13a}$$

to be understood similarly as before, with the manifestly finite form (note (II.16e)) simplest from (A.10)

$$\overset{0}{\underline{I}} = (\overset{0}{B} - \overset{0}{B}{}^{0})(\mathbf{1} + \overset{0}{W}) + \overset{0}{\underline{I}}{}^{0}\,\overset{U}{\boldsymbol{\Gamma}}(\mathbf{1} - \overset{U}{R})\,. \tag{III.13b}$$

The PDE is obtained by applying $\hat{\mathcal{O}}\!\not{p}_{4,0}$ on (A.10) written as

$$\overset{0}{\underline{I}} = -\overset{0}{\underline{I}}{}^{0}\,\boldsymbol{\Pi}\,{}^{0}\overset{0}{\underline{I}} + (\overset{0}{B} - \overset{0}{B}{}^{0})(\mathbf{1} + \overset{0}{W}) + \overset{0}{\underline{I}}{}^{0}\,\boldsymbol{\Gamma}(\mathbf{1} - R)$$

and using (II.18), as

$$\hat{\mathcal{O}}\!\not{p}_{4,0}\overset{0}{\underline{I}} = i\hat{\kappa}(V)\,\overset{0}{\underline{I}}{}^{0}\;{}^{0}\overset{0}{\underline{I}}\,. \tag{III.13c}$$

Finally, in view of (A.8) we set

$$\overset{0}{\underline{\boldsymbol{\Gamma}}} \equiv \text{“}\overset{0}{\boldsymbol{\Gamma}} - \overset{0}{\boldsymbol{\Pi}}\;{}^{0}\overset{0}{\underline{I}}\text{”} \tag{III.14a}$$

with the finite form

$$\overset{0}{\underline{\boldsymbol{\Gamma}}} = \overset{U}{\boldsymbol{\Gamma}}(\mathbf{1} - \overset{U}{R})\,. \tag{III.14b}$$

The PDE is obtained applying $\hat{\mathcal{O}}\!\not{p}_{2,1}$ on (A.8) written as

$$\overset{0}{\underline{\boldsymbol{\Gamma}}} = \boldsymbol{\Gamma}(\mathbf{1} - R) - \boldsymbol{\Pi}\;{}^{0}\overset{0}{\underline{I}}$$

---

[16] From (A.6) follows that the RHS in (III.12b) is independent of the direction of that momentum.

and using (II.18), as

$$\hat{\mathcal{O}}\not{p}_{2,1}\, \overset{0}{\underline{\boldsymbol{\Gamma}}} = i\hat{\kappa}(V)\; {}^{0}\overset{0}{\underline{I}} \,. \tag{III.14c}$$

The formula before (III.12c), and (A.11) and (A.8) can now be written

$$\overset{p}{\mathsf{I}}{}^{0} = \overset{0}{\underline{I}}{}^{0}\; \overset{p}{\boldsymbol{\Gamma}}{}^{0} + \{(\overset{0}{\mathsf{B}} - \overset{0}{\mathsf{B}}{}^{0})\,\overset{p}{\mathsf{W}}{}^{0} + (\overset{p}{\mathsf{B}} - \overset{0}{\mathsf{B}})\,(\mathbf{1} + \overset{p}{\mathsf{G}}\,\overset{p}{\mathsf{I}})^{0}\} \tag{III.15}$$

$$\overset{p}{\mathsf{I}} = \overset{0}{\underline{I}} + \overset{0}{\underline{I}}{}^{0}\, \overset{p}{\boldsymbol{\Pi}}\; {}^{0}\overset{0}{\underline{I}} + \{(\overset{0}{\mathsf{B}} - \overset{0}{\mathsf{B}}{}^{0})\,(\overset{p}{\mathsf{W}} - \overset{0}{\mathsf{W}}) + (\overset{p}{\mathsf{B}} - \overset{0}{\mathsf{B}})\,(\mathbf{1} + \overset{p}{\mathsf{G}}\,\overset{p}{\mathsf{I}}) + \overset{0}{\underline{I}}{}^{0}\, \overset{p}{\boldsymbol{\Gamma}}\, \overset{p}{R}\}\,, \tag{III.16}$$

$$\overset{p}{\boldsymbol{\Gamma}} = \overset{0}{\underline{\boldsymbol{\Gamma}}} + \overset{p}{\boldsymbol{\Pi}}\; {}^{0}\overset{0}{\underline{I}} + \{\overset{p}{\boldsymbol{\Gamma}}\,\overset{p}{R}\}\,. \tag{III.17}$$

These formulae display the UR singularity structure, as $p \to 0$, of the LHSs, with the curly brackets being $O(p)$ then. As to $\overset{p}{\boldsymbol{\Gamma}}{}^{0}$ and $\overset{p}{\boldsymbol{\Pi}}$, from the integrated form of (II.18), analogous to (II.10), follows for $p^2 \to 0$

$$\overset{p}{\boldsymbol{\Gamma}}{}^{0} = \hat{h}(V)^{-1}\, b_0^{-\frac{1}{3}}\, [-\ln(-p^2 - i\varepsilon)]^{-\frac{1}{3}} + O((\ln p^2)^{-\frac{4}{3}} \ln\ln p^2) \tag{III.18}$$

and

$$\begin{aligned}\overset{p}{\boldsymbol{\Pi}} = &-i\hat{a}(V)^{-2}\, \hat{h}(V)^{-2}\, \{b_0^{\frac{1}{3}}[-\ln(-p^2 - i\varepsilon)]^{\frac{1}{3}} + \hat{k}(V)\} \\ &+ O((\ln p^2)^{-\frac{2}{3}} \ln\ln p^2)\,.\end{aligned} \tag{III.19}$$

Here the UR divergence appears in the imaginary, dispersive, part only, the absorptive part is (we omit the error estimates)

$$2\,\mathcal{Re}\, \overset{p}{\Pi} \approx \tfrac{2}{3}\pi\hat{a}(V)^{-2}\, \hat{h}(V)^{-2}\, b_0^{\frac{1}{3}}\, \Theta(p^2)\, [-\ln p^2]^{-\frac{2}{3}}$$

which equals the one obtained using (III.18) bilinearly, the Γ_0 analog of (II.12), and inserting the two-massless-particle phase space factor:

$$2\,\mathcal{Re}\, \overset{p}{\Pi} \approx \hat{a}(V)^{-2}\, |\hat{h}(V)^{-1}\, b_0^{-\frac{1}{3}}\, [\ln(-p^2)]^{-\frac{1}{3}}|^2 \cdot \{\tfrac{1}{2}\, 2^{-3}\, \pi^{-1}\, \Theta(p^2)\}\,.$$

