Research Article

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Inhomogeneous conformable abstract Cauchy problem

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Abstract: In this paper, we discuss the solution of the inhomogeneous conformable abstract Cauchy problem. The homogeneous problem is also studied. The analysis of conformable fractional calculus and fractional semigroups is also given. Existence, uniqueness and regularity of a mild solution for the conformable abstract Cauchy problem are studied. Applications illustrating our main abstract results are also given.

Keywords: α-semigroup of operators, conformable derivative, inhomogeneous Cauchy problem

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1 Introduction

It is well known that fractional differential equations play an important role in describing many phenomena and processes in various fields of science such as physics, chemistry, control systems, electrodynamics and aerodynamics. For examples and details, the reader can see [1–6].

In 2014, Khalil et al. introduced, in [7], a new well-behaved simple fractional derivative, namely "the conformable fractional derivative." The definition goes as follows:

Given a function $f : [0, \infty) \longrightarrow \mathbb{R}$. Then for all $t > 0$, $\alpha \in (0, 1]$, let

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.$$

T_α is called the conformable fractional derivative of f of order α. Let $f^{(\alpha)}(t)$ stands for $T_\alpha(f)(t)$.

If f is α-differentiable in some $(0, b)$, $b > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t).$$

The conformable derivative satisfies the following properties:

1. $T_\alpha(1) = 0$,
2. $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$,
3. $T_\alpha(\sin at) = at^{1-\alpha} \cos at,\ a \in \mathbb{R}$,
4. $T_\alpha(\cos at) = -at^{1-\alpha} \sin at,\ a \in \mathbb{R}$,
5. $T_\alpha(e^{at}) = at^{1-\alpha} e^{at},\ a \in \mathbb{R}$.

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Furthermore, many functions behave as in the usual derivative. Here are some formulas:

$$T_\alpha\Big(\frac{1}{\alpha}t^\alpha\Big) = 1,$$
$$T_\alpha\Big(e^{\frac{1}{\alpha}t^\alpha}\Big) = e^{\frac{1}{\alpha}t^\alpha},$$
$$T_\alpha\Big(\sin\frac{1}{\alpha}t^\alpha\Big) = \cos\Big(\frac{1}{\alpha}t^\alpha\Big),$$
$$T_\alpha\Big(\cos\frac{1}{\alpha}t^\alpha\Big) = -\sin\Big(\frac{1}{\alpha}t^\alpha\Big).$$

For more properties on the conformable derivative see [1,3,7–16]. Furthermore, in [17–21] the authors proved the existence and uniqueness of solutions of initial value problems and boundary value problems for conformable fractional differential equations. Analogously, if X is a Banach space, we write:

Definition 1.1. Let $u : [0, \infty) \longrightarrow X$ be an X valued function. The conformable derivative of u of order $\alpha \in (0, 1]$, at $t > 0$ is defined by

$$\frac{\mathrm{d}^\alpha u(t)}{\mathrm{d}t^\alpha} = \lim_{\varepsilon\to 0} \frac{u(t + \varepsilon t^{1-\alpha}) - u(t)}{\varepsilon}.$$

When the limit exists, we say that u is α-differentiable at t.

If u is α-differentiable in some $(0, a]$, $a > 0$ and $\lim_{t\to 0^+} u^{(\alpha)}(t)$ exists in X, then we define $u^{(\alpha)}(0) = \lim_{t\to 0^+} u^{(\alpha)}(t)$.

The α-fractional integral of a function u is given by

$$I^a_\alpha(u)(t) = \int_a^t \frac{1}{s^{1-\alpha}} u(s)\mathrm{d}s.$$

Theorem 1.1. *If a function $u : [0, \infty) \longrightarrow X$ is α-differentiable at $t > 0$, $\alpha \in (0, 1]$, then u is continuous at t. If, in addition, u is differentiable, then $\frac{\mathrm{d}^\alpha u(t)}{\mathrm{d}t^\alpha} = t^{1-\alpha}\frac{\mathrm{d}u(t)}{\mathrm{d}t}$.*

Lemma 1.1. *Let $u : [0, \infty) \longrightarrow X$ be differentiable and $\alpha \in (0, 1]$. Then, for all $t > 0$, we have*

$$I^0_\alpha\Big(\frac{\mathrm{d}^\alpha u}{\mathrm{d}t^\alpha}\Big)(t) = u(t) - u(0).$$

Therefore, if u is continuous, then

$$\frac{\mathrm{d}^\alpha I^0_\alpha(u)(t)}{\mathrm{d}t^\alpha} = u(t).$$

Based on the conformable fractional derivative, Abdeljawad et al. in [22] introduced the so-called C_0-α-semigroup $(T(t))_{t\geq 0}$, which is a generalization of the classical strongly continuous semigroup. More precisely:

Definition 1.2. Let $\alpha \in (0, a]$ for any $a > 0$. For a Banach space X, a family $\{T(t)\}_{t\geq 0} \subseteq \mathcal{L}(X, X)$ is called a fractional α-semigroup (or α-semigroup) of operators if

(i) $T(0) = I$,

(ii) $T(t + s)^{\frac{1}{\alpha}} = T(t^{\frac{1}{\alpha}})T(s^{\frac{1}{\alpha}})$ for all $t, s \in [0, \infty)$.

Clearly, if $\alpha = 1$, then 1-semigroups are just the usual semigroups.

Definition 1.3. An α-semigroup $T(t)$ is called a C_0-semigroup, if for each $x \in X$, $T(t)x \to x$ as $t \to 0^+$.

The conformable α-derivative of $T(t)$ at $t = 0$ is called the α-infinitesimal generator of the fractional α-semigroup $T(t)$, with domain equals:

$$\{x \in X, \lim_{t\to 0^+} T^{(\alpha)}(t)x \ \text{exists}\}.$$

We will write A for such a generator.

Example 1.2. (see [22])

(i) For a bounded linear operator A, define $T(t) = e^{2\sqrt{t}A}$, then $(T(t))_{t\geq 0}$ is $\frac{1}{2}$-semigroup.

(ii) Let $X = C([0, \infty) : \mathbb{R})$ be the Banach space of bounded uniformly continuous functions on $[0, \infty)$ with supremum norm. For $f \in X$ we define $(T(t)f)(s) = f(s + \sqrt{t})$. It is easy to check that $T(t)$ is a C_0-$\frac{1}{2}$-semigroup of operators.

In this paper, we will study the following fractional Cauchy problem:

$$\begin{cases} \dfrac{d^\alpha u(t)}{dt^\alpha} = Au(t) + f(t), \quad t > 0, \\ u(0) = x, \end{cases} \tag{1}$$

where $\frac{d^\alpha u(t)}{dt^\alpha}$ is the conformable fractional derivative of order $\alpha \in (0, 1]$, $(A, D(A))$ is a linear operator which generates a fractional C_0-α-semigroup $(T(t))_{t\geq 0}$ on a Banach space $(X, \|.\|)$ and $x \in X$, $f : [0, S) \longrightarrow X$. We denote by $C([0, S]; X)$, the Banach space of continuous functions from $[0, S]$ into X with the supremum norm $\|u\|_\infty = \sup_{t\in[0,T]}\|u(t)\|$. Moreover, the solution of the homogeneous part was also discussed. In fact, the homogeneous problem was studied in [22], but in this paper we discuss the solution under weaker conditions. For this purpose, we use the semigroup technique. We should mention here that the solution of such a problem was studied using tensor product technique, see [23].

