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Inheriting geodesic flows

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Abstract

We investigate the propagation equations for the expansion, vorticity and shear for perfect fluid spacetimes which are geodesic. It is assumed that spacetime admits a conformal Killing vector which is inheriting so that fluid flows lines are mapped conformally. We establish that the vorticity and the electric part of the Weyl tensor cannot coexist, i.e. they cannot be simultaneously nonzero. For a nonhomothetic vector field the propagation of the quantity $ln(R_{ab}u^a u^b)$ along the the integral curves of the symmetry vector is homogeneous.

Keywords: conformal motions, relativistic fluids, propagation equations

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1 Introduction

Many of the classical exact solutions in general relativity are spacetimes possessing a high degree of symmetry. In this paper we impose the condition that the spacetime manifold admits a conformal Killing vector. The advantage of this symmetry requirement are two-fold: it provides a deeper insight into the spacetime geometry, and it facilitates generation of exact solutions to the field equations, some of which may be new, e.g. the conformally invariant models of Herrera et al. [1], Maartens and Maharaj [2], Castejon Amenedo and Coley [3] and Maartens and Mellin [4]. Conformal Killing vectors generate constants of the motion along null geodesics for massless particles; this associates the conformal symmetry with a well-defined, physically meaningful conserved quantity.

The geometric and dynamic features, with perfect fluid and anisotropic matter tensors, for spacetimes with a conformal symmetry $\mathbf{X}$ have been extensively studied by Maartens et al. [5] and Coley and Tupper [6, 7]. An additional requirement imposed, by Coley and Tupper [6, 7], on the fluid 4-velocity $\mathbf{u}$ is that it must be inheriting so that fluid flow lines are mapped conformally onto fluid flow lines. These analyses indicate that perfect fluid spacetimes admitting inheriting conformal Killing vectors are rare. This has led Coley and Tupper to conjecture that the only perfect fluid spacetimes, with reasonable physical conditions, admitting inheriting conformal Killing vectors are: spacetimes conformal to flat spacetime, spacetimes with $\mathbf{X}$ parallel to $\mathbf{u}$, and spacetimes with $\mathbf{X}$ orthogonal to $\mathbf{u}$, apart from Robertson-Walker spacetimes and some other simple, symmetric exceptional cases. This suggests that there is room for further investigation of perfect fluids with an inheriting conformal symmetry.

Our specific objective in this study is to investigate the behaviour of the kinematic and dynamic variables of spacetimes that admit an inheriting conformal Killing vector. We assume that the matter distribution is that of a perfect fluid but we do not specify the spacetime geometry. The propagation equations are considered in particular for geodesic flows. In Section 2 we define an inheriting conformal symmetry. The Lie derivative of the kinematical and dynamical quantities are found. In Section 3 we take the Lie derivative of the expansion, shear and vorticity propagation equations. A number of results pertaining to the active gravitational mass, the shear and the vorticity are found. In particular we establish that the vorticity and the electric part of the Weyl tensor cannot coexist. Finally in Section 4 we discuss the significance of the results obtained, and briefly consider possibilities for future work.

2 Inheriting perfect fluids

For the fluid 4-velocity $\mathbf{u}$ we write

$$u_{a;b} = \sigma_{ab} + \tfrac{1}{3}\Theta h_{ab} + \omega_{ab} - \dot{u}_a u_b$$

where $\Theta = u^a{}_{;a}$ is the rate of expansion, $h_{ab} = g_{ab} + u_a u_b$ is the symmetric projection tensor ($h_{ab}u^b = 0$), $\sigma_{ab} = \frac{1}{2}(u_{a;c}h^c{}_b + u_{b;c}h^c{}_a) - \frac{1}{3}\Theta h_{ab}$ is the symmetric shear tensor ($\sigma_{ab}u^a = 0 = \sigma^a{}_a$), $\omega_{ab} = h^c{}_a h^d{}_b u_{[c;d]}$ is the skew-symmetric vorticity tensor ($\omega_{ab}u^b = 0$), and $\dot{u}_a = u_{a;b}u^b$ is the acceleration vector ($\dot{u}_a u^a = 0$). The overhead dot denotes covariant differentiation along a fluid particle worldline. Square brackets denote skew-symmetrisation. We can decompose the matter tensor in terms of $\mathbf{u}$, and consequently the Einstein field equations take the form

$$R_{ab} - \tfrac{1}{2}R g_{ab} = (\mu + p)u_a u_b + p g_{ab} \qquad (1)$$

for a perfect fluid with energy density μ and isotropic pressure p.