Similarly, the absorptive part of $\overset{p}{\Gamma}{}^{0}$ is

$$\begin{aligned}2\,\mathcal{Im}\, \overset{p}{\Gamma}{}^{0} &\approx -i\tfrac{2}{3}\pi\hat{h}(V)^{-1}\, b_0^{-\frac{1}{3}}\, \Theta(p^2)\, [-\ln p^2]^{-\frac{4}{3}} \\ &= \hat{a}(V)^{-2}\, [\hat{h}(V)^{-1}\, b_0^{-\frac{1}{3}}\, (-\ln p^2)^{-\frac{1}{3}}] \\ &\quad \cdot [-i\hat{a}(V)^2\, b_0^{-1} (-\ln p^2)^{-1}] \cdot \{\tfrac{1}{2}\, 2^{-3} \pi^{-1}\, \Theta(p^2)\}\end{aligned}$$

where the last square bracket is the (nonexceptional) UR form of the zero-mass four point vertex.

The $p \to 0$ singularities in (III.15–17) are quantitatively related to the ones at exceptional momenta for vanishing mass, as comparison with the formulae (III.5), (III.8), and (III.21) of SD 2 shows: it is merely necessary to replace $\ln(-p^2)$ by $\ln\lambda^{-2} = \ln(m^2\lambda^{-2}/m^2)$, $\lambda \to \infty$. Related to this is the following: Choose for the momenta in (III.16–17), and in the more general equation from (A.6)

$$I = \overset{0}{\underline{I}}{}^0 \Gamma + \{(\overset{0}{B} - \overset{0}{B}{}^0)(1+W) + (B - \overset{0}{B})(1+GI)\}$$

instead of (III.15), the momenta appearing in (III.4–6). Inspection of the correction terms shows that they have for $\lambda \to \infty$ the same order $O(\lambda^{-1}(\ln\lambda)^c)$ as in (III.4–6). This means that (III.4–6) are also valid in the Γ_0 theory provided one replaces those Γ_0 functions not possessing the restrictions required in (III.4–6) by Γ_0 functions. The results of the next section will even allow us to pass from the Γ_0 to the Γ_{as} functions.

Formulae (III.4–6) relate to the small-distance behaviour of amputated functions. The Wilson expansion in the narrow sense involves unamputated functions. Concerning these, we only remark that e.g. (A.3) is in the massless theory not adequate in coordinate space even after using (A.7) and (III.12a): Due to factors $\overset{0}{G}$ the two terms on the RHS are then not separately UR finite. The appropriate version[17] of (A.3) is

$$1 + GI = [(1 + \overset{0}{G}\overset{0}{I})^0 (\overset{0}{I}{}^0)^{-1}]\, \Gamma + (1 + W - 1^0 \langle 1 - 1^0 \langle W)$$

which in coordinate space corresponds to Zimmermann's[18]

$$\begin{aligned} &N_0(A(x+\xi)A(x-\xi)) \\ &\quad = H(\xi)\tfrac{1}{2}N_0(A^2(x)) + M_2(A(x+\xi)A(x-\xi)) \end{aligned}$$

in view of (A.7). Here (suitable matrix elements of) the N_0 and M_2 products are finite also in the massless theory for fixed x and ξ, $\xi^2 \neq 0$, such that also the square bracket in the penultimate equation is finite.

Finally, we mention that Eqs. (III.12c–14c) allow to determine the precise small-momenta behaviour of the underlined functions, written here under omission of the error estimate:

$$\begin{aligned} {}^{p}\overset{0}{\underline{I}}{}^0 &\approx -i\hat{a}(V)^2\,\hat{h}(V)\, b_0^{-\frac{2}{3}}[-\ln(-p^2 - i\varepsilon)]^{-\frac{2}{3}}, \\ {}^{\lambda p}\overset{0}{\underline{I}}{}^{\lambda q} &\approx -2i\hat{a}(V)^{+2}\, b_0^{-1}(\ln\lambda^{-2})^{-1}, \\ \overset{0}{\underline{\Gamma}}{}^p &\approx 2\hat{h}(V)^{-1}\, b_0^{-\frac{1}{3}}[-\ln(-p^2 - i\varepsilon)]^{-\frac{1}{3}}. \end{aligned}$$

III.3 Asymptotic Forms Obtained from Zero-Mass Theory

Comparison of the limit formulae analogous to (II.9), cited in Section III.1, for the functions considered there with the formulae (III.12a,

[17] The author thanks H.-J. Thun for a helpful discussion in this connection.
[18] See last Ref. of [5], Eq. (6.31).

13a, 14a), recalling also (II.20), shows that we have

$$\Gamma_{\underline{as}}(p(-p)\,0\,0,\,;m^2,g)=X(g)\,\underline{\Gamma}_0(p(-p)\,0\,0,\,;m^2,V(g))\,, \tag{III.20}$$

$$\begin{aligned}\Gamma_{\underline{as}}(p(-p)\,q(-q),\,;m^2,g)&=Z_1(g)^{-2}\,\underline{\Gamma}_0(p(-p)\,q(-q),\,;m^2,V(g))\\&+iY(g)\,\underline{\Gamma}_0(p(-p)\,0\,0,\,;m^2,V(g))\,\underline{\Gamma}_0(q(-q)\,0\,0,\,;m^2,V(g))\end{aligned} \tag{III.21}$$

and

$$\begin{aligned}\Gamma_{\underline{as}}(p(-p),0;m^2,g)&=Z_1(g)^{-1}\,Z_2(g)^{-1}\,\underline{\Gamma}_0(p(-p),0;m^2,V(g))\\&+iY(g)\,X(g)^{-1}\,\underline{\Gamma}_0(p(-p),\,;m^2,V(g))\,.\end{aligned} \tag{III.22}$$

Inserting (III.20) into (III.1) and using (III.12c) and (II.23) yields

$$X(g)=Z_1(g)^{-1}\,Z_2(g) \tag{III.23}$$

with the multiplicative integration constant obtained from $g\to 0$, using (II.22) and the "elementary recipe". Inserting (III.21) into (III.2) and using (III.13c), (II.23), and (III.12c) gives

$$Y(g)=h(g)^{-2}\,a(g)^{-2}\,X(g)^2\,[k(g)-\hat{k}(V(g))]\,, \tag{III.24}$$

and then (III.22) is seen to be consistent with (III.3) and (III.14c). It is easily seen, using $g\to 0$ limites, that (III.21–24) is the only acceptable solution of the relevant PDEs.

