

Initial and Final Value Theorem on Fractional Hankel Transform

¹A.S.Gudadhe, ²R.D.Taywade, ³V.N.Mahalle

¹Deptt.of Mathematics, Govt. Vidarbha Institute of Science & Humanities, Amravati.

²Deptt.of Mathematics, Prof. Ram Meghe Institute of Technology & Research, Badnera, Amravati.

³Deptt.of Mathematics, Bar. R.D.I.K.N.K.D. College, Badnera, Amravati.

Abstract: The fractional Hankel transform which is a generalization of the Hankel transform has many applications. In this paper we establish the initial and final value theorem for the generalized fractional Hankel transform.

Keywords: Fourier transform, Hankel transform, fractional Hankel transform.

I. Introduction:

Namias [1, 2] introduced the concept of Fourier transform of fractional order by the method of eigen values and open the path of defining the number of fractional integral transform having applications in quantum mechanics [1], Optics [3] and Signal processing [4]. Hankel transform is also generalized to its fractional version. Namias [2] himself has presented the formula for fractional Hankel transform as

$$H_{\alpha}[f(x)](y) = \frac{e^{-i(v+1)\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)}}{\sin \frac{\alpha}{2}} \int_0^{\infty} x e^{-\frac{i}{2}(x^2+y^2)\cot \frac{\alpha}{2}} J_{\nu}\left(\frac{xy}{\sin \frac{\alpha}{2}}\right) f(x) dx,$$

Which is a generalization of Hankel transform,

$$H_{\alpha}[f(x)](y) = \int_0^{\infty} x f(x) J_{\nu}(xy) dx.$$

Kerr [5] introduced the fractional Hankel transform as,

$$H_{\alpha}[f(x)](y) = F(y) = \int_0^{\infty} f(x) K_{\alpha}(x, y) dx, \quad (1.1)$$

where $K_{\alpha}(x, y) = A_{v, \alpha} e^{-\frac{i}{2}(x^2+y^2)\cot \frac{\alpha}{2}} \left(\frac{xy}{\sin \frac{\alpha}{2}}\right)^{\frac{1}{2}} J_{\nu}\left(\frac{xy}{\sin \frac{\alpha}{2}}\right)$

$$= \delta(x - y) \text{ for } \alpha = 0 \text{ \& } \alpha = 2\pi,$$

and $A_{v, \alpha} = \left|\sin \frac{\alpha}{2}\right|^{-\frac{1}{2}} e^{i\left(\frac{\pi}{2}\hat{\alpha} - \frac{\alpha}{2}\right)(v+1)}$, $\hat{\alpha} = \text{sgn } \alpha$, $f \in L^2(R^+)$, $\alpha \in R$ and $v > -1$,

which is a generalization of the Hankel transform,

$$H_{\alpha}[f(x)](y) = \int_0^{\infty} \sqrt{xy} f(x) J_{\nu}(xy) dx.$$

We have extended (1.1) to the Zemanian spaces and obtained inverse in [6, 7]. For the parameter $\alpha = \pi$ the fractional Hankel transform reduces to the above Hankel transform. Initial and final value theorems are proved for Hankel type transformation in [8]. The paper is organized as follows. In section II, Initial value theorem and in section III final value theorem on fractional Hankel transform are given, where as section IV concludes the paper.

II. Initial Value Theorem:

We first prove the classical initial-value theorem.

Theorem: 2.1:

Let $|v| < \frac{1}{2}$ and a function $f(x)$ satisfies the following conditions,

i) $f(x) \rightarrow 0$ as $x \rightarrow \infty$

ii) $x^{-v-\frac{1}{2}} f(x)$ is absolutely integrable on every compact subsets $X < x < \infty$ and

iii) $\lim_{x \rightarrow 0} x^{-v-\frac{1}{2}} f(x) = \rho$.