Manifolds with structure may admit groups of transformations which preserve this structure. A conformal motion preserves the metric up to a factor. A conformal Killing vector $\mathbf{X}$ is defined by

$$\mathcal{L}_{\mathbf{X}} g_{ab} = 2\psi g_{ab} \tag{2}$$

where $\psi = \psi(x^a)$ is the conformal factor. The existence of a conformal Killing vector $\mathbf{X}$ is subject to the integrability condition

$$\mathcal{L}_{\mathbf{X}} C_{abcd} = 0 \tag{3}$$

which indicates that the Weyl tensor $\mathbf{C}$ is conformally invariant. Equation (3) is identically satisfied for conformally flat spacetimes, e.g. Robertson–Walker spacetimes [8]. A vector $\mathbf{X}$ is said to be an inheriting conformal Killing vector if, in addition to (2), it satisfies

$$\mathcal{L}_{\mathbf{X}} u_a = \psi u_a \tag{4}$$

Hence inheriting conformal Killing vectors map fluid flow lines onto fluid flow lines. The physical significance of the assumption (4) has been extensively investigated by Maartens et al. [5] and Coley and Tupper [6, 7]. As a consequence of (2) and (4) we observe that

$$\mathcal{L}_{\mathbf{X}} h_{ab} = 2\psi h_{ab} \tag{5}$$

so that the inheriting vector $\mathbf{X}$ is a conformal motion of the projection tensor. If $\mathbf{u}$ is also orthogonal to $\mathbf{X}$ then the conformal vector is intrinsic to the projected hypersurfaces containing $\mathbf{h}$ as the metric tensor. The role of the intrinsic symmetries will be considered elsewhere.

If $\mathbf{X}$ is an inheriting conformal Killing vector then (2) and (4) hold, and we can establish the following relations

$$\mathcal{L}_{\mathbf{X}} \dot{u}_a \quad = \quad \psi_{,a} + u_a \dot{\psi} \tag{6}$$

$$\mathcal{L}_{\mathbf{X}}\Theta = -\psi\Theta + 3\dot{\psi} \tag{7}$$

$$\mathcal{L}_{\mathbf{X}}\sigma_{ab} = \psi\sigma_{ab} \tag{8}$$

$$\mathcal{L}_{\mathbf{X}}\omega_{ab} = \psi\omega_{ab} \tag{9}$$

The inheriting vector $\mathbf{X}$ is a conformal motion of the shear and the vorticity, but not of the acceleration and expansion in general. Equations (6)-(9) govern the evolution of the kinematical quantities for an inheriting conformal symmetry $\mathbf{X}$. We find the Lie derivative of the Ricci tensor

$$\mathcal{L}_{\mathbf{X}}R_{ab} = -2\psi_{;ab} - g_{ab}\Box\psi \tag{10}$$

where $\Box\psi = g^{ab}\psi_{;ab}$. Contracting the Lie derivative of the Ricci tensor in (10) gives the Lie derivative of the Ricci scalar

$$\mathcal{L}_{\mathbf{X}}R = -2\psi R - 6\Box\psi \tag{11}$$

Then we find that the the Lie derivative of the Einstein field equations (1) becomes

$$u_a u_b \mathcal{L}_{\mathbf{X}}\mu + h_{ab}\mathcal{L}_{\mathbf{X}}p + 2\psi(\mu u_a u_b + p h_{ab}) = 2\Box\psi g_{ab} - 2\psi_{;ab}$$

on utilising (10) and (11). Contracting this equation with $u^a u^b$, h^{ab}, $u^a h^b{}_c$, $h^{ac}h^{bd} - \frac{1}{3}h^{ab}h^{cd}$, yields the following set of equations:

$$\mathcal{L}_{\mathbf{X}}\mu = -2\psi\mu - 2\Box\psi - 2u^a u^b \psi_{;ab} \tag{12}$$

$$3\mathcal{L}_{\mathbf{X}}p = 4\Box\psi - 6\psi p - 2u^a u^b \psi_{;ab} \tag{13}$$

$$0 = 2u^a\psi_{;ac} + 2u^a u^b u_c \psi_{;ab} \tag{14}$$

$$0 = \psi_{;ab}(h^{ac}h^{bd} - \tfrac{1}{3}h^{ab}h^{cd}) \tag{15}$$

The inheriting vector $\mathbf{X}$ is not a conformal motion of the energy density and pressure in general. Equations (12)-(15) govern the evolution of the dynamical quantities for a geodesic inheriting symmetry $\mathbf{X}$. This system severely restricts the behaviour of the gravitational field $\mathbf{g}$. It may be noted that if $\mathbf{X}$ is homothetic, i.e. $\psi = \text{constant} \neq 0$, then the acceleration $\dot{u}^a$ is conserved along the integral curves, and the other kinematic and dynamic quantities turn inheriting. However observe that $\dot{u}^a$ is conserved even for $\psi = \psi(t)$.