Formulae (III.20–22) can be written more symmetrically as ($V=V(g)$ understood)

$$\begin{aligned}&a(g)^{-2}\,h(g)^{-1}\,\Gamma_{\underline{as}}(p(-p)\,0\,0,\,;m^2,g)\\&\quad=\hat{a}(V)^{-2}\,\hat{h}(V)^{-1}\,\underline{\Gamma}_0(p(-p)\,0\,0,\,;m^2,V)\end{aligned} \tag{III.25}$$

$$\begin{aligned}&a(g)^{-2}\,\Gamma_{\underline{as}}(p(-p)\,q(-q),\,;m^2,g)\\&\quad-ia(g)^{-4}\,h(g)^{-2}\,k(g)\,\Gamma_{\underline{as}}(p(-p)\,0\,0,\,;m^2,g)\,\Gamma_{\underline{as}}(q(-q)\,0\,0,\,;m^2,g)\\&=\hat{a}(V)^{-2}\,\underline{\Gamma}_0(p(-p)\,q(-q),\,;m^2,V)\\&\quad-i\hat{a}(V)^{-4}\,\hat{h}(V)^{-2}\,\hat{k}(V)\,\underline{\Gamma}_0(p(-p)\,0\,0,\,;m^2,V)\,\underline{\Gamma}_0(q(-q)\,0\,0,\,;m^2,V)\end{aligned} \tag{III.26}$$

$$\begin{aligned}&h(g)\,\Gamma_{\underline{as}}(p(-p),0;m^2,g)\\&\quad-ia(g)^{-2}\,h(g)^{-1}\,k(g)\,\Gamma_{\underline{as}}(p(-p)\,0\,0,\,;m^2,g)\\&=\hat{h}(V)\,\underline{\Gamma}_0(p(-p),0;m^2,V)\\&\quad-i\hat{a}(V)^{-2}\,\hat{h}(V)^{-1}\,\hat{k}(V)\,\underline{\Gamma}_0(p(-p)\,0\,0,\,;m^2,V)\end{aligned} \tag{III.27}$$

where both sides are solutions of the homogeneous PDEs to

$$\mathcal{O}\!\!\not{p}_{0,0}=\hat{\mathcal{O}}\!\!\not{p}_{0,0}\quad(U^2=m^2,\ V=V(g))$$

that is, in conventional terminology [21], the expressions (III.25–27) are renormalization group invariant.

The formulae (III.20–24) solve the problem of computing the $\Gamma_{\underline{as}}$ from the $\underline{\Gamma}_0$ and thus, by (III.12b, 13b, 14b), from the Γ_0 in the most economical way. These formulae are simpler than those of SD 2 B and free of any limiting processes.

Inserting (III.20–24) into the large-λ versions of (II.15–17) described at the end of the last section and using (II.20) and (II.23) we find that (III.4–6) hold also for the functions Γ_{as} provided one writes $\Gamma_{\underline{as}}$ where in (III.15–17) there appears a $\underline{\Gamma}_0$ instead of a Γ_0 function. It follows that in (III.4–6) for large λ the LHSs do not approach the corresponding Γ_{as} functions, the difference residing, however, only in the λ-free factors on the RHSs.

IV. Infrared Singularities in Non-massless Theories

IV.1 Definition of Parquet Approximation

The PA[19] VF $\Gamma_p(p_1 \ldots p_4, ; U^2, V) \equiv \Gamma_p$ is defined by the crossing symmetric BS equation Fig. 1 (letters and the dotted lines disregarded) with propagators $i(k^2+i\varepsilon)^{-1}$. The notation is self-explanatory, the function marked i being two-particle irreducible in all three channels. In Fig. 1c, the PA in the narrow sense uses on the RHS only the first term (a constant in momentum space); we shall see that the UR behaviour is unaffected by the higher terms.

All the integrals in Fig. 1 are (at least in the iterative solution, see Appendix B) logarithmically divergent. As is well known, the BS- and the i-kernel each must contain an undetermined (logarithmically divergent) constant to allow a finite vertex function. These constants are fixed implicitly by prescribing the value of the vertex Γ_p at some subtraction point, e.g.

$$\Gamma_p(p_1 \ldots p_4, ; U^2, V)|_{\text{s.pt to}-U^2} = -iV. \tag{IV.1}$$

Then all functions in Fig. 1 can be constructed, *modulo* the mentioned constants, as power series in V (see Appendix B for a computational scheme). Under change of subtraction point we obviously have

$$\Gamma_p(p_1 \ldots p_4, ; U^2, V) = \Gamma_p(p_1 \ldots p_4, ; U_0^2, \Gamma_p(p_1' \ldots p_4', ; U^2, V)|_{\text{s.pt.to}-U_0^2}). \tag{IV.2}$$

Differentiating with respect to U_0^2 at $U_0^2 = U^2$ yields

$$\{U^2[\partial/\partial U^2] + \beta_p(V)[\partial/\partial V]\}\,\Gamma_p(p_1 \ldots p_4, ; U^2, V) \equiv \mathcal{O}\!\!/_p\Gamma_p(\cdots) = 0 \tag{IV.3}$$

---

[19] This approximation was introduced by Diatlov, Sudakov, and Ter-Martirosian [2] to study meson-meson scattering at large energy in pseudoscalar meson theory. While the PA is not suitable for that purpose (unless $V<0$ in (IV.1), cp. Ref. [14]), its use for UR behaviour studies is legitimate [3]. We here always suppose $0<V<V_\infty$ where V_∞ is the first positive zero of $\beta_p(V)$.

where (cp. (I.4a))

$$\beta_p(V) = \tfrac{1}{2} i \sum_{j=1}^{3} p_j\,[\partial/\partial p_j]\; \Gamma_p(p_1 p_2 p_3(-p_1-p_2-p_3),\,;\,U^2, V)|_{\text{s.pt.to}-U^2}$$
$$= b_0 V^2 + b_{1p} V^3 + \cdots . \tag{IV.4}$$

(IV.3) allows to write (IV.2) in the form

$$\Gamma_p(p_1 \ldots p_4,\,;\,U^2, V) = \Gamma_p(p_1 \ldots p_4,\,;\,U^2\lambda^2, V_p(\lambda)) \tag{IV.5}$$

defining $V_p(\lambda)$ analogously to (II.27). From (IV.5) the small-momenta behaviour of Γ_p follows by setting $p_i \to \lambda p_i$ and using the analog of (II.3)

$$V_p(\lambda) = b_0^{-1}(\ln\lambda^{-2})^{-1} - b_0^{-3} b_{1p}(\ln\lambda^{-2})^{-2}\ln\ln\lambda^{-2} + O((\ln\lambda)^{-2}) \tag{IV.6}$$

as $\lambda \to 0$.

