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Article

Inhomogeneous Long-Range Percolation for Real-Life Network Modeling

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Abstract: The study of random graphs has become very popular for real-life network modeling, such as social networks or financial networks. Inhomogeneous long-range percolation (or scale-free percolation) on the lattice $\mathbb{Z}^d$, $d \geq 1$, is a particular attractive example of a random graph model because it fulfills several stylized facts of real-life networks. For this model, various geometric properties, such as the percolation behavior, the degree distribution and graph distances, have been analyzed. In the present paper, we complement the picture of graph distances and we prove continuity of the percolation probability in the phase transition point. We also provide an illustration of the model connected to financial networks.

Keywords: network modeling; stylized facts of real-life networks; small-world effect; long-range percolation; scale-free percolation; graph distance; phase transition; continuity of percolation probability; inhomogeneous long-range percolation; infinite connected component

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1. Introduction

Random graph theory has become very popular to model real-life networks. Real-life networks may be understood as sets of particles that are possibly linked with each other. Such networks appear, for example, as virtual social networks, see [1], as financial networks such as the banking system, see [2,3], or the network of interbank transactions, see [4,5]. In the latter example, banks are modeled by particles and two banks are linked if one bank transacts a payment to the other one. The connectivity of the network plays a crucial role on the spread of information and the development of default cascades, the latter being crucial for macroeconomic stability, see [6]. It is, therefore, of major interest to understand the geometry of such networks. Using empirical data one has observed several stylized facts about large real-life networks, for a detailed outline we refer to [1,7], and Section 1.3 in [8]:

- Distant particles are typically connected by very few links, *i.e.*, although there are possibly a lot of particles in the network, any two particles are typically connected through only a few other particles. This is called the "small-world effect". For example, there is the observation that most particles in real-life networks are connected by at most six links, see also [9]. For the Facebook network with 721 million users, where there is a link between two users if they are "friends" on Facebook, the average number of minimal links that connect any two users is around 4.5, while around 99% of all users are connected by at most six links, see [10]. For the movie actor network, where there is a link between two actors if they appeared in the same film, the average number of minimal links that connect any two actors is also around 4.5, while the number of actors in the network is over two hundred thousand. See [7] for more examples.
- Linked particles tend to have common friends, which is called the "clustering property". For instance, if x is friend of both y and z, then it is likely that y and z are also friends. As an example, [10] discovers the following in the Facebook network: given a user with 100 friends, about 14% of the possible friendships among his friends exist.
- The degree distribution, that is, the distribution of the number of links of a given particle, is heavy-tailed, *i.e.*, its survival probability has a power law decay. It is observed that in real-life networks the (power law) tail parameter τ is often between 1 and 2. For instance, for the movie actor network τ is estimated to be around 1.3. For more explicit examples we refer to [7,8].

Since it is too complicated to model large networks particle by particle, many different random graph models have been developed and their properties were analyzed. A well studied model in the literature is the homogeneous long-range percolation model on $\mathbb{Z}^d$, $d \geq 1$. In this model, the particles are the vertices of $\mathbb{Z}^d$. For fixed $\lambda, \alpha > 0$, any two particles $x, y \in \mathbb{Z}^d$ are linked with probability p_{xy} which behaves as $\lambda|x-y|^{-\alpha}$ for $|x-y| \to \infty$, *i.e.*, the closer particles are the more likely they are connected. This model has therefore a local clustering property. Moreover, for values of α not too large, the graph distance between $x, y \in \mathbb{Z}^d$, that is, the minimal number of links that connect x and y, behaves roughly logarithmically as $|x-y|$ tends to infinity, see [11]. This behavior can be interpreted as a version of the small-world effect in the sense that if two particles are separated by large (Euclidean) distance $|x|$, they are connected by only roughly $\log|x|$ links. But homogeneous long-range percolation does not fulfill the stylized fact of having heavy-tailed degree distributions, which makes this model less attractive for real-life network modeling. Therefore, [12] introduced the inhomogeneous long-range percolation model (also known as scale-free percolation model) on $\mathbb{Z}^d$ which extends the homogeneous model in such a way that the degree distributions turn out to be heavy-tailed. In this extended model one assigns to each particle $x \in \mathbb{Z}^d$ a positive random weight W_x whose distribution is heavy-tailed with tail parameter $\beta > 0$. Given the weights $(W_x)_{x\in\mathbb{Z}^d}$, two particles $x, y \in \mathbb{Z}^d$ are then linked with probability p_{xy} being approximately $\lambda W_x W_y |x - y|^{-\alpha}$ for large $|x - y|$. Note that p_{xy} is decreasing in the distance between particles x and y and increasing in their weights. This implies that the weights make particles more or less attractive, *i.e.*, if a given particle x has a large weight W_x, it plays the role of a hub in the network. This extension of the homogeneous model is very natural since the existence of hubs in real-life networks is often observed. For instance, the number of friends of a famous person on Facebook is typically clearly above average, or large banks do much more transactions than small banks. Therefore, this model can be used to model real-life networks where the possibility of a link mainly depends on the "sizes" of particles and their separations. As a concrete example, we illustrate a financial network in Section 4. The inhomogeneous model has by definition a local clustering property. Moreover, depending on the values of α and β, the degree distribution is heavy-tailed with tail parameter $\tau = \tau(\alpha, \beta) > 1$, see Theorem 2.2 in [12]. Hence, in contrast to the homogeneous model, it fulfills the stylized fact of having heavy-tailed degree distributions. For real-life applications the interesting case is $\tau \in (1, 2)$ and in this case the graph distance between two particles is of doubly logarithmical order as their separation tends to infinity, see [12] and Theorem 8 below. This is again a version of the small-world effect and it says, for instance, that if we increase the (Euclidean) distance between two particles by a factor 1000, their graph distance only grows by roughly 2. One goal of this paper is to complement the picture about graph distances of [12] by providing analogous results to [11,13–15] for inhomogeneous long-range percolation.

In homogeneous long-range percolation it is known that there is a critical constant $\lambda_c = \lambda_c(\alpha, d)$ such that there is an infinite connected component of particles for $\lambda > \lambda_c$, and there is no such component for $\lambda < \lambda_c$, *i.e.*, in the former case there is an infinite connected network in $\mathbb{Z}^d$. Such phase transitions appear in many random graph models and they play an important role in the stability of networks, see for instance the banking system modeled in [6]. The phase transition picture in homogeneous long-range percolation can be traced back to the work of [16–18]. Later work concentrated more on the geometrical properties of percolation like graph distances, see [11,14,15,19,20]. A good overview of the literature for long-range percolation is provided in [11,21]. For homogeneous long-range percolation it is known that for $\alpha \leq d$ there is an infinite connected component for all $\lambda > 0$, and therefore $\lambda_c = 0$. This infinite connected component contains all particles of $\mathbb{Z}^d$, a.s., *i.e.*, in that case we have a completely connected network of all particles of $\mathbb{Z}^d$, which is not of interest for real-life network applications. In the case $\alpha \in (d, 2d)$ we have $\lambda_c > 0$ and there is no infinite connected component at criticality λ_c, see [22]. In view of the banking system modeled in [6], this means that the banking system is still stable at criticality. This result combined with Proposition 1.3 of [23] shows continuity of the percolation probability, that is, the probability that a given particle belongs to an infinite connected component is continuous in the choice of parameter λ. In particular, given that there exists an infinite connected network, the probability that a given particle belongs to this network can still be arbitrarily small for appropriate choices of λ. This is important for network modeling as the following example illustrates. Assume we model a population where a link between two individuals has the following interpretation: if one of the two individuals has a disease, it infects the other individual. Assume that initially none of the individuals have a disease and choose uniformly at random one individual in a given (arbitrarily large) finite area to have a disease. Then, the probability that this individual infects infinitely many other individuals is still small for $\lambda > \lambda_c$ close to λ_c. For $\alpha \geq 2d$, the problem is still open, except in the case $d = 1$ and $\alpha > 2$ because in that latter case there does not exist an infinite connected component for any choice $\lambda > 0$.

In inhomogeneous long-range percolation the conditions for the existence of a non-trivial critical percolation constant $\lambda_c \in (0, \infty)$ were derived in [12], see also Theorems 1 and 2 below. The continuity of the percolation probability was conjectured in that article. One main goal of the present work is to prove this conjecture for $\alpha \in (d, 2d)$. The crucial technique to prove this conjecture is the renormalization method presented in [22]. This technique will also allow to complement the picture of graph distances provided in [12], which in particular allows to analyze the small-world effects for different networks.

Organization of this article. In Section 2, we describe the model assumptions and notations. We also state the conditions that are required for a non-trivial phase transition. In Section 3, we state the main results of the article. Namely, we show the continuity of the percolation function in Theorem 5 which is based on a finite box estimate stated in Theorem 6. We also complement the picture about graph distances of [12], see Theorem 8 below. In Section 4, we discuss a concrete financial application of the model, compare the results to homogeneous long-range percolation model results and we discuss open problems. Finally, we provide all proofs of our results in Section 5.

2. Model Assumptions and Phase Transition Picture

We define the inhomogeneous long-range percolation model of [12] in a slightly modified version. The reason for this modification is that the model becomes easier to handle but it keeps the essential features of inhomogeneous long-range percolation. In particular, all results of [12] only depend on the asymptotic behavior of survival probabilities. Therefore, we choose an explicit distributional example which on the one hand has the right asymptotic behavior and on the other hand is easy to handle. This, of course, does not harm the generality of the results.

