# INITIAL-BOUNDARY VALUE PROBLEMS <br> FOR THE EQUATION $u_{t t}=\left(\sigma\left(u_{x}\right)\right)_{x}+\left(\alpha\left(u_{x}\right) u_{x t}\right)_{x}+f^{*}$ 

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Abstract. Existence and uniqueness theorems are proved for global weak solutions of initial-boundary value problems corresponding to the equation

$$
u_{t t}=\left(\sigma\left(u_{x}\right)\right)_{x}+\left(\alpha\left(u_{x}\right) u_{x t}\right)_{x}+f
$$

under assumptions that do not require smoothness or monotonicity of $\sigma$. The initial data are not assumed to be smooth, the boundary data are allowed to be time dependent, and $f$ is only assumed to be in $L^{2}$.

1. Introduction. An equation used to model the longitudinal displacement in a homogeneous bar of uniform cross section and unit length is

$$
\begin{equation*}
u_{t t}=\left(\sigma\left(u_{x}\right)\right)_{x}+\delta^{2} u_{x x t}, \quad x \in(0,1), t \in[0, T] . \tag{1.1}
\end{equation*}
$$

The question of the existence and uniqueness of solutions to (1.1) augmented with boundary conditions and initial conditions of various types has been dealt with in several papers. Among these are [4-11]. These papers all establish the existence and uniqueness of global, classical solutions to (1.1). In each paper, $\sigma$ is assumed to be either quite smooth or monotone. The initial data are also taken to be very smooth. More recently, (1.1) has been considered under assumptions that allow $\sigma$ to possess corners and fail to be monotone [ $2,3,18$ ]. The initial data are also much rougher in these papers, which obtain weak rather than classical solutions to (1.1).

In this paper we consider a slightly more general equation,

$$
\begin{equation*}
u_{t t}=\left(\sigma\left(u_{x}\right)\right)_{x}+\left(\alpha\left(u_{x}\right) u_{x t}\right)_{x}+f, \tag{1.2}
\end{equation*}
$$

where $\alpha(V) \geq \delta$ for all $V$ and $\alpha$ is continuous, the function $f$ being a body force which is only assumed to be in $L^{2}((0,1) \times(0, T))$. The assumptions made on $\sigma$ are similar to those used in $[2,3]$ and the initial data are also the same.

It makes good sense physically to consider the situation where a nonzero timedependent force is applied to one or both ends of the material. Also, this force need not be continuous. The mathematical model for this realistic physical problem would be (1.2), initial data

$$
\begin{equation*}
u(0, x)=u_{0}(x) \in W^{1 \infty}(0,1), \quad u_{t}(0, x)=u_{1}(x) \in L^{2}(0,1) \tag{1.3}
\end{equation*}
$$

[^0]along with boundary conditions of the form
\[

$$
\begin{gather*}
u(t, 1)=0  \tag{1.4.1}\\
\sigma\left(u_{x}(t, 0)\right)+\alpha\left(u_{x}(t, 0)\right) u_{x t}(t, 0)=k_{0}(t) \tag{1.4.2}
\end{gather*}
$$
\]

or of the form

$$
\begin{align*}
& \sigma\left(u_{x}(t, 0)\right)+\alpha\left(u_{x}(t, 0)\right) u_{x t}(t, 0)=k_{0}(t)  \tag{1.5.1}\\
& \sigma\left(u_{x}(t, 1)\right)+\alpha\left(u_{x}(t, 1)\right) u_{x t}(t, 1)=k_{1}(t) \tag{1.5.2}
\end{align*}
$$

where $k_{i}$ is only assumed to be bounded and measurable. The existence and uniqueness theory for (1.2), (1.3), and either (1.4) or (1.5) is currently unsolved in the available literature [20]. The solution of this problem when $\alpha(V) \geq \delta>0$ is the main goal of this paper.

One might also consider the boundary conditions

$$
\begin{equation*}
u(t, 1)=b_{1}(t), \quad u(t, 0)=b_{0}(t) \tag{1.6}
\end{equation*}
$$

If $\sigma$ is assumed to be globally Lipschitz, these boundary conditions may be obtained [12], but this assumption is overly restrictive for us in this paper. In the special case where $b_{i}(t)=0$, the problem (1.2), (1.3), (1.6) may be shown to be well posed and the details are worked out in [15] without an assumption of a positive lower bound for $\alpha$. Therefore, we omit the boundary conditions (1.6) and note that these type of conditions are less interesting than (1.4) and (1.5) because the thing that can actually be controlled at the ends of the material is not the displacement but the force.

Because of the discontinuities of $k_{i}$, the roughness of the initial data, and the lack of smoothness of $\sigma$ and $\alpha$, the traditional methods of obtaining estimates on local classical solutions are not easily applicable. Instead, we have chosen to estimate global weak solutions of an approximate problem obtained by adding the assumption that $\sigma$ is globally Lipschitz. This approximate problem may be dealt with by using elliptic regularization and the Galerkin method. The solution to it has just barely enough regularity to allow us to obtain estimates on $u_{x}$. The basic ideas used are generalizations of those in [2], but because of the weakness of the solutions to this approximate problem, the implementation of these ideas requires careful attention to measurability questions.

Section 2 discusses a variational formulation for the approximate initial-boundary value problems and gives a proof of existence and uniqueness which is based on an abstract theorem whose proof is found in Sec. 5. Section 3 obtains the necessary estimates on the solutions to the approximate problem. Section 4 uses these estimates to obtain existence and uniqueness for the initial-boundary value problems of interest. For $E$ a Banach space, $L^{2}(a, b ; E)$ is the space of strongly measurable, square-integrable functions. The notation $\rightarrow$ will denote weak or weak* convergence, while $\rightarrow$ will mean strong convergence.
2. The approximate problem. In this section, we prove existence and uniqueness theorems for global weak solutions of initial-boundary value problems for the equation

$$
\begin{equation*}
u_{t t}=\left(\sigma\left(u_{x}\right)\right)_{x}+\left(\alpha\left(u_{x}\right) u_{x t}\right)_{x}+f \tag{2.1}
\end{equation*}
$$

with initial conditions

$$
\begin{align*}
u(0, x) & =u_{0}(x), & & u_{0} \in W^{1 \infty}(0,1)  \tag{2.2.1}\\
u_{t}(0, x) & =u_{1}(x), & & u_{1} \in L^{2}(0,1) \tag{2.2.2}
\end{align*}
$$

and boundary conditions which are sufficiently general to include

$$
\begin{gather*}
u(1, t)=0  \tag{2.3.1}\\
\sigma\left(u_{x}(0, t)\right)+\alpha\left(u_{x}(0, t)\right) u_{x t}(0, t)=k_{0}(t) \tag{2.3.2}
\end{gather*}
$$

or

$$
\begin{align*}
& \sigma\left(u_{x}(0, t)\right)+\alpha\left(u_{x}(0, t)\right) u_{x t}(0, t)=k_{0}(t)  \tag{2.4.1}\\
& \sigma\left(u_{x}(1, t)\right)+\alpha\left(u_{x}(1, t)\right) u_{x t}(1, t)=k_{1}(t) \tag{2.4.2}
\end{align*}
$$

where $k_{i}$ is only assumed to be bounded and measurable. The problem solved is called the approximate problem because

$$
\begin{align*}
& \text { For some } \sigma_{0}, W(V)=\int_{0}^{V}\left(\sigma(s)+\sigma_{0}\right) d s \text { is bounded below, }  \tag{2.5.1}\\
& \sigma \text { is bounded, and }\left|\sigma\left(V_{1}\right)-\sigma\left(V_{2}\right)\right| \leq K\left|V_{1}-V_{2}\right| \tag{2.5.2}
\end{align*}
$$

assumptions that will be weakened later in Sec. 3. We also assume that $\alpha$ is continuous and that

$$
\begin{equation*}
0 \leq \delta \leq \alpha(V)<M_{0}<\infty \tag{2.5.3}
\end{equation*}
$$

