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Initial Coefficient Bounds for a General Class of Bi-Univalent Functions

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Abstract. Recently, Srivastava et al. [22] reviewed the study of coefficient problems for bi-univalent functions. Inspired by the pioneering work of Srivastava et al. [22], there has been triggering interest to study the coefficient problems for the different subclasses of bi-univalent functions (see, for example, [1, 3, 6, 7, 27, 29],). Motivated essentially by the aforementioned works, in this paper we propose to investigate the coefficient estimates for a general class of analytic and bi-univalent functions. Also, we obtain estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new class. Further, we discuss some interesting remarks, corollaries and applications of the results presented here.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

For analytic functions f and g in $\mathbb{U}$, f is said to be subordinate to g if there exists an analytic function w such that (see, for example, [13])

$$w(0) = 0, \qquad |w(z)| < 1 \quad \text{and} \quad f(z) = g(w(z)) \qquad (z \in \mathbb{U}).$$

This subordination will be denoted here by

$$f \prec g \qquad (z \in \mathbb{U})$$

or, conventionally, by

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}).$$

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In particular, when g is univalent in $\mathbb{U}$,

$$f \prec g \qquad (z \in \mathbb{U}) \;\Leftrightarrow\; f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α $(0 \leqq \alpha < 1)$ in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α $(0 \leqq \alpha < 1)$ in $\mathbb{U}$, the class $\mathcal{S}_{\varphi}^{\beta}(\alpha)$ of $\beta-$spirallike functions of order α $(0 \leqq \alpha < 1; |\beta| < \frac{\pi}{2})$, the class $\mathcal{S}^*(\varphi)$ of Ma-Minda starlike functions and the class $\mathcal{K}(\varphi)$ of Ma-Minda convex functions (φ is an analytic function with positive real part in $\mathbb{U}$, $\varphi(0) = 1$, $\varphi'(0) > 0$ and φ maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis) (see [5, 11, 24]). The above-defined function classes have recently been investigated rather extensively in (for example) [9, 17, 25, 26] and the references therein.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \qquad (|w| < r_0(f);\; r_0(f) \geqq \frac{1}{4}),$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots.$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). For a brief history and interesting examples of functions which are in (or which are not in) the class Σ, together with various other properties of the bi-univalent function class Σ one can refer the work of Srivastava et al. [22] and references therein. In fact, the study of the coefficient problems involving bi-univalent functions was reviewed recently by Srivastava et al. [22]. Various subclasses of the bi-univalent function class Σ were introduced and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1) were found in several recent investigations (see, for example, [1–4, 6–8, 12, 14, 16, 19–21, 23, 27, 29]). The aforecited all these papers on the subject were actually motivated by the pioneering work of Srivastava et al. [22]. However, the problem to find the coefficient bounds on $|a_n|$ $(n = 3, 4, \ldots)$ for functions $f \in \Sigma$ is still an open problem.

Motivated by the aforementioned works (especially [22] and [3, 7]), we define the following subclass of the function class Σ.

Definition 1.1. *Let $h : \mathbb{U} \to \mathbb{C}$, be a convex univalent function such that*

$$h(0) = 1 \quad \text{and} \quad h(\bar{z}) = \overline{h(z)} \qquad (z \in \mathbb{U} \text{ and } \Re(h(z)) > 0).$$

Suppose also that the function $h(z)$ is given by

$$h(z) = 1 + \sum_{n=1}^{\infty} B_n z^n \qquad (z \in \mathbb{U}).$$

A function $f(z)$ given by (1) is said to be in the class $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, h)$ if the following conditions are satisfied:

$$f \in \Sigma,\; e^{i\beta}\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) \prec h(z)\cos\beta + i\sin\beta \quad (z \in \mathbb{U}), \tag{2}$$

and

$$e^{i\beta}\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right) \prec h(w)\cos\beta + i\sin\beta \qquad (w \in \mathbb{U}), \tag{3}$$

where $\beta \in (-\pi/2, \pi/2)$, $\lambda \geqq 1$, $\mu \geqq 0$ *and the function* g *is given by*

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots \tag{4}$$

the extension of f^{-1} *to* $\mathbb{U}$.