Consider the lattice $\mathbb{Z}^d$ for fixed $d \geq 1$ with vertices $x \in \mathbb{Z}^d$ and edges (x, y) for $x, y \in \mathbb{Z}^d$. Assume $(W_x)_{x \in \mathbb{Z}^d}$ are i.i.d. Pareto distributed weights with parameters $\theta = 1$ and $\beta > 0$, *i.e.*, the weights W_x have i.i.d. survival probabilities

$$\mathbb{P}\left[W_x > w\right] = w^{-\beta}, \qquad \text{for } w \geq 1$$

Conditionally given these weights $(W_x)_{x \in \mathbb{Z}^d}$, we assume that edges (x, y) are independently from each other either occupied or vacant. The conditional probability of an occupied edge (or link) between x and y is chosen as

$$p_{xy} = 1 - \exp\left(-\frac{\lambda W_x W_y}{|x-y|^\alpha}\right), \qquad \text{for fixed given parameters } \alpha, \lambda \in (0, \infty) \tag{1}$$

For $|\cdot|$ we choose the Euclidean norm. Note that p_{xy} is approximately $\lambda W_x W_y |x-y|^{-\alpha}$ for large $|x-y|$. If there is an occupied edge between x and y we write $x \Leftrightarrow y$; if there is a finite connected path of occupied edges between x and y we write $x \leftrightarrow y$ and we say that x and y are connected. Clearly $\{x \Leftrightarrow y\} \subset \{x \leftrightarrow y\}$. We define the cluster of $x \in \mathbb{Z}^d$ to be the connected component

$$\mathcal{C}(x) = \{y \in \mathbb{Z}^d;\ x \leftrightarrow y\}$$

which denotes all particles $y \in \mathbb{Z}^d$ that can be reached within the network. Our aim is to study the size of the cluster $\mathcal{C}(x)$ and to investigate its percolation properties as a function of $\lambda > 0$ and $\alpha > 0$, that is, as a function of the edge probabilities $(\lambda, \alpha) \mapsto p_{xy} = p_{xy}(\lambda, \alpha)$. The percolation probability is defined by

$$\theta(\lambda, \alpha) = \mathbb{P}\left[|\mathcal{C}(0)| = \infty\right]$$

which is the probability that the connected component of a given particle contains infinitely many particles. This is non-decreasing in λ and non-increasing in α. For given $\alpha > 0$, the critical value $\lambda_c(\alpha)$ is defined as

$$\lambda_c = \lambda_c(\alpha) = \inf\{\lambda > 0;\ \theta(\lambda, \alpha) > 0\}$$

Note that $\theta(\lambda, \alpha)$ and $\lambda_c(\alpha)$ also depend on β, but this parameter will be kept fixed.
Trivial case. For $\min\{\alpha, \beta\alpha\} \leq d$, we have $\lambda_c = 0$. This comes from the fact that for any $\lambda > 0$

$$\mathbb{P}\left[\left|\{y \in \mathbb{Z}^d;\ 0 \Leftrightarrow y\}\right| = \infty\right] = 1$$

see Theorem 2.1 in [12]. This says that the degree distribution of a given vertex is infinite, a.s., and therefore there is an infinite connected component, a.s. In the trivial case, all particles have infinitely many links which is, of course, not interesting for real-life network applications. For this reason we only consider the non-trivial case $\min\{\alpha, \beta\alpha\} > d$. In this latter case the degree distribution is heavy-tailed with tail parameter $\tau = \beta\alpha/d > 1$, see Theorem 2.2 of [12], which is in line with the stylized facts. In particular, if $\alpha > d$ and $d < \beta\alpha < 2d$, we have that the degree distribution is heavy-tailed with (power law) tail parameter $\tau \in (1, 2)$. According to the stylized facts this latter case is of special interest for real-life network applications. Theorems 1 and 2 give the phase transition pictures for $d \geq 1$ in the non-trivial case $\min\{\alpha, \beta\alpha\} > d$, see Figure 1 for an illustration.

Theorem 1 (upper bounds)**.** *Fix $d \geq 1$. Assume $\min\{\alpha, \beta\alpha\} > d$.*

(a) If $d \geq 2$, then $\lambda_c < \infty$.

(b) If $d = 1$ and $\alpha \in (1, 2]$, then $\lambda_c < \infty$.

(c) If $d = 1$ and $\min\{\alpha, \beta\alpha\} > 2$, then $\lambda_c = \infty$.

Since $W_x \geq 1$, a.s., the edge probability stochastically dominates a configuration with independent edges being occupied with probabilities $1 - \exp(-\lambda|x - y|^{-\alpha})$. The latter is the homogeneous long-range percolation model on $\mathbb{Z}^d$ and it is well known that this model percolates (for $d \geq 2$ see [22]; for $d = 1$ and $\alpha \in (1, 2]$ see [17]). For part (c) of the theorem we refer to Theorem 3.1 of [12]. Note that in this latter case the network only contains connected components of finite size, a.s., for all $\lambda > 0$. This means that in this case there is no phase transition. The next theorem follows from Theorems 4.2 and 4.4 of [12].

Theorem 2 (lower bounds)**.** *Fix* $d \geq 1$*. Assume* $\min\{\alpha, \beta\alpha\} > d$*.*

(a) If $\tau = \beta\alpha/d < 2$*, then* $\lambda_c = 0$*.*

(b) If $\tau = \beta\alpha/d > 2$*, then* $\lambda_c > 0$*.*

The phase transition pictures differ for $d = 1$ and $d \geq 2$ in that the former has a region where $\lambda_c = \infty$ and the latter does not. Note that $\tau = \beta\alpha/d < 2$ corresponds to infinite variance of the degree distribution and $\tau = \beta\alpha/d > 2$ to finite variance of the degree distribution. In particular, for the interesting case $\tau = \beta\alpha/d \in (1, 2)$ for real-life network applications we have $\lambda_c = 0$. This implies that there is no phase transition and the network will always have an infinite connected component, a.s., for any $\lambda > 0$.

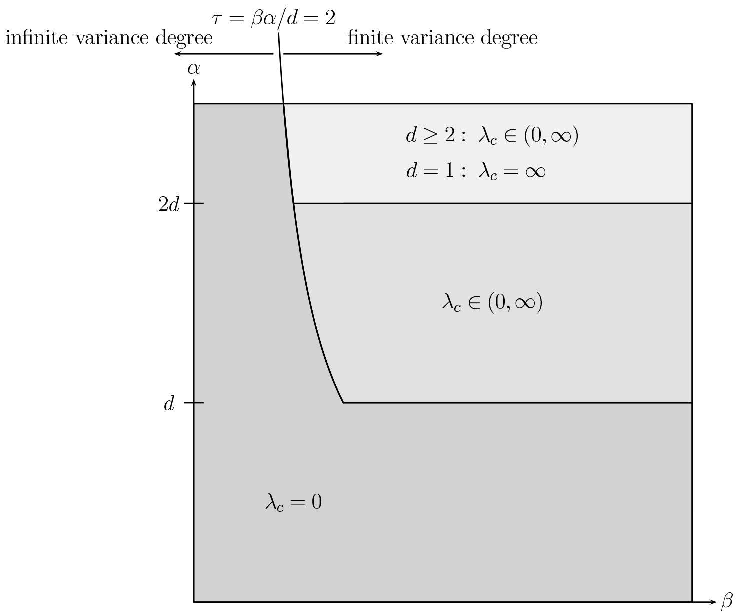

Figure 1. phase transition picture for $d \geq 1$.

3. Main Results

3.1. Continuity of Percolation Probability

We say that there exists an infinite cluster $\mathcal{C}$ if there is an infinite connected component $\mathcal{C}(x)$ for some $x \in \mathbb{Z}^d$. Since the model is translation invariant and ergodic, the event of having an infinite cluster $\mathcal{C}$ is a zero-one event. Thus, for $\lambda > \lambda_c$ there exists an infinite cluster, a.s. Moreover, from Theorem 1.3 in [22] we know that an infinite cluster is unique, a.s. This justifies the notation $\mathcal{C}$ for *the* infinite cluster in the case of percolation $\theta(\lambda, \alpha) > 0$ and implies that we have a *unique* infinite connected network, a.s. As an example, the largest connected component in the Facebook network studied by [10] covers 99.91% of all 721 million users while the second largest component covers only around 2000 users. The following theorem shows that there is no infinite connected component at criticality λ_c whenever $\min\{\alpha, \beta\alpha\} > d$ and $\alpha \in (d, 2d)$.

Theorem 3. *Assume* $\min\{\alpha, \beta\alpha\} > d$ *and* $\alpha \in (d, 2d)$. *Choose* $\lambda \in (0, \infty)$ *with* $\theta(\lambda, \alpha) > 0$. *There exist* $\lambda' \in (0, \lambda)$ *and* $\alpha' \in (\alpha, 2d)$ *such that*

$$\theta(\lambda', \alpha') > 0$$

In particular, $\{\lambda \in (0, \infty);\ \theta(\lambda, \alpha) > 0\}$ *is an open interval in* $(0, \infty)$*, and there does not exist an infinite cluster* $\mathcal{C}$ *at criticality* λ_c*.*

Note that for $\beta\alpha < 2d$ we have $\lambda_c = 0$, hence there is no infinite connected component at criticality. Theorems 2 and 3 therefore imply the following corollary.

Corollary 4. *Assume* $\alpha \in (d, 2d)$ *and* $\tau = \beta\alpha/d > 2$. *There is no infinite cluster* $\mathcal{C}$ *at criticality* $\lambda_c > 0$.

Next we state continuity of the percolation probability in λ which was conjectured in [12].

Theorem 5. *For* $\min\{\alpha, \beta\alpha\} > d$ *and* $\alpha \in (d, 2d)$*, the percolation probability* $\lambda \mapsto \theta(\lambda, \alpha)$ *is continuous.*

This theorem exactly supports the example of spread of disease mentioned in the introduction.

3.2. Percolation on Finite Boxes

For integers $n \geq 1$ and $x \in \mathbb{Z}^d$ define the box centered at x with total side length $2n$ by $\Lambda_n(x) = x + [-n, n]^d$ and abbreviate $\Lambda_n = \Lambda_n(0)$. Let $\mathcal{C}_n$ be the largest connected component in box Λ_n (with a fixed deterministic rule if there is more than one largest connected component in Λ_n).

Theorem 6. *Assume* $\min\{\alpha, \beta\alpha\} > d$ *and* $\alpha \in (d, 2d)$. *Choose* $\lambda \in (0, \infty)$ *with* $\theta(\lambda, \alpha) > 0$. *For each* $\alpha' \in (\alpha, 2d)$ *there exist* $\rho > 0$ *and* $N_0 < \infty$ *such that for all* $n \geq N_0$ *we have*

$$\mathbb{P}\left[|\mathcal{C}_n| \geq \rho|\Lambda_n|\right] \geq 1 - e^{-\rho n^{2d-\alpha'}}$$

This is the analog to the statement in homogeneous long-range percolation, see Theorem 3.2 in [11]. It says that in case of percolation largest connected components in finite boxes cover a positive fraction of these box sizes with high probability for large n, or in other words, the number of particles belonging to the largest connected network in Λ_n is proportional to n^d. For instance, assume we model a population where a link between two individuals has the following interpretation: if one of the two individuals has a disease, it transmits its disease to the other individual. Assume that initially all individuals in a large finite area do not have a disease and choose uniformly at random one individual to have a disease. Then, the above result implies that the probability that a positive fraction of all individuals in this area will get infected is strictly positive.