By the formulae of Appendix A we may now define PA functions

$$\Gamma_p(p_1 p_2, q_1\,;\,U^2, V) \equiv \Gamma_p \qquad (\overset{U}{\Gamma}{}_p^0 = 1) \tag{IV.7a}$$

and

$$\Gamma_p(,\,q(-q);\,U^2, V) \equiv \Pi_p \qquad (\overset{U}{\Pi}_p = 0) \tag{IV.7b}$$

and then by skeleton expansions also all functions

$$\Gamma_p(p_1 \ldots p_{2n}, q_1 \ldots q_2\,;\,U^2, V)\,.$$

The point of interest presently is that the considerations of Section III.2 can be taken over *in toto* merely all functions to be read as the PA ones. Since $\mathcal{O}\!\!\!/\!\!\!\mathcal{h}_p$ annihilates I_p as well as the propagator, from (A.1) follows that $\mathcal{O}\!\!\!/\!\!\!\mathcal{h}_p$ also annihilates $B_p - B'_p$ i.e. B_p subtracted arbitrarily (but U-independently). Since U enters the definitions of Γ_p and Π_p in (IV.7), however, we finally obtain the PDEs

$$[\mathcal{O}\!\!\!/\!\!\!\mathcal{h}_p + l\eta_p(V)]\,\Gamma_p(p_1 \ldots p_{2n}, q_1 \ldots q_l;\,U^2, V) = -i\delta_{n0}\delta_{l2}\kappa_p(V) \tag{IV.8}$$

with

$$\eta_p(V) = \tfrac{1}{3} b_0 V + \cdots V^2 + \cdots, \tag{IV.9a}$$

$$\kappa_p(V) = \tfrac{1}{3} b_0 + \cdots V + \cdots \tag{IV.9b}$$

analogous to (II.18). Thus, the UR singularity structure of the functions $h_p(V)^l\,\Gamma_p(p_1 \ldots p_{2n}, q_1 \ldots q_l;\,U^2, V)$ with $h_p(V)$ defined in analogy to (II.25b), is the same as the one of the functions

$$\hat{a}(V')^{-n+l}\,\hat{h}(V')^l\,\Gamma_0(p_1 \ldots p_{2n}, q_1 \ldots q_l;\,U^2, V')$$

for any V resp. V' in the allowed ranges. The higher terms on the RHS of Fig. 1c hereby have no effect (except on the functions with q-arguments in view of the factor $h_p(V)^l$) since they change $\beta_p(V)$ in (IV.4) only by terms $O(V^4)$ that are, in view of (II.3), without interest for UR behaviour.

IV.2. Application of Parquet Approximation

Consider a theory that describes besides self-coupled (pseudo) scalar neutral massless particles also (arbitrary) massive particles. We will prove that under a certain assumption the UR singularities of the VFs of this theory are the same as the ones of the PA functions, up to overall normalizations.

Let M^2 and G stand for the "large" masses and for the couplings constants related to vertex functions with fields to those particles. Then the four-point vertex of the zero-mass-particle field still obeys the equations Fig. 1, provided

1) if the pole of the propagator of the zero-mass-particle comes out as $i(k^2+i\varepsilon)^{-1} f(M^{-2}U^2, V, G)$ we consider the function

$$\tilde{\Gamma} = f(M^{-2}U^2, V, G)^2 \Gamma . \qquad \text{(IV.10)}$$

2) All corrections relative to the parquet approximation functions are lumped into the i-kernel, which are the corrections to the free zero-mass propagator from self-energy effects by the massive as well as the massless particles, and the contributions from all intermediate states other than the two-massless-particle ones, giving terms to be added to the RHS of Fig. 1c. This has the effect that, keeping[20] (IV.1) for $\tilde{\Gamma}$, (IV.3) is replaced by

$$\{U^2[\partial/\partial U^2] + \beta(V, M^{-2}U^2, G)[\partial/\partial V]\}\, \tilde{\Gamma}(p_1 \dots p_4, ; U^2, V, M^2, G) = 0$$

where (IV.11)

$$\beta(V, M^{-2}U^2, G) = \tfrac{1}{2} i \sum_{j=1}^{3} p_j[\partial/\partial p_j]\, \tilde{\Gamma}(p_1 p_2 p_3(-p_1-p_2-p_3), ; U^2, V, M^2, G)|_{\text{s.pt.to} - U^2} . \qquad \text{(IV.12)}$$

The mentioned assumption now is

$$\lim_{U\to 0} \beta(V, M^{-2}U^2, G) = \beta_p(V) \qquad \text{(IV.13)}$$

which holds certainly in all orders of G, and which means that the momenta dependence near the normalization point is independent of the large masses and related coupling constants if the normalization point is chosen at sufficiently small momenta. Note that the assumption is formally analogous, with small and large momenta interchanged, to the one usually made [21] to extract nontrivial consequences from the

[20] Due to (IV.10) we may have to introduce a new V only implicitly defined in terms of the old one, but this does not alter the conclusions. Note also that the present BS equations may not be UV finite upon subtraction, due to two-massive-bosons intermediate states in B and the i-kernel. This can be remedied, e.g. by restricting the integrations to small momenta, and does not affect the UR singularity structure.

exact renormalization group equations of a massive theory. That assumption is proven to hold to all orders of renormalized perturbation theory by e.g. the mass-vertex-insertion technique (SD $1\frac{1}{2}$). It is likely that also for (IV.13) one can give a formal proof in a large class of renormalizable theories. The main difference between the two situations compared here is that in the UR case, the true behaviour is ultimately calculable, while in the UV case it is not (except in separate orders of perturbation theory, or upon solution of the Gell-Mann-Low limit problem [17]). – There is a subtle point, however, with the U-dependence from self-energy corrections to the zero-mass propagator, which will be discussed in detail in Section IV.3: to have (IV.13) hold one must calculate correctly rather than in unsuitable expansions in V. This precaution is unnecessary if (IV.13) is replaced by the weaker

$$\lim_{U\to 0}\lim_{V\to 0} V^{-2}\beta(V, M^{-2}U^2, G) = b_0 \tag{IV.14}$$

which actually suffices as far as the strongest UR singularity is concerned.