Suppose $F(y) = H^{\alpha}[f(x)](y)$ is fractional Hankel transform of $f(x)$ then

$$\lim_{y \rightarrow \infty} e^{-\frac{i}{2}y^2 \tan \frac{\alpha}{2}} y^{-v-\frac{1}{2}} F(y) = \rho B_{v, \alpha}, \text{ where } B_{v, \alpha} = \frac{\sqrt{\sin \frac{\alpha}{2}}}{(\cos \frac{\alpha}{2})^{v+1}}$$

Proof:- Consider $\left| e^{-\frac{i}{2}y^2 \tan \frac{\alpha}{2}} y^{-v-\frac{1}{2}} F(y) - \rho B_{v, \alpha} \right| = I$, then using equation (1.1), and the result from [9], (p. no. 30.(8.6), result (12)), we obtain,

$$e^{-\frac{i}{2}y^2 \tan \frac{\alpha}{2} y^{-v-\frac{1}{2}} A_{v,\alpha}} e^{-\frac{i}{2}y^2 \cot \frac{\alpha}{2}} \int_0^\infty x^{v+\frac{1}{2}} e^{-\frac{i}{2}x^2 \cot \frac{\alpha}{2}} \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right)^{\frac{1}{2}} J_\nu \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right) dx = B_{v,\alpha} = \frac{\sqrt{|\sin \frac{\alpha}{2}|}}{(\cos \frac{\alpha}{2})^{v+1}},$$

which gives,

$$I = \left| e^{-\frac{i}{2}y^2 \tan \frac{\alpha}{2} y^{-v-\frac{1}{2}} A_{v,\alpha}} \int_0^\infty e^{-\frac{i}{2}(x^2+y^2) \cot \frac{\alpha}{2}} \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right)^{\frac{1}{2}} J_\nu \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right) f(x) dx - \rho e^{-i2y^2 \tan \alpha 2y^{-v-12} A_{v,\alpha} e^{-i2 y^2 \cot \alpha 20 \infty x v + 12} e^{-i2 x^2 \cot \alpha 2xy \sin \alpha 212 / \nu (xy \sin \alpha 2)} dx, \right. \\ \left. (2.1) \right.$$

where $0 \leq \alpha \leq \pi$ and $-1 < \text{Re } v < \frac{1}{2}$

$$= \left| e^{-\frac{i}{2}y^2 \tan \frac{\alpha}{2} y^{-v-\frac{1}{2}} A_{v,\alpha}} \int_0^\infty e^{-\frac{i}{2}(x^2+y^2) \cot \frac{\alpha}{2}} \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right)^{\frac{1}{2}} J_\nu \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right) [f(x) - \rho x^{v+\frac{1}{2}}] dx \right| \\ \leq \left| e^{-\frac{i}{2}y^2 \tan \frac{\alpha}{2} y^{-v-\frac{1}{2}} A_{v,\alpha}} \int_0^X e^{-\frac{i}{2}(x^2+y^2) \cot \frac{\alpha}{2}} \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right)^{\frac{1}{2}} J_\nu \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right) [f(x) - \rho x^{v+\frac{1}{2}}] dx \right| + \\ \left| e^{-\frac{i}{2}y^2 \tan \frac{\alpha}{2} y^{-v-\frac{1}{2}} A_{v,\alpha}} \int_X^\infty e^{-\frac{i}{2}(x^2+y^2) \cot \frac{\alpha}{2}} \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right)^{\frac{1}{2}} J_\nu \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right) [f(x) - \rho x^{v+\frac{1}{2}}] dx \right| \\ \leq I_1 + I_2 \text{ for some } X, 0 < X < \infty.$$

$$\text{Consider } I_1 = \left| e^{-\frac{i}{2}y^2 \tan \frac{\alpha}{2} y^{-v-\frac{1}{2}} A_{v,\alpha}} \int_0^X e^{-\frac{i}{2}(x^2+y^2) \cot \frac{\alpha}{2}} \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right)^{\frac{1}{2}} J_\nu \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right) [f(x) - \rho x^{v+\frac{1}{2}}] dx \right| \\ \leq \left| y^{-v-\frac{1}{2}} A_{v,\alpha} \int_0^X \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right)^{\frac{1}{2}} J_\nu \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right) 0 < x < X \left[\frac{f(x)}{x^{v+\frac{1}{2}}} - \rho \right] x^{v+\frac{1}{2}} dx \right|.$$

By hypothesis (iii) as $x \rightarrow 0$, $\left| \frac{f(x)}{x^{v+\frac{1}{2}}} - \rho \right| < \epsilon$.