Our approach is abstract. To motivate the definitions that we will use, let $b(t, x)=$ $k_{0}(t)(1-x)+k_{1}(t) x$ and suppose

$$
\begin{gather*}
-\int_{0}^{T} \int_{0}^{1} u_{t}(t, x) \varphi_{t}(t, x) d x d t+\int_{0}^{T} \int_{0}^{1}\left[\sigma\left(u_{x}(t, x)\right)-b(t, x)\right] \varphi_{x}(t, x) d x d t \\
+\int_{0}^{T} \int_{0}^{1} \alpha\left(u_{x}(t, x)\right) u_{x t}(t, x) \varphi_{x}(t, x) d x d t=\int_{0}^{T} \int_{0}^{1}\left(f(t, x)+b_{x}(t, x)\right) \varphi(t, x) d x d t  \tag{2.6.1}\\
u, u_{t}, u_{x}, u_{x t} \quad \text { are in } L^{2}((0, T) \times(0,1)) \tag{2.6.2}
\end{gather*}
$$

for all $\varphi \in C_{0}^{\infty}(0, T ; E)$ where $E$ is a closed subspace of $H^{1}(0,1)$ containing $C_{0}^{\infty}(0,1)$. Thus, in particular, (2.6) holds if $\varphi \in C_{0}^{\infty}((0, T) \times(0,1))$, which implies that $u$ is a weak solution of (2.1). Integrating (2.6.1) by parts formally yields the variational boundary conditions

$$
\begin{align*}
& \int_{0}^{T}\left(\sigma\left(u_{x}(t, 1)\right)-k_{1}(t)+\alpha\left(u_{x}(t, 1)\right) u_{x t}(t, 1)\right) \varphi(t, 1) d t=0  \tag{2.7.1}\\
& \int_{0}^{T}\left(\sigma\left(u_{x}(t, 0)-k_{0}(t)+\alpha\left(u_{x}(t, 0)\right) u_{x t}(t, 0)\right) \varphi(t, 0) d t=0\right. \tag{2.7.2}
\end{align*}
$$

with (2.7) holding for all $\varphi \in C_{0}^{\infty}(0, T ; E)$. If $E=H^{1}(0,1)$ these conditions yield (2.4). If $E=\left\{u \in H^{1}(0,1): u(1)=0\right\}$, the boundary conditions (2.3) are obtained. These formal manipulations may all be made precise ([19] or [16]) but we shall regard (2.6) along with the initial data as the fundamental object of our study.

Letting $E$ be the closed subspace of $H^{1}(0,1)$ just described, we shall let $H=$ $L^{2}(0,1)$ and identify $H$ and $H^{\prime}$. Therefore,

$$
\begin{equation*}
E \subseteq H=H^{\prime} \subseteq E^{\prime} \tag{2.8}
\end{equation*}
$$

For $u \in E$ and $t \in[0, T]$, let $Q(u), N(t, \cdot)$, and $M$ be operators mapping $E$ to $E^{\prime}$ given by

$$
\begin{align*}
\langle Q(u) w, v\rangle & =\left(\alpha\left(u_{x}\right) w_{x}, v_{x}\right)_{H}  \tag{2.9.1}\\
\langle N(t, w), v\rangle & =\left(\sigma\left(w_{x}\right)-b(t, \cdot), v_{x}\right)_{H}  \tag{2.9.2}\\
\langle M w, v\rangle & =\left(\beta\left(w_{x}\right), v_{x}\right)_{H} \tag{2.9.3}
\end{align*}
$$

where

$$
\begin{equation*}
\beta(V)=\int_{0}^{V} \alpha(s) d s \tag{2.10}
\end{equation*}
$$

For $h \in L^{1}\left(0, T ; E^{\prime}\right), h^{\prime}$ may be defined as an $E^{\prime}$-valued distribution by the rule

$$
\begin{equation*}
h^{\prime}(\varphi)=-\int_{0}^{T} h(t) \varphi^{\prime}(t) d t \tag{2.11}
\end{equation*}
$$

for all $\varphi \in C_{0}^{\infty}(0, T)$. Also let $\mathscr{V}=L^{2}(0, T ; E)$ and if $u \in \mathscr{V}$ consider $Q(u), N$, and $M$ to be operators from $\mathscr{V}$ to $\mathscr{V}^{\prime}=L^{2}\left(0, T ; E^{\prime}\right)$ defined by

$$
\begin{align*}
(Q(u) w)(t) & =Q(u(t)) w(t),  \tag{2.12.1}\\
(N w)(t) & =N(t, w(t)),  \tag{2.12.2}\\
(M w)(t) & =M(w(t)) . \tag{2.12.3}
\end{align*}
$$

With the above conventions and definitions, we can state the following theorem.

Theorem 1. Let $u_{1} \in H, u_{0} \in E$, and $g \in \mathscr{V}^{\prime}$. Then there exists a unique solution $u$ to

$$
\begin{gather*}
u^{\prime \prime}+N u+Q(u) u^{\prime}=g \quad \text { in } \mathscr{V}^{\prime},  \tag{2.13.1}\\
u(0)=u_{0},  \tag{2.13.2}\\
u^{\prime}(0)=u_{1},  \tag{2.13.3}\\
u \in L^{\infty}(0, T ; E), \quad u^{\prime} \in \mathscr{V} \cap L^{\infty}(0, T ; H) . \tag{2.13.4}
\end{gather*}
$$

A theorem similar to Theorem 1 is proved in [13] but there are some differences here and so we have chosen to present a proof. This is done in Sec. 5 to avoid obscuring the main ideas.

Lemma 1. Suppose $u, u^{\prime} \in L^{2}(a, b ; E)$ and let $\bar{u}_{t}(\cdot, \cdot)$ and $\bar{u}_{t x}(\cdot, \cdot)$ be Borel measurable representatives for $u^{\prime}(\cdot)(\cdot)$ and $\left(\partial u^{\prime}(\cdot) / \partial x\right)(\cdot)$, respectively. Also define

$$
\begin{align*}
\bar{u}_{x}(t, x) & =\int_{a}^{t} \bar{u}_{t x}(s, x) d s+u_{0}^{\prime}(x)  \tag{2.14.1}\\
\bar{u}(t, x) & =\int_{a}^{t} \bar{u}_{t}(s, x) d s+u_{0}(x) \tag{2.14.2}
\end{align*}
$$

Then
(1) $\bar{u}(t, \cdot)=u(t)$ a.e., $\bar{u}_{t}(t, \cdot)=u^{\prime}(t)$ a.e.,
(2) $\bar{u}_{x}(t, \cdot)=\frac{\partial u(t)}{\partial x}$ a.e., $\bar{u}_{t x}(t, \cdot)=\frac{\partial u^{\prime}(t)}{\partial x}$ a.e.,
(3) $\bar{u}_{x}=\frac{\partial}{\partial x}(\bar{u})$ in sense of distributions,
(4) $\bar{u}_{t x}=\frac{\partial}{\partial x}\left(\bar{u}_{t}\right)$ in sense of distributions.

Lemma 2. Let $\bar{u}$ be the measurable representative of the solution of Theorem 1 with $g=f+b_{x} \in L^{2}(0, T ; H)$ described in Lemma 1. Then (2.6) holds for each $\varphi \in C_{0}^{\infty}(0, T ; E)$.