This paper is organized as follows. In Section 2, we introduce an important analysis of conformable fractional calculus and fractional semigroups including their important properties. In Section 3, we study the homogeneous part of problem (1). In Section 4, we prove the existence, uniqueness and regularity of a mild solution for the initial value problem (1). In Section 5, we introduce some applications illustrating our main abstract results.

2 Analysis of the fractional semigroups

In this section, we introduce a new analysis of fractional semigroups, which is very important for solving abstract fractional Cauchy problems.

Theorem 2.1. *Let $T(t)$ be a C_0-α-semigroup where $\alpha \in (0, 1]$. There exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t^\alpha} \quad \text{for } 0 \leq t \leq \infty.$$

Proof. Clearly, $\|T(t)\|$ is bounded on $[0, \eta]$ for some $\eta > 0$. If this is false, then there is a sequence (t_n) satisfying $t_n \geq 0$, $\lim_{n\to\infty} t_n = 0$ and $\|T(t_n)\| > n$. From the uniform boundedness theorem, it then follows that for some $x \in X$, $\|T(t)x\|$ is unbounded contrary to the definition of C_0-α-semigroup. Thus, $\|T(t)\| \leq M$ for $0 \leq t \leq \eta$. Since $\|T(0)\| = 1$, then $M \geq 1$. Given $t \geq 0$, we can write t^α as $t^\alpha = n\eta^\alpha + \delta^\alpha$, where $0 \leq \delta < \eta$. By the α-semigroup property, we have

$$\begin{aligned} \|T(t)\| &= \|T(n\eta^\alpha + \delta^\alpha)^{\frac{1}{\alpha}}\| = \|T(n\eta^\alpha)^{\frac{1}{\alpha}} T(\delta^\alpha)^{\frac{1}{\alpha}}\| = \|T^n(\eta^\alpha)^{\frac{1}{\alpha}} T(\delta)\| \\ &\leq \|T(\eta)\|^n \|T(\delta)\| \leq MM^n \leq MM^{\frac{t^\alpha}{\eta^\alpha}} = Me^{t^\alpha(\eta^{-\alpha}\log M)} = Me^{\omega t^\alpha}, \end{aligned}$$

where $\omega = \eta^{-\alpha}\log M$. □

Remark 2.1. If $\{T(t)\}_{t\geq 0} \subseteq \mathcal{L}(X, X)$ is a C_0-α-semigroup, then $h(t) = T(t^{\frac{1}{\alpha}})$ is a C_0-semigroup.

Corollary 2.1. *If $T(t)$ is a C_0-α-semigroup, then for every $x \in X$, $t \to T(t)x$ is a continuous function from $\mathbb{R}_0^+$ (the nonnegative real line) into X.*

Proof. Let $t, h > 0$. Since

$$T(t+h) = T((t+h)^\alpha - t^\alpha + t^\alpha)^{\frac{1}{\alpha}} = T((t+h)^\alpha - t^\alpha)^{\frac{1}{\alpha}} T(t).$$

The right continuity at t follows from

$$\|T(t+h)x - T(t)x\| \le \|T(t)\| \left\| T((t+h)^\alpha - t^\alpha)^{\frac{1}{\alpha}} x - x \right\| \le M e^{\omega t^\alpha} \left\| T((t+h)^\alpha - t^\alpha)^{\frac{1}{\alpha}} x - x \right\|.$$

Since $\lim_{t\to 0^+} T(t)x = x$ and $\lim_{h\to 0^+}((t+h)^\alpha - t^\alpha)^{\frac{1}{\alpha}} = 0$, then

$$\lim_{h\to 0^+} \left\| T((t+h)^\alpha - t^\alpha)^{\frac{1}{\alpha}} x - x \right\| = 0.$$

Similarly, we can prove the left continuity. □

Theorem 2.2. *Let $T(t)$ be a C_0-α-semigroup where $\alpha \in (0,1]$ and let A be its α-infinitesimal generator. Then*
(*a*) *For $x \in X$*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon t^{1-\alpha}} \frac{1}{s^{1-\alpha}} T(s)x\,\mathrm{d}s = T(t)x \quad \text{for every } t > 0.$$

(*b*) *For $x \in X$, $\int_0^t \frac{1}{s^{1-\alpha}} T(s)x\,\mathrm{d}s \in D(A)$ and*

$$A\left(\int_0^t \frac{1}{s^{1-\alpha}} T(s)x\,\mathrm{d}s\right) = T(t)x - x.$$

(*c*) *For $x \in D(A)$, $T(t)x \in D(A)$ and*

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} T(t)x = AT(t)x = T(t)Ax. \tag{2}$$

(*d*) *For $x \in D(A)$*

$$T(t)x - T(s)x = \int_s^t \frac{1}{u^{1-\alpha}} T(u)Ax\,\mathrm{d}u = \int_s^t \frac{1}{u^{1-\alpha}} AT(u)x\,\mathrm{d}u.$$

Proof. (a): Let $x \in X$. Then we have

$$\begin{aligned}
&\left\| \frac{1}{\varepsilon} \int_t^{t+\varepsilon t^{1-\alpha}} \frac{1}{s^{1-\alpha}} T(s)x\,\mathrm{d}s - T(t)x \right\| \\
&= \left\| \frac{1}{\varepsilon} \int_t^{t+\varepsilon t^{1-\alpha}} \frac{1}{s^{1-\alpha}} T(s)x\,\mathrm{d}s - \frac{1}{\varepsilon} \int_t^{t+\varepsilon t^{1-\alpha}} \frac{1}{s^{1-\alpha}} T(t)x\,\mathrm{d}s + \frac{1}{\varepsilon} \int_t^{t+\varepsilon t^{1-\alpha}} \frac{1}{s^{1-\alpha}} T(t)x\,\mathrm{d}s - T(t)x \right\| \qquad (3) \\
&\le \left\| \frac{1}{\varepsilon} \int_t^{t+\varepsilon t^{1-\alpha}} \frac{1}{s^{1-\alpha}} T(s)x\,\mathrm{d}s - \frac{1}{\varepsilon} \int_t^{t+\varepsilon t^{1-\alpha}} \frac{1}{s^{1-\alpha}} T(t)x\,\mathrm{d}s \right\| + \left\| \frac{1}{\varepsilon} \int_t^{t+\varepsilon t^{1-\alpha}} \frac{1}{s^{1-\alpha}} T(t)x\,\mathrm{d}s - T(t)x \right\|.
\end{aligned}$$

Since

$$\frac{1}{\varepsilon} \int_t^{t+\varepsilon t^{1-\alpha}} \frac{1}{s^{1-\alpha}} T(t)x\,\mathrm{d}s = \frac{1}{\alpha} \frac{(t+\varepsilon t^{1-\alpha})^\alpha - t^\alpha}{\varepsilon} T(t)x,$$

then

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon t^{1-\alpha}}\frac{1}{s^{1-\alpha}}T(t)x\mathrm{d}s=\frac{\mathrm{d}^\alpha}{\mathrm{d}t}\left(\frac{t^\alpha}{\alpha}\right)T(t)x=T(t)x.$$

Therefore,

$$\lim_{\varepsilon\to 0}\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon t^{1-\alpha}}\frac{1}{s^{1-\alpha}}T(t)x\mathrm{d}s-T(t)x\right\|=0. \tag{4}$$

For the first term in the right side of inequality (3) we have

$$\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon t^{1-\alpha}}\frac{1}{s^{1-\alpha}}T(s)x\mathrm{d}s-\frac{1}{\varepsilon}\int_t^{t+\varepsilon t^{1-\alpha}}\frac{1}{s^{1-\alpha}}T(t)x\mathrm{d}s\right\|\le\frac{1}{\varepsilon}\int_t^{t+\varepsilon t^{1-\alpha}}\frac{1}{s^{1-\alpha}}\|T(s)x-T(t)x\|\mathrm{d}s.$$