3 Geodesic flows

We make the assumption that the acceleration vanishes so that fluid flow is geodesic. When $\dot{u}^a = 0$ we observe from (6) that two cases arise:
(a) $\psi = \text{constant} \neq 0$, (b) $\psi_{,a} \neq 0$.
When $\psi_{,a} \neq 0$, then

$$u_a = -\frac{\psi_{,a}}{\dot{\psi}} \tag{16}$$

and the 4-velocity $\mathbf{u}$ is specified completely by the conformal factor. We note that (16) is consistent with the unit, timelike requirement $u^a u_a = -1$.

We now analyse the geodesic propagation equations for the expansion Θ, the shear σ_{ab}, and the vorticity ω_{ab}, in an attempt to obtain general results governing the evolution of the relativistic perfect fluid. The expansion propagation equation is given by

$$\Theta_{,a}u^a = -\tfrac{1}{3}\Theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{ab}u^a u^b \tag{17}$$

This propagation equation is also called the Raychaudhuri equation. The shear propagation equation can be written as

$$\begin{aligned} \sigma_{ab,c}u^c &= -\tfrac{2}{3}\Theta\sigma_{ab} - \sigma_{ac}\sigma^c{}_b - \omega_{ac}\omega^c{}_b + C_{cbad}u^c u^d \\ &\quad +\tfrac{1}{3}h_{ab}\left(\sigma_{cd}\sigma^{cd} - \omega_{cd}\omega^{cd}\right) + \tfrac{1}{2}R^{cd}\left(h_{ac}h_{bd} - \tfrac{1}{3}h_{ab}h_{cd}\right) \end{aligned} \tag{18}$$

The vorticity propagation equation is given by

$$\omega_{ab,c}u^c = -\tfrac{2}{3}\Theta\omega_{ab} + \omega_{bc}\sigma^c{}_a - \omega_{ac}\sigma^c{}_b \tag{19}$$

In (17)-(19) we have followed the motivation and conventions of Wald [9].

3.1 Expansion propagation

We first consider the expansion propagation equation. Taking the Lie derivative of (17) along an inheriting conformal Killing vector $\mathbf{X}$, and using equations (4), (7)-(9) we obtain

$$\mathcal{L}_{\mathbf{X}} R_{ab} u^a u^b = -3\ddot{\psi} - \dot{\psi}\Theta + 2\psi\left(\dot{\Theta} + \frac{1}{3}\Theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab}\right) + \Box\psi$$

Then on substituting the Raychaudhuri equation (17) into the above equation we generate the result

$$\mathcal{L}_{\mathbf{X}} R_{ab} u^a u^b = -2\psi R_{ab} u^a u^b - 3\ddot{\psi} - \dot{\psi}\Theta + \Box\psi \qquad (20)$$

The quantity

$$R_{ab} u^a u^b = \frac{1}{2}(\mu + 3p)$$

is related to the active gravitational mass. Consequently (20) provides us with an indication of how the active gravitational mass changes along the integral curves of the inheriting conformal Killing vector $\mathbf{X}$.

We observe that

$$\begin{aligned} \mathcal{L}_{\mathbf{X}} R_{ab} u^a u^b &= -2\psi R_{ab} u^a u^b \\ \Leftrightarrow \quad 3\ddot{\psi} + \dot{\psi}\Theta - \Box\psi &= 0 \end{aligned}$$

which provides the condition for $\mathbf{X}$ to be a conformal motion of the active gravitational mass $R_{ab}u^a u^b$. For a homothetic vector the condition $3\ddot{\psi}+\dot{\psi}\Theta-\Box\psi = 0$ is identically satisfied. When $\psi_{,a} \neq 0$, vorticity vanishes, as will be shown in equation (23) below, and this condition provides an additional constraint on the expansion. Then the 4-velocity becomes comoving and hypersurface orthogonal leading to $\psi = \psi(t)$, and the conformal factor is explicitly determined by

$(\sqrt{{}^3g}\psi^{;0})_{,0} = 0$, and $u^a = \delta^a_0$. The line element can then be written in the orthogonal synchronous form

$$ds^2 = -dt^2 + H_{\alpha\beta}dx^\alpha dx^\beta$$

where the $H_{\alpha\beta}$ can depend on the spacetime coordinates x^a but $|H| = \det H_{\alpha\beta}$ is a function of t alone. Note that Coley and Tupper [6] have demonstrated that orthogonal synchronous perfect fluid spacetimes (other than Robertson-Walker) do not admit any proper ($\psi_{;ab} \neq 0$) inheriting conformal symmetry. Our argument shows that the propagation of $ln(\mu + 3p)$ along the intergal curves of $\mathbf{X}$ is homogeneous.