Proof. Let $\varphi \in C_{0}^{\infty}(0, T ; E)$ and use (2.13.1) to write

$$
\begin{equation*}
\left\langle u^{\prime \prime}, \varphi\right\rangle_{\mathcal{Y}^{\prime \prime}}+\langle N u, \varphi\rangle_{\mathscr{Y}, \mathscr{V}}+\left\langle Q(u) u^{\prime}, \varphi\right\rangle_{\mathscr{V}^{\prime} \mathscr{V}}=\left\langle f+b_{x}, \varphi\right\rangle_{\mathscr{Y}^{\prime} \mathscr{Y}} . \tag{2.15}
\end{equation*}
$$

Consider the first term of (2.15):

$$
\begin{align*}
\left\langle u^{\prime \prime}, \varphi\right\rangle_{7} \cdots \neq & =\int_{0}^{T} \int_{0}^{t}\left\langle u^{\prime \prime}(t), \varphi^{\prime}(s)\right\rangle d s d t \\
& =\int_{0}^{T} \int_{s}^{T}\left\langle u^{\prime \prime}(t), \varphi^{\prime}(s)\right\rangle d t d s \\
& =-\int_{0}^{T}\left\langle u^{\prime}(s), \varphi^{\prime}(s)\right\rangle d s+\int_{0}^{T}\left\langle u^{\prime}(T), \varphi^{\prime}(s)\right\rangle d s \\
& =-\int_{0}^{T}\left\langle u^{\prime}(s), \varphi^{\prime}(s)\right\rangle d s \\
& =-\int_{0}^{T} \int_{0}^{1} \bar{u}_{t}(s, x) \varphi_{t}(s, x) d x d s \quad \text { from (2.13.4) and (2.8). } \tag{2.16}
\end{align*}
$$

Note that $u^{\prime}(T)$ makes sense because $u^{\prime}$ and $u^{\prime \prime}$ are both in $\mathscr{V}^{\prime}=L^{2}\left(0, T ; E^{\prime}\right)$. This yields the first term of (2.6) and it is clear from (2.8), (2.9), and (2.13) that formula (2.6) is obtained.

Earlier we demonstrated that integrating (2.6) by parts and choosing $E$ appropriately will yield boundary conditions of various sorts. It remains to discuss the initial conditions. For $u$ the solution of Theorem $1, u \in \mathscr{V}, u^{\prime} \in \mathscr{V}$, and $u^{\prime \prime} \in \mathscr{V}^{\prime}$. Therefore [14] we may conclude that $u$ equals a function in $C(0, T ; E)$ a.e. and $u^{\prime}$ equals a function in $C(0, T ; H)$ a.e. Therefore, the manner in which the initial conditions are satisfied may be described by

$$
\begin{gather*}
u(t)=v_{0}(t) \quad \text { and } \quad u^{\prime}(t)=v_{1}(t) \quad \text { if } t \notin D, m(D)=0,  \tag{2.17}\\
v_{0} \in C(0, T ; E), \quad v_{1} \in C(0, T ; H),  \tag{2.18.1}\\
 \tag{2.18.2}\\
\lim _{t \rightarrow 0}\left|v_{1}(t)-u_{1}\right|_{H}+\left\|v_{0}(t)-u_{0}\right\|_{E}=0 .
\end{gather*}
$$

If $\bar{u}$ is the measurable representative of $u$ from Lemma 1 it follows that

$$
\begin{equation*}
v_{0}(t)=\tilde{u}(t, \cdot), \quad \bar{u}_{t}(t, \cdot)=v_{1}(t) \quad \text { a.e. } \tag{2.18.3}
\end{equation*}
$$

Lemma 3. If $\bar{u}$ is Borel measurable and satisfies (2.18), (2.6), then if $v(t)=\bar{u}(t, \cdot)$,

$$
\begin{gather*}
v^{\prime}(t)=\bar{u}_{t}(t, \cdot) \quad \text { a.e., }  \tag{2.19.1}\\
v^{\prime \prime} \in \mathscr{V}^{\prime}, \quad \text { and } v \text { solves }(2.13) \text { with } g=f+b_{x} \tag{2.19.2}
\end{gather*}
$$

Proof. Let $\Psi \in C_{0}^{\infty}(0, T)$ and let $w \in E$.

$$
\begin{aligned}
& \left\langle\int_{0}^{T} v(t) \Psi^{\prime}(t) d t, w\right\rangle_{E^{\prime} E} d t=\int_{0}^{T} \Psi^{\prime}(t)\langle v(t), w\rangle_{E^{\prime} E} d t \\
& \quad=\int_{0}^{T} \Psi^{\prime}(t)(v(t), w)_{H} d t=\int_{0}^{T} \int_{0}^{1} \bar{u}(t, x) w(x) \Psi^{\prime}(t) d x d t \\
& \quad=-\int_{0}^{T} \int_{0}^{1} \bar{u}_{t}(t, x) w(x) \Psi(t) d x d t \\
& \quad=\left\langle-\int_{0}^{T} \Psi(t) \bar{u}_{t}(t, \cdot) d t, w\right\rangle_{E^{\prime} E}
\end{aligned}
$$

Since $w \in E$ is arbitrary, (2.19.1) is proved. Now apply (2.6) to $\varphi(t, x)=\Psi(t) w(x)$,

$$
\begin{aligned}
\left\langle\int_{0}^{T} v^{\prime}(t) \Psi^{\prime}(t) d t, w\right\rangle & =\int_{0}^{T}\langle N v, \varphi\rangle d t+\int_{0}^{T}\left\langle Q(v) v^{\prime}, \varphi\right\rangle d t-\int_{0}^{T}\langle g, \varphi\rangle d t \\
& =\left\langle\int_{0}^{T}\left(N v(t)+Q(v) v^{\prime}(t)-g(t)\right) \Psi(t) d t, w\right\rangle
\end{aligned}
$$

where $g=f+b_{x}$. Since $w$ is arbitrary

$$
-v^{\prime \prime}=N v+Q(v) v^{\prime}-g
$$

This proves (2.19.2).
Corollary 1. There exists a unique solution $u$ to the problem (2.18) and (2.6).
Proof. The existence comes from Lemma 2. To obtain uniqueness, use Lemma 3 and the uniqueness part of Theorem 1.
3. Estimates for the approximate problem. In this section we will estimate the solution of Theorem 1 assuming that for some constants $\sigma_{0}$ and $R>0$,

$$
\begin{equation*}
\left(\sigma(V)+\sigma_{0}\right) \beta(V) \geq 0 \quad \text { if }|V| \geq R \tag{3.1}
\end{equation*}
$$

Let $\left|k_{1}(t)\right|+\left|k_{0}(t)\right| \leq L$ for all $t$ and from now on let $g=f+b_{x}$. Also define

$$
\begin{equation*}
W(V)=\int_{0}^{V}\left(\sigma(s)+\sigma_{0}\right) d s \tag{3.2}
\end{equation*}
$$

and assume

$$
\begin{equation*}
a \leq u_{0}^{\prime}(x) \leq b \tag{3.3}
\end{equation*}
$$

Lemma 4. Let $q$ and $q_{t}$ both be in $L^{2}((0, T) \times(0,1))$. Then there exists a measurable set $D$ with $m(D)=0$ such that for $x \notin D, t \rightarrow q(t, x)$ is equal to a continuous function a.e. $t$ and if $q(0, x)$ is the value of this continuous function at $t=0$,

$$
\begin{equation*}
q(t, x)=q(0, x)+\int_{0}^{t} q_{t}(s, x) d s \quad \text { a.e. } t \tag{3.4}
\end{equation*}
$$

Lemma 5. For $u$ the solution of Theorem 1 .

$$
\begin{align*}
\left|u^{\prime}(t)\right|_{H}^{2}+ & 2 \int_{0}^{1} W\left(\bar{u}_{x}(t, x)\right) d x+2 \delta \int_{0}^{t}\left|\bar{u}_{t x}(s, \cdot)\right|_{H}^{2} d s-2 \int_{0}^{1} W\left(u_{0}^{\prime}(x)\right) d x-\left|u_{1}\right|_{H}^{2} \\
& -2\left(L+\left|\sigma_{0}\right|\right) \int_{0}^{t}\left|\bar{u}_{t x}(s, \cdot)\right|_{H} d s \leq 2\|g\|_{L^{2}(0, T ; H)}\left(\int_{0}^{t}\left|u^{\prime}(s)\right|_{H}^{2} d s\right)^{1 / 2} \tag{3.5}
\end{align*}
$$

Proof. Multiply (2.13.1) by $u^{\prime}$ and integrate by parts. Let $-J \leq W(V)$ for all $V$. Such a constant exists because of (3.1) and (3.2).