Since $s\to T(s)x$ is continuous for $s\ge 0$, then $s\to\frac{1}{s^{1-\alpha}}\|T(s)x-T(t)x\|$ is continuous for $t\le s\le t+\varepsilon t^{1-\alpha}$ and therefore

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon t^{1-\alpha}}\frac{1}{s^{1-\alpha}}\|T(s)x-T(t)x\|\mathrm{d}s=\frac{t^{1-\alpha}}{t^{1-\alpha}}\|T(t)x-T(t)x\|=0.$$

Consequently,

$$\lim_{\varepsilon\to 0}\left\|\frac{1}{\varepsilon}\int_t^{t+\varepsilon t^{1-\alpha}}\frac{1}{s^{1-\alpha}}T(s)x\mathrm{d}s-\frac{1}{\varepsilon}\int_t^{t+\varepsilon t^{1-\alpha}}\frac{1}{s^{1-\alpha}}T(t)x\mathrm{d}s\right\|=0. \tag{5}$$

Finally, from (4) and (5) we prove part (a).

For part (b), let $x\in X$. Then we have

$$\frac{T(\tau+\varepsilon\tau^{1-\alpha})-T(\tau)}{\varepsilon}\left(\int_0^t\frac{1}{s^{1-\alpha}}T(s)x\mathrm{d}s\right)=\frac{1}{\varepsilon}\int_0^t\frac{1}{s^{1-\alpha}}T(\tau+\varepsilon\tau^{1-\alpha})T(s)x\mathrm{d}s-\frac{1}{\varepsilon}\int_0^t\frac{1}{s^{1-\alpha}}T(\tau)T(s)x\mathrm{d}s. \tag{6}$$

Now, using the α-semigroup property yields

$$T(\tau+\varepsilon\tau^{1-\alpha})T(s)=T((\tau+\varepsilon\tau^{1-\alpha})^\alpha)^{\frac{1}{\alpha}}T((s)^\alpha)^{\frac{1}{\alpha}}=T((\tau+\varepsilon\tau^{1-\alpha})^\alpha+s^\alpha)^{\frac{1}{\alpha}}.$$

Similarly,

$$T(\tau)T(s)=T(\tau^\alpha+s^\alpha)^{\frac{1}{\alpha}}.$$

Then equation (6) becomes

$$\frac{T(\tau+\varepsilon\tau^{1-\alpha})-T(\tau)}{\varepsilon}\left(\int_0^t\frac{1}{s^{1-\alpha}}T(s)x\mathrm{d}s\right)=\frac{1}{\varepsilon}\int_0^t\frac{1}{s^{1-\alpha}}T((\tau+\varepsilon\tau^{1-\alpha})^\alpha+s^\alpha)^{\frac{1}{\alpha}}x\mathrm{d}s-\frac{1}{\varepsilon}\int_0^t\frac{1}{s^{1-\alpha}}T(\tau^\alpha+s^\alpha)^{\frac{1}{\alpha}}x\mathrm{d}s. \tag{7}$$

Now, using the change of variables $u=((\tau+\varepsilon\tau^{1-\alpha})^\alpha+s^\alpha)^{\frac{1}{\alpha}}$ on the first term on the right side of (7), we get

$$\frac{T(\tau+\varepsilon\tau^{1-\alpha})-T(\tau)}{\varepsilon}\left(\int_0^t\frac{1}{s^{1-\alpha}}T(s)x\mathrm{d}s\right)=\frac{1}{\varepsilon}\int_{\tau+\varepsilon\tau^{1-\alpha}}^{((\tau+\varepsilon\tau^{1-\alpha})^\alpha+t^\alpha)^{\frac{1}{\alpha}}}\frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u-\frac{1}{\varepsilon}\int_0^t\frac{1}{s^{1-\alpha}}T(\tau^\alpha+s^\alpha)^{\frac{1}{\alpha}}x\mathrm{d}s.$$

Similarly, by change of variables $u = (\tau^\alpha + s^\alpha)^{\frac{1}{\alpha}}$ on the second term on the right side of (7) yields

$$\frac{T(\tau + \varepsilon\tau^{1-\alpha}) - T(\tau)}{\varepsilon}\left(\int_0^t \frac{1}{s^{1-\alpha}}T(s)x\mathrm{d}s\right) = \frac{1}{\varepsilon}\int_{\tau+\varepsilon\tau^{1-\alpha}}^{((\tau+\varepsilon\tau^{1-\alpha})^\alpha+t^\alpha)^{\frac{1}{\alpha}}} \frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u - \frac{1}{\varepsilon}\int_\tau^{(\tau^\alpha+t^\alpha)^{\frac{1}{\alpha}}} \frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u. \tag{8}$$

Since

$$\int_{\tau+\varepsilon\tau^{1-\alpha}}^{((\tau+\varepsilon\tau^{1-\alpha})^\alpha+t^\alpha)^{\frac{1}{\alpha}}} \frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u = \int_{(\tau^\alpha+t^\alpha)^{\frac{1}{\alpha}}}^{((\tau+\varepsilon\tau^{1-\alpha})^\alpha+t^\alpha)^{\frac{1}{\alpha}}} \frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u + \int_{\tau+\varepsilon\tau^{1-\alpha}}^{(\tau^\alpha+t^\alpha)^{\frac{1}{\alpha}}} \frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u,$$

and

$$\int_{\tau+\varepsilon\tau^{1-\alpha}}^{(\tau^\alpha+t^\alpha)^{\frac{1}{\alpha}}} \frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u - \int_{\tau}^{(\tau^\alpha+t^\alpha)^{\frac{1}{\alpha}}} \frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u = -\int_{\tau}^{\tau+\varepsilon\tau^{1-\alpha}} \frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u,$$

then equation (8) becomes

$$\frac{T(\tau + \varepsilon\tau^{1-\alpha}) - T(\tau)}{\varepsilon}\left(\int_0^t \frac{1}{u^{1-\alpha}}T(s)x\mathrm{d}s\right) = \frac{1}{\varepsilon}\int_{(\tau^\alpha+t^\alpha)^{\frac{1}{\alpha}}}^{((\tau+\varepsilon\tau^{1-\alpha})^\alpha+t^\alpha)^{\frac{1}{\alpha}}} \frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u - \frac{1}{\varepsilon}\int_\tau^{\tau+\varepsilon\tau^{1-\alpha}} \frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u. \tag{9}$$

Now, if we take $g(t) = \int_0^t \frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u$, then

$$\frac{g(t + \varepsilon t^{1-\alpha}) - g(t)}{\varepsilon} = \frac{1}{\varepsilon}\left\{\int_0^{t+\varepsilon t^{1-\alpha}} \frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u - \int_0^t \frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u\right\} = \frac{1}{\varepsilon}\int_t^{t+\varepsilon t^{1-\alpha}} \frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u.$$

By part (a), $g(t)$ is α-differentiable and

$$g^{(\alpha)}(t) = \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}g(t) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon t^{1-\alpha}} \frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u = T(t)x.$$