Moreover $J_\nu(x) = O(x^\nu)$ as $x \rightarrow 0$.

Therefore above inequality becomes,

$$I_1 \leq \left| y^{-v-\frac{1}{2}} A_{v,\alpha} \epsilon \int_0^X M \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right)^{\frac{1}{2}} \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right)^\nu x^{v+\frac{1}{2}} dx \right| \\ \leq \left| A_{v,\alpha} \epsilon \frac{M}{|\sin \frac{\alpha}{2}|^{v+\frac{1}{2}}} \int_0^X x^{2v+1} dx \right|, \tag{2.2}$$

integral on R.H.S. exists as $x \rightarrow 0$ if $v > -1$.

Therefore, because of R.H.S. is independent of y , I_1 can be made less than $\frac{\epsilon}{2}$ if $v > -1$.

$$\text{Now } I_2 = \left| e^{-\frac{i}{2}y^2 \tan \frac{\alpha}{2} y^{-v-\frac{1}{2}} A_{v,\alpha}} \int_X^\infty e^{-\frac{i}{2}(x^2+y^2) \cot \frac{\alpha}{2}} \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right)^{\frac{1}{2}} J_\nu \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right) [f(x) - \rho x^{v+\frac{1}{2}}] dx \right| \\ = \left| y^{-v-\frac{1}{2}} A_{v,\alpha} \int_X^\infty \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right)^{\frac{1}{2}} J_\nu \left(\frac{xy}{|\sin \frac{\alpha}{2}|}\right) [f(x) - \rho x^{v+\frac{1}{2}}] dx \right|. \tag{2.3}$$

Since $\sqrt{z} J_\nu(z)$ is bounded on $0 < z < \infty$, moreover $[f(x) - \rho x^{v+\frac{1}{2}}]$ is absolutely integrable in $X < x < \infty$, hence I_2 can be made $< \frac{\epsilon}{2}$ as $y \rightarrow \infty$ provided $v > -\frac{1}{2}$.

Therefore $I_2 < \frac{\epsilon}{2}$ if $v > -\frac{1}{2}$.

We get $I \leq I_1 + I_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2}$ if $|v| < \frac{1}{2}$, from (2.1), (2.2), and (2.3).

Hence the proof.

III. Final Value Theorem:

Theorem: 3.1: Let $|v| < \frac{1}{2}$ and a function $f(x)$ satisfies the following conditions,

- i) $f(x) \rightarrow 0$ as $x \rightarrow \infty$
- ii) $x^{v+\frac{1}{2}}f(x)$ is absolutely integrable on every compact subsets $X \leq x < \infty$ and
- iii) $\lim_{x \rightarrow \infty} x^{-v-\frac{1}{2}}f(x) = \delta$.

Suppose $F(y) = H^\alpha[f(x)](y)$ is fractional Hankel transform of $f(x)$ then

$$\lim_{y \rightarrow 0} e^{-\frac{i}{2}y^2 \tan \frac{\alpha}{2}} y^{-v-\frac{1}{2}} F(y) = \delta B_{v,\alpha}$$

Proof :- Consider

$$I = \left| e^{-\frac{i}{2}y^2 \tan \frac{\alpha}{2}} y^{-v-\frac{1}{2}} A_{v,\alpha} \int_0^\infty e^{-\frac{i}{2}(x^2+y^2) \cot \frac{\alpha}{2}} \left(\frac{xy}{|\sin \frac{\alpha}{2}|} \right)^{\frac{1}{2}} J_v \left(\frac{xy}{|\sin \frac{\alpha}{2}|} \right) [f(x) - \delta x^{v+\frac{1}{2}}] dx \right| \tag{3.1}$$

where $0 \leq \alpha \leq \pi$ and $-1 < \operatorname{Re} v < \frac{1}{2}$

$$\leq \left| e^{-\frac{i}{2}y^2 \tan \frac{\alpha}{2}} y^{-v-\frac{1}{2}} A_{v,\alpha} \int_0^X e^{-\frac{i}{2}(x^2+y^2) \cot \frac{\alpha}{2}} \left(\frac{xy}{|\sin \frac{\alpha}{2}|} \right)^{\frac{1}{2}} J_v \left(\frac{xy}{|\sin \frac{\alpha}{2}|} \right) [f(x) - \delta x^{v+\frac{1}{2}}] dx \right| +$$