Let $\mathcal{C}_n(x)$ be the vertices in $\Lambda_n(x)$ that are connected with x within box $\Lambda_n(x)$. For $\ell < n$ and $\rho > 0$ we denote by

$$\mathcal{D}_n^{(\rho,\ell)} = \{x \in \Lambda_n;\ |\mathcal{C}_\ell(x)| \geq \rho|\Lambda_\ell(x)|\}$$

the set of vertices $x \in \Lambda_n$ which are (ρ, ℓ)-dense, *i.e.*, surrounded by sufficiently many connected vertices in $\Lambda_\ell(x)$, see also Definition 2 in [11].

Corollary 7. *Under the assumptions of Theorem 6 we have the following.*

(i) *There exists* $\rho > 0$ *such that for any* $x \in \mathbb{Z}^d$

$$\lim_{n\to\infty} \mathbb{P}\left[|\mathcal{C}_n(x)| \geq \rho|\Lambda_n(x)| \big| x \in \mathcal{C}\right] = 1$$

(ii) *For any* $\alpha' \in (\alpha, 2d)$ *there exist* $\rho > 0$ *and* ℓ_0 *such that for any* ℓ *and* n *with* $\ell_0 \leq \ell \leq n/\ell_0$

$$\mathbb{P}\left[|\mathcal{D}_n^{(\rho,\ell)}| \geq \rho|\Lambda_n|\right] \geq 1 - e^{-\rho n^{2d-\alpha'}}$$

This result can be interpreted as local clustering in that with high probability (for large n) particles are surrounded by many other particles belonging to the same connected network. In the sense of the above example this says that an infected individual will transmit its disease to a positive fraction of all individuals in his (Euclidean) neighborhood. Corollary 7 is the analog to Corollaries 3.3 and 3.4 in [11]. Once the proofs of Theorem 6 and Lemma 10 (a), below, are established it follows from the derivations in [11].

3.3. Graph Distances

For $x, y \in \mathbb{Z}^d$ we define $d(x, y)$ to be the minimal number of occupied edges which connect x and y, and we set $d(x, y) = \infty$ for $y \notin \mathcal{C}(x)$. The value $d(x, y)$ is called graph distance or chemical distance between x and y, and it denotes the minimal number of occupied edges that need to be crossed from x to y (and vice versa). If typically $d(x, y)$ is small for distant x and y, then we say that the network has the small-world effect.

Theorem 8. *Assume* $\min\{\alpha, \beta\alpha\} > d$.

(a) *(infinite variance of degree distribution). Assume* $\tau = \beta\alpha/d < 2$. *For any* $\lambda > \lambda_c = 0$ *there exists* $\eta_1 > 0$ *such that for every* $\epsilon > 0$

$$\lim_{|x|\to\infty} \mathbb{P}\left[\eta_1 \leq \frac{d(0,x)}{\log\log|x|} \leq (1+\epsilon)\frac{2}{|\log(\beta\alpha/d - 1)|} \middle| 0, x \in \mathcal{C}\right] = 1$$

(b1) *(finite variance of degree distribution case 1). Assume* $\tau = \beta\alpha/d > 2$ *and* $\alpha \in (d, 2d)$. *For any* $\lambda > \lambda_c$ *and any* $\epsilon > 0$

$$\lim_{|x|\to\infty} \mathbb{P}\left[1 - \varepsilon \leq \frac{\log d(0,x)}{\log\log|x|} \leq (1+\epsilon)\frac{\log 2}{\log(2d/\alpha)} \middle| 0, x \in \mathcal{C}\right] = 1$$

(b2) *(finite variance of degree distribution case 2). Assume* $\min\{\alpha, \beta\alpha\} > 2d$. *There exists* $\eta_2 > 0$ *such that*

$$\lim_{|x|\to\infty} \mathbb{P}\left[\eta_2 < \frac{d(0,x)}{|x|}\right] = 1$$

From Theorem 8 (a) we conclude that in the case $\tau \in (1, 2)$, which is the interesting one for real-life network applications due to the stylized facts, we have a small-world effect and the graph distance is of order $\log \log |x|$ as $|x| \to \infty$. This, for instance, says that if we increase the (Euclidean) distance between two particles in the network by a factor 1000, their graph distance only grows by roughly of order 2. In the case $\tau > 2$ and $\alpha \in (d, 2d)$ (for $\lambda > \lambda_c$) the small-world effect is less pronounced in that the graph distance is conjectured to be of order $(\log |x|)^{\Delta}$ for $|x| \to \infty$. Note that this is a conjecture because the bounds in Theorem 8 (b1) are not sufficiently sharp to obtain the exact power $\Delta > 0$. Finally, in the case $\min\{\alpha, \beta\alpha\} > 2d$ we do not have the small-world effect and graph distance behaves linearly in the Euclidean distance. In Figure 2 we illustrate Theorem 8 and we complete the conjectured picture about the graph distances.

Case (a) of Theorem 8 was proved in Theorems 5.1 and 5.3 of [12]. Statement (b1) proves upper and lower bounds in case 1 of finite variance of the degree distribution. The lower bound was proved in Theorem 5.5 of [12]. The upper bound will be proved below in Proposition 11. Finally, the lower bound in (b2) improves the one given in Theorem 5.6 of [12].

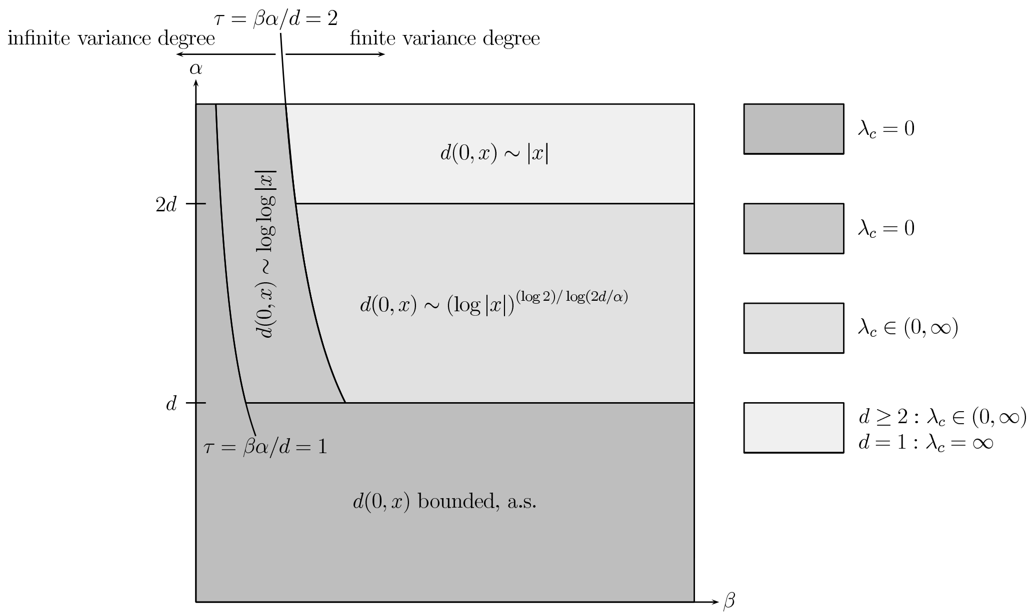

Figure 2. picture about the graph distances (partly as conjecture).

4. Example and Discussion

As an application of the inhomogeneous long-range percolation model we consider the interbank network studied in [4]. This network analyzes the interbank payments transferred between 7584 commercial banks over the Fedwire Funds Service in the United States in the first quarter of 2004. 62 daily networks were studied, where particles represent banks and where there is a link between two banks if at least one transaction between these banks takes place during the day considered. To model such a daily network we use inhomogeneous long-range percolation in $\mathbb{Z}^2$, where particles and links have the above interpretation. The observations of [4] allow to calibrate model parameters and to interpret the results of Section 3.

In fact, [4] considers directed links, meaning that each link also indicates which of the two connected banks sends the payment and which bank receives the payment. This leads to two types of degrees of a given bank: out-degrees (number of banks to which it sends a payment to) and in-degrees (number of banks from which it receives a payment). [4] observes that the distribution of each type is heavy-tailed with estimated tail parameters $\hat{\tau}_{\text{out}} = 1.11$ and $\hat{\tau}_{\text{in}} = 1.15$, respectively. In particular, the degree distribution is heavy-tailed with a tail parameter τ which can directly be estimated from the data (ignoring the direction of the links). The estimates of $\hat{\tau}_{\text{out}}$ and $\hat{\tau}_{\text{in}}$ suggest that the tail parameter $\tau = \beta\alpha/2$ is between 1 and 2, which is in line with the stylized fact of having heavy-tailed degree distributions with finite mean and infinite variance.

Moreover, [4] observes that the total number of payments sent from a given bank to its business partners is heavy-tailed with estimated tail parameter $\hat{\beta}_{\text{out}} = 0.8$. We relate this quantity to the size of a bank and we assume that the weight of a bank has the same tail behavior as the total number of payments it sends out. Therefore, we choose $\hat{\beta} = 0.8$ as a calibration of β. Data on the asset sizes of the banks would allow to calibrate β differently, but the number of transactions can also serve as a good measure for the size of the bank, and the role of a hub function in the network.

The largest connected component consists of around 6490 banks on average, *i.e.*, the network has a giant connected component that contains about 86% of the individual banks on average. This observation supports Theorem 6. Moreover, the average directed graph distance between any two banks is around 2.6 on average, while the maximal directed graph distance is at most 7 in all of the 62 daily networks. Note that the directed graph distance dominates the graph distance of our model and, therefore, we observe that the network shares the stylized fact of having a small-world effect. Theorem 8 allows us to calibrate α in such a way that we get reasonable upper bounds on the graph distances. The theorem states that the graph distances are at most $3.15/|\log(\tau - 1)|$ if we assume that all banks are in a box of side length $88 \approx \sqrt{7584}$. Observe that this bound is sensitive in the choice of $\tau = \beta\alpha/2 \in (1, 2)$ for fixed $\beta = 0.8$. If we choose $\alpha = 2.75$ such that $\tau = 1.1$, the bound is 1.37, while it is 4.54 for $\alpha = 3.75$ and $\tau = 1.5$. Choosing $\alpha = 3.25$ gives $\tau = 1.3$ and the graph distances are bounded by roughly 2.6, which provides a calibration that fits to the observations in [4].