Similarly,

$$\frac{g(((\tau + \varepsilon\tau^{1-\alpha})^\alpha + t^\alpha)^{\frac{1}{\alpha}}) - g((\tau^\alpha + t^\alpha)^{\frac{1}{\alpha}})}{\varepsilon} = \frac{1}{\varepsilon}\int_{(\tau^\alpha+t^\alpha)^{\frac{1}{\alpha}}}^{((\tau+\varepsilon\tau^{1-\alpha})^\alpha+t^\alpha)^{\frac{1}{\alpha}}} \frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u.$$

Now, since $\tau \to g((\tau^\alpha + t^\alpha)^{\frac{1}{\alpha}})$ is α-differentiable, then

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{(\tau^\alpha+t^\alpha)^{\frac{1}{\alpha}}}^{((\tau+\varepsilon\tau^{1-\alpha})^\alpha+t^\alpha)^{\frac{1}{\alpha}}} \frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u = \frac{\mathrm{d}^\alpha}{\mathrm{d}\tau^\alpha}g((\tau^\alpha + t^\alpha)^{\frac{1}{\alpha}}).$$

Using the fact that $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}f(t) = t^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}f(t) = t^{1-\alpha}f'(t)$, and the chain rule we get

$$\begin{aligned}
\frac{\mathrm{d}^\alpha}{\mathrm{d}\tau^\alpha}g\left((\tau^\alpha + t^\alpha)^{\frac{1}{\alpha}}\right) &= \tau^{1-\alpha}\alpha\tau^{\alpha-1}\frac{1}{\alpha}(\tau^\alpha + t^\alpha)^{\frac{1}{\alpha}-1}g'\left((\tau^\alpha + t^\alpha)^{\frac{1}{\alpha}}\right) \\
&= \left((\tau^\alpha + t^\alpha)^{\frac{1}{\alpha}}\right)^{1-\alpha}g'\left((\tau^\alpha + t^\alpha)^{\frac{1}{\alpha}}\right) \\
&= g^{(\alpha)}\left((\tau^\alpha + t^\alpha)^{\frac{1}{\alpha}}\right) \\
&= T(\tau^\alpha + t^\alpha)^{\frac{1}{\alpha}}x.
\end{aligned}$$

This implies that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\limits_{(\tau^\alpha+t^\alpha)^{\frac{1}{\alpha}}}^{((\tau+\varepsilon\tau^{1-\alpha})^\alpha+t^\alpha)^{\frac{1}{\alpha}}}\frac{1}{u^{1-\alpha}}T(u)x\mathrm{d}u = T(\tau^\alpha+t^\alpha)^{\frac{1}{\alpha}}x. \tag{10}$$

From (9) and (10) and part (a), we get

$$\lim_{\varepsilon\to 0}\frac{T(\tau+\varepsilon\tau^{1-\alpha})-T(\tau)}{\varepsilon}\left(\int\limits_0^t\frac{1}{u^{1-\alpha}}T(s)x\mathrm{d}s\right) = T^{(\alpha)}(\tau)\left(\int\limits_0^t\frac{1}{u^{1-\alpha}}T(s)x\mathrm{d}s\right) = T(\tau^\alpha+t^\alpha)^{\frac{1}{\alpha}}x - T(\tau)x = T(\tau)(T(t)x-x).$$

Finally, letting $\tau\to 0^+$ yields

$$A\left(\int\limits_0^t\frac{1}{s^{1-\alpha}}T(s)x\mathrm{d}s\right) = T(t)x - x.$$

Part (c) is proved in [22]. Part (d) is obtained by applying I_α^0 to (2), where $I_\alpha^0(v)(t) = \int_s^t \frac{1}{u^{1-\alpha}}v(u)\mathrm{d}u$ and using the fact that $\mathrm{I}_\alpha^s\left(\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}v\right)(t) = v(t) - v(s)$. □

Corollary 2.2. *Let A be an α-infinitesimal generator of a C_0-α-semigroup $T(t)$. Then A is closed and densely defined linear operator.*

Proof. Let $x_\varepsilon = \frac{1}{\varepsilon}\int_t^{t+\varepsilon t^{1-\alpha}}\frac{1}{s^{1-\alpha}}T(s)x\mathrm{d}s$, where $x\in X$. By part (a) of Theorem 2.2, $x_\varepsilon \in D(A)$ and $x_\varepsilon \to T(t)x$. Thus, $T(t)x\in\overline{D(A)}$ for every $t>0$. Since x is the limit of $T(t)x$ when $t\to 0^+$, it follows that $x\in\overline{D(A)}$, thus $\overline{D(A)} = X$.

The linearity of A is evident. To prove the closedness of A, let $x_n\in D(A)$ such that $x_n\to x$ and $Ax_n\to y$ as $n\to\infty$. From part (d) of Theorem 2.2, we have

$$\left(\frac{T(t+\varepsilon t^{1-\alpha})-T(t)}{\varepsilon}\right)x_n = \frac{1}{\varepsilon}\int\limits_t^{t+\varepsilon t^{1-\alpha}}\frac{1}{s^{1-\alpha}}T(s)Ax_n\mathrm{d}s.$$

Since for every $t\le s\le t+\varepsilon t^{1-\alpha}$

$$\begin{aligned}\left\|\frac{1}{s^{1-\alpha}}T(s)Ax_n - \frac{1}{s^{1-\alpha}}T(s)y\right\| &\le \frac{1}{s^{1-\alpha}}\|T(s)(Ax_n-y)\| \\ &\le \frac{1}{s^{1-\alpha}}\|T(s)\|\,\|Ax_n-y\| \\ &\le \frac{1}{s^{1-\alpha}}Me^{\omega s^\alpha}\|Ax_n-y\| \\ &\le \sup_{t\le s\le t+\varepsilon t^{1-\alpha}}\left\{\frac{M}{s^{1-\alpha}}e^{\omega s^\alpha}\right\}\|Ax_n-y\|,\end{aligned}$$

then $\frac{1}{s^{1-\alpha}}T(s)Ax_n$ converges uniformly to $\frac{1}{s^{1-\alpha}}T(s)y$, and

$$\lim_{n\to 0}\frac{1}{\varepsilon}\int\limits_t^{t+\varepsilon t^{1-\alpha}}\frac{1}{s^{1-\alpha}}T(s)Ax_n\mathrm{d}s = \frac{1}{\varepsilon}\int\limits_t^{t+\varepsilon t^{1-\alpha}}\frac{1}{s^{1-\alpha}}T(s)y\mathrm{d}s.$$

This implies that

$$\left(\frac{T(t+\varepsilon t^{1-\alpha})-T(t)}{\varepsilon}\right)x = \frac{1}{\varepsilon}\int\limits_t^{t+\varepsilon t^{1-\alpha}}\frac{1}{s^{1-\alpha}}T(s)y\mathrm{d}s.$$

Letting $\varepsilon\to 0$ and $t\to 0^+$ yield $x\in D(A)$ and $Ax = y$. □

Theorem 2.3. *Let $T(t)$ and $U(t)$ be fractional C_0-α-semigroups of bounded linear operators, where A and B are their infinitesimal generators, respectively. If $A = B$, then $T(t) = U(t)$ for $t \geq 0$.*

Proof. For $x \in D(A) = D(B)$, the function $s \to T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} U(s)x$ is α-differentiable. Furthermore,

$$\begin{aligned}\frac{\mathrm{d}^\alpha}{\mathrm{d}s^\alpha} T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} U(s)x &= -AT(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} U(s)x + T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} BU(s)x \\ &= -T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} AU(s)x + T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} BU(s)x = 0.\end{aligned} \tag{11}$$

Now, applying I_α^0 to both sides of (11) yields

$$\begin{aligned} I_\alpha^0\left(\frac{\mathrm{d}^\alpha}{\mathrm{d}s^\alpha} T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} U(s)x\right) &= 0 \\ \left[T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} U(s)x\right]_0^t &= 0 \\ U(t)x - T(t)x &= 0. \end{aligned}$$

Thus, $U(t)x = T(t)x$ for $t \geq 0$. Since $D(A)$ is dense in X and $T(t)$, $U(t)$ are bounded, then $T(t)x = U(t)x$ for every $x \in X$. □

3 Homogeneous initial value problem

Let X be a Banach space, and A be a densely defined linear operator from $D(A) \subset X$ into X. Given $x \in X$ we consider the homogeneous initial value problem:

$$\begin{cases} \dfrac{\mathrm{d}^\alpha u(t)}{\mathrm{d}t^\alpha} = Au(t), \quad t > 0, \\ u(0) = x. \end{cases} \tag{12}$$

Definition 3.1. An X valued function $u : \mathbb{R}^+ \longrightarrow X$ is a solution of (12) if u is continuous for $t \geq 0$, α-continuously differentiable, $u(t) \in D(A)$ for $t > 0$ and (12) is satisfied.