To solve (IV.11) one must consider

$$U^{-2}\,dU^2 = \beta(V, M^{-2}, G)^{-1}\,dV = (b_0 V^2 + \text{correction})^{-1}\,dV \tag{IV.15}$$

yielding

$$\ln U^2 = -b_0^{-1} V^{-1} + \text{const} + \text{correction} \tag{IV.16}$$

from which follows that as $\lambda \to 0$

$$\tilde{\Gamma}(\lambda p_1 \ldots \lambda p_4, ; U^2, V, M^2, G) = -i[b_0 \ln \lambda^{-2} + \text{correction}]^{-1} \tag{IV.17}$$

where the correction vanishes relative to the first term as $\lambda \to 0$. A sample calculation hereto is given in Section IV.3.

The other VFs with zero-mass-particle arguments alone are treated analogously; in particular, the UR singularities at exceptional momenta for the $\tilde{\Gamma}$ are the ones of the PA (upon appropriate overall normalization, see the discussion after (IV.9)). It is straightforward to analyze the UR singularities of VFs with (also) massive particles arguments, on the basis of their many-particle structure [22]. It follows that if in the present class of theories a mass term for the zero-mass particles is switched on, one obtains the same type of nonregular behaviour in the mass as in (III.9), the result for ϕ^4 theory.

Models to which these considerations apply are those where a (pseudo) scalar neutral[21] particle is forced to have zero mass by *ad hoc* renormalization condition: From renormalization theory we know that such particles are necessarily self-coupled as in ϕ^4 theory.

[21] The modifications for massless-particle multiplets are obvious. For electrically charged massless "particles" additional UR-effects set in making these "particles" presumably nondiscrete (cp. SD $1\frac{1}{2}$).

Attention is required, though, since (IV.11), (IV.1) also have the isolated solution $\tilde{\Gamma} \equiv 0$ corresponding to $V=0$. This solution is realized in the σ-model in the Goldstone mode [23]. There the irreducible vertex is a contact term plus a bare σ-propagator (plus other terms) and as known from the axial vector Ward identity the four-point vertex (IV.17) vanishes as $\lambda \to 0$ of order λ^2 rather than as slowly as the RHS. Then the small-momenta form of $\tilde{\Gamma}$ involves the mass parameter f_π already [24]. Also the UR behaviour of other VFs will now differ from the one found in Sections III.2 and IV.1, in particular, $\overset{0}{\Gamma}{}^0$ and, related to it by (III.12a), $\overset{0}{l}{}^0$ no longer vanish. Likewise, the effect of switching-on a mass term[22] will not be of the type (III.9), we have not worked it out, however.

IV.3. Corrections to Parquet Approximation

The reader's appreciation of the reasoning in the previous section may be helped by presenting here a comparison of the UR singularities in full ϕ^4 theory with the PA ones, and proving how a systematic correction of the latter brings refined agreement with the former.

In the Γ_0 theory of Section II.3 the propagator pole is $\hat{a}(V)^{-1} i(k^2+i\varepsilon)^{-1}$ and thus the nonexceptional small-momenta-behaviour of the adjusted four point vertex (cp. (IV.10)) is

$$\tilde{\Gamma}_0(\lambda p_1 \ldots \lambda p_4, ; m^2, V) = -i b_0^{-1} (\ln \lambda^{-2})^{-1} - i b_0^{-3} p_1 (\ln \lambda^{-2})^{-2} \ln \ln \lambda^{-2} + O((\ln \lambda)^{-2}) \tag{IV.18}$$

as $\lambda \to 0$. Deviation from the PA result, cp. (IV.6), occurs first in the term with

$$b_1 = b_{1p} + 4c_0 \mp b_{1p} \tag{IV.19}$$

as one easily sees from the defining formula of $\hat{b}(v)$ (Section II.3).

We now note that the equations Fig. 1 differ from the corresponding ones of ϕ^4 theory only by having a free rather than a self-energy-corrected propagator. Thus, according to the discussion in Section IV.2, the largest correction of the i-kernel to be considered is the one in Fig. 2. The correction to the propagator is, from (II.12),

$$\Delta G(p) = 2i(p^2+i\varepsilon)^{-1} b_0^{-2} c_0 (\ln[m^2(-p^2-i\varepsilon)^{-1}])^{-1} + \cdots . \tag{IV.20}$$

[22] Note that the mass-switch-on considered in Section III.1 differs from the usual one induced by a term in the Lagrangian linear in the σ-field [23]. However, in the Goldstone mode, both are equivalent with respect to the singularity in question. A discussion in the one-loop approximation was recently given by Guralnik, Tsao, and Wong [25].

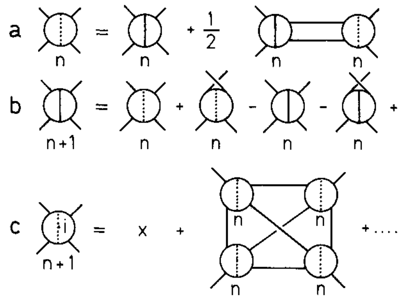

Fig. 1. Crossing symmetric Bethe-Salpeter equation

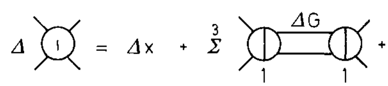

Fig. 2. Propagator correction to the irreducible kernel

In the spirit of the PA this is simplest obtained from its absorptive part

$$2\operatorname{Re}\Delta G(p) \approx 4\pi b_0^{-2} c_0 \theta(p^2)(p^2)^{-1}(\ln[m^2(p^2)^{-1}])^{-2}$$
$$\approx |i(p^2)^{-1}|^2 \, |-ib_0^{-1}\ln[m^2(-p^2)^{-1}]|^2 \, \{\tfrac{1}{6}2^{-8}\pi^{-3}\theta(p^2)p^2\}$$

the last braquet being the three-particle phase space factor. Inserting (IV.20) into the equation Fig. 2 and computing[23] the contribution to (IV.12) gives

$$\beta(V, m^{-2}U^2) = b_0 V^2 + b_{1p} V^3 + 4b_0^{-1} c_0 [\ln(U^{-2}m^2)]^{-1} V^2 + \cdots \qquad \text{(IV.21)}$$

which satisfies (IV.13) and (IV.14). Inserting (IV.21) into (IV.15) it suffices to treat the last term in (IV.21) as a perturbation, which is equivalent to inserting into the square bracket $b_0^{-1}V^{-1}$ due to (IV.16). This reduces (IV.21), in view of (II.26) and (IV.19), to (II.19a).