Although one might think that the physical distances between banks have no direct influence on transactions because transactions are done over the Fedwire Funds Service, [5] observes that the physical separation of banks has an influence on the transactions. Namely, [5] splits the network into different districts and finds that there are much more (in terms of value) transactions within a district or transactions between districts that share a common border, compared to transactions between districts that do not share a common border. It is observed that small banks lend money to regional banks which then lend money to big banks. In other words, short links are typically the result of short physical distances, while long links typically arise between large players in the network, and the fact that big banks play the role of hubs. This is in line with the definition of p_{xy} and its local clustering property. In particular, this justifies the use of a model where also physical distances between particles are incorporated.

We conclude that our model can be calibrated to the network of transactions over the Fedwire Funds Service. We choose $d = 2$, $\beta = 0.8$ and $\alpha = 3.25$ which gives tail parameter $\tau = \beta\alpha/d = 1.3$. These parameters provide percolation for any $\lambda > 0$, *i.e.*, there is a giant connected component that contains a positive fraction of all banks. Graph distances are then bounded by roughly 2.6 which coincides with the observations of [4], *i.e.*, we have the small-world effect. Moreover, we have local clustering and power law degrees with $\tau \in (1, 2)$.

In this article, we complemented the picture of graph distances provided in [12] and proved continuity of the percolation probability for $\alpha \in (d, 2d)$ which was conjectured in that article. We see that the model fulfills the stylized fact of having a small-world effect for appropriate model parameters. Moreover, we proved that in case of percolation, the largest connected components in large finite boxes cover a positive faction of these box sizes with high probability. This shows that the model, restricted to a finite box, has a giant connected component. We also showed that the model exhibits a local clustering property in the sense that with high probability, particles are surrounded by many other particles belonging to the same connected component. An important difference in the inhomogeneous long-range percolation model compared to the homogeneous model is that the former fulfills the stylized fact of having heavy-tailed degree distributions. Therefore, the inhomogeneous model is an appealing framework for real-life network modeling, in particular for $\tau \in (1, 2)$ where we obtain an infinite connected network for any $\lambda > 0$ and graph distances in the infinite connected network behave doubly logarithmically. Moreover, the above example shows how the model can be applied to financial networks.

In percolation theory one important problem is to understand the behavior of the model at criticality λ_c. In nearest-neighbor Bernoulli bond percolation on $\mathbb{Z}^d$, where nearest-neighbor edges are vacant or occupied with probability $p \in (0, 1)$, it is known that for $d = 2$ and for $d \geq 19$ there is no percolation at criticality and hence the percolation function is continuous at the critical value (see [24,25] for more details). In cases $3 \leq d \leq 18$ this question is still open. In the homogeneous long-range percolation model it was shown by [22] that there is no percolation at criticality for $\alpha \in (d, 2d)$. It is believed that the long-range percolation model behaves similarly to the nearest-neighbor Bernoulli percolation model when $\alpha > 2d$ and, thus, showing continuity for such values and $d > 1$ remains a difficult problem. In addition, in our model the case $\min\{\alpha, \beta\alpha\} > 2d$ for $d > 1$ is still open which is conjectured to behave as nearest-neighbor Bernoulli percolation, and hence is not of interest for real-life network modeling.

Another problem which remains to be answered in both homogeneous and inhomogeneous long-range percolation is the continuity of the critical parameter $\lambda_c(\alpha)$ as a function of α and also as a function of parameter β, the exponent of the power law in weights (in case of the inhomogeneous model).

There was quite some work done to understand the geometry of the homogeneous long-range percolation model. In particular, there are five different behaviors depending on $\alpha < d, \alpha = d, \alpha \in (d, 2d), \alpha = 2d$ and $\alpha > 2d$, for a review of existing results see discussion in [21]. In some of these cases, like $\alpha = 2d$ (for $d \geq 1$) and $\alpha > 2d$, the results are not yet fully known. The case $d = 1$ and $\alpha = 2$ was resolved recently in [26]. It is clear that in the case of inhomogeneous long-range percolation the complexity even increases due to having more parameters and, hence, degrees of freedom. For instance, the understanding of the graph distance behavior is still poor for $\min\{\alpha, \beta\alpha\} > 2d$, though we believe that it should behave similarly to nearest-neighbor Bernoulli bond percolation.

Moreover, for real-life network applications it will be important to (at least) get reasonable bounds on the percolation probability $\theta(\lambda, \alpha)$ and the optimal constants in Theorem 8. This will allow for model calibration of real-life networks so that (asymptotic) network properties can be studied.

5. Proofs

5.1. Bounds on Percolation on Finite Boxes

The basis for all the proofs of the previous statements is Lemma 9 below which determines large connected components on finite boxes. For integers $m \geq 1$ and $x \in \mathbb{Z}^d$ we define the box of size m^d and lower left corner x by $B_m(x) = x + [0, m-1]^d$, and we abbreviate $B_m = B_m(0)$. Let C_m be the largest connected component in box B_m (with a fixed deterministic rule if there is more than one largest connected component in B_m).

Lemma 9. *Assume* $\min\{\alpha, \beta\alpha\} > d$ *and* $\alpha \in (d, 2d)$. *Choose* $\lambda \in (0, \infty)$ *with* $\theta(\lambda, \alpha) > 0$ *and let* $\alpha' \in [\alpha, 2d)$. *For every* $\varepsilon \in (0,1)$ *and* $\rho > 0$ *there exists* $N_0 \geq 1$ *such that for all* $m \geq N_0$

$$\mathbb{P}\left[|C_m| \geq \rho m^{\alpha'/2}\right] \geq 1 - \varepsilon$$

where C_m *is the largest connected component in box* $B_m = [0, m-1]^d$.

Sketch of proof of Lemma 9. This lemma corresponds to Lemma 2.3 of [22] in our model. Its proof is based on renormalization arguments which only depend on the fact that $\alpha \in (d, 2d)$ and that the edge probabilities are bounded from below by $1 - \exp(-\lambda|x-y|^{-\alpha})$ for any $x, y \in \mathbb{Z}^d$. Using that $W_x \geq 1$ for all $x \in \mathbb{Z}^d$, a.s., we see by stochastic dominance that the renormalization holds also true for our model. Renormalization shows that for m sufficiently large, the probability of $\{B_m$ contains at least a positive fraction of m^d vertices that are connected within a fixed enlargement of $B_m\}$ is bounded by a multiple of the probability of the same event but on a much smaller scale. To bound the latter probability we then use the fact that the model is percolating, and from this we can conclude Lemma 9. We skip the details of the proof of Lemma 9 and refer to the proof of Lemma 2.3 of [27] for the details, in particular, the bound on ψ_n in our homogeneous percolation model (see proof of Lemma 2.3 in [27]) also applies to the inhomogeneous percolation model. □

Although the above lemma does not allow the connected component C_m to have size proportional to the size of box B_m, it is useful because it allows to start a new renormalization scheme to improve these bounds. This results in our Theorem 6 and is done similar as in Section 3 of [11]. For the proof of Theorem 6 we use the following lemma which has two parts. The first one gives the initial step of the renormalization and the second one gives a standard site-bond percolation model result. Once the lemma is established the proof of Theorem 6 becomes a routine task.

Let $C_m(x)$ denote the largest connected component in box $B_m(x)$ (with a fixed deterministic rule if there is more than one largest connected component in $B_m(x)$). For $x, y \in m\mathbb{Z}^d$, we say that boxes $B_m(x)$ and $B_m(y)$ are *pairwise attached*, write $B_m(x) \Leftrightarrow B_m(y)$, if there is an occupied edge between a vertex in $C_m(x)$ and a vertex in $C_m(y)$.

Lemma 10.

(a) Assume $\min\{\alpha, \beta\alpha\} > d$ *and* $\alpha \in (d, 2d)$*. Choose* $\lambda \in (0, \infty)$ *such that* $\theta(\lambda, \alpha) > 0$*. For each* $\xi < \infty$ *and* $r \in (0, 1)$ *there exist* $m < \infty$ *and an integer* $\delta > 0$ *such that*

$$\mathbb{P}\left[|C_m(x)| < \delta|B_m(x)|\right] \le 1 - r,$$
$$\mathbb{P}\left[B_m(x) \Leftrightarrow B_m(y) \middle| |C_m(x)| \ge \delta|B_m(x)|,\ |C_m(y)| \ge \delta|B_m(y)|\right] \ge 1 - e^{-\xi\left(\frac{|x-y|}{m}\right)^{-\alpha}},$$

for all $x \ne y \in m\mathbb{Z}^d$*.*

(b) [Lemma 3.6, [11]] Let $d \ge 1$ *and consider the site-bond percolation model on* $\mathbb{Z}^d$ *with sites being alive with probability* $r \in [0, 1]$ *and sites* $x, y \in \mathbb{Z}^d$ *are attached with probability* $\widetilde{p}_{x,y} = 1 - \exp(-\xi|x - y|^{-\alpha})$ *where* $\alpha \in (d, 2d)$ *and* $\xi \ge 0$*. Let* $|\widetilde{\mathcal{C}}_N|$ *be the size of the largest attached cluster* $\widetilde{\mathcal{C}}_N$ *of living sites in box* B_N*. For each* $\alpha' \in (\alpha, 2d)$ *there exist* $N_0 \ge 1$*,* $\nu > 0$ *and* $\xi_0 < \infty$ *such that*

$$\mathbb{P}_{\xi,r}\left[|\widetilde{\mathcal{C}}_N| \ge \nu|B_N|\right] \ge 1 - e^{-\nu\xi N^{2d-\alpha'}}$$

holds for all $N \ge N_0$ *whenever* $\xi \ge \xi_0$ *and* $r \ge 1 - e^{-\nu\xi}$*.*