Remark 1.2. *If we set* $h(z) = \frac{1+Az}{1+Bz}$, $-1 \leqq B < A \leqq 1$, *in the class* $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, h)$, *we have* $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, \frac{1+Az}{1+Bz})$ *and defined as*

$$f \in \Sigma,\ e^{i\beta}\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) \prec \frac{1+Az}{1+Bz}\cos\beta + i\sin\beta \qquad (z \in \mathbb{U})$$

and

$$e^{i\beta}\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right) \prec \frac{1+Aw}{1+Bw}\cos\beta + i\sin\beta \qquad (w \in \mathbb{U}),$$

where $\beta \in (-\pi/2, \pi/2)$, $\lambda \geqq 1$, $\mu \geqq 0$ *and the function* g *is given by (4).*

Remark 1.3. *Taking* $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 \leqq \alpha < 1$ *in the class* $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, h)$, *we have* $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, \alpha)$ *and* $f \in \mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, \alpha)$ *if the following conditions are satisfied:*

$$f \in \Sigma,\ \Re\left(e^{i\beta}\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)\right) > \alpha\cos\beta \qquad (z \in \mathbb{U})$$

and

$$\Re\left(e^{i\beta}\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right)\right) > \alpha\cos\beta \qquad (w \in \mathbb{U}),$$

where $\beta \in (-\pi/2, \pi/2)$, $0 \leqq \alpha < 1$, $\lambda \geqq 1$, $\mu \geqq 0$ *and the function* g *is given by (4). It is interest to note that the class* $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(0, \alpha) := \mathcal{N}_{\Sigma}^{\mu,\lambda}(\alpha)$ *the class was introduced and studied by Çağlar et al. [3].*

Remark 1.4. *Taking* $\lambda = 1$ *and* $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 \leqq \alpha < 1$ *in the class* $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, h)$, *we have* $\mathcal{NP}_{\Sigma}^{\mu,1}(\beta, \alpha)$ *and* $f \in \mathcal{NP}_{\Sigma}^{\mu,1}(\beta, \alpha)$ *if the following conditions are satisfied:*

$$f \in \Sigma,\ \Re\left(e^{i\beta} f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) > \alpha\cos\beta \qquad (z \in \mathbb{U})$$

and

$$\Re\left(e^{i\beta} g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right) > \alpha\cos\beta \qquad (w \in \mathbb{U}),$$

where $\beta \in (-\pi/2, \pi/2)$, $0 \leqq \alpha < 1$, $\mu \geqq 0$ *and the function* g *is given by (4).We notice that the class* $\mathcal{NP}_{\Sigma}^{\mu,1}(0, \alpha) := \mathcal{F}_{\Sigma}(\mu, \alpha)$ *was introduced by Prema and Keerthi [16].*

Remark 1.5. *Taking* $\mu + 1 = \lambda = 1$ *and* $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 \leqq \alpha < 1$ *in the class* $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, h)$, *we have* $\mathcal{NP}_{\Sigma}^{0,1}(\beta, \alpha)$ *and* $f \in \mathcal{NP}_{\Sigma}^{0,1}(\beta, \alpha)$ *if the following conditions are satisfied:*

$$f \in \Sigma,\ \Re\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > \alpha\cos\beta \qquad (z \in \mathbb{U})$$

and

$$\Re\left(e^{i\beta}\frac{wg'(w)}{g(w)}\right) > \alpha\cos\beta \qquad (w \in \mathbb{U}),$$

where $\beta \in (-\pi/2, \pi/2)$, $0 \leqq \alpha < 1$ *and the function* g *is given by (4). In addition, the class* $\mathcal{NP}_{\Sigma}^{0,1}(0,\alpha) := \mathcal{S}_{\Sigma}^{*}(\alpha)$ *was studied by Li and Wang [10] and considered by others.*