Corollary 2. For $u$ the solution of Theorem 1

$$
\begin{equation*}
\left|u^{\prime}(t)\right|_{H}^{2} \leq C \tag{3.6.1}
\end{equation*}
$$

where

$$
\begin{equation*}
C=\left(2 J+2 \int_{0}^{1} W\left(u_{0}^{\prime}(x)\right) d x+\left|u_{1}\right|_{H}^{2}+\|g\|^{2}+T \frac{\left(L+\left|\sigma_{0}\right|\right)^{2}}{2 \delta}\right) e^{T} \tag{3.6.2}
\end{equation*}
$$

Proof. This follows from (3.5) and Gronwall's inequality.
For the rest of the section we shall assume that

$$
\begin{equation*}
\left\{u \in H^{1}(0,1): u(1)=0\right\} \subseteq E \tag{3.7}
\end{equation*}
$$

Lemma 6. Let $\bar{u}$ be the measurable representative of the solution of Theorem 1 described in Lemma 1 and let

$$
\begin{equation*}
q(t, x)=\int_{0}^{x} \bar{u}_{t}(t, z) d z+\int_{0}^{t} k_{0}(s) d s-\int_{0}^{t} \int_{0}^{x} f(s, z) d z d s-\beta\left(\bar{u}_{x}(t, x)\right)+\sigma_{0} t \tag{3.8}
\end{equation*}
$$

Then

$$
\begin{equation*}
q_{t}(t, x)=\sigma_{0}+\sigma\left(\bar{u}_{x}(t, x)\right) \tag{3.9}
\end{equation*}
$$

Proof. Let $\varphi \in C_{0}^{\infty}((0, T) \times(0,1))$ and consider $-\int_{0}^{T} \int_{0}^{1} q(t, x) \varphi_{t}(t, x) d x d t$. The use of Fubini's theorem and integration by parts along with (2.6.1) applied with $\Psi(t, x)=\int_{x}^{1} \varphi(t, z) d z$ in place of $\varphi$ will yield

$$
-\int_{0}^{T} \int_{0}^{1} q(t, x) \varphi_{t}(t, x) d x d t=\int_{0}^{T} \int_{0}^{1}\left(\sigma\left(\bar{u}_{x}(t, x)\right)+\sigma_{0}\right) \varphi(t, x) d x d t
$$

and this proves the lemma. It is because of (3.7) that $\Psi(t, x)$ given above may be used in (2.6.1).

Using Lemmas 4 and 6 , there exists a set of measure zero, $D$, such that for $x \notin D$,

$$
\begin{equation*}
q(t, x)=\int_{0}^{t} \sigma_{0}+\sigma\left(\bar{u}_{x}(s, x)\right) d s+q(0, x) \quad \text { a.e. } t \tag{3.10.1}
\end{equation*}
$$

where

$$
\begin{equation*}
q(0, x)=\int_{0}^{x} u_{1}(z) d z-\beta\left(u_{0}^{\prime}(x)\right) \tag{3.10.2}
\end{equation*}
$$

From (3.6.1) and Lemma 1,

$$
\begin{align*}
\left|\int_{0}^{x} \bar{u}_{t}(t, z) d z+\int_{0}^{t} k_{0}(s) d s-\int_{0}^{t} \int_{0}^{x} f(s, z) d z d s+\sigma_{0} t\right| \\
\leq C+L T+\|f\|_{L^{2}(0, T ; H)} \sqrt{T}+\left|\sigma_{0}\right| T \quad \text { a.e. } \tag{3.11}
\end{align*}
$$

Let

$$
\begin{equation*}
C_{1}=2\left(\left|u_{1}\right|_{H}+|\beta(a)|+|\beta(b)|+\beta(R)-\beta(-R)+C+L T+\|f\| \sqrt{T}+\left|\sigma_{0}\right| T\right) \tag{3.12}
\end{equation*}
$$

Thus $\left|q(t, x)+\beta\left(\bar{u}_{x}(t, x)\right)\right|<C_{1} / 2$ a.e. $t$.
Lemma 7. For all $x \notin D,|q(t, x)| \leq C_{1}$ a.e. and

$$
\begin{equation*}
\left|\beta\left(\bar{u}_{x}(t, x)\right)\right| \leq \frac{3 C_{1}}{2} \quad \text { for all } t . \tag{3.13}
\end{equation*}
$$

Proof. Let $x \notin D$ and let $\tilde{q}(t, x)=\int_{0}^{t} \sigma_{0}+\sigma\left(\bar{u}_{x}(s, x)\right) d s+q(0, x)$ for all $t$. Thus $|\tilde{q}(0, x)|<C_{1}$. If $\tilde{q}(t, x)>C_{1}$ for some $t$, there exists $a \in[0, T]$ with $\tilde{q}(a, x)=C_{1}$ and $\tilde{q}(t, x)>C_{1}$ for all $t \in[a, a+\varepsilon]$. Therefore, for $t \in[a, a+\varepsilon]$,

$$
\tilde{q}(t, x)=C_{1}+\int_{a}^{t} \sigma_{0}+\sigma\left(\bar{u}_{x}(s, x)\right) d s
$$

and it follows that, in a subset of $[a, a+\varepsilon]$ having positive measure, the inequality (3.11) holds and also

$$
\begin{gather*}
\sigma_{0}+\sigma\left(\bar{u}_{x}(t, x)\right)>0  \tag{3.14.1}\\
\tilde{q}(t, x)=q(t, x) \tag{3.14.2}
\end{gather*}
$$

Since (3.11) holds, $-\beta\left(\bar{u}_{x}(t, x)\right) \geq C_{1} / 2$ and therefore

$$
\begin{equation*}
\beta\left(\bar{u}_{x}(t, x)\right) \leq-C_{1} / 2 \leq \beta(-R)<0 . \tag{3.15}
\end{equation*}
$$

But (3.15) implies (3.1), which requires that $\sigma\left(\bar{u}_{x}(t, x)\right)+\sigma_{0} \leq 0$, contradicting (3.14.1). Therefore, $\tilde{q}(t, x) \leq C_{1}$ for all $t \in[0, T]$. A similar argument shows $\tilde{q}(t, x) \geq-C_{1}$ for all $t$. From (3.11) and (3.10.1), $\left|\beta\left(\bar{u}_{x}(t, x)\right)\right| \leq 3 C_{1} / 2$ a.e. From (2.14.1) of Lemma $1,\left|\beta\left(\bar{u}_{x}(t, x)\right)\right| \leq 3 C_{1} / 2$ for all $t \in[0, T]$ and this completes the proof of Lemma 7.
4. The exact problem. In this section we use the results of the previous section to obtain existence and uniqueness theorems for (2.1), (2.2) and boundary conditions which include (2.3) and (2.4) under the assumptions

$$
\begin{align*}
& \alpha \text { is continuous and } 0<\delta \leq \alpha(V) \leq M_{0}  \tag{4.1}\\
& \sigma \text { is Lipschitz on every bounded interval, } \tag{4.2.1}
\end{align*}
$$

and for some constant $\sigma_{0}$ and $R>0$,

$$
\begin{equation*}
\left(\sigma(V)+\sigma_{0}\right) \beta(V) \geq 0 \quad \text { if }|V| \geq R \tag{4.2.2}
\end{equation*}
$$

Also assume

$$
\begin{align*}
& a \leq u_{0}^{\prime}(x) \leq b \quad \text { for all } x \in(0,1)  \tag{4.3.1}\\
& \left|k_{1}(t)\right|+\left|k_{0}(t)\right| \leq L \text { for all } t  \tag{4.3.2}\\
& k_{0} \text { and } k_{1} \text { are measurable, }  \tag{4.3.3}\\
& u_{1} \in L^{2}(0,1), \quad u_{0} \in E \tag{4.3.4}
\end{align*}
$$

where $E \subseteq H^{1}(0,1)$ is a closed subspace satisfying

$$
\begin{equation*}
E \supseteq\left\{u \in H^{1}(0,1): u(1)=0\right\} \tag{4.3.5}
\end{equation*}
$$