That (IV.21) satisfies (IV.13) is due to our using the estimate (IV.20) rather than an expansion of $\Delta G(p)$ in powers of V, which would in (IV.21) yield terms like $V^4 \ln(U^{-2}m^2)$ violating (IV.13) and (unless one performs an infinite summation) not leading to the correction sought, while (IV.14), which is satisfied, guarantees only the correctness of the strongest UR singularity.

In all of this Section IV we needed not specify whether or not the nontrivial skeleton terms on the RHS of Fig. 1c were considered also.

[23] The calculation to lowest order in inverse logarithm, the accuracy sufficient here, is trivial and therefore omitted.

What was said at the end of Section IV.1 holds also for the more general theories: the nonconstant terms are $O(V^4)$ and thus also add to $\beta(V, \ldots)$ only terms of this order, and therefore modify only singularities weaker than considered above. In a word, those terms on the RHS of Fig. 1c are UR-soft due to being "massless-particle dressed"; their only effect is via overall normalization factors such as in (IV.10) and those displayed after (IV.9).

V. Conclusions

The new results of this paper are: 1) The technique to obtain the AFs at Euclidean exceptional momenta from the directly-constructed zero-mass theory without any limiting processes needed (Section III.3), completing thereby the consistency argument for certain assumptions concerning asymptotic limits in Section IV.3 of SD 2. 2) A detailed study [26] of the UR singularity structure of zero-mass-theory VFs near exceptional momenta (Section III.2). 3) The proof that these UR singularities are present also in theories with massive particles besides the massless ones (Section IV.2). 4) Determination of the mass singularity of finite-mass theory VFs near vanishing mass (Section III.1). 5) Universality [14] of the coefficients b_0, b_1, c_0, h_0, and k_0 (Section II.3).

For the details of the application of results on AFs to the large-momenta-behaviour problem in perturbation theory (e.g., summing leading logarithms, next-to-leading ones etc.) and beyond perturbation theory (introducing certain apparently consistent assumptions as mentioned before) we refer the reader to SD 2 Section IV. Here we only remark that the possible occurrence of logarithms besides power laws, as exemplified in SD 2 Section IV.4, is a manifestation of not-fully-reducible representations of the dilatation group emphasized by Otterson and Zimmermann [27], and Dell'Antonio [28]. The degeneracy of dimension leading hereto occurs in the cited example if the operator $N_2(\phi^2)$ has canonical asymptotic dimension. (The consistency test of SD 2 Section IV.3 requires its dimension to lie between zero, or one for positivity reasons, and four.)

The extension of the considerations and results of Section III to other functions and larger sets of Euclidean exceptional momenta leads only to more complexity but not to essentially new problems. Thus we consider the problem of the large-Euclidean-momenta behaviour of VFs solved in principle as far as it can be solved on a formal level, i.e. without also solving the problem[24] of existence of Gell-Mann-Low eigenvalues and of computation of anomalous dimensions. We hope to discuss the problem of exceptional Minkowskian momenta in later papers.

[24] See Ref. [29] and references given therein.

Acknowledgment. The author is indebted to K. G. Wilson for suggesting to him that the infrared singularities in massless-particle theories be present also in theories with additional massive particles, and for pointing out to him the work of Larkin and Khmel'nitskiĭ.

Appendix A. Algebraic Deductions from the Bethe-Salpeter Equation

We here derive some formulae, using the formalism of SD 2 A, which are simpler than those used in SD 2 B for similar purposes. The matrix notation employed is

$$I = B + BGI = B + IGB \tag{A.1}$$

for the BS equation. Here I is the four point vertex (ϕ arguments only), B the BS kernel (which in the ϕ^4 model involves a logarithmically divergent constant so that for finiteness one should first consider a regularized theory and go to the limit only in the subtracted equations), and G stands for the pair of propagators. Momenta are indicated as

$$\boldsymbol{\Gamma}((-\tfrac{1}{2}q - p)(-\tfrac{1}{2}q + p)(\tfrac{1}{2}q + p')(\tfrac{1}{2}q - p'),) \Leftrightarrow {}_{p}\overset{q}{I}_{p'}$$

$$(2\pi)^4\,[\delta(p-p') + \delta(p+p')]\,G(-\tfrac{1}{2}q - p)\,G(-\tfrac{1}{2}q + p) \Leftrightarrow \overset{q}{G}$$

$$(2\pi)^4\,[\delta(p-p') + \delta(p+p')] \Leftrightarrow \mathbf{1}$$

etc., and are suppressed if internal (and then integrated over with a factor $\frac{1}{2}(2\pi)^{-4}$ supplied), or if external (left relative, right relative, and throughgoing or total) and kept general. In (A.1) it is understood that internal relative momenta fit unless a momentum associated with the factor to the left or right is indicated to be fixed (usually at zero) such that that factor, if the relative momentum is fixed, is constant in the internal momentum integration, amounting to let the arguments of the G-link coalesce in coordinate space, indicated by a bracket $\rangle$ or $\langle$.

From (A.1) we derived (SD 2 (A.9 a))

$$\mathbf{1} + GI = (\mathbf{1} + \overset{0}{G}\overset{0}{I}{}^{0})(\mathbf{1} + W) \tag{A.2a}$$

with

$$W \equiv -\mathbf{1} + [\mathbf{1} - \overset{0}{G}(\overset{0}{B} - \overset{0}{B}{}^{0})]^{-1}[\mathbf{1} + \overset{0}{G}(B - \overset{0}{B})(\mathbf{1} + GI) + (G - \overset{0}{G})I], \tag{A.2b}$$

inverses defined by expansion. As shown in SD 2 A,

$$GI = \overset{0}{G}\overset{0}{I}{}^{0} \langle (\mathbf{1} + W) + W \tag{A.3}$$

from (A.2a) is the simplest Wilson expansion [17, 5] formula.