Remark 1.6. *Taking* $\mu = 1$ *and* $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 \leqq \alpha < 1$ *in the class* $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta,h)$, *we have* $\mathcal{NP}_{\Sigma}^{1,\lambda}(\beta,\alpha)$ *and* $f \in \mathcal{NP}_{\Sigma}^{1,\lambda}(\beta,\alpha)$ *if the following conditions are satisfied:*

$$f \in \Sigma, \ \Re\left(e^{i\beta}\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right)\right) > \alpha\cos\beta \qquad (z \in \mathbb{U})$$

and

$$\Re\left(e^{i\beta}\left((1-\lambda)\frac{g(w)}{w} + \lambda g'(w)\right)\right) > \alpha\cos\beta \qquad (w \in \mathbb{U}),$$

where $\beta \in (-\pi/2, \pi/2)$, $0 \leqq \alpha < 1$, $\lambda \geqq 1$ *and the function* g *is given by (4). Further, the class* $\mathcal{NP}_{\Sigma}^{1,\lambda}(0,\alpha) := \mathcal{B}_{\Sigma}(\alpha,\lambda)$ *was introduced and discussed by Frasin and Aouf [6]*

Remark 1.7. *Taking* $\mu = \lambda = 1$ *and* $h(z) = \frac{1+(1-2\alpha)z}{1-z}$, $0 \leqq \alpha < 1$ *in the class* $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta,h)$, *we have* $\mathcal{NP}_{\Sigma}^{1,1}(\beta,\alpha)$ *and* $f \in \mathcal{NP}_{\Sigma}^{1,1}(\beta,\alpha)$ *if the following conditions are satisfied:*

$$f \in \Sigma, \ \Re\left(e^{i\beta} f'(z)\right) > \alpha\cos\beta \qquad (z \in \mathbb{U})$$

and

$$\Re\left(e^{i\beta} g'(w)\right) > \alpha\cos\beta \qquad (w \in \mathbb{U}),$$

where $\beta \in (-\pi/2, \pi/2)$, $0 \leqq \alpha < 1$ *and the function* g *is given by (4). Also, the class* $\mathcal{NP}_{\Sigma}^{1,1}(0,\alpha) := \mathcal{H}_{\Sigma}^{\alpha}$ *was introduced and studied by Srivastava et al. [22].*

In order to derive our main result, we have to recall here the following lemmas.

Lemma 1.8. *[15] If* $p \in \mathcal{P}$, *then* $|p_i| \leqq 2$ *for each* i, *where* $\mathcal{P}$ *is the family of all functions* p, *analytic in* $\mathbb{U}$, *for which*

$$\Re\{p(z)\} > 0 \qquad (z \in \mathbb{U}),$$

where

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \qquad (z \in \mathbb{U}).$$

Lemma 1.9. *[18, 28] Let the function* $\varphi(z)$ *given by*

$$\varphi(z) = \sum_{n=1}^{\infty} B_n z^n \qquad (z \in \mathbb{U})$$

be convex in $\mathbb{U}$. *Suppose also that the function* $h(z)$ *given by*

$$\psi(z) = \sum_{n=1}^{\infty} \psi_n z^n \qquad (z \in \mathbb{U})$$

is holomorphic in $\mathbb{U}$. *If*

$$\psi(z) \prec \varphi(z) \qquad (z \in \mathbb{U})$$

then

$$|\psi_n| \leqq |B_1| \qquad (n \in \mathbb{N} = \{1, 2, 3, \dots\}).$$

The object of the present paper is to introduce a general new subclass $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, h)$ of the function class Σ and obtain estimates of the coefficients $|a_2|$ and $|a_3|$ for functions in this new class $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, h)$.

2. Coefficient Bounds for the Function Class $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, h)$

In this section we find the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, h)$.

