# Initial Value Problems for Systems of Ordinary First and Second Order Differential Equations with a Singularity of the First Kind

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# INITIAL VALUE PROBLEMS FOR SYSTEMS OF ORDINARY FIRST AND SECOND ORDER DIFFERENTIAL EQUATIONS WITH A SINGULARITY OF THE FIRST KIND[1]

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**Abstract:** Analytical properties like existence, uniqueness and smoothness of bounded solutions of nonlinear singular initial value problems for ordinary differential equations of first and second order are considered. Particular attention is paid to the structure of initial conditions which are necessary and sufficient for the solution to be continuous.

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## 1 Introduction

We investigate analytical properties of the following class of nonlinear systems of initial value problems of first order:

$$y'(t) = \frac{M(t)}{t} y(t) + f(t, y(t)), \quad t \in (0, 1], \tag{1.1a}$$

$$B_0 y(0) = \beta, \tag{1.1b}$$

$$y \in C[0, 1], \tag{1.1c}$$

where $y, f$ are vector-valued functions of dimension $n$, $M$ is an $n \times n$ matrix, $B_0$ is an $m \times n$ matrix and $\beta$ is a vector of dimension $m \leq n$. The results for (1.1) can also be used in the analysis of second order systems of the form

$$y''(t) = \frac{A_1(t)}{t} y'(t) + \frac{A_0(t)}{t^2} y(t) + f(t, y(t)), \quad t \in (0, 1], \tag{1.2a}$$

$$B_0 y(0) = \beta, \tag{1.2b}$$

$$y \in C[0, 1], \tag{1.2c}$$

where $A_0, A_1$ are $n \times n$ matrices, $B_0$ is an $m \times n$ matrix and $\beta$ is a vector of dimension $m \leq n$. The linear transformation $z(t) := (y(t), ty'(t))^T$ applied to (1.2a) yields a $2n$-dimensional system of the form (1.1a). We briefly comment on the results for (1.2), a more thorough discussion can be found in [11]. For (1.1a) we will also consider "terminal value problems" where the boundary conditions are posed at $t = 1$ instead of $t = 0$.

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Mathematical models of various applications from physics, chemistry and mechanics (e. g. Thomas-Fermi differential equation, see [3], Ginzburg-Landau equation, see [16], problems in shell buckling, see [5]) take the form of boundary value problems for (1.2a) when, due to symmetries in the geometry of the problem data, the underlying PDEs can be reduced to systems of ODEs. Singular problems arise also in other research fields. Problems as different as the solution of differential equations posed on unbounded intervals (see [15]), the computation of connecting orbits or invariant manifolds for dynamical systems (see [14]), differential-algebraic equations (see [12]) or Sturm-Liouville eigenvalue problems (see [6]) are in the scope of techniques for singular boundary value problems. Often, numerical methods are used to approximate the solution, cf. [2] or [7]. Initial value problems of the form (1.1) are also encountered in the computation of avalanche run-up (see [13]).
Research activities in the above areas are a strong motivation for the search for a method to be used as a basis for a reliable standard code designed especially for solving singular boundary value problems, and taking into account the specific difficulties caused by singularities. Initial value methods are often considered an attractive alternative to the computationally expensive application of direct discretization methods, see for example [1]. Multiple shooting seems to be of particular interest, because within its framework one can use different controlling mechanisms close to and away from the singular point. Also, higher order methods often show unsatisfactory convergence properties for singular problems, for results on collocation schemes see [10] and [17]. Moreover, the standard acceleration techniques based on low-order methods do not work efficiently either, since a proper asymptotic error expansion for the basic scheme does not exist, in general, cf. [8].
Our aim is to develop a theoretical background for the shooting procedure based on the numerical solution of *singular initial value problems* and this paper provides the study of the analytical properties of (1.1) and (1.2). This knowledge is necessary for the convergence analysis of the underlying one-step or multi-step methods. Here, existence and uniqueness of bounded solutions of (1.1) and (1.2), as well as their smoothness, will be discussed. In the analysis of the linear case in Section 2 we heavily rely on techniques developed in [9]. For the nonlinear problem treated in Section 3 techniques from [9] cannot be applied, a modified standard technique for initial value problems is the proper tool here. In Section 4 the analogous results for the second order problems are formulated. In order to avoid repetitions, we restrict our attention to the important case of the initial value problem and refer to [11] for details of the terminal value problem.