For $m>0$ define

$$
\sigma_{m}(V)= \begin{cases}\sigma(V) & \text { if }|V| \leq m  \tag{4.4}\\ \sigma(m) & \text { if } V>m \\ \sigma(-m) & \text { if } V<-m\end{cases}
$$

Thus $\sigma_{m}$ is globally Lipschitz.
In this section let $u_{m}$ be the solution of Theorem 1 with $\sigma$ replaced with $\sigma_{m}$ in the definition of $N$ and let $\bar{u}_{m}$ be the measurable representative of Lemma 1. Assume

$$
\begin{equation*}
m>R \tag{4.5}
\end{equation*}
$$

and define

$$
\begin{equation*}
W_{m}(V)=\int_{0}^{V}\left(\sigma_{m}(s)+\sigma_{0}\right) d s \tag{4.6}
\end{equation*}
$$

Therefore, $W_{m}$ is decreasing for $V<-R$ and increasing for $V>R$. Since $W_{m}$ coincides with $W$ on $[-R, R]$, it follows that $W_{m}$ has a minimum which does not depend on $m$. Therefore

$$
\begin{equation*}
\left|u_{m}^{\prime}(t)\right|_{H}^{2} \leq C \tag{4.7}
\end{equation*}
$$

where $C$ is given in (3.6.2) and does not depend on $m$. From Lemma 7 there exists a set $D_{m}$ with $m\left(D_{m}\right)=0$ such that for all $x \notin D_{m}$,

$$
\begin{equation*}
\left|\beta\left(\bar{u}_{m x}(t, x)\right)\right|<\frac{3}{2} C_{1} \quad \text { for all } t \tag{4.8}
\end{equation*}
$$

where $C_{1}$ depends only on $u_{1}, u_{0}, R, \min (W(V)), f, k_{0}$, and $k_{1}$. Thanks to (4.1), (4.8) implies

$$
\begin{equation*}
\left|\bar{u}_{m x}(t, x)\right| \leq \frac{3}{2} C_{1} / \delta \quad \text { a.e. } \tag{4.9}
\end{equation*}
$$

Now assume

$$
\begin{equation*}
m>\frac{3 C_{1}}{2 \delta} \tag{4.10}
\end{equation*}
$$

Theorem 2. Suppose (4.1), (4.2), and (4.3) hold. Then there exist a unique solution, $u$, to (2.6), (2.18) and a constant $N$ such that $\left|u_{x}(t, x)\right| \leq N$ a.e.

Proof. Let $m>\max \left(R, 3 C_{1} /(2 \delta)\right)$ and consider $\bar{u}_{m}$. The function $\bar{u}_{m}$ satisfies (2.6) and (2.18) with $\sigma_{m}$ in place of $\sigma$. But by (4.9) $\left|\bar{u}_{m x}(t, x)\right| \leq 3 C_{1} /(2 \delta)<m$ a.e. and so $\sigma_{m}\left(\bar{u}_{m x}(t, x)\right)=\sigma\left(\bar{u}_{m x}(t, x)\right)$ a.e. Let $u=\bar{u}_{m}$.
Corollary 3. The conclusion of Theorem 2 remains true if $(4.1)$ is replaced with the weaker assumption

$$
\begin{equation*}
\alpha \text { is continuous, } \quad \delta \leq \alpha(V) \tag{4.11}
\end{equation*}
$$

Proof. Let $M \geq \max \left\{\alpha(V): V \in\left[-3 C_{1} /(2 \delta), 3 C_{1} /(2 \delta)\right]\right\}$ and let $\alpha_{M}(V)=$ $\min (\alpha(V), M)$. If $u$ is the solution of (2.6), (2.18) described in Theorem 2 after replacing $\alpha$ with $\alpha_{M}$, then (4.9) implies $\left|u_{x}(t, x)\right| \leq 3 C_{1} /(2 \delta)$ a.e. and so $\alpha_{M}\left(u_{x}(t, x)\right)=$ $\alpha\left(u_{x}(t, x)\right)$ a.e. This proves Corollary 3.

We shall now give the examples of initial-boundary value problems obtained from Theorem 2.

Example 1. Let $E=H^{1}(0,1)$. Then if $u$ is the solution described in Theorem 2, $u$ is a weak solution of

$$
\begin{align*}
u_{t t} & =\sigma\left(u_{x}\right)_{x}+\left(\alpha\left(u_{x}\right) u_{x t}\right)_{x}+f,  \tag{4.12.1}\\
u(0, x) & =u_{0}(x),  \tag{4.12.2}\\
u_{t}(0, x) & =u_{1}(x),  \tag{4.12.3}\\
\sigma\left(u_{x}(t, 0)\right) & +\alpha\left(u_{x}(t, 0)\right) u_{x t}(t, 0)=k_{0}(t),  \tag{4.12.4}\\
\sigma\left(u_{x}(t, 1)\right) & +\alpha\left(u_{x}(t, 1)\right) u_{x t}(t, 1)=k_{1}(t) . \tag{4.12.5}
\end{align*}
$$

Example 2. Let $E=\left\{u \in H^{1}(0,1): u(1)=0\right\}$. Then $u$ solves (4.12.1)-(4.12.4) and

$$
\begin{equation*}
u(t, 1)=0 . \tag{4.13}
\end{equation*}
$$

5. Proof of Theorem 1. Let $N(t, \cdot), M$, and $Q$ be described in Sec. 2.

Lemma 8. For $v, u \in E$,

$$
\begin{align*}
\|N(t, u)-N(t, v)\|_{E^{\prime}} & \leq K\|u-v\|_{E}  \tag{5.1.1}\\
\|M u-M v\|_{E^{\prime}} & \leq M_{0}\|u-v\|_{E}  \tag{5.1.2}\\
\delta\|u-v\|_{E}^{2} & \leq M u-M v, u-v\rangle \leq M_{0}\|u-v\|_{E^{\prime}}^{2} \tag{5.1.3}
\end{align*}
$$

and if $u, u^{\prime} \in L^{2}\left(a, a+T_{1} ; E\right)$ where $T_{1}>0$ and $u^{\prime}$ is understood in the sense of $E^{\prime}$-valued distributions $((2.11),(2.8))$, then

$$
\begin{equation*}
M(u(t))=M(u(a))+\int_{a}^{t} Q(u(s)) u^{\prime}(s) d s . \tag{5.1.4}
\end{equation*}
$$

(Note that $u(a)$ has meaning because $u, u^{\prime}$ are in $L^{2}\left(a, a+T_{1} ; E\right)$.)
Proof. Parts (5.1.1)-(5.1.3) are obvious. Let $\varphi \in C_{0}^{\infty}\left(a, a+T_{1}\right)$ and let $v \in E$. Also let $\bar{u}$ and $\bar{u}_{x}$ be given in Lemma 1, and let $b=a+T_{1}$.

$$
\begin{align*}
\left\langle\int_{a}^{b} M(u(t)) \varphi^{\prime}(t) d t, v\right\rangle & =\int_{a}^{b}\langle M(u(t)), v\rangle \varphi^{\prime}(t) d t \\
& =\int_{a}^{b}\left(\beta\left(\frac{\partial u(t)}{\partial x}\right), v_{x}\right)_{H} \varphi^{\prime}(t) d t \\
& =\int_{a}^{b} \int_{0}^{1} \beta\left(\bar{u}_{x}(t, x)\right) v_{x}(x) \varphi^{\prime}(t) d x d t \\
& =\int_{0}^{1} \int_{a}^{b} \beta\left(\bar{u}_{x}(t, x)\right) \varphi^{\prime}(t) v_{x}(x) d t d x \\
& =-\int_{a}^{b} \int_{0}^{1} \alpha\left(\bar{u}_{x}(t, x)\right) \bar{u}_{t x}(t, x) \varphi(t) v_{x}(x) d x d t \\
& =-\int_{a}^{b}\left\langle Q(\bar{u}(t, \cdot)) \bar{u}_{t}(t, \cdot), v\right\rangle \varphi(t) d t \\
& =\left\langle-\int_{a}^{b} Q(u(t)) u^{\prime}(t) \varphi(t) d t, v\right\rangle \tag{5.2}
\end{align*}
$$