Proof of Lemma 10 (a). We adapt the proof of Lemma 3.5 of [11] to our model. Fix $r \in (0, 1)$ and $\xi < \infty$. Choose $\rho > 0$ such that

$$\lambda\left(2\sqrt{d} + 1\right)^{-\alpha}\rho^2 = \xi$$

note that this differs from choice (3.13) in [11]. Lemma 9 then provides that there exists $N_0 \ge 1$ such that for all $m \ge N_0$

$$\mathbb{P}\left[|C_m| < \rho m^{\alpha/2}\right] \le 1 - r$$

For the choice $\delta = \rho m^{\alpha/2-d}$ the first part of the result follows. For the second part we choose $x \ne y \in m\mathbb{Z}^d$. For $x' \in B_m(x)$ and $y' \in B_m(y)$ we have upper bound, using that $W_z \ge 1$ for all $z \in \mathbb{Z}^d$, a.s.,

$$1 - p_{x'y'} \ \le\ \exp\left(-\lambda|x' - y'|^{-\alpha}\right) \ \le\ \exp\left(-\lambda\left(2\sqrt{d} + 1\right)^{-\alpha}|x - y|^{-\alpha}\right) \tag{2}$$

a.s., where the latter no longer depends on the weights $(W_z)_{z\in\mathbb{Z}^d}$. For our choices of δ and ρ, Equation (2) implies

$$\begin{aligned}
&\mathbb{P}\left[B_m(x) \not\Leftrightarrow B_m(y) \middle| |C_m(x)| \ge \delta|B_m(x)|,\ |C_m(y)| \ge \delta|B_m(y)|\right] \\
&\quad = \mathbb{E}\left[\prod_{x'\in C_m(x), y'\in C_m(y)}(1 - p_{x'y'}) \middle| |C_m(x)| \ge \delta|B_m(x)|,\ |C_m(y)| \ge \delta|B_m(y)|\right] \\
&\quad \le \exp\left(-\lambda\left(2\sqrt{d} + 1\right)^{-\alpha}|x - y|^{-\alpha}\rho^2 m^\alpha\right) = \exp\left(-\xi\left(\frac{|x - y|}{m}\right)^{-\alpha}\right)
\end{aligned}$$

This shows the second inequality of part (a). For part (b) we refer to Lemma 3.6 in [11]. □

Proof of Theorem 6. The proof follows as in Theorem 3.2 of [11], we briefly sketch the main argument. Choose the constants $N_0 \geq 1$, $\nu > 0$, $\xi > \xi_0$, $r \geq 1 - e^{-\nu\xi}$ and $\delta > 0$ as in Lemma 10, and note that it is sufficient to prove the theorem for $L = mN$, where $N \geq N_0$ and m is chosen (fixed) as in Lemma 10 (a). In this set up B_L can be viewed as a disjoint union of $B_m(x)$ for $x \in (m\mathbb{Z}^d \cap B_L)$. There are N^d such disjoint boxes. We call $B_m(x)$ alive if $|C_m(x)| \geq \delta|B_m|$ and we say that disjoint $B_m(x)$ and $B_m(y)$ are pairwise attached if their largest connected components $C_m(x)$ and $C_m(y)$ share an occupied edge. Part (a) of Lemma 10 provides that $B_m(x)$ is alive with probability exceeding r and $B_m(x)$ and $B_m(y)$ are pairwise attached with probability exceeding $\widetilde{p}_{x,y}$ for living boxes $B_m(x)$ and $B_m(y)$ with $x, y \in m\mathbb{Z}^d$ (note that in this site-bond percolation model the attachedness property is only considered between living vertices because these form the clusters). For any $N \geq N_0$, let $A_{N,m}$ be the event that box B_L contains a connected component formed by attaching at least $\nu|B_N|$ of the living boxes. On event $A_{N,m}$ we have for the largest connected component in B_L

$$|C_L| \geq (\nu|B_N|)(\delta|B_m|) = \nu\delta|B_L|$$

thus, the volume of the largest connected component C_L in box B_L is proportional to the volume of that box and there remains to show that this occurs with sufficiently large probability. Part (b) of Lemma 10 and stochastic dominance provide (note that we scale $x, y \in m\mathbb{Z}^d$ from Lemma 10 (a) to the site-bond percolation model on $\mathbb{Z}^d$ in Lemma 10 (b))

$$\begin{aligned} \mathbb{P}\left[|C_L| \geq \nu\delta|B_L|\right] \;&\geq\; \mathbb{P}\left[A_{N,m}\right] \;\geq\; \mathbb{P}_{\xi,r}\left[|\widetilde{\mathcal{C}}_N| \geq \nu|B_N|\right] \\ &\geq\; 1 - e^{-\nu\xi N^{2d-\alpha'}} \;=\; 1 - e^{-\nu\xi m^{\alpha'-2d} L^{2d-\alpha'}} \end{aligned}$$

Choosing $\rho \leq \min\{\nu\delta, \nu\xi m^{\alpha'-2d}\}$ provides

$$\mathbb{P}\left[|C_L| \geq \rho|B_L|\right] \geq 1 - e^{-\rho L^{2d-\alpha'}}$$

This finishes the proof of Theorem 6. □

Proof of Corollary 7. The proofs of (i) and (ii) of Corollary 7 follow completely analogous to the proofs of Corollaries 3.3 and 3.4 in [11] (note that Lemma 10 (a) replaces Lemma 3.5 of [11] and Theorem 6 replaces Theorem 3.2 of [11]). □

5.2. Proof of Continuity of the Percolation Probability

The key to the proofs of the continuity statements is again Lemma 9.

Proof of Theorem 3. Note that $\min\{\alpha, \beta\alpha\} > d$ and $\alpha \in (d, 2d)$ imply that $\lambda_c < \infty$. Therefore, there exists $\lambda \in (\lambda_c, \infty)$ with $\theta = \theta(\lambda, \alpha) > 0$. For these choices of $\lambda > 0$ we have a unique infinite cluster $\mathcal{C}$, a.s., and we can apply Lemma 9.

We consider the same site-bond percolation model on $\mathbb{Z}^d$ as in Lemma 10 (b). Choose $\alpha' \in (\alpha, 2d)$, $0 < \chi < 1 - \varepsilon < 1$ and $\kappa > 0$ and define the model as follows: the following events are independent and every site $x \in \mathbb{Z}^d$ is alive with probability $r = 1 - \varepsilon - \chi \in (0, 1)$ and sites $x, y \in \mathbb{Z}^d$ are attached with probability $\widetilde{p}_{xy} = 1 - \exp(-\kappa(1-\chi)|x-y|^{-\alpha'})$. For given $\alpha' \in (\alpha, 2d)$ we choose the parameters

$\varepsilon, \chi, \kappa$ such that there exists an infinite attached cluster of living vertices, a.s., which is possible (see proof of Theorem 2.5 in [22]).

The proof is now similar to the one of Theorem 6. Choose $\rho > 0$ such that $\lambda \left(2\sqrt{d}+1\right)^{-\alpha'} \rho^2 = \kappa$. From Lemma 9 we know that for all m sufficiently large and any $x \in m\mathbb{Z}^d$

$$\mathbb{P}\left[|C_m(x)| \geq \rho m^{\alpha'/2}\right] \geq 1 - \varepsilon > 1 - \varepsilon - \chi = r$$

where $C_m(x)$ denotes the largest connected component in $B_m(x)$. The latter events define alive vertices x on the lattice $m\mathbb{Z}^d$ (which due to scaling is equivalent to the above aliveness in the site-bond percolation model on $\mathbb{Z}^d$). Note that this aliveness property is independent between different vertices $x \in m\mathbb{Z}^d$. Attachedness $B_m(x) \Leftrightarrow B_m(y)$, for $x \neq y \in m\mathbb{Z}^d$, is then used as in the proof of Theorem 6 and we obtain in complete analogy to the proof of the latter theorem

$$\begin{aligned}&\mathbb{P}\left[B_m(x) \Leftrightarrow B_m(y) \Big| |C_m(x)| \geq \rho m^{\alpha'/2}, |C_m(y)| \geq \rho m^{\alpha'/2}\right] \\ &\quad \geq 1 - \exp\left(-\lambda\left(2\sqrt{d}+1\right)^{-\alpha} |x-y|^{-\alpha}\rho^2 m^{\alpha'}\right) \geq 1 - \exp\left(-\kappa\left(\frac{|x-y|}{m}\right)^{-\alpha'}\right)\end{aligned}$$

where in the last step we used the choice of ρ and the fact that $\alpha < \alpha'$. Since $\kappa > \kappa(1-\chi)$ we get percolation and there exists an infinite cluster $\mathcal{C}$, a.s., which implies $\theta(\lambda, \alpha) > 0$. Of course, this is no surprise because of the choice $\lambda > \lambda_c$ with $\theta(\lambda, \alpha) > 0$.

Note that the probability of a vertex $x \in m\mathbb{Z}^d$ being alive depends only on finitely many edges of maximal distance $\sqrt{d}m$ (they all lie in the box $B_m(x)$) and therefore this probability is a continuous function of λ and α. This implies that we can choose $\delta \in (0, \chi\lambda)$ and $\gamma \in (0, \alpha' - \alpha)$ so small that

$$\mathbb{P}_{\lambda-\delta,\alpha+\gamma}\left[|C_m(x)| \geq \rho m^{\alpha'/2}\right] \geq 1 - \varepsilon - \chi = r$$

where $\mathbb{P}_{\lambda-\delta,\alpha+\gamma}$ is the measure where for occupied edges we replace parameters λ by $\lambda - \delta \in (0, \lambda)$ and α by $\alpha + \gamma \in (\alpha, \alpha')$. As above we obtain, note $\alpha + \gamma < \alpha'$,

$$\begin{aligned}&\mathbb{P}_{\lambda-\delta,\alpha+\gamma}\left[B_m(x) \Leftrightarrow B_m(y) \Big| |C_m(x)| \geq \rho m^{\alpha'/2}, |C_m(y)| \geq \rho m^{\alpha'/2}\right] \\ &\quad \geq 1 - \exp\left(-(\lambda-\delta)\left(2\sqrt{d}+1\right)^{-(\alpha+\gamma)} |x-y|^{-(\alpha+\gamma)}\rho^2 m^{\alpha'}\right) \\ &\quad \geq 1 - \exp\left(-\kappa\left(1-\delta/\lambda\right)\left(\frac{|x-y|}{m}\right)^{-\alpha'}\right)\end{aligned}$$

Since $\delta/\lambda < \chi$ we get percolation and there exists an infinite cluster $\mathcal{C}$, a.s., which implies that $\theta(\lambda - \delta, \alpha + \gamma) > 0$. This finishes the proof of Theorem 3. □

Proof of Theorem 5. We need to modify Proposition 1.3 of [23] because in our model, edges are not occupied independently induced by the random choices of weights $(W_x)_{x \in \mathbb{Z}^d}$.

(i) From Theorem 3 it follows that $\theta(\lambda, \alpha) = 0$ for all $\lambda \in (0, \lambda_c]$, which proves continuity of $\lambda \mapsto \theta(\lambda, \alpha)$ on $(0, \lambda_c]$.