# 2 Analytic results for linear systems of first order

## 2.1 Linear problems with constant coefficient matrix $M$

Here, we consider initial value problems

(2.1a) $$y'(t) = \frac{M}{t}y(t) + f(t), \quad t \in (0,1],$$

(2.1b) $$B_0 y(0) = \beta,$$

(2.1c) $$y \in C[0,1],$$

and "terminal value problems" (with the initial condition posed at $t = 1$)

(2.2a) $$y'(t) = \frac{M}{t}y(t) + f(t), \quad t \in (0,1],$$

$$B_1 y(1) = \beta, \tag{2.2b}$$

$$y \in C[0,1]. \tag{2.2c}$$

We discuss this case in a comprehensive and complete manner since it is the key to the understanding of the rest of the theory. Particular attention is paid to the structure of $n$ linearly independent initial conditions (2.1b), (2.1c) and (2.2b), (2.2c) necessary and sufficient for a unique bounded solution to exist.
We first construct the general solution of (2.1a). Let us denote by $J$ the Jordan canonical form of $M$ and by $E$ the associated matrix of generalized eigenvectors of $M$. Moreover, let

$$\begin{aligned} v(t) &:= E^{-1}y(t), \\ g(t) &:= E^{-1}f(t). \end{aligned}$$

Then we can rewrite (2.1a) and obtain

$$v'(t) = \frac{J}{t}v(t) + g(t). \tag{2.3}$$

To simplify matters, we assume $J \in \mathbb{C}^{n\times n}$ to consist of only one block,

$$J = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ 0 & & & \lambda \end{pmatrix}, \quad \lambda = \sigma + \mathrm{i}\rho \in \mathbb{C}. \tag{2.4}$$

**LEMMA 2.1** *Every solution of* (2.3) *has the form*

$$v(t) = \Phi(t)c + \Phi(t)\int_1^t \Phi^{-1}(\tau)g(\tau)\,d\tau, \tag{2.5}$$

*where $c \in \mathbb{C}^n$ is an arbitrary vector and*

$$\Phi(t) = t^J := \exp(J\ln(t))$$

*is the fundamental solution matrix which satisfies*[2]

$$\Phi'(t) = \frac{J}{t}\Phi(t), \;\; \Phi(1) = I, \quad t \in (0,1]. \tag{2.6}$$

**Proof:** See [4].

We note that the fundamental solution matrix has the form

$$t^J = t^\lambda \begin{pmatrix} 1 & \ln(t) & \frac{\ln(t)^2}{2} & \cdots & \frac{\ln(t)^{n-1}}{(n-1)!} \\ 0 & 1 & \ln(t) & \cdots & \frac{\ln(t)^{n-2}}{(n-2)!} \\ 0 & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ln(t) \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}. \tag{2.7}$$

From the structure of this matrix and from Lemma 2.1 it is clear that the solution $v(t)$ given by (2.5) is not continuous on $[0,1]$ in general, and its smoothness depends on the eigenvalues of $M$. Consequently, we treat separately the cases[3] $\sigma < 0$, $\lambda = 0$, $\sigma > 0$.

[2] $I$ denotes the identity matrix in $\mathbb{R}^{n\times n}$.

[3] We exclude the case of purely imaginary eigenvalues of $M$, $\lambda = \mathrm{i}\sigma$, leading to solutions of the form $t^{\mathrm{i}\sigma} = \cos(\sigma\ln t) + \mathrm{i}\sin(\sigma\ln t)$.

**Eigenvalues with negative real parts**

Before formulating the main result of this section, we state the following lemma. Here and in the following,

$$\|f\|_\delta := \max_{0\le t\le\delta} |f(t)|,$$

where $|\cdot|$ denotes the maximum norm in $\mathbb{C}^n$. For $\delta = 1$, we define $\|f\| := \max_{0\le t\le 1} |f(t)|$.

**LEMMA 2.2** *Let $\gamma > 0$ and in $J$ from (2.4) assume either $\sigma < 0$ or $\lambda = 0$. Then for*

$$u(t) := t^\gamma \int_0^1 s^{-J} s^{\gamma-1} f(st)\, ds$$

*the following estimate holds:*

$$|u(t)| \le \text{const.}\, t^\gamma \|f\|_\delta, \quad t \in [0,\delta].$$

**Proof:** Clearly,

$$\begin{aligned}\int_0^1 s^{\gamma-1}|s^{-J}|\, ds &\le \int_0^1 \sum_{i=0}^{n-1} s^{\gamma-\sigma-1}\frac{(-\ln(s))^i}{i!}\, ds = \sum_{i=0}^{n-1}\sum_{k=0}^{i} \frac{s^{\gamma-\sigma}(-\ln(s))^k}{k!(\gamma-\sigma)^{i+1-k}}\Bigg|_0^1 \\ &= \sum_{i=0}^{n-1} \frac{1}{(\gamma-\sigma)^{i+1}} < \infty.\end{aligned}$$