Since $v \in E$ was arbitrary, it follows that

$$
\begin{equation*}
(M u)^{\prime}=Q(u) u^{\prime} \tag{5.3}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
M(u(t))=M(u(a))+\int_{0}^{t} Q(u(s)) u^{\prime}(s) d s \quad \text { a.e. } \tag{5.4}
\end{equation*}
$$

We need a few more definitions. Let $T_{1}$ satisfy

$$
\begin{equation*}
0<T_{1}<1, \quad \delta e^{-\delta T_{1}}-K T_{1} \geq \delta / 2 \tag{5.5}
\end{equation*}
$$

and let $\left\{E_{n}\right\}$ be a sequence of finite-dimensional subspaces of $E$ satisfying

$$
\begin{equation*}
E=\overline{\bigcup_{n<\infty} E_{n}}, \quad E_{n+1} \supseteq E_{n} \quad \text { for all } n \tag{5.6}
\end{equation*}
$$

Let $u_{0} \in E$ and $u_{1} \in H$ be given and let $\left\{u_{0 n}\right\}$ and $\left\{u_{1 n}\right\}$ be sequences satisfying

$$
\begin{gather*}
u_{0 n}, u_{1 n} \in E_{n}  \tag{5.7.1}\\
\lim _{n \rightarrow \infty}\left\|u_{0 n}-u_{0}\right\|_{E}+\left|u_{1 n}-u_{1}\right|_{H}=0 \tag{5.7.2}
\end{gather*}
$$

Also, for $g$ given in $L^{2}\left(a, a+T_{1} ; E^{\prime}\right)$, let

$$
\begin{align*}
f_{n}(t) & =\int_{a}^{t} g(s) d s+M\left(u_{0 n}\right)+u_{1 n}  \tag{5.8.1}\\
f(t) & =\int_{a}^{t} g(s) d s+M\left(u_{0}\right)+u_{1} \tag{5.8.2}
\end{align*}
$$

Thus $\lim _{n \rightarrow \infty}\left\|f_{n}-f\right\|_{L^{2}\left(a, a+T_{1} ; E^{\prime}\right)}=0$. Let $\left\{w_{1}, \ldots, w_{n}\right\}$ be a basis for $E_{n}$ satisfying $\left(w_{k}, w_{r}\right)_{H}=\delta_{k r}$.
Lemma 9. There exists a unique function $u_{n}$ satisfying

$$
\begin{equation*}
u_{n}, u_{n}^{\prime} \in L^{2}\left(a, a+T_{1} ; E_{n}\right) \tag{5.9.1}
\end{equation*}
$$

and for all $w \in E_{n}$,

$$
\begin{gather*}
\left(u_{n}^{\prime}(t), w\right)_{H}+\left\langle\int_{a}^{t} N\left(s, u_{n}(s)\right) d s, w\right\rangle+\left\langle M u_{n}(t), w\right\rangle=\left\langle f_{n}(t), w\right\rangle \quad \text { a.e., }  \tag{5.9.2}\\
u_{n}(a)=u_{0 n} \tag{5.9.3}
\end{gather*}
$$

Proof. Let $u_{n}(t)=\sum_{k=1}^{n} y_{k}(t) w_{k}$ and let $\mathscr{A}: L^{2}\left(a, a+T_{1} ; R^{n}\right) \rightarrow L^{2}\left(a, a+T_{1} ; R^{n}\right)$ be defined by

$$
\begin{equation*}
(\mathscr{A} \mathbf{y})_{r}(t)=\int_{a}^{t}\left\langle N\left(s, \sum_{k=1}^{n} y_{k}(s) w_{k}\right), w_{r}\right\rangle d s+\left\langle M\left(\sum_{k=1}^{n} y_{k}(t) w_{k}\right), w_{r}\right\rangle \tag{5.10}
\end{equation*}
$$

Thus, $u_{n}$ is a solution to (5.9) if and only if $\mathbf{y}(t)$ solves

$$
\begin{gather*}
\mathbf{y}^{\prime}+\mathscr{A} \mathbf{y}=\mathbf{F} \text { in } L^{2}\left(a, a+T_{1} ; R^{n}\right)  \tag{5.11.1}\\
\mathbf{y}(a)=\mathbf{y}_{0},  \tag{5.11.2}\\
\mathbf{y}, \mathbf{y}^{\prime} \in L^{2}\left(a, a+T_{1} ; R^{n}\right), \tag{5.11.3}
\end{gather*}
$$

where

$$
\begin{equation*}
y_{0 r}=\left(u_{0 n}, w_{r}\right)_{H}, \quad F_{r}(t)=\left\langle f_{n}(t), w_{r}\right\rangle . \tag{5.12}
\end{equation*}
$$

Letting $\mathbf{y}(t)=e^{\lambda(t-a)} \mathbf{z}(t)$, it follows that $\mathbf{y}$ solves (5.11) if and only if $z$ solves

$$
\begin{gather*}
\mathbf{z}^{\prime}+\lambda \mathbf{z}+\mathscr{A}_{\lambda} \mathbf{z}=e^{-\lambda(\cdot-a)} \mathbf{F}  \tag{5.13.1}\\
\mathbf{z}(a)=\mathbf{y}_{0}  \tag{5.13.2}\\
\mathbf{z}, \mathbf{z}^{\prime} \in L^{2}\left(a, a+T_{1} ; R^{n}\right) \tag{5.13.3}
\end{gather*}
$$

where

$$
\begin{equation*}
\mathscr{A}_{\lambda}(\mathbf{z})(t)=e^{-\lambda(t-a)} \mathscr{A}\left(e^{\lambda(\cdot-a)} \mathbf{z}\right)(t) . \tag{5.14}
\end{equation*}
$$

Using the equivalence of all norms on $R^{n}$ and (5.5), a little analysis yields for all $\mathbf{y}, \mathbf{z} \in L^{2}\left(a, a+T_{1} ; R^{n}\right)$,

$$
\begin{equation*}
\left\langle\left(\lambda I+\mathscr{A}_{\lambda}\right) \mathbf{y}-\left(\lambda I+\mathscr{A}_{\lambda}\right) \mathbf{z}, \mathbf{y}-\mathbf{z}\right\rangle_{L^{2}\left(a, a+T_{i} ; R^{n}\right)} \geq \eta\|\mathbf{y}-\mathbf{z}\|^{2} \tag{5.15}
\end{equation*}
$$

for some $\eta>0$, the norm in (5.15) being the norm of $L^{2}\left(a, a+T_{1} ; R^{n}\right)$.
It follows from [14, 17] that there exists a unique solution $\mathbf{z}$ to (5.13). Therefore there exists a unique solution to (5.11) and consequently there exists a unique solution to (5.9). This proves Lemma 9.