SD 2 (A.5) can be written

$$I = \overset{0}{I}[1 + \overset{0}{G}(B - \overset{0}{B})(1 + GI) + (G - \overset{0}{G})I] + (B - \overset{0}{B})(1 + GI)\,. \quad \text{(A.4)}$$

Inserting here SD 2 (A.7)

$$\overset{0}{I} = (\overset{0}{I}{}^{0} + \overset{0}{B} - \overset{0}{B}{}^{0})\,[1 - \overset{0}{G}(\overset{0}{B} - \overset{0}{B}{}^{0})]^{-1} \quad \text{(A.5)}$$

yields

$$I = \overset{0}{I}{}^{0} \langle (1 + W) + (\overset{0}{B} - \overset{0}{B}{}^{0})(1 + W) + (B - \overset{0}{B})(1 + GI) \quad \text{(A.6)}$$

which is the Wilson expansion formula analogous to (A.3) for the amputated function. Multiplying (A.2) from the left with the bare vertex γ and writing

$$\Gamma((\tfrac{1}{2}q + p)(\tfrac{1}{2}q - p), (-q)) \Leftrightarrow \overset{q}{\Gamma}{}^{p}$$

yields

$$\Gamma = \overset{0}{\Gamma}{}^{0} \langle (1 + W) \quad \text{(A.7)}$$

showing the structure of Γ, given $\overset{0}{\Gamma}{}^{0} = 1$ in the massive theory.

From the subtracted BS equation for Γ

$$\Gamma - \overset{0}{\Gamma} = \Gamma G B - \overset{0}{\Gamma}\overset{0}{G}\overset{0}{B} = (\Gamma - \overset{0}{\Gamma})\overset{0}{G}\overset{0}{B} + \Gamma(G - \overset{0}{G})\overset{0}{B} + \Gamma G(B - \overset{0}{B})$$

follows, using (A.1)

$$\Gamma - \overset{0}{\Gamma} = \Gamma(G - \overset{0}{G})\overset{0}{I} + \Gamma G(B - \overset{0}{B})(1 + \overset{0}{G}\overset{0}{I})$$

and herefrom, using (A.6) at zero total momentum

$$\begin{aligned}\overset{0}{\Gamma} = {} & \Gamma\{1 - G(B - \overset{0}{B}) - [G - \overset{0}{G} + G(B - \overset{0}{B})\overset{0}{G}](1 + \overset{0}{W}{}^{T})(\overset{0}{B} - {}^{0}\overset{0}{B})\} \\ & - \Gamma[G - \overset{0}{G} + G(B - \overset{0}{B})\overset{0}{G}](1 + \overset{0}{W}{}^{T}) \rangle {}^{0}\overset{0}{I}\end{aligned}$$

which we abbreviate as

$$\overset{0}{\Gamma} = \Gamma(1 - R) - (\Pi - \overset{0}{\Pi})(\overset{0}{\Gamma}{}^{0})^{-1}\, {}^{0}\overset{0}{I} \quad \text{(A.8)}$$

in view of (A.7), the notation

$$\Gamma(\,, q(-q)) \Leftrightarrow \overset{q}{\Pi}$$

and

$$\begin{aligned}\Pi - \overset{0}{\Pi} &= \Gamma G \gamma^{T} - \overset{0}{\gamma}\overset{0}{G}\overset{0}{\Gamma}{}^{T} = \Gamma G \overset{0}{\gamma}{}^{T} - \gamma \overset{0}{G}\overset{0}{\Gamma}{}^{T} \\ &= \Gamma[G - \overset{0}{G} + G(B - \overset{0}{B})\overset{0}{G}]\overset{0}{\Gamma}{}^{T}\,.\end{aligned} \quad \text{(A.9)}$$

Inserting (A.8) with (A.7) into (A.6) taken at zero total momentum yields

$$\begin{aligned}\overset{0}{I} = & -\overset{0}{I}{}^{0}(\overset{0}{\Gamma}{}^{0})^{-1}(\Pi - \overset{0}{\Pi})(\overset{0}{\Gamma}{}^{0})^{-1}\,{}^{0}\overset{0}{I} \\ & + (\overset{0}{B} - \overset{0}{B}{}^{0})(1 + \overset{0}{W}) + \overset{0}{I}{}^{0}(\overset{0}{\Gamma}{}^{0})^{-1}\,\Gamma(1 - R)\,.\end{aligned} \tag{A.10}$$

Subtracting from (A.6) the same equation with zero throughgoing momentum and using (A.7) and (A.8) yields

$$\begin{aligned}I = & \overset{0}{I} - \overset{0}{I}{}^{0}(\overset{0}{\Gamma}{}^{0})^{-1}\overset{0}{\Pi}(\overset{0}{\Gamma}{}^{0})^{-1}\,{}^{0}\overset{0}{I} + \overset{0}{I}{}^{0}(\overset{0}{\Gamma}{}^{0})^{-1}\,\Pi(\overset{0}{\Gamma}{}^{0})^{-1}\,{}^{0}\overset{0}{I} \\ & + (\overset{0}{B} - \overset{0}{B}{}^{0})(W - \overset{0}{W}) + (B - \overset{0}{B})(1 + GI) + \overset{0}{I}{}^{0}(\overset{0}{\Gamma}{}^{0})^{-1}\,\Gamma R\,.\end{aligned} \tag{A.11}$$

Appendix B. Computational Scheme for BS Equation

We here briefly describe the algorithm by which one could verify the well-known result, used in Section IV.1, that the solution of the *BS* equation (A.1) is fixed knowing G and the subtracted B, and prescribing I at one point in momentum space, as ${}^{1}\overset{2}{I}{}^{3}$ say.