(ii) Next we show that $\lambda \mapsto \theta(\lambda, \alpha)$ is left-continuous on $\lambda > \lambda_c$, that is,

$$\lim_{\lambda' \uparrow \lambda} \theta(\lambda', \alpha) = \theta(\lambda, \alpha) \tag{3}$$

To prove this we couple all percolation realization as λ varies. This is achieved by randomizing the percolation constant λ, see [23,28]. Conditionally given the i.i.d. weights $(W_x)_{x\in\mathbb{Z}^d}$, define a collection of independent exponentially distributed random variables $\phi_{(x,y)}$, indexed by the edges (x, y), which have conditional distribution

$$\mathbf{P}\left[\phi_{(x,y)} \le \ell \,\middle|\, (W_x)_{x\in\mathbb{Z}^d}\right] = 1 - \exp\left(-\frac{\ell W_x W_y}{|x-y|^\alpha}\right), \qquad \ell \in (0,\infty) \tag{4}$$

compare to Equation (1). We denote the probability measure of $(\phi_{(x,y)})_{x,y\in\mathbb{Z}^d}$ by $\mathbf{P}$ in order to distinguish this coupling model. We say that an edge (x, y) is ℓ-open if $\phi_{(x,y)} < \ell$, and we define the connected cluster $C_\ell(0)$ of the origin to be the set of all vertices $x \in \mathbb{Z}^d$ which are connected to the origin by an ℓ-open path. Note that we have a natural ordering in ℓ, *i.e.*, for $\ell_1 < \ell_2$ we obtain $C_{\ell_1}(0) \subset C_{\ell_2}(0)$. Moreover for $\ell = \lambda > 0$, the λ-open edges are exactly the occupied edges in this coupling (note that the exponential distribution Equation (4) is absolutely continuous). This implies for $\ell = \lambda$

$$\theta(\lambda, \alpha) = \mathbb{P}\left[|\mathcal{C}(0)| = \infty\right] = \mathbf{P}\left[|C_\lambda(0)| = \infty\right]$$

By countable subadditivity of $\mathbf{P}$ and the increasing property of $C_\ell(0)$ in ℓ we have

$$\lim_{\lambda'\uparrow\lambda} \theta(\lambda', \alpha) = \mathbf{P}\left[|C_{\lambda'}(0)| = \infty \text{ for some } \lambda' < \lambda\right]$$

Moreover, the increasing property of $C_\ell(0)$ in ℓ provides $\{|C_{\lambda'}(0)| = \infty \text{ for some } \lambda' < \lambda\} \subset \{|C_\lambda(0)| = \infty\}$. Therefore, to prove Equation (3) it suffices to show that

$$\mathbf{P}\left[\{|C_{\lambda'}(0)| < \infty \text{ for all } \lambda' < \lambda\} \cap \{|C_\lambda(0)| = \infty\}\right] = 0$$

Choose $\lambda_0 \in (\lambda_c, \lambda)$. Since there is a unique infinite cluster for $\lambda_0 > \lambda_c$, a.s., there exists an infinite cluster $C_{\lambda_0} \subset C_\lambda(0)$ on the set $\{|C_\lambda(0)| = \infty\}$. If the origin belongs to C_{λ_0} then the proof is done. Otherwise, because C_{λ_0} is a subgraph of $C_\lambda(0)$, there exists a finite path π of λ-open edges connecting the origin with an edge in C_{λ_0}. By the definition of λ-open edges we have $\phi_{(x,y)} < \lambda$ for all edges $(x, y) \in \pi$. Since π is finite we obtain the strict inequality $\lambda_1 = \max_{(x,y)\in\pi} \phi_{(x,y)} < \lambda$. Choose $\lambda' \in (\lambda_0 \vee \lambda_1, \lambda)$ and it follows that $|C_{\lambda'}(0)| = \infty$. This completes the proof for the left-continuity in λ.
(iii) Finally, we need to prove right-continuity of $\lambda \mapsto \theta(\lambda, \alpha)$ on $\lambda \ge \lambda_c$. For integers $n > 1$ we consider boxes $\Lambda_n = [-n, n]^d$ centered at the origin, see also Theorem 6. We define the events $A_n = \{\mathcal{C}(0) \cap \Lambda_n^c \ne \emptyset\}$, *i.e.*, the connected component $\mathcal{C}(0)$ of the origin leaves box Λ_n. Note that $\theta(\lambda, \alpha)$ is the decreasing limit of $\mathbb{P}[A_n]$ as $n \to \infty$. Therefore, it suffices to show that $\mathbb{P}[A_n]$ is a continuous function in λ. We write $\mathbb{P}_\lambda = \mathbb{P}$ to indicate on which parameter λ the probability law depends. We again denote by $\mathcal{C}_n(0)$ the connected component of the origin connected within box Λ_n, see Corollary 7. Then, we have

$$A_n = \{\mathcal{C}(0) \cap \Lambda_n^c \ne \emptyset\} = \{\mathcal{C}_n(0) \Leftrightarrow \Lambda_n^c\}$$

Choose $\delta_0 \in (0, \lambda)$, then we have for all $\lambda' \in (\lambda - \delta_0, \lambda + \delta_0)$ and all $n' > n$

$$\begin{aligned}
|\mathbb{P}_\lambda[A_n] - \mathbb{P}_{\lambda'}[A_n]| &= |\mathbb{P}_\lambda[\mathcal{C}_n(0) \Leftrightarrow \Lambda_n^c] - \mathbb{P}_{\lambda'}[\mathcal{C}_n(0) \Leftrightarrow \Lambda_n^c]| \\
&\le |\mathbb{P}_\lambda[\mathcal{C}_n(0) \Leftrightarrow (\Lambda_n^c \cap \Lambda_{n'})] - \mathbb{P}_{\lambda'}[\mathcal{C}_n(0) \Leftrightarrow (\Lambda_n^c \cap \Lambda_{n'})]| \\
&\quad +2\, \mathbb{P}_{\lambda+\delta_0}[\mathcal{C}_n(0) \Leftrightarrow \Lambda_{n'}^c] \\
&\le |\mathbb{P}_\lambda[\mathcal{C}_n(0) \Leftrightarrow (\Lambda_n^c \cap \Lambda_{n'})] - \mathbb{P}_{\lambda'}[\mathcal{C}_n(0) \Leftrightarrow (\Lambda_n^c \cap \Lambda_{n'})]| \\
&\quad + 2(2n+1)^d \sup_{x\in\Lambda_n} \mathbb{P}_{\lambda+\delta_0}[x \Leftrightarrow \Lambda_{n'}^c]
\end{aligned} \tag{5}$$

We bound the two terms on the right-hand side of Equation (5).
(a) First we prove that for all $\varepsilon > 0$ there exists $n' > n$ such that for all $x \in \Lambda_n$

$$\mathbb{P}_{\lambda+\delta_0}\left[x \Leftrightarrow \Lambda_{n'}^{\mathrm{c}}\right] < \varepsilon(2n+1)^{-d}/4 \tag{6}$$

This is done as follows. For $m > n$ we define the following events

$$L_m = \{x \Leftrightarrow \partial\Lambda_{m+1}\} = \{x \Leftrightarrow (\Lambda_{m+1} \setminus \Lambda_m)\}$$

This implies for $n' > n$ that

$$E_{n'} \stackrel{\text{def.}}{=} \{x \Leftrightarrow \Lambda_{n'}^{\mathrm{c}}\} = \bigcup_{m \geq n'} L_m$$

Moreover, note that $E_{n'}$ is decreasing in n',

$$\limsup_{n'\to\infty} \mathbb{P}_{\lambda+\delta_0}\left[E_{n'}\right] = \lim_{n'\to\infty} \mathbb{P}_{\lambda+\delta_0}\left[E_{n'}\right] = \mathbb{P}_{\lambda+\delta_0}\left[\bigcap_{n'>n} E_{n'}\right] = \mathbb{P}_{\lambda+\delta_0}\left[\bigcap_{n'>n}\left(\bigcup_{m\geq n'} L_m\right)\right]$$

We prove Equation (6) by contradiction. Assume that Equation (6) does not hold true, *i.e.*, $\limsup_{n'\to\infty} \mathbb{P}\left[E_{n'}\right] > 0$. Then the first lemma of Borel-Cantelli implies

$$\infty = \sum_{m>n} \mathbb{P}_{\lambda+\delta_0}\left[L_m\right] = \sum_{m>n} \mathbb{P}_{\lambda+\delta_0}\left[x \Leftrightarrow (\Lambda_{m+1} \setminus \Lambda_m)\right] = \mathbb{E}_{\lambda+\delta_0}\left[\sum_{m>n} 1_{\{x\Leftrightarrow(\Lambda_{m+1}\setminus\Lambda_m)\}}\right]$$

The latter implies that the degree distribution $D_x = |\{y \in \mathbb{Z}^d; x \Leftrightarrow y\}|$ has an infinite mean. This is a contradiction to Theorem 2.2 of [12] saying that for $\min\{\alpha, \beta\alpha\} > d$ the survival function of the degree distribution has a power-law decay with rate $\alpha\beta/d > 1$ which provides a finite mean. Therefore, Equation (6) holds true.
(b) For all $\varepsilon > 0$ and all $n' > n$ there exists $\delta_1 \in (0, \delta_0)$ such that for all $\lambda' \in (\lambda - \delta_1, \lambda + \delta_1)$

$$\left|\mathbb{P}_\lambda\left[\mathcal{C}_n(0) \Leftrightarrow (\Lambda_n^{\mathrm{c}} \cap \Lambda_{n'})\right] - \mathbb{P}_{\lambda'}\left[\mathcal{C}_n(0) \Leftrightarrow (\Lambda_n^{\mathrm{c}} \cap \Lambda_{n'})\right]\right| < \varepsilon/2 \tag{7}$$

Note that $\Lambda_{n'}$ only contains finitely many edges of finite distance. Therefore, continuity in λ is straightforward which provides Equation (7).

Combining Equations (6) and (7) provides continuity of $\mathbb{P}_\lambda[A_n]$ in λ for all n, see also Equation (5). Therefore, right-continuity of $\lambda \mapsto \theta(\lambda, \alpha)$ follows. This finishes the proof of Theorem 5. □

5.3. Proofs of the Graph Distances

In this section, we prove Theorem 8. Statement (a) of Theorem 8 is proved in Theorems 5.1 and 5.3 of [12], the lower bound of statement (b1) is proved in Theorem 5.5 of [12]. Therefore, there remain the proofs of the upper bound in (b1) and of the lower bound in (b2) of Theorem 8.