Since $y, y^{\prime}$ are in $L^{2}\left(a, a+T_{1} ; R^{n}\right), y$ is absolutely continuous and $\mathbf{y}(t)=\mathbf{y}_{0}+$ $\int_{a}^{t} \mathbf{y}^{\prime}(s) d s$. It follows that for $w \in E_{n}$

$$
\begin{equation*}
\left\langle M u_{n}(t), w\right\rangle=\int_{a}^{t}\left\langle Q u_{n}(s) u_{n}^{\prime}(s), w\right\rangle d s+\left\langle M\left(u_{0 n}\right), w\right\rangle . \tag{5.16}
\end{equation*}
$$

Therefore, from (5.8.1), (5.12), (5.10), (5.11.1) yields

$$
\begin{equation*}
y_{r}^{\prime}=\int_{a}^{(\cdot)} I_{r}(s) d s+\left(u_{1 n}, w_{r}\right)_{H} \tag{5.17}
\end{equation*}
$$

where $I_{r}(\cdot) \in L^{2}\left(a, a+T_{1}\right)$ is given by

$$
\begin{equation*}
\left\langle g(s), w_{r}\right\rangle-\left\langle Q\left(u_{n}(s)\right) u_{n}^{\prime}(s), w_{r}\right\rangle-\left\langle N\left(s, u_{n}(s)\right), w_{r}\right\rangle \tag{5.18}
\end{equation*}
$$

It follows that $\mathbf{y}^{\prime \prime}$ exists and is in $L^{2}\left(a, a+T_{1} ; R^{n}\right)$. Thus $y_{r}^{\prime}(a)$ makes sense and equals ( $\left.u_{1 n}, w_{r}\right)_{H}$. Differentiating (5.17), we have proved the following lemma.

Lemma 10. The function, $u_{n}$, of Lemma 9 satisfies for all $w \in E_{n}$ :

$$
\left.\begin{array}{rl}
\left(u_{n}^{\prime \prime}(t), w\right)_{H}+\left\langle N\left(t, u_{n}(t)\right), w\right\rangle+\left\langle Q\left(u_{n}(t)\right) u_{n}^{\prime}(t), w\right\rangle=\langle g(t), w\rangle \text { a.e., } \\
u_{n}^{\prime}(a) & =u_{1 n} \\
u_{n}(a) & =u_{0 n} \\
u_{n}^{\prime \prime} & \in L^{2}\left(a, a+T_{1} ; H\right) \subseteq L^{2}\left(a, a+T_{1} ; E^{\prime}\right) \\
u_{n}, & u_{n}^{\prime} \tag{5.19.5}
\end{array}\right) L^{2}\left(a, a+T_{1} ; E\right) . ~ \$
$$

For $t \in\left[a, a+T_{1}\right]$ and $w(\cdot)$ a simple function, (5.19.1) implies

$$
\begin{array}{r}
\int_{a}^{t}\left(u_{n}^{\prime \prime}(s), w(s)\right)_{H} d s+\int_{a}^{t}\left\langle N\left(s, u_{n}(s)\right), w(s)\right\rangle d s+\int_{a}^{t}\left\langle Q\left(u_{n}(s)\right) u_{n}^{\prime}(s), w(s)\right\rangle d s \\
=\int_{a}^{t}\langle g(s), w(s)\rangle d s \tag{5.20}
\end{array}
$$

Therefore (5.20) holds for all $w \in L^{2}\left(a, a+T_{1} ; E_{n}\right)$. Replacing $w$ by $u_{n}^{\prime}$ and integrating by parts,

$$
\begin{align*}
&\left|u_{n}^{\prime}(t)\right|_{H}^{2}+2 \int_{0}^{1} W\left(\bar{u}_{n x}(t, x)\right) d x+2 \delta \int_{a}^{t}\left|\frac{\partial u_{n}^{\prime}(s)}{\bar{\partial} x}\right|_{H}^{2} d s-\left|u_{1 n}\right|^{2} \\
&-2 \int_{0}^{1} W\left(u_{0 n}^{\prime}(x)\right) d x-2\left(L+\left|\sigma_{0}\right|\right) \int_{a}^{t}\left|\frac{\partial u_{n}^{\prime}(s)}{\partial x}\right|_{H} d s \\
& \leq 2\|g\|_{L^{2}\left(a, a+T_{1} ; E^{\prime}\right)}\left(\left\|u_{n}^{\prime}\right\|_{L^{2}(a, t ; H)}+\left\|\frac{\partial u_{n}^{\prime}}{\partial x}\right\|_{L^{2}(a, t ; H)}\right) \tag{5.21}
\end{align*}
$$

Let $-J$ be the lower bound of (2.5.1). The boundedness of $\sigma$ implies $\left|\int_{0}^{1} W\left(u_{0 n}^{\prime}(x)\right) d x\right|$ $\leq C_{1}\left\|u_{0 n}\right\|_{E}$, a quantity which is bounded independent of $n$. Therefore, there exists a constant $C_{2}$ which does not depend on $t \in\left[a, a+T_{1}\right]$ or $n$ such that

$$
\begin{align*}
&\left|u_{n}^{\prime}(t)\right|_{H}^{2}+2 \delta\left\|\frac{\partial u_{n}^{\prime}}{\partial x}\right\|_{L^{2}(a, i ; H)}^{2}-2\left(L+\left|\sigma_{0}\right|+\|g\|\right)\left\|\frac{\partial u_{n}^{\prime}}{\partial x}\right\|_{L^{2}(a, t ; H)} \\
& \leq C_{2}+\int_{a}^{t}\left|u_{n}^{\prime}(s)\right|_{H}^{2} d s \tag{5.22}
\end{align*}
$$

Here $C_{2} \geq 2 J+\|g\|_{L^{2}\left(a, a+T_{1} ; E^{\prime}\right)}^{2}+\left|\int_{0}^{1} W\left(u_{0 n}^{\prime}(x)\right) d x\right|+\left|u_{1}\right|_{H}^{2}$.
It follows from (5.22) and Gronwall's inequality that $\left|u_{n}^{\prime}(t)\right|_{H}$ is bounded independent of $n$ and $t$. Therefore (5.22) implies that $\left\|\partial u_{n}^{\prime} / \partial x\right\|_{L^{2}\left(a, a+T_{1} ; H\right)}$ is bounded independent of $n$. Summarizing this, there exists a constant $C_{3}$ which does not depend on $n$ or $t$ such that

$$
\begin{equation*}
\left|u_{n}^{\prime}(t)\right|_{H}+\left\|\frac{\partial u_{n}^{\prime}}{\partial x}\right\|_{L^{2}\left(a, a+T_{1} ; H\right)} \leq C_{3} . \tag{5.23}
\end{equation*}
$$

This implies

$$
\begin{equation*}
\left\|u_{n}^{\prime}\right\|_{L^{2}\left(a, a+T_{1} ; E\right)}+\left\|u_{n}\right\|_{L^{\infty}\left(a, a+T_{1} ; E\right)}+\left\|u_{n}^{\prime}\right\|_{L^{\infty}\left(a, a+T_{1} ; H\right)} \leq C_{4} \tag{5.24}
\end{equation*}
$$

where $C_{4}$ does not depend on $n$.
Let $A: L^{2}\left(a, a+T_{1} ; E\right) \rightarrow L^{2}\left(a, a+T_{1} ; E^{\prime}\right)$ be given by

$$
\begin{equation*}
A u(t)=\int_{a}^{t} N(s, u(s)) d s+M u(t) \tag{5.25}
\end{equation*}
$$

Thanks to the size of $T_{1}, A$ is monotone. It is also clear that $A$ is hemicontinuous $\left(\lim _{t \rightarrow 0}\langle A(u+t w), v\rangle=\langle A u, v\rangle\right)$ and bounded. For convenience, define

$$
\begin{equation*}
\mathscr{U}=L^{2}\left(a, a+T_{1} ; E\right), \quad \mathscr{U}_{n}=L^{2}\left(a, a+T_{1} ; E_{n}\right), \quad \mathscr{H}=L^{2}\left(a, a+T_{1} ; H\right) . \tag{5.26}
\end{equation*}
$$

By (5.9), the following holds for all $w \in \mathscr{U}_{n}$ :

$$
\begin{gather*}
\left(u_{n}^{\prime}, w\right)_{\mathscr{H}}+\left\langle A u_{n}, w\right\rangle_{\mathscr{U}}=\left\langle f_{n}, w\right\rangle_{\mathscr{W}},  \tag{5.27.1}\\
u_{n}(a)=u_{0 n} . \tag{5.27.2}
\end{gather*}
$$

Because of (5.24) there is a subsequence still denoted by $u_{n}$ satisfying

$$
\begin{align*}
u_{n} & \rightarrow u \text { in } L^{\infty}\left(a, a+T_{1} ; E\right) \\
u_{n}^{\prime} & \rightarrow u^{\prime} \quad \text { in } \mathscr{U} \\
u_{n}^{\prime} & \rightarrow u^{\prime} \quad \text { in } L^{\infty}\left(a, a+T_{1} ; H\right)  \tag{5.28}\\
A u_{n} & -\xi \quad \text { in } \mathscr{U}^{\prime} \\
u_{n}(a) & -u(a) \text { in } H .
\end{align*}
$$

Now $u_{n}(a)=u_{0 n}$ and $u_{0 n} \rightarrow u_{0}$ in $E$. Since $u_{n}(a) \rightarrow u(a)$ in $H$, it follows that $u_{0}=u(a)$ and $\left\|u_{n}(a)-u(a)\right\|_{E} \rightarrow 0$.