From (A.5) follows

$${}^{1}\overset{2}{I} = ({}^{1}\overset{2}{I}{}^{3} + {}^{1}\overset{2}{B} - {}^{1}\overset{2}{B}{}^{3})\,[1 - \overset{2}{G}(\overset{2}{B} - \overset{2}{B}{}^{3})]^{-1}\,. \tag{B.1}$$

Using this in the transposed form of (A.6) gives

$$\overset{2}{I} = [1 - (\overset{2}{B} - {}^{1}\overset{2}{B})\overset{2}{G}]^{-1}\,({}^{1}\overset{2}{I} + \overset{2}{B} - {}^{1}\overset{2}{B}) \tag{B.2}$$

and finally from (A.4)

$$I = [1 - (1 + \overset{2}{I}\overset{2}{G})(B - \overset{2}{B})G - \overset{2}{I}(G - \overset{2}{G})]^{-1}\,[\overset{2}{I} + (1 + \overset{2}{I}\overset{2}{G})(B - \overset{2}{B})]\,. \tag{B.3}$$

The I so constructed solves (A.1) if we set

$${}^{1}\overset{2}{B}{}^{3} = (1 + {}^{1}\overset{2}{I}\overset{2}{G} >)^{-1}\,[{}^{1}\overset{2}{I}{}^{3} - {}^{1}\overset{2}{I}\overset{2}{G}(\overset{2}{B}{}^{3} - {}^{1}\overset{2}{B}{}^{3})] \tag{B.4}$$

where numerator and denominator both diverge in perturbation theory, ${}^{1}\overset{2}{B}{}^{3}$ being there divergent in second and higher order. (B.3) is the unique solution of the initial problem if the inverses used in its construction are unique.

The expansion in powers of B of the I in (B.3) is the same as one obtains applying Bogoliubov-Parasiuk-Hepp (BPH) subtraction prescriptions [30] to the iteration solution $I = B + BGB + \cdots$ of (A.1), whereby B is (final-) subtracted to have the value ${}^{1}\overset{2}{I}{}^{3}$ at the corresponding

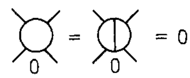

Fig. 3. Starting approximation for calculation Fig. 1

momenta, and reducible (sub)diagrams are subtracted to have the value zero at those momenta.

The computation of I_p, referred to in Section IV.1, could be organized as shown in Figs. 1 and 3, letters denoting the order of approximation, whereby the dotted line indicates orientation since the approximate I_p are not crossing symmetric. Each time the equation Fig. 1a is solved, the procedure described before could be applied. To generate the power series solution in V, of course only quadratures need be performed, with crossing symmetric result in each order. Again there is a correspondence, similar to the one described before, to BPH-subtracted vertex parts.

References

1. Symanzik, K.: Commun. math. Phys. **18**, 227 (70) (to be cited as SD 1), Springer Tracts in Modern Physics **57**, 222 (1971) (SD $1\frac{1}{2}$), Commun. math. Phys. **23**, 49 (1971) (SD 2)
2. Diatlov, I.T., Sudakov, V.V., Ter-Martirosian, K.A.: Zh. E.T.F. **32**, 767 (1967), transl. JETP **5**, 631 (1967)
3. Larkin, A.I., Khmel'nitskiĭ, D.E.: Zh. E.T.F. **56**, 2087 (1969), transl. JETP **29**, 1123 (1969)
4. Callan Jr., C.G.: Phys. Rev. **D2**, 1541 (1970)
5. Zimmermann, W.: in: Lectures on Elementary Particles and Quantum Field Theory, 1970 Brandeis Summer Institute, Eds. Deser, S., Grisaru, M., Pendleton, H.: Cambridge Mass.: MIT Press 1971. Ann. Phys. (N.Y.) **77**, 536, 570 (1973)
6. Coleman, S., Jackiw, R.: Ann. Phys. (N.Y.) **67**, 552 (1971), Appendix D
7. Christ, N., Hasslacher, B., Mueller, A.H.: Phys. Rev. **D6**, 3543 (1972)
8. Mason, A.L.: Phys. Rev. **D 7**, 3799 (1973)
9. Mitter, P.K.: Phys. Rev. **D 7**, 2927 (1973). See also: Schroer, B.: Lecture Notes in Physics, Vol. 17, Berlin: Springer 1973
10. Lowenstein, J.H.: Commun. math. Phys. **24**, 1 (1971)
11. Weinberg, S.: Phys. Rev. **118**, 8, 38 (1960)
12. Fink, J.P.: J. Math. Phys. **9**, 1389 (1968)
13. Westwater, M.J.: Fortschr. d. Phys. **17**, 1 (1969)
14. Symanzik, K.: Lettere al Nuovo Cimento **6**, 77 (1973)
15. Mack, G., Symanzik, K.: Commun. math. Phys. **27**, 247 (1972)
16. Blanchard, P., Sénéor, R.: Th. 1420, CERN, Nov. 1971
17. Gell-Mann, M., Low, F.E.: Phys. Rev. **95**, 1300 (1954)
18. Wilson, K.G.: Phys. Rev. **179**, 1499 (1969)
19. Speer, E., Westwater, M.J.: Ann. de l'Inst. H. Poincaré A **14**, 1 (1971)
20. Schwinger, J.: Proc. Nat. Acad. Sc. (US) **37**, 452, 455 (1951)
21. Bogoliubov, N.N., Shirkov, D.V.: Introduction to the Theory of Quantized Fields. New York: Interscience Publ., 1959

22. Symanzik, K.: J. Math. Phys. **1**, 249 (1960)
23. E.g., Symanzik, K.: Lettere al Nuovo Cimento **2**, 10 (1969)
24. E.g., Adler, S. L., Dashen, R. F.: Current Algebras. New York: Benjamin Inc., 1968
25. Guralnik, G. S., Tsao, H. S., Wong, T. F.: Phys. Rev. **D7**, 2467 (1973)
26. Cp. Symanzik, K.: in Renormalization of Yong-Mills Fields and Application to Particle Physics, C.N.R.S. Marseille, 72/p. 470
27. Otterson, P., Zimmermann, W.: Commun. math. Phys. **24**, 107 (1972)
28. Dell'Antonio, G. F.: Nuovo Cimento **12A**, 756 (1972)
29. Wilson, K. G.: Phys. Rev. **D 7**, 2911 (1973)
30. E.g. Hepp, K.: Théorie de la rénormalization. Berlin Heidelberg New York: Springer 1969

K. Symanzik
Deutsches Elektronensynchrotron (DESY)
D-2000 Hamburg 52
Notkestieg 1
Federal Republic of Germany