Recently, Abdeljawad et al. in [22] showed that if A is the infinitesimal generator of a C_0-α-semigroup $T(t)$, then the fractional Abstract Cauchy problem (12) has a unique solution of the form $u(t) = T(t)x$, for every $x \in D(A)$. Actually, uniqueness of solutions of the initial value problem (12) follows from weaker assumptions as we will see the next theorem.

Lemma 3.1. [24] *Let $u(t)$ be a continuous X valued function on $[0, S]$. If*

$$\left\| \int_0^S e^{ns} u(s)\mathrm{d}s \right\| \leq M \quad \text{for } n = 1, 2, \ldots$$

Then $u \equiv 0$ on $[0, S]$.

Theorem 3.1. *For a densely defined linear operator A, if $R(\lambda : A)$ exists for all real $\lambda \geq \lambda_0$ and*

$$\limsup_{\lambda \to \infty} \lambda^{-1} \log \|R(\lambda : A)\| \leq 0, \tag{13}$$

then the initial value problem (12) has at most one solution for every $x \in X$.

Proof. Note first that $u(t)$ is solution of (12) if and only if $e^{z\frac{t^\alpha}{\alpha}}u(t)$ is a solution of the initial value problem

$$\frac{d^\alpha v}{dt^\alpha} = (A + zI)v, \quad v(0) = x.$$

Thus, we may translate A by a constant multiple of the identity and assume that $R(\lambda : A)$ exists for all real λ, $\lambda \geq 0$ and (13) is satisfied. To prove the uniqueness of the solution of (12), let $u(t)$ be a solution of (12) satisfying $u(0) = 0$. We prove that $u \equiv 0$. Let $t \to R(\lambda : A)u(t)$ for $\lambda \geq 0$ be an X valued function, where $u(t)$ is the solution of (12). Then we have

$$\frac{d^\alpha}{dt^\alpha}R(\lambda : A)u(t) = R(\lambda : A)Au(t) = \lambda R(\lambda : A)u(t) - u(t). \tag{14}$$

Now, multiplying (14) by $e^{-\lambda\frac{t^\alpha}{\alpha}}$ yields

$$e^{-\lambda\frac{t^\alpha}{\alpha}}\frac{d^\alpha}{dt^\alpha}R(\lambda : A)u(t) - e^{-\lambda\frac{t^\alpha}{\alpha}}\lambda R(\lambda : A)u(t) = -e^{-\lambda\frac{t^\alpha}{\alpha}}u(t).$$

Since

$$\frac{d^\alpha}{dt^\alpha}(e^{-\lambda\frac{t^\alpha}{\alpha}}R(\lambda : A)u(t)) = e^{-\lambda\frac{t^\alpha}{\alpha}}\frac{d^\alpha}{dt^\alpha}R(\lambda : A)u(t) - e^{-\lambda\frac{t^\alpha}{\alpha}}\lambda R(\lambda : A)u(t),$$

then

$$\frac{d^\alpha}{dt^\alpha}(e^{-\lambda\frac{t^\alpha}{\alpha}}R(\lambda : A)u(t)) = -e^{-\lambda\frac{t^\alpha}{\alpha}}u(t). \tag{15}$$

Now, by applying I_α^0 to equation (15) and using the fact that $I_\alpha^0\left(\frac{d^\alpha}{dt^\alpha}f\right)(t) = f(t) - f(0)$, we get

$$I_\alpha^0\left(\frac{d^\alpha}{dt^\alpha}\left(e^{-\lambda\frac{t^\alpha}{\alpha}}R(\lambda : A)u(t)\right)\right) = -\int_0^t \frac{1}{s^{1-\alpha}}e^{-\lambda\frac{s^\alpha}{\alpha}}u(s)ds,$$

$$e^{-\lambda\frac{t^\alpha}{\alpha}}R(\lambda : A)u(t) - e^0 R(\lambda : A)u(0) = -\int_0^t \frac{1}{s^{1-\alpha}}e^{-\lambda\frac{s^\alpha}{\alpha}}u(s)ds.$$

Since $u(0) = 0$, it follows that

$$R(\lambda : A)u(t) = -e^{\lambda\frac{t^\alpha}{\alpha}}\int_0^t \frac{1}{s^{1-\alpha}}e^{-\lambda\frac{s^\alpha}{\alpha}}u(s)ds = -\int_0^t \frac{1}{s^{1-\alpha}}e^{\frac{\lambda}{\alpha}(t^\alpha - s^\alpha)}u(s)ds. \tag{16}$$

Now, from Assumption (13) it follows that for every $\sigma > 0$

$$\lim_{\lambda\to\infty} e^{-\lambda\frac{\sigma}{\alpha}}\|R(\lambda : A)\| = \lim_{\lambda\to\infty} e^{\lambda\left(-\frac{\sigma}{\alpha}+\lambda^{-1}\log\|R(\lambda:A)\|\right)} = 0.$$

Consequently, from (16) it follows that

$$\lim_{\lambda\to\infty}\left\{-\int_0^t \frac{1}{s^{1-\alpha}}e^{\frac{\lambda}{\alpha}(t^\alpha-\sigma-s^\alpha)}u(s)ds\right\} = \lim_{\lambda\to\infty} e^{-\lambda\frac{\sigma}{\alpha}}R(\lambda : A)u(t) = 0. \tag{17}$$

Now, since

$$\lim_{\lambda\to\infty}\left\{-\int_0^t \frac{1}{s^{1-\alpha}}e^{\frac{\lambda}{\alpha}(t^\alpha-\sigma-s^\alpha)}u(s)ds\right\} = \lim_{\lambda\to\infty}\left\{-\int_0^{(t^\alpha-\sigma)^{\frac{1}{\alpha}}} \frac{1}{s^{1-\alpha}}e^{\frac{\lambda}{\alpha}(t^\alpha-\sigma-s^\alpha)}u(s)ds - \int_{(t^\alpha-\sigma)^{\frac{1}{\alpha}}}^t \frac{1}{s^{1-\alpha}}e^{\frac{\lambda}{\alpha}(t^\alpha-\sigma-s^\alpha)}u(s)ds\right\} \tag{18}$$

and for $(t^\alpha - \sigma)^{\frac{1}{\alpha}} < s < t$, yields $t^\alpha - \sigma - s^\alpha < 0$, then $\lim_{\lambda\to\infty}\frac{1}{s^{1-\alpha}}e^{\frac{\lambda}{\alpha}(t^\alpha-\sigma-s^\alpha)}u(s) = 0$. Consequently, it follows from the dominated convergence theorem that

$$\lim_{\lambda\to\infty} \int_{(t^\alpha-\sigma)^{\frac{1}{\alpha}}}^{t} \frac{1}{s^{1-\alpha}} e^{\frac{\lambda}{\alpha}(t^\alpha-\sigma-s^\alpha)} u(s)\mathrm{d}s = \int_{(t^\alpha-\sigma)^{\frac{1}{\alpha}}}^{t} \lim_{\lambda\to\infty} \frac{1}{s^{1-\alpha}} e^{\frac{\lambda}{\alpha}(t^\alpha-\sigma-s^\alpha)} u(s)\mathrm{d}s = 0. \tag{19}$$