By density of $\bigcup_{n<\infty} \mathscr{U}_{n}$ in $\mathscr{U},(5.27 .1),(5.8)$, and (5.28) imply $\xi=f-u^{\prime}$. Also

$$
\begin{align*}
\left\langle A u_{n}, u_{n}\right\rangle_{\mathscr{U}} \leq\left\langle f_{n}, u_{n}\right\rangle_{\mathscr{U}}+\frac{1}{2}\left|u_{n}(a)-u(a)\right|_{H}^{2} & \\
& +\left(u^{\prime}, u\right)_{\mathscr{H}}-\left(u^{\prime}, u_{n}\right)_{\mathscr{H}}-\left(u_{n}^{\prime}, u\right)_{\mathscr{H}} \tag{5.29}
\end{align*}
$$

Therefore $\overline{\lim }_{n \rightarrow \infty}\left\langle A u_{n}, u_{n}\right\rangle_{\mathscr{U}} \leq\langle f, u\rangle_{\mathscr{U}}-\left(u^{\prime}, u\right)_{\mathscr{H}}=\langle\xi, u\rangle_{\mathscr{U}}$. Since $A$ is type $M$ [17], it follows that $A u=\xi=f-u^{\prime}$.

Lemma 11. Let $u_{0} \in E, u_{1} \in H$, and let $g \in \mathscr{U}^{\prime}=L^{2}\left(a, a+T_{1} ; E^{\prime}\right)$. Then there exists a unique solution to

$$
\begin{gather*}
u^{\prime \prime}+N u+Q(u) u^{\prime}=g \quad \text { in } \mathscr{U}^{\prime},  \tag{5.30.1}\\
u(a)=u_{0},  \tag{5.30.2}\\
u^{\prime}(a)=u_{1},  \tag{5.30.3}\\
u, u^{\prime} \in \mathscr{U}, \quad u \in L^{\infty}\left(a, a+T_{1} ; E\right), \quad u^{\prime} \in L^{\infty}\left(a, a+T_{1} ; H\right) . \tag{5.30.4}
\end{gather*}
$$

Proof. The discussion preceding Lemma 11 has shown the existence of a solution $u$ satisfying (5.30.4) and

$$
\begin{equation*}
u^{\prime}+A u=f, \quad u(0)=u_{0} \tag{5.31}
\end{equation*}
$$

Since $A$ is strictly monotone, the solution to (5.31) is unique [14]. (This is even true in a larger class of functions than those of (5.30.4).) However, since $u^{\prime} \in \mathscr{U}$, Lemma 8 implies

$$
\begin{equation*}
A u(t)=\int_{a}^{t} N(s, u(s)) d s+\int_{a}^{t} Q(u(s)) u^{\prime}(s) d s+M\left(u_{0}\right) \tag{5.32}
\end{equation*}
$$

and it follows that $u^{\prime \prime} \in \mathscr{U}^{\prime}$ and (5.30) holds. This proves Lemma 11.
Proof of Theorem 1. Choose $T_{1}$ small enough that $\delta e^{-\delta T_{\mathrm{I}}}-K T_{1} \geq \delta / 2$ and use Lemma 11 to solve (5.30.1)-(5.30.3) on [0, T1], obtaining a solution which satisfies (5.30.4) with $a=0$. Next use $u\left(T_{1}\right) \in E$ and $u^{\prime}\left(T_{1}\right) \in H$ as new initial data and solve (5.30.1)-(5.30.3) with these new initial data on $\left[T_{1}, 2 T_{1}\right]$. Continuing in this way proves Theorem 1.

## References

[1] E. C. Aifantis and J. B. Serrin, The mechanical theory of fluid interfaces and Maxwell's rule, J. Colloid Interface Sci. 96, 517-529 (1983)
[2] G. Andrews, On the existence of solutions to the equation $u_{t t}=u_{x x t}+\sigma\left(u_{x}\right)_{x}$, J. Differential Equations 35, 200-231 (1980)
[3] G. Andrews and J. M. Ball, Asymptotic behaviour and changes of phase in one-dimensional nonlinear viscoelasticity, J. Differential Equations 44, 306-341 (1982)
[4] C. Dafermos, The mixed initial-boundary value problem for the equations of nonlinear one-dimensional viscoelasticity, J. Differential Equations 6, 71-86 (1969)
[5] J. M. Greenberg, On the existence, uniqueness, and stability of solutions of the equation $\rho_{0} X_{t l}=$ $E\left(X_{x}\right) X_{x x}+\lambda X_{x x t}$, J. Math. Anal. Appl. 25, 575-591 (1969)
[6] J. M. Greenberg and R. C. MacCamy, On the exponential stability of solutions of $E\left(u_{x}\right) u_{x x}+\lambda u_{x t x}=$ $\rho u_{t t}$, J. Math. Anal. Appl. 31, 406-417 (1970)
[7] J. M. Greenberg, R. C. MacCamy, and V. J. Mizel, On the existence, uniqueness, and stability of solutions of the equation $\sigma^{\prime}\left(u_{x}\right) u_{x x}+\lambda u_{x t x}=\rho_{0} u_{t t}$, J. Math. Mech. 17, 707-728 (1968)
[8] Ya Kanel, Certain systems of quasi-linear parabolic equations of divergence type, USSR Comput. Math. and Math. Phys. 6, 74-88 (1966)
[9] Ya Kanel, A model system of equations for the one-dimensional motion of a gas, Differential Equations 4, 374-380 (1968) (English translation)
[10] A. V. Kazhikhov, Correctness "in the large" of mixed-boundary value problems for a model system of equations of a viscous gas, Dinamika Splos̆n. Sredy 21, 18-47 (1975) (in Russian)
[11] A. V. Kazhikhov and V. B. Nikolaev, Correctness of boundary value problems for equations of gas dynamics with a nonmonotone state function, Čisl. Metody Meh. Splošn. Sredy 10, 77-84 (1979) (in Russian)
[12] K. Kuttler and D. Hicks, Initial-boundary value problems for some nonlinear conservation laws, Applicable Anal. 24, 1-12 (1987)
[13] K. Kuttler, Regularity of weak solutions of some nonlinear conservation laws, Applicable Analysis, Vol. 26, October 1987
[14] K. Kuttler, Time dependent implicit evolution equations, Nonlinear Anal. Theory, Methods Appl. 10(5), 447-463 (1986)
[15] K. Kuttler, Initial boundary value problems for the displacement in an isothermal viscous gas, submitted
[16] K. Kuttler and E. Aifantis, Existence and uniqueness in nonclassical diffusion, Quart. Appl. Math. 45, 549-560 (1987)
[17] J.-L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris, 1969
[18] R. Pego, Phase transitions in one-dimensional nonlinear visco-elasticity: admissibility and stability, Arch. Rat. Mech. Anal. V. 97(4), 353-394 (1987)
[19] R. E. Showalter, Hilbert space methods for partial differential equations, Pittman, 1977
[20] V. A. Solonnikov and A. V. Kazhikhov, Existence theorems for the equations of motion of a compressible viscous fluid, Annual Rev. Fluid Mech. 13, 79-95 (1981)
[21] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, 1977


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