$$\left| e^{-\frac{i}{2}y^2 \tan \frac{\alpha}{2}} y^{-v-\frac{1}{2}} A_{v,\alpha} \int_X^\infty e^{-\frac{i}{2}(x^2+y^2) \cot \frac{\alpha}{2}} \left(\frac{xy}{|\sin \frac{\alpha}{2}|} \right)^{\frac{1}{2}} J_v \left(\frac{xy}{|\sin \frac{\alpha}{2}|} \right) [f(x) - \delta x^{v+\frac{1}{2}}] dx \right|,$$

$\leq I_1 + I_2$.

Now $I_1 = \left| e^{-\frac{i}{2}y^2 \tan \frac{\alpha}{2}} y^{-v-\frac{1}{2}} A_{v,\alpha} \int_0^X e^{-\frac{i}{2}(x^2+y^2) \cot \frac{\alpha}{2}} \left(\frac{xy}{|\sin \frac{\alpha}{2}|} \right)^{\frac{1}{2}} J_v \left(\frac{xy}{|\sin \frac{\alpha}{2}|} \right) [f(x) - \delta x^{v+\frac{1}{2}}] dx \right|,$

moreover $J_v(z) = O(z^v)$ as $z \rightarrow 0$, $I_1 = \left| y^{-v-\frac{1}{2}} A_{v,\alpha} \int_0^X M \left(\frac{xy}{|\sin \frac{\alpha}{2}|} \right)^{v+\frac{1}{2}} [f(x) - \delta x^{v+\frac{1}{2}}] dx \right|$

$$= \left| \frac{A_{v,\alpha}}{|\sin \frac{\alpha}{2}|^{v+\frac{1}{2}}} \int_0^X x^{v+\frac{1}{2}} [f(x) - \delta x^{v+\frac{1}{2}}] dx \right| \tag{3.2}$$

The term is independent of y and $x^{v+\frac{1}{2}}f(x)$ is integrable according to condition (ii), therefore integral converges if x^{2v+2} converges as if $v > -1$, hence I_1 can be made less than $\frac{\epsilon}{2}$.

Now $I_2 = \left| e^{-\frac{i}{2}y^2 \tan \frac{\alpha}{2}} y^{-v-\frac{1}{2}} A_{v,\alpha} \int_X^\infty e^{-\frac{i}{2}(x^2+y^2) \cot \frac{\alpha}{2}} \left(\frac{xy}{|\sin \frac{\alpha}{2}|} \right)^{\frac{1}{2}} J_v \left(\frac{xy}{|\sin \frac{\alpha}{2}|} \right) [f(x) - \delta x^{v+\frac{1}{2}}] dx \right|$

$$= \left| y^{-v-\frac{1}{2}} A_{v,\alpha} \int_X^\infty \left(\frac{xy}{|\sin \frac{\alpha}{2}|} \right)^{\frac{1}{2}} J_v \left(\frac{xy}{|\sin \frac{\alpha}{2}|} \right) [f(x) - \delta x^{v+\frac{1}{2}}] dx \right|,$$

but $\sqrt{z} J_v(z) = O(1)$ as $z \rightarrow \infty$

$$I_2 \leq \left| y^{-v-\frac{1}{2}} A_{v,\alpha} M \int_X^\infty x^{v+\frac{1}{2}} \left[\frac{f(x)}{x^{v+\frac{1}{2}}} - \delta \right] dx \right|, \tag{3.3}$$

by condition (iii) in hypothesis, R.H.S. converges in $X \leq x < \infty$. Also as $y \rightarrow 0$, R.H.S. converges if $v < -\frac{1}{2}$, therefore $I \leq I_1 + I_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2}$ provided $-1 < v < -\frac{1}{2}$, using (3.1), (3.2) and (3.3),

Hence the proof.

IV. Conclusion:

We have proved the Initial and Final value theorems for fractional Hankel transform. Further we plan to apply these theorems for solving some definite integrals.

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