Now, (17), (18) and (19) imply that

$$\lim_{\lambda\to\infty} \int_{0}^{(t^\alpha-\sigma)^{\frac{1}{\alpha}}} \frac{1}{s^{1-\alpha}} e^{\frac{\lambda}{\alpha}(t^\alpha-\sigma-s^\alpha)} u(s)\mathrm{d}s = 0. \tag{20}$$

In (20), let $\tau = (t^\alpha - \sigma - s^\alpha)$ to get

$$\int_{0}^{(t^\alpha-\sigma)^{\frac{1}{\alpha}}} \frac{1}{s^{1-\alpha}} e^{\frac{\lambda}{\alpha}(t^\alpha-\sigma-s^\alpha)} u(s)\mathrm{d}s = \frac{1}{\alpha} \int_{0}^{t^\alpha-\sigma} e^{\frac{\lambda}{\alpha}\tau} u((t^\alpha - \sigma - \tau)^{\frac{1}{\alpha}})\mathrm{d}\tau.$$

Consequently,

$$\lim_{\lambda\to\infty} \int_{0}^{t^\alpha-\sigma} e^{\frac{\lambda}{\alpha}\tau} u(t^\alpha - \sigma - \tau)^{\frac{1}{\alpha}}\mathrm{d}s = 0. \tag{21}$$

Finally, using Lemma (3.1), (21) yields

$$u(t^\alpha - \sigma - \tau) = 0 \quad 0 \le \tau \le t^\alpha - \sigma.$$

Since t and σ are arbitrary, we conclude that $u(t) = 0$ for $t \ge 0$. □

Remark 3.1. If A is the generator of a C_0-α-semigroup $(T(t))_{t\ge0}$ in $\mathcal{L}(X, X)$, where $\|T(t)\| \le Me^{\omega\frac{t^\alpha}{\alpha}}$, for some $M \ge 0$ and $\omega \in \mathbb{R}$, then the resolvent operator $R(\lambda, A) = (\lambda I - A)^{-1}$ is given by

$$R(\lambda, A)x = \int_{0}^{\infty} \frac{1}{t^{1-\alpha}} e^{-\lambda\frac{t^\alpha}{\alpha}} T(t)x\mathrm{d}t \tag{22}$$

for $x \in X$ and $\lambda \in \rho(A)$, $Re\ \lambda > \omega$. Moreover, $R(\lambda, A)$ satisfies for all $x \in X$

$$\|R(\lambda, A)x\| \le \frac{M}{Re\ \lambda - \omega}\|x\|.$$

Theorem 3.2. *For a densely defined linear operator A with a nonempty resolvent set $\rho(A)$, the following statements are equivalent*:

(h_1) *The initial value problem* (12) *has a unique solution $u(t)$, for every initial value condition $u(0) = x \in D(A)$.*

(h_2) *A is the infinitesimal generator of a C_0-α-semigroup $T(t)$.*

Proof. The implication $(h_2) \Rightarrow (h_1)$ was proved in [22].

So we prove only $(h_1) \Rightarrow (h_2)$. Since $\rho(A) \neq \emptyset$, then A is closed and $([D(A)], \|x\|_A = \|x\| + \|Ax\|)$ is a Banach space. Now we define $V : [D(A)] \to C([0, S], [D(A)])$, where $((C([0, S], [D(A)]), \|.\|_\infty)$ is the Banach space of continuous functions from $[0, S]$ into $[D(A)]$ with the usual supremum norm $\|u\|_\infty = \sup_{t\in[0,S]}\|u(t)\|_A)$, by $V(x) = u_x$, where u_x is the unique solution of (12) related to the initial condition x. By linearity of (12) and the uniqueness of the solution, V is a linear operator. The closedness of V follows from the fact that if $x_n \to x \in [D(A)]$, and $u_{x_n} \to v \in C([0, S], [D(A)])$, then $Au_{x_n}(t) \to Av(t)$ in X uniformly in t. Therefore, from

$$u_{x_n}(t) = x_n + \int_{0}^{t} \frac{1}{s^{1-\alpha}} Au_{x_n}(s)\mathrm{d}s$$

we obtain as $n \to \infty$

$$v(t) = x + \int_0^t \frac{1}{s^{1-\alpha}} Av(s)\mathrm{d}s.$$

Since, for every $t > 0$

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} v(t) = \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\left(\int_0^t \frac{1}{s^{1-\alpha}} Av(s)\mathrm{d}s\right),$$
$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}(I_\alpha^0 Av(t)) = Av(t)$$

and $v(0) = x$, then the uniqueness of solution gives that $v = u_x$ and so V is closed. By closed graph theorem V is bounded and

$$\sup_{0\le t\le S} \|u_x(t)\|_A \le C\|x\|_A. \tag{23}$$

Now, we define a linear operator $U(t) : [D(A)] \to [D(A)]$ by $U(t)x = u_x(t)$. For $x \in D(A)$, $U(t^\alpha + s^\alpha)^{\frac{1}{\alpha}}x$ and $U(t)U(s)x$ are solutions of (12) with initial condition $U(s)x$. The uniqueness of the solution gives us the equality

$$\forall x \in D(A), \quad U(t+s)^{\frac{1}{\alpha}}x = U(t)^{\frac{1}{\alpha}}U(s)^{\frac{1}{\alpha}}x, \quad t, s \ge 0.$$

Then $U(t)$ has the α-semigroup property. From (23) it follows that $U(t)$ is uniformly bounded for $t \in [0, S]$. For $nS^\alpha \le t^\alpha \le (n+1)S^\alpha$, we have $U(t)x = U(t^\alpha - nS^\alpha)^{\frac{1}{\alpha}}U(S)^n x$. Then $U(t)$ can be extended to an α-semigroup on $[D(A)]$ satisfying $\|U(t)x\|_A \le Me^{\omega t^\alpha}\|x\|_A$.

Now, we will show that

$$U(t)Ay = AU(t)y \quad \text{for} \quad y \in D(A^2). \tag{24}$$

Consider $v(t) = y + \int_0^t \frac{1}{s^{1-\alpha}} u_{Ay}(s)\mathrm{d}s$, where u_{Ay} is a solution of (12) satisfying $u_{Ay}(0) = Ay$. Now using the fact that $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}(I_\alpha^0 f(t)) = f(t)$, we get

$$v^{(\alpha)} = \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\left(\int_0^t \frac{1}{s^{1-\alpha}} u_{Ay}(s)\mathrm{d}s\right) = \frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}(I_\alpha^0 u_{Ay}(t)) = u_{Ay}(t).$$

Since u_{Ay} is a solution of (12), it follows that

$$\begin{aligned} u_{Ay}(t) - u_{Ay}(0) &= I_\alpha^0\left(\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha} u_{Ay}(t)\right), \\ u_{Ay}(t) - Ay &= \int_0^t \frac{1}{s^{1-\alpha}} \frac{\mathrm{d}^\alpha}{\mathrm{d}s^\alpha} u_{Ay}(s)\mathrm{d}s. \end{aligned} \tag{25}$$

From (25) and the closedness of A, we get

$$v^{(\alpha)}(t) = u_{Ay}(t) = Ay + \int_0^t \frac{1}{s^{1-\alpha}} \frac{\mathrm{d}^\alpha}{\mathrm{d}s^\alpha} u_{Ay}(s)\mathrm{d}s = A\left(y + \int_0^t \frac{1}{s^{1-\alpha}} u_{Ay}(s)\mathrm{d}s\right) = Av(t).$$

But $v(0) = y$. So the uniqueness of the solution of (12) gives $v(t) = u_y(t)$. Consequently, $Au_y(t) = v^{(\alpha)}(t) = u_{Ay}(t)$, which is the same as in (24).