Theorem 2.1. *Let $f(z)$ given by (1) be in the class $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, h)$, $\lambda \geqq 1$ and $\mu \geqq 0$, then*

$$|a_2| \leqq \sqrt{\frac{2|B_1|\cos\beta}{(1+\mu)(2\lambda+\mu)}} \tag{5}$$

and

$$|a_3| \leqq \frac{2|B_1|cos\beta}{(2\lambda+\mu)(1+\mu)}, \tag{6}$$

where $\beta \in (-\pi/2, \pi/2)$.

Proof. It follows from (2) and (3) that there exists $p, q \in \mathcal{P}$ such that

$$e^{i\beta}\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) = p(z)\cos\beta + i\sin\beta \tag{7}$$

and

$$e^{i\beta}\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right) = p(w)\cos\beta + i\sin\beta, \tag{8}$$

where $p(z) \prec h(z)$ and $q(w) \prec h(w)$ have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots \qquad (z \in \mathbb{U}) \tag{9}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + \dots \qquad (w \in \mathbb{U}). \tag{10}$$

Equating coefficients in (7) and (8), we get

$$e^{i\beta}(\lambda+\mu)a_2 = p_1\cos\beta \tag{11}$$

$$e^{i\beta}\left[\frac{a_2^2}{2}(\mu-1) + a_3\right](2\lambda+\mu) = p_2\cos\beta \tag{12}$$

$$-e^{i\beta}(\lambda+\mu)a_2 = q_1\cos\beta \tag{13}$$

and

$$e^{i\beta}\left[(\mu+3)\frac{a_2^2}{2} - a_3\right](2\lambda+\mu) = q_2\cos\beta. \tag{14}$$

From (11) and (13), we get

$$p_1 = -q_1 \tag{15}$$

and

$$2e^{i2\beta}(\lambda+\mu)^2 a_2^2 = (p_1^2+q_1^2)\cos^2\beta. \tag{16}$$

Also, from (12) and (14), we obtain

$$a_2^2 = \frac{e^{-i\beta}(p_2+q_2)\cos\beta}{(1+\mu)(2\lambda+\mu)}. \tag{17}$$

Since $p, q \in h(\mathbb{U})$, applying Lemma 1.9, we immediately have

$$|p_m| = \left|\frac{p^{(m)}(0)}{m!}\right| \leqq |B_1| \qquad (m \in \mathbb{N}), \tag{18}$$

and

$$|q_m| = \left|\frac{q^{(m)}(0)}{m!}\right| \leqq |B_1| \qquad (m \in \mathbb{N}). \tag{19}$$

Applying (18), (19) and Lemma 1.9 for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_2| \leqq \sqrt{\frac{2|B_1|\cos\beta}{(1+\mu)(2\lambda+\mu)}}.$$

This gives the bound on $|a_2|$ as asserted in (5).

Next, in order to find the bound on $|a_3|$, by subtracting (14) from (12), we get

$$2(a_3-a_2^2)(2\lambda+\mu) = e^{-i\beta}(p_2-q_2)\cos\beta. \tag{20}$$

It follows from (17) and (20) that

$$a_3 = \frac{e^{-i\beta}\cos\beta(p_2+q_2)}{(1+\mu)(2\lambda+\mu)} + \frac{e^{-i\beta}(p_2-q_2)\cos\beta}{2(2\lambda+\mu)}. \tag{21}$$

Applying (18), (19) and Lemma 1.9 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leqq \frac{2|B_1|cos\beta}{(2\lambda+\mu)(1+\mu)}.$$

This completes the proof of Theorem 2.1. □

3. Corollaries and Consequences

In view of Remark 1.2, if we set

$$h(z) = \frac{1+Az}{1+Bz} \qquad (-1 \leqq B < A \leqq 1;\ z \in \mathbb{U})$$

and

$$h(z) = \frac{1+(1-2\alpha)z}{1-z} \qquad (0 \leqq \alpha < 1;\ z \in \mathbb{U}),$$

in Theorem 2.1, we can readily deduce Corollaries 3.1 and 3.2, respectively, which we merely state here without proof.