3.2 Shear propagation

We now consider the shear propagation equation. Taking the Lie derivative of (18) along an inheriting conformal Killing vector $\mathbf{X}$, and using (3)-(5), (7)-(10) we obtain after some simplification

$$3\dot{\psi}\sigma_{ab} = -2\psi C_{cbad}u^c u^d - \tfrac{1}{2}\left(2\psi^{;cd} + g^{cd}\Box\psi\right)\left[h_{ac}h_{bd} - \tfrac{1}{3}h_{ab}h_{cd}\right]$$

Substituting the dynamical equation (15) into the above we obtain

$$3\dot{\psi}\sigma_{ab} = -2\psi C_{cbad}u^c u^d$$

for perfect fluids. We note from Wald [9] that

$$E_{ab} = C_{acbd}u^c u^d$$

is the electric part of the Weyl tensor. Hence we can write

$$3\dot{\psi}\sigma_{ab} = 2\psi E_{ab} \qquad (21)$$

for geodesic flows.

For a homothetic vector we must have $E_{ab} = 0$; however the magnetic part of the Weyl tensor may not be zero. When $\psi_{,a} \neq 0$, we obtain from (21) the expression

$$\sigma_{ab} = \frac{2\psi}{3\dot{\psi}} E_{ab} \qquad (22)$$

for the shear as measured by an observer with 4-velocity (16). Thus the shear propagation equation for a geodesic inheriting symmetry relates shear with the electric part of the Weyl tensor by (22).

3.3 Vorticity propagation

Finally we consider the vorticity propagation equation. Taking the Lie derivative of (19) along an inheriting conformal Killing vector $\mathbf{X}$, and using (4), (7)-(9) we generate the result

$$\dot{\psi}\omega_{ab} = 0 \qquad (23)$$

for perfect fluid spacetimes. Equation (23) is identically satisfied for homothetic vectors leaving ω_{ab} free. When $\psi_a \neq 0$, we have

$$\omega_{ab} = 0$$

so that the vorticity vanishes as measured by an observer with 4-velocity (16). Thus the vorticity propagation equation is highly restrictive for a geodesic inheriting symmetry.

We can summarise our results in terms of the following theorem:

Theorem I: For inheriting geodesic flows the propagation equations imply:

(i) For homothetic vectors

$$E_{ab} = 0$$

with shear and vorticity being free.

(ii) For nonhomothetic vectors

$$\omega_{ab} = 0,\ E_{ab} = \frac{3\dot{\psi}}{2\psi}\sigma_{ab},\ u_a = -\frac{\psi_{,a}}{\dot{\psi}}$$

and thus implying that both E_{ab} and ω_{ab} cannot be both nonzero simultaneously.

This is an important result bringing out a kind of complementary relation between E_{ab} and ω_{ab}.

4 Discussion

We have studied the propagation equations for geodesic flows in perfect fluid spacetimes. The restrictions that the inheriting conformal symmetry $\mathbf{X}$ places on spacetime manifold are given in *Theorem I*, which essentially states that the electric part of the Weyl tensor and the vorticity cannot coexist. We observe that the electric part of the Weyl tensor is constrained, however the magnetic part is not directly affected by the inheriting conformal symmetry. We have demonstrated that general results may be found, without specifying the spacetime geometry, by utilising the propagation equations. In future work we intend to generalise these results for nongeodesic flows. This is a nontrivial task as the propagation equations are not as easy to cope with when $\dot{u}^a \neq 0$. Also, the matter tensor could be generalised to include anisotropic terms. The analyses of Maartens et al. [5] and Coley and Tupper [6, 7, 10, 11] indicate that a wider range of possibilities are permitted if the condition of a perfect fluid is relaxed.

Finally we briefly comment on the condition $E_{ab} = 0$ for pure magnetic fields. Our results indicate that solutions with $E_{ab} = 0$ are possible in perfect fluid spacetimes with a homothetic vector. In this context we observe that recently Lozanovski and Aarons [12] found a purely magnetic solution for a perfect fluid which satisfies the weak, strong and dominant energy conditions. In this class of spacetimes, the expansion and the shear are nonvanishing but the fluid is irrotational and nonaccelerating. In addition there have been various attempts to study the dynamics of purely magnetic spacetimes in the case of dust [13, 14, 15]. It has been shown that such "anti-Newtonian universes" are subject to severe integrability conditions, and it is conjectured that no physically acceptable solution exists. Mars [16] has comprehensively studied the existence of magnetic solutions in Petrov type I vacuum spacetimes, and established a uniqueness result.

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