From the fact that $\overline{D(A)} = X$ and $\rho(A) \ne \emptyset$ we get $\overline{D(A^2)} = X$. Let $\lambda_0 \in \rho(A)$, $\lambda_0 \ne 0$, and let $y \in D(A^2)$. Let $x = (\lambda_0 I - A)y$. Then, by (24), $U(t)x = (\lambda_0 I - A)U(t)y$ and

$$\|U(t)x\| = \|(\lambda_0 I - A)U(t)y\| \le K\|U(t)y\|_A \le K_1 e^{\omega t^\alpha}\|y\|_A.$$

But

$$\|y\|_A = \|y\| + \|Ay\| \le K_2\|x\|.$$

So

$$\|U(t)x\| \le K_2 e^{\omega t^\alpha}\|y\|.$$

Therefore, $U(t)$ can be extended by continuity to all of X. Thus, $U(t)$ becomes a C_0-α-semigroup of X.

Now it suffices to show that A is the α-infinitesimal generator of $U(t)$. Denoted by A_1 the α-infinitesimal generator of $U(t)$. Since

$$\frac{d^\alpha}{dt^\alpha}U(t)x = AU(t)x, \quad \text{for } t \ge 0.$$

Then, $\frac{d^\alpha}{dt^\alpha}U(t)x\,|_{t=0} = Ax$. Hence, $A_1 \supseteq A$.

For $Re\ \lambda > \omega$, and $y \in D(A^2)$, we have

$$\frac{1}{t^{1-\alpha}}e^{-\lambda\frac{t^\alpha}{\alpha}}AU(t)y = \frac{1}{t^{1-\alpha}}e^{-\lambda\frac{t^\alpha}{\alpha}}U(t)Ay = \frac{1}{t^{1-\alpha}}e^{-\lambda\frac{t^\alpha}{\alpha}}U(t)A_1y. \tag{26}$$

Using (22) and integrating (26) from 0 to ∞ we get $AR(\lambda, A_1)y = R(\lambda, A_1)A_1y$. But $R(\lambda, A_1)A_1y = A_1R(\lambda, A_1)y$. Thus, $AR(\lambda, A_1)y = A_1R(\lambda, A_1)y$ for every $y \in D(A^2)$. From the fact that $A_1R(\lambda, A_1)$ is bounded, A is closed and $\overline{D(A^2)} = X$, it follows that $AR(\lambda, A_1)y = A_1R(\lambda, A_1)y$ for every $y \in X$, which means $D(A) \supseteq \text{Range}\,R(\lambda : A_1) = D(A_1)$ and $A \supseteq A_1$. Thus, $A = A_1$. □

4 Inhomogeneous initial value problem

Let X be a Banach space and A be a densely defined linear operator from $D(A) \subset X$ into X. Given $x \in X$, we consider the inhomogeneous initial value problem:

$$\begin{cases} \dfrac{d^\alpha u(t)}{dt^\alpha} = Au(t) + f(t), \quad t > 0, \\ u(0) = x, \end{cases} \tag{27}$$

where $f : [0, S) \longrightarrow X$.

Let

$$L^p_\alpha(0, S; X) = \left\{ f : [0, S] \to X \text{ is measurable } X \text{ valued function such that } : \int_0^S \frac{1}{s^{1-\alpha}}\|f(s)\|^p ds < \infty \right\}.$$

It is a classical procedure to prove that $L^p_\alpha(0, S; X)$ is a Banach space under the norm

$$\|f\|_p = \left(\int_0^S \tfrac{1}{s^{1-\alpha}}\|f(s)\|^p ds\right)^{\frac{1}{p}}.$$

Definition 4.1. A function $u : [0, S) \longrightarrow X$ is a solution of (27) on $[0, S]$ if u is continuous on $[0, S)$, α-continuously differentiable on $(0, S)$, $u(t) \in D(A)$ for $0 < t < S$ and (27) is satisfied on $[0, S)$.

Let $T(t)$ be the C_0-α-semigroup generated by A and let u be a solution of 27. Then the X valued function $g(s) = T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}}u(s)$ is α-differentiable for $0 < s < t$ and

$$\begin{aligned} \frac{d^\alpha g}{ds^\alpha}(s) &= -T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}}Au(s) + T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}}u^\alpha(s) \\ &= -T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}}Au(s) + T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}}Au(s) + T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}}f(s) \\ &= T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}}f(s). \end{aligned} \tag{28}$$

Now applying I_α^0 to (28), for $f \in L_\alpha^1(0, S : X)$ we get $T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} \frac{1}{s^{1-\alpha}} f(s)$ is integrable and

$$I_\alpha^0(g^{(\alpha)})(t) = T(t^\alpha - t^\alpha)^{\frac{1}{\alpha}} u(t) - T(t^\alpha)^{\frac{1}{\alpha}} u(0) = \int_0^t \frac{1}{s^{1-\alpha}} T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f(s) ds.$$

So

$$u(t) = T(t)x + \int_0^t \frac{1}{s^{1-\alpha}} T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f(s) ds. \tag{29}$$

From the previous results we have:

Corollary 4.1. *If $f \in L_\alpha^1(0, S : X)$, then for every $x \in X$ the initial value problem* (27) *has at most one solution. If it has a solution, this solution is given by* (29).

Definition 4.2. Let A be the generator of a C_0-α-semigroup $T(t)$. For $x \in X$ and $f \in L_\alpha^1(0, S : X)$, the function $u \in C([0, S] : X)$ given by

$$u(t) = T(t)x + \int_0^t \frac{1}{s^{1-\alpha}} T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f(s) ds, \quad 0 \le t \le S$$

is called the mild solution of (27).

Theorem 4.1. *Let $T(t)$ be a C_0-α-semigroup with generator A, $f \in L_\alpha^1(0, S : X)$ be continuous on $(0, S]$ and let*

$$v(t) = \int_0^t \frac{1}{s^{1-\alpha}} T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f(s) ds, \quad 0 < t \le S. \tag{30}$$

The initial value problem (27) *has a solution u on $[0, S)$ for every $x \in D(A)$ if one of the following conditions is satisfied*:
(*i*) $v(t)$ *is continuously α-differentiable on* $(0, S)$.
(*ii*) $v(t) \in D(A)$ *for* $0 < t < S$ *and $Av(t)$ is continuous on* $(0, S)$.