Corollary 3.1. *Let $f(z)$ given by (1) be in the class $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, \frac{1+Az}{1+Bz})$, then*

$$|a_2| \leqq \sqrt{\frac{2(A-B)\cos\beta}{(1+\mu)(2\lambda+\mu)}} \tag{22}$$

and

$$|a_3| \leqq \frac{2(A-B)cos\beta}{(2\lambda+\mu)(1+\mu)}, \tag{23}$$

where $\beta \in (-\pi/2, \pi/2)$, $\mu \geqq 0$ and $\lambda \geqq 1$.

Corollary 3.2. *Let $f(z)$ given by (1) be in the class $\mathcal{NP}_{\Sigma}^{\mu,\lambda}(\beta, \alpha)$, $0 \leqq \alpha < 1$, $\mu \geqq 0$ and $\lambda \geqq 1$, then*

$$|a_2| \leqq \sqrt{\frac{4(1-\alpha)\cos\beta}{(1+\mu)(2\lambda+\mu)}} \tag{24}$$

and

$$|a_3| \leqq \frac{4(1-\alpha)cos\beta}{(2\lambda+\mu)(1+\mu)}, \tag{25}$$

where $\beta \in (-\pi/2, \pi/2)$.

Remark 3.3. *When $\beta = 0$ the estimates of the coefficients $|a_2|$ and $|a_3|$ of the Corollary 3.2 are improvement of the estimates obtained in [3, Theorem 3.1].*

Corollary 3.4. *Let $f(z)$ given by (1) be in the class $\mathcal{NP}_{\Sigma}^{\mu,1}(\beta, \alpha)$, $0 \leqq \alpha < 1$ and $\mu \geqq 0$, then*

$$|a_2| \leqq \sqrt{\frac{4(1-\alpha)\cos\beta}{(1+\mu)(2+\mu)}} \tag{26}$$

and

$$|a_3| \leqq \frac{4(1-\alpha)cos\beta}{(2+\mu)(1+\mu)}, \tag{27}$$

where $\beta \in (-\pi/2, \pi/2)$.

Corollary 3.5. *Let $f(z)$ given by (1) be in the class $\mathcal{NP}_{\Sigma}^{0,1}(\beta, \alpha)$, $0 \leqq \alpha < 1$, then*

$$|a_2| \leqq \sqrt{2(1-\alpha)\cos\beta} \tag{28}$$

and

$$|a_3| \leqq 2(1-\alpha)cos\beta, \tag{29}$$

where $\beta \in (-\pi/2, \pi/2)$.

Remark 3.6. *Taking $\beta = 0$ in Corollary 3.5, the estimate (28) reduces to $|a_2|$ of [10, Corollary 3.3] and (29) is improvement of $|a_3|$ obtained in [10, Corollary 3.3].*

Corollary 3.7. *Let $f(z)$ given by (1) be in the class $\mathcal{NP}_{\Sigma}^{1,\lambda}(\beta,\alpha)$, $0 \leqq \alpha < 1$ and $\lambda \geqq 1$, then*

$$|a_2| \leqq \sqrt{\frac{2(1-\alpha)\cos\beta}{2\lambda+1}} \tag{30}$$

and

$$|a_3| \leqq \frac{2(1-\alpha)cos\beta}{2\lambda+1}, \tag{31}$$

where $\beta \in (-\pi/2, \pi/2)$.

Remark 3.8. *Taking $\beta = 0$ in Corollary 3.7, the inequality (31) improves the estimate of $|a_3|$ in [6, Theorem 3.2].*

Corollary 3.9. *Let $f(z)$ given by (1) be in the class $\mathcal{NP}_{\Sigma}^{1,1}(\beta,\alpha)$, $0 \leqq \alpha < 1$, then*

$$|a_2| \leqq \sqrt{\frac{2(1-\alpha)\cos\beta}{3}} \tag{32}$$

and

$$|a_3| \leqq \frac{2(1-\alpha)cos\beta}{3}, \tag{33}$$

where $\beta \in (-\pi/2, \pi/2)$.

Remark 3.10. *For $\beta = 0$ the inequality (33) improves the estimate $|a_3|$ of [22, Theorem 2].*

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