The proof of the upper bound in Theorem 8 (b1) follows from the following proposition and the fact that $\alpha \mapsto \Delta(\alpha, 2d) = \log 2/\log(2d/\alpha)$ is a continuous function. The following proposition corresponds to Proposition 4.1 in [11] in the homogeneous long-range percolation model.

Proposition 11. *Let $\alpha \in (d, 2d)$ and $\tau = \beta\alpha/d > 2$ and $\lambda > \lambda_c$. For each $\Delta' > \Delta = \Delta(\alpha, 2d) = \log 2/\log(2d/\alpha)$ and each $\varepsilon > 0$, there exists $N_0 < \infty$ such that*

$$\mathbb{P}\left[d(x,y) \geq (\log|x-y|)^{\Delta'}, x, y \in \mathcal{C}\right] \leq \varepsilon$$

holds for all $x, y \in \mathbb{Z}^d$ with $|x - y| \geq N_0$.

Sketch of proof of Proposition 11. We only sketch the proof because it is almost identical to the one in [11]. Definition 1 and Figure 1 of [11] define for $x, y \in \mathbb{Z}^d$ a hierarchy of depth $m \in \mathbb{N}$ connecting x and y as the following collection of vertices:

$$\mathcal{H}_m(x,y) = \left\{z_\sigma \in \mathbb{Z}^d;\ \sigma \in \{0,1\}^k \text{ for } k = 1, \ldots, m\right\}$$

is a hierarchy of depth $m \in \mathbb{N}$ connecting x and y if

(1) $z_0 = x$ and $z_1 = y$,
(2) $z_{\sigma 00} = z_{\sigma 0}$ and $z_{\sigma 11} = z_{\sigma 1}$ for all $k = 0, \ldots, m-2$ and $\sigma \in \{0,1\}^k$,
(3) for all $k = 0, \ldots, m-2$ and and $\sigma \in \{0,1\}^k$ such that $z_{\sigma 01} \neq z_{\sigma 10}$ the edge between $z_{\sigma 01}$ and $z_{\sigma 10}$ is occupied,
(4) each bond $(z_{\sigma 01}, z_{\sigma 10})$ specified in (3) appears only once in $\mathcal{H}_k(x,y)$.

The pairs of vertices $(z_{\sigma 00}, z_{\sigma 01})$ and $(z_{\sigma 10}, z_{\sigma 11})$ are called gaps. The proof is then based on the fact that for large distances $|x - y|$ the event $\mathcal{B}_m$ of the existence of a hierarchy $\mathcal{H}_m(x,y)$ of depth m that connects x and y through points z_σ which are dense is very likely (m appropriately chosen), see Lemma 4.3 in [11], in particular formula (4.18) in [11] (where the key is Corollary 7 (ii)). On this likely event $\mathcal{B}_m$, Lemma 4.2 of [11] then proves that the graph distance cannot be too large, see (4.8) in [11]. We can now almost literally translate Lemmas 4.2 and 4.3 of [11] to our situation. The only changes are that in formulas (4.16) and (4.21) of [11] we need to replace $\beta > 0$ of [11]'s notation by λ in our notation and we need to use that the weights W_x are at least one, a.s. We refrain from giving more details. □

There remains the proof of the lower bound in (b2) of Theorem 8. We use a renormalization technique which is based on a scheme introduced by [13]. Choose an integer valued sequence $a_n > 1$, $n \in \mathbb{N}_0$, and define the box lengths $(m_n)_{n \in \mathbb{N}_0}$ as follows: set $m_0 = a_0$ and for $n \in \mathbb{N}$,

$$m_n = a_n m_{n-1} = m_0 \prod_{i=1}^{n} a_i = \prod_{i=0}^{n} a_i$$

Define the n-stage boxes, $n \in \mathbb{N}_0$, by

$$B_{m_n}(x) = x + [0, m_n - 1]^d, \qquad \text{for } x \in \mathbb{Z}^d$$

For $n \geq 1$, the children of n-stage box $B_{m_n}(x)$ are the a_n^d disjoint $(n-1)$-stage boxes

$$B_{m_{n-1}}(x + ym_{n-1}) = x + ym_{n-1} + [0, m_{n-1} - 1]^d \subset \mathbb{Z}^d \qquad \text{with } y \in ([0, a_n - 1]^d \cap \mathbb{Z}^d)$$

We are going to define *good* n-stage boxes $B_{m_n}(\cdot)$, note that we need a different definition from Definition 2 of [13].

Definition 12 (good n-stage boxes)**.** *Choose $n \in \mathbb{N}_0$ and $x \in \mathbb{Z}^d$ fixed.*

- *0-stage box $B_{m_0}(x)$ is good under a given edge configuration if there is no occupied edge in $B_{m_0}(x)$ with size larger than $m_0/100$.*
- *n-stage box $B_{m_n}(x)$, $n \geq 1$, is good under a given edge configuration if for all $j \in \{-1,0,1\}^d$*
  - *(a) there is no occupied edge in $B_{m_n}\left(x + j\frac{m_{n-1}}{2}\right)$ with size larger than $m_{n-1}/100$; and*
  - *(b) among the children of $B_{m_n}\left(x + j\frac{m_{n-1}}{2}\right)$ there are at most 3^d that are not good.*

Lemma 13. *Assume $\min\{\alpha, \beta\alpha\} > d$. For all $\delta \in (0, \alpha(\beta \wedge 1) - d)$ there exist $t_0 \geq 1$ and a constant $c_1 > 0$ such that for all $t \geq t_0$ and all $s \geq 1$,*

$$\mathbb{P}\left[\textit{there is an occupied edge in } [0, s-1]^d \textit{ with size larger than } t\right] \leq c_1 s^d t^{d-\alpha(\beta\wedge 1)+\delta}$$

Proof of Lemma 13. Let W_1 and W_2 be two independent random variables each having a Pareto distribution with parameters $\theta = 1$ and $\beta > 0$. For $u \geq 1$ we have, using integration by parts in the first step,

$$\begin{aligned}
\mathbb{E}\left[\frac{W_1 W_2}{u} \wedge 1\right] &= \frac{1}{u} + \frac{1}{u}\int_1^u \mathbb{P}[W_1 W_2 > v]dv \;=\; \frac{1}{u} + \frac{1}{u}\int_1^u v^{-\beta}(1+\beta\log v)dv \\
&\leq (1+\beta\log u)\left(u^{-(\beta\wedge 1)} + \frac{1}{u}\int_1^u v^{-\beta}dv\right) \\
&\leq \max\{1+\log u, 1 + 1_{\{\beta\neq 1\}}/|\beta - 1|\}\,(1+\beta\log u)\,u^{-(\beta\wedge 1)}
\end{aligned}$$

where the last step follows by distinguishing between the cases $\beta = 1$, $\beta > 1$ and $\beta < 1$. Choose t_0 so large that $\lambda^{-1}t_0^\alpha \geq 1$ which, together with the above calculations, implies that for all $t \geq t_0$ and $x, y \in \mathbb{Z}^d$ with $|x-y| > t \geq t_0$,

$$\begin{aligned}
\mathbb{E}\left[\frac{\lambda W_x W_y}{|x-y|^\alpha} \wedge 1\right] &\leq \left(1 + 1_{\{\beta\neq 1\}}/|\beta-1|\right)\left(1 + \max\{1,\beta\}\log(\lambda^{-1}|x-y|^\alpha)\right)^2 \left(\lambda^{-1}|x-y|^\alpha\right)^{-(\beta\wedge 1)} \\
&\leq |x-y|^{-\alpha(\beta\wedge 1)+\delta}
\end{aligned}$$

where the second inequality holds for all $|x-y| > t \geq t_0$ with t_0 large enough. It follows that for all $t \geq t_0$, using $1 - e^{-x} \leq x \wedge 1$,

$$\begin{aligned}
&\mathbb{P}\left[\text{there is an occupied edge in } [0,s-1]^d \text{ with size larger than } t\right] \leq \sum_{\substack{x,y\in[0,s-1]^d:\\ |x-y|>t}} \mathbb{E}\left[\frac{\lambda W_x W_y}{|x-y|^\alpha} \wedge 1\right] \\
&\leq \sum_{\substack{x,y\in[0,s-1]^d:\\ |x-y|>t}} |x-y|^{-\alpha(\beta\wedge 1)+\delta} \leq s^d \sum_{y\in\mathbb{Z}^d:\, |y|>t} |y|^{-\alpha(\beta\wedge 1)+\delta}
\end{aligned}$$

Hence, for an appropriate constant $c_1 > 0$ and for all $t \geq t_0$ with t_0 sufficiently large,

$$\mathbb{P}\left[\text{there is an occupied edge in } [0,s-1]^d \text{ with size larger than } t\right] \leq c_1 s^d t^{d-\alpha(\beta\wedge 1)+\delta}$$

which finishes the proof of Lemma 13. □

Lemma 14. *Assume* $\min\{\alpha, \beta\alpha\} > 2d$*. For* $a_n = n^2$*,* $n \geq 1$*, and* a_0 *sufficiently large we have*

$$\sum_{n\geq 0} \mathbb{P}\left[B_{m_n}(0) \text{ is not good }\right] < \infty$$

This lemma is the analog in our model to Lemma 1 of [13] and provides a Borel-Cantelli type of result that eventually the boxes $B_{m_n}(0)$ are good, a.s., for all n sufficiently large.