If (27) *has a solution u on $[0, S)$ for some $x \in D(A)$, then v satisfies both conditions.*

Proof. If (27) has a solution u for some $x \in D(A)$, then this solution is given by (29). Furthermore, since $v(t) = u(t) - T(t)x$ is the difference of two α-differentiable functions, then v is α-differentiable and $v^{(\alpha)}(t) = u^{(\alpha)}(t) - T(t)Ax$ is obviously continuous on $(0, S)$. Thus, (i) is satisfied. Also, if $x \in D(A)$, then $T(t)x \in D(A)$ for $t \ge 0$ and therefore $v(t) = u(t) - T(t)x \in D(A)$ for $t \ge 0$ and $Av(t) = Au(t) - AT(t)x = u^{(\alpha)}(t) - f(t) - T(t)Ax$ is continuous on $(0, S)$. Thus, (*ii*) is satisfied.

Now, it is easy for $\varepsilon > -\tau^\alpha$ to verify the identity:

$$\frac{T(\tau + \varepsilon\tau^{1-\alpha}) - T(\tau)}{\varepsilon} v(t) = T(\tau)\left\{ \frac{v((\tau + \varepsilon\tau^{1-\alpha})^\alpha - \tau^\alpha + t^\alpha)^{\frac{1}{\alpha}} - v(t)}{\varepsilon} \right. \\ \left. - \frac{1}{\varepsilon} \int_t^{((\tau+\varepsilon\tau^{1-\alpha})^\alpha - \tau^\alpha + t^\alpha)^{\frac{1}{\alpha}}} \frac{1}{s^{1-\alpha}} T((\tau + \varepsilon\tau^{1-\alpha})^\alpha - \tau^\alpha + t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f(s) ds \right\}. \tag{31}$$

The continuity of f yields

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_t^{((\tau+\varepsilon\tau^{1-\alpha})^\alpha - \tau^\alpha + t^\alpha)^{\frac{1}{\alpha}}} \frac{1}{s^{1-\alpha}} T((\tau + \varepsilon\tau^{1-\alpha})^\alpha - \tau^\alpha + t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f(s) ds = f(t).$$

If $v(t)$ is continuously α-differentiable on $(0, S)$, then by letting $\varepsilon \to 0$ in (31) we get

$$T^{(\alpha)}(\tau)v(t) = T(\tau)\{v^{(\alpha)}(t) - f(t)\}. \tag{32}$$

It follows from (32) that if $\tau \to 0$, then $v(t) \in D(A)$ for $0 < t < S$ and $Av(t) = v^{(\alpha)}(t) - f(t)$. Since $v(0) = 0$ it follows that $u(t) = T(t)x + v(t)$ is solution of (27) for $x \in D(A)$.

If $v(t) \in D(A)$, then by (32), $v(t)$ is continuously α-differentiable and $v^{(\alpha)}(t) = Av(t) + f(t)$. Since $v(0) = 0$, then $u(t) = T(t)x + v(t)$ is the solution of (27) for $x \in D(A)$. □

Corollary 4.2. *Let $T(t)$ be a C_0-α-semigroup with generator A. If f is continuously α-differentiable on $[0, S]$, then the initial value problem (27) has a solution on $[0, S)$ for every $x \in D(A)$.*

Proof. We have by change of variables

$$v(t) = \int_0^t \frac{1}{s^{1-\alpha}} T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f(s)\mathrm{d}s = \int_0^t \frac{1}{s^{1-\alpha}} T(s) f((t^\alpha - s^\alpha)^{\frac{1}{\alpha}})\mathrm{d}s.$$

Then it is clear that $v(t)$ is α-differentiable for $t > 0$ and

$$v^{(\alpha)}(t) = T(t)f(0) + \int_0^t \frac{1}{s^{1-\alpha}} T(s) f^{(\alpha)}((t^\alpha - s^\alpha)^{\frac{1}{\alpha}})\mathrm{d}s = T(t)f(0) + \int_0^t \frac{1}{s^{1-\alpha}} T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f^{(\alpha)}(s)\mathrm{d}s.$$

Then $v^{(\alpha)}(t)$ is continuous on $(0, S)$ and therefore the result follows from condition (i) in Theorem 4.1. □

Corollary 4.3. *Let $T(t)$ be a C_0-α-semigroup with generator A, and let $f \in L^1_\alpha(0, S; X)$ be continuous on $(0, S)$. If $f(s) \in D(A)$ for $0 < s < S$ and $Af(s) \in L^1_\alpha(0, S; X)$, then for every $x \in D(A)$ the initial value problem (27) has a solution on $[0, S)$.*

Proof. From the assumptions, it follows that for $s > 0$, $T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f(s) \in D(A)$ and $AT(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f(s) = T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} Af(s) \in L^1_\alpha(0, S; X)$. Therefore, $v(t)$ defined by (30) satisfies $v(t) \in D(A)$ for $t > 0$ and

$$Av(t) = A\int_0^t \frac{1}{s^{1-\alpha}} T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} f(s)\mathrm{d}s = \int_0^t \frac{1}{s^{1-\alpha}} T(t^\alpha - s^\alpha)^{\frac{1}{\alpha}} Af(s)\mathrm{d}s$$

is continuous. Now, the result follows from condition (ii) in Theorem 4.1. □

5 Examples

Example 5.1. Consider the homogeneous initial value problem

$$\begin{cases} \dfrac{\partial^\alpha v(t,x)}{\partial t^\alpha} = \dfrac{\partial v(t,x)}{\partial x}, \quad t > 0, \quad -\infty < x < +\infty, \\ v(0,x) = g(x). \end{cases} \tag{33}$$

Let $X = BU(-\infty, +\infty)$ be the Banach space of bounded uniformly continuous functions with usual supremum norm $\|f\|_\infty = \sup_{-\infty<x<+\infty} \|f(x)\|$. Define the operator A by

$$A = \frac{\partial}{\partial x}(.), \quad D(A) = \{\psi \in X : \psi' \in X\}.$$

The operator A generates a C_0-α-semigroup $(T(t))_{t\geq 0}$ on X (see [22]), defined by $(T(t)f)(x) = f\left(x + \frac{t^\alpha}{\alpha}\right)$.

Let $u(t)(x) = v(t,x)$ and $u(0)(x) = g(x)$. Then the initial value problem (33) takes the form :

$$\begin{cases} \dfrac{\partial^\alpha u(t)}{\partial t^\alpha} = Au(t), \quad t > 0, \\ u(0) = g. \end{cases}$$

If $g \in D(A)$, then by Theorem 3.2 the above initial value problem has a unique solution given by $v(t,x) = (T(t)g)(x) = g\left(x + \frac{t^\alpha}{\alpha}\right)$.

Example 5.2. Consider the inhomogeneous conformable system with initial value condition

$$\begin{cases} \dfrac{d^\alpha u}{dt^\alpha}(t) = Au(t) + f(t), \quad t > 0, \\ u(0) = x, \end{cases} \tag{34}$$

where $A \in M_n(\mathbb{R})$ is a square matrix on $\mathbb{R}^n$, $x \in \mathbb{R}^n$ and $f : [0,\infty) \to \mathbb{R}^n$ is a continuously α-differentiable function.

The matrix A is a linear operator on $D(A) = \mathbb{R}^n$, which is a generator of a C_0-α-semigroup $(T(t))_{t\geq 0}$ defined by

$$T(t)y = e^{\frac{t^\alpha}{\alpha}A}y = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{t^{n\alpha}}{\alpha^n}A^n y.$$

One can easily see that all assumptions of Corollary 4.2 are satisfied and so (34) has a unique solution

$$u(t) = e^{\frac{t^\alpha}{\alpha}A}x + \int_0^t \frac{1}{s^{1-\alpha}} e^{A\left(\frac{t^\alpha}{\alpha} - \frac{s^\alpha}{\alpha}\right)} f(s)ds.$$

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