Proof of Lemma 14. We prove by induction that $\psi_n = \mathbb{P}\left[B_{m_n}(0) \text{ is not good}\right]$ is summable. Choose $\delta \in (0, \alpha(\beta \wedge 1) - 2d)$ and set $\gamma = \min\{\alpha, \beta\alpha\} - 2d - \delta > 0$. For m_0 sufficiently large we obtain by Lemma 13,

$$\begin{aligned} \psi_0 &= \mathbb{P}\left[\text{there is an occupied edge in } B_{m_0}(0) \text{ with size larger than } m_0/100\right] \\ &\leq c_1 m_0^d \left(\frac{m_0}{100}\right)^{d-\alpha(\beta\wedge 1)+\delta} < 3^{-d}2^{-4d-1}e^{-2} \end{aligned} \tag{8}$$

where the last step holds true for m_0 sufficiently large. Because $B_{m_1}(0)$ has only one child (because $a_1 = 1$) we get for m_0 sufficiently large

$$\psi_1 \leq 3^d\psi_0 \leq c_1 3^d m_0^d \left(\frac{m_0}{100}\right)^{d-\alpha(\beta\wedge 1)+\delta} < 3^{-d}2^{-8d-1}e^{-4} \tag{9}$$

For the induction step we note that n-stage box $B_{m_n}(0)$ is not good if at least one of the 3^d translations $B_{m_n}(0 + j\frac{m_{n-1}}{2})$, $j \in \{-1, 0, 1\}^d$, fails to have property (a) or (b) of Definition 12. Using translation invariance and Lemma 13 we get for all $n \geq 2$ and for all m_0 sufficiently large, set $c_2 = c_1 100^{\alpha(\beta\wedge 1)-d-\delta}$,

$$\psi_n \leq 3^d \left(c_2 a_n^{\alpha(\beta\wedge 1)-d-\delta} m_n^{-\gamma} + \mathbb{P}\left[\text{there are at least } 3^d+1 \text{ children of } B_{m_n}(0) \text{ that are not good}\right]\right)$$

Note that the event in the probability above ensures that there are at least two children $B_{m_{n-1}}(y)$ and $B_{m_{n-1}}(z)$ of $B_{m_n}(0)$ that are not good and are separated by at least Euclidean distance $2m_{n-1}$. Therefore, using $m_i = a_0(i!)^2$, $i \geq 0$, the two boxes $B_{m_{n-1}}(y)$ and $B_{m_{n-1}}(z)$ are well separated in the sense that the events $\{B_{m_{n-1}}(y) \text{ is not good}\}$ and $\{B_{m_{n-1}}(z) \text{ is not good}\}$ are independent. Note that for the latter we need to make sure that $B_{m_{n-1}}(y + jm_{n-2}/2)$ and $B_{m_{n-1}}(z + lm_{n-2}/2)$ are disjoint for all $j, l \in \{-1, 0, 1\}^d$, which is the case because $B_{m_{n-1}}(y)$ and $B_{m_{n-1}}(z)$ have at least distance $2m_{n-1}$. The independence implies the following bound

$$\begin{aligned} \psi_n &\leq 3^d \left(c_2 a_n^{\alpha(\beta\wedge 1)-d-\delta} m_n^{-\gamma} + \binom{a_n^d}{2}\psi_{n-1}^2\right) \leq 3^d \left(c_2 a_n^{\alpha(\beta\wedge 1)-d-\delta} m_n^{-\gamma} + a_n^{2d}\psi_{n-1}^2\right) \\ &= 3^d \left(c_2 n^{2(\gamma+d)} \left(m_0(n!)^2\right)^{-\gamma} + n^{4d}\psi_{n-1}^2\right) = c_2 3^d m_0^{-\gamma} n^{2(\gamma+d)}(n!)^{-2\gamma} + 3^d n^{4d}\psi_{n-1}^2 \end{aligned}$$

It follows that there is $n_0 < \infty$ such that for all for all $n \geq n_0$ and m_0 large enough

$$\psi_n \leq 3^{-d}2^{-4d-2}e^{-2}(n+1)^{-4d}e^{-2n} + 3^d n^{4d}\psi_{n-1}^2 \tag{10}$$

and we can choose m_0 so large that Equation (10) holds true also for all $2 \leq n < n_0$. We claim that for all $a_0 = m_0$ sufficiently large and all $n \geq 0$,

$$\psi_n \leq 3^{-d}2^{-4d-1}e^{-2}(n+1)^{-4d}e^{-2n} \tag{11}$$

which will imply Lemma 14 because the right-hand side is summable. Indeed, Equation (11) is true for $n \in \{0, 1\}$ by Equations (8) and (9). Assuming that Equation (11) holds for all $n-1$ with $n \geq 2$ we get, using Equation (10),

$$\begin{aligned} \psi_n &\leq 3^{-d}2^{-4d-2}e^{-2}(n+1)^{-4d}e^{-2n} + 3^d n^{4d}\psi_{n-1}^2 \\ &\leq 3^{-d}2^{-4d-2}e^{-2}(n+1)^{-4d}e^{-2n} + 3^{-d}n^{-4d}2^{-8d-2}e^{-4}e^{-4n+4} \\ &= 3^{-d}2^{-4d-1}e^{-2}(n+1)^{-4d}e^{-2n}\left(2^{-1} + \left(\frac{n+1}{n}\right)^{4d} 2^{-4d-1}e^{-2n+2}\right) \\ &\leq 3^{-d}2^{-4d-1}e^{-2}(n+1)^{-4d}e^{-2n}\left(2^{-1}+2^{-1}\right) \end{aligned}$$

where the last step follows since $(n+1)/n \leq 2$ and $e^{-2n+2} \leq 1$. □

The following lemma is the analog of Proposition 3 of [13] and it depends on Lemma 2 of [13] and Lemma 14. Since its proof is completely similar to the one of Proposition 3 of [13] once Lemma 14 has been established we skip this proof.

Lemma 15 (Proposition 3 of [13])**.** *Choose $a_n = n^2$ for $n \geq 1$. There exists a constant $c_3 > 0$ such that for every n sufficiently large, if for every $j \in \{-1, 0, 1\}^d$ the n-stage box $B_{m_n}\left(0 + j\frac{m_n}{2}\right)$ is good and for every $l > n$ the l-stage boxes $\widehat{B}_{m_l}$ centered at $B_{m_n}(0)$ are good, then if $x, y \in B_{m_n}(0)$ satisfy $|x-y| > m_n/8$ then $d(x, y) \geq c_3|x-y|$.*

Proof of Theorem 8 (b2). Lemma 14 says that, a.s., the l-stage boxes $\widehat{B}_{m_l}$ are eventually good for all $l \geq n$. Moreover, from Lemma 15 we obtain the linearity in the distance for these good boxes which says that, a.s., for n sufficiently large and $|x| > m_n/8$ we have $d(0, x) \geq c_3|x|$. □

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Author Contributions

All authors contributed significantly to all aspects of this work.

Conflicts of Interest

The authors declare no conflict of interest.

References

1. Newman, M.E.J.; Watts, D.J.; Strogatz, S.H. Random graph models of social networks. *Proc. Natl. Acad. Sci. USA* **2002**, *99*, 2566–2572.
2. Amini, H.; Cont, R.; Minca, A. Stress testing the resilience of financial networks. *Int. J. Theor. Appl. Finance* **2012**, *15*, 1250006–1250020.

3. Cont, R.; Moussa, A.; Santos, E.B. Network structure and systemic risk in banking system. *SSRN Server.* **2010**, doi:10.2139/ssrn.1733528.
4. Soramäki, K.; Bech, M.; Arnold, J.; Glass, R.; Beyeler, W. The topology of interbank payment flows. *Phys. A* **2007**, *379*, 317–333.
5. Bech, M.; Atalay, E. The topology of the federal funds market. *Phys. A* **2010**, *389*, 5223–5246.
6. Grilli, R.; Tedeschi, G.; Gallegati, M. Bank interlinkages and macroeconomic stability. *Int. Rev. Econ. Finance* **2014**, *34*, 72–88.
7. Albert, R.; Barabási, A.L. Statistical mechanics of complex networks. *Rev. Modern Phys.* **2002**, *74*, 47–97.
8. Durrett, R. *Random Graph Dynamics*; Cambridge University Press: New York, NY, USA, 2007.
9. Watts, D.J. *Six Degrees: The Science of a Connected Age*; W.W. Norton: New York, NY, USA, 2003.
10. Ugander, J.; Karrer, K.; Backstrom, L.; Marlow, C. The anatomy of the Facebook social graph. **2011**, arXiv:1111.4503v1.
11. Biskup, M. On the scaling of the chemical distance in long-range percolation models. *Ann. Probab.* **2004**, *32*, 2938–2977.
12. Deijfen, M.; van der Hofstad, R.; Hooghiemstra, G. Scale-free percolation. *Ann. Inst. Henri Poincaré Probab. Stat.* **2013**, *49*, 817–838.
13. Berger, N. A lower bound for the chemical distance in sparse long-range percolation models. **2008**, arXiv:math/0409021v1.
14. Biskup, M. Graph diameter in long-range percolation. *Random Struct. Algorithms* **2011**, *39*, 210–227.
15. Trapman, P. The growth of the infinite long-range percolation cluster. *Ann. Probab.* **2010**, *38*, 1583–1608.
16. Aizenman, M.; Newman, C.M. Discontinuity of the percolation density in one-dimensional $1/|x-y|^2$ percolation models. *Comm. Math. Phys.* **1986**, *107*, 611–647.
17. Newman, C.M.; Schulman, L.S. One dimensional $1/|j-i|^s$ percolation models: the existence of a transition for $s \leq 2$. *Comm. Math. Phys.* **1986**, *104*, 547–571.
18. Schulman, L.S. Long-range percolation in one dimension. *J. Phys. A* **1983**, *16*, L639–L641.
19. Benjamini, I.; Berger, N. The diameter of long-range percolation clusters on finite cycles. *Random Struct. Algorithms* **2001**, *19*, 102–111.
20. Coppersmith, D.; Gamarnik, D.; Sviridenko, M. The diameter of a long-range percolation graph. *Random Struct. Algorithms* **2002**, *21*, 1–13.
21. Chatterjee, S.; Dey, P.S. Multiple phase transitions in long-range first-passage percolation on square lattices. **2013**, arXiv:1309.5757.
22. Berger, N. Transience, recurrence and critical behavior for long-range percolation. *Comm. Math. Phys.* **2002**, *226*, 531–558.
23. Aizenman, M.; Kesten, H.; Newman, C.M. Uniqueness of the infinite cluster and continuity of connectivity functions for short- and long-range percolation. *Comm. Math. Phys.* **1987**, *111*, 505–531.

24. Hara, T.; Slade, G. Mean-field critical behavior for percolation in high dimensions. *Comm. Math. Phys.* **1990**, *128*, 333–391.
25. Grimmett, G.R. *Percolation*, 2nd ed.; Springer: New York, NY, USA, 1999.
26. Ding, J.; Sly, A. Distances in critical long range percolation. **2013**, arXiv:1303.3995.
27. Berger, N. Transience, recurrence and critical behavior for long-range percolation. **2014**, arXiv:math/0110296v3.
28. Van den Berg, J.; Keane, M. On the continuity of the percolation probability function. *Contemp. Math.* **1984**, *26*, 61–65.