**LEMMA 2.3** *Let all eigenvalues of $M$ have negative real parts. Then for every $f \in C^p[0,1]$, $p \ge 0$, there exists a unique solution $y \in C[0,1]$ of* (2.1a). *This solution has the form*

$$y(t) = t\int_0^1 s^{-M} f(st)\, ds =: t\eta(t), \tag{2.8}$$

*and satisfies*[4] *$y(0) = 0$. Moreover, $y \in C^{p+1}[0,1]$ and the following estimates hold:*

$$|y(t)| \le \text{const.}\, t\|f\|, \tag{2.9}$$

$$|y'(t)| \le \text{const.}\, \|f\|. \tag{2.10}$$

**Proof:** We first rewrite (2.5) and obtain[5]

$$\begin{aligned} v(t) &= t^J\left(c - \int_0^1 \tau^{-J} g(\tau)\, d\tau\right) + t^J\int_0^t \tau^{-J} g(\tau)\, d\tau \\ &= t^J\tilde{c} + t\int_0^1 s^{-J} g(st)\, ds =: v_h(t) + v_p(t). \end{aligned} \tag{2.11}$$

Since the function $G(t,s) := s^{-J}g(st)$ is continuous on $[0,1]\times[0,1]$, $v_p \in C[0,1]$ follows, and it is clear from (2.7) that $v \in C[0,1]$ iff $\tilde{c} = 0$. Thus the unique solution $y \in C[0,1]$ of (2.1a) is given by (2.8). The smoothness result follows by substituting (2.8) into (2.1a), on noting that $\eta \in C^p[0,1]$ if $f \in C^p[0,1]$. The estimate for $y(t)$ can be derived from Lemma 2.2 for $\gamma = \delta = 1$, and the estimate for $y'(t)$ follows by substituting (2.8) into (2.1a).

[4]This condition is necessary and sufficient for $y$ to be continuous.
[5]Here, $J$ may consist of more than one Jordan block.

**Eigenvalue $\lambda = 0$**

Let $X_0$ be the eigenspace of $M$ associated with the eigenvalue $\lambda = 0$, and let us denote by $R$ the orthogonal projection onto $X_0$. Let $\tilde{R} \in \mathbb{C}^{n\times r}$ be the matrix consisting of the linearly independent columns of $R$, and denote by $\mathcal{R}(R)$ the range of $R$. In order to simplify the subsequent analysis we select a basis in which $M$ is reduced to Jordan form and use this basis to construct the projections.

**LEMMA 2.4** *Let all eigenvalues of $M$ be zero. Then for every $f \in C^p[0,1]$, $p \geq 0$, and every $\gamma \in \mathcal{R}(R)$, there exists a solution $y \in C[0,1]$ of* (2.1a)*. This solution has the form*

$$y(t) = \gamma + t\int_0^1 s^{-M} f(st)\, ds \tag{2.12}$$

*and satisfies $My(0) = 0$. Let $m = r$ and assume that the $r \times r$ matrix $B_0\tilde{R}$ is nonsingular. Then there exists a unique solution $y \in C[0,1]$ satisfying* (2.1)*. This solution is given by* (2.12) *with $\gamma = \tilde{R}(B_0\tilde{R})^{-1}\beta$. Moreover, $y \in C^{p+1}[0,1]$ and the following estimates hold:*

$$|y(t)| \leq \text{const.}\, t\|f\| + |\tilde{R}(B_0\tilde{R})^{-1}\beta|, \tag{2.13}$$

$$|y'(t)| \leq \text{const.}\, \|f\|. \tag{2.14}$$

**Proof:** We first show that $v_p(t)$ in the representation of the solution (2.11) is continuous. Define functions

$$h_m(t) := \int_{\frac{1}{m}}^1 s^{-J} g(st)\, ds, \quad m \in \mathbb{N},$$

and

$$h_\infty(t) := \int_0^1 s^{-J} g(st)\, ds.$$

Then, see the proof of Lemma 2.2,

$$\lim_{m\to\infty} |h_\infty(t) - h_m(t)| \leq \lim_{m\to\infty} \text{const.} \sum_{i=0}^{n-1}\sum_{k=0}^{i} \frac{\frac{1}{m}|\ln(m)|^k}{k!} = 0$$

and hence $h_\infty$ is continuous as the uniform limit of continuous functions.
It is obvious from (2.7) that $v \in C[0,1]$ iff $J\tilde{c} = 0$ in (2.11). This is equivalent to $v(0) = \tilde{c}$ and yields (2.12) with $y(0) = \gamma \in \ker M = \mathcal{R}(R)$. Consequently, $My(0) = 0$ is a necessary and sufficient condition for the solution of (2.1a) to be continuous. The smoothness results for the solution follow immediately from the smoothness of $f$ and (2.12).
Finally, we derive a condition which is sufficient for $y$ from (2.12) to satisfy (2.1b). We substitute $y$ into (2.1b) and obtain $B_0\gamma = \beta$. This yields $B_0\tilde{R}\alpha = \beta$ on noting that for every $\gamma \in \mathcal{R}(R)$ there exists a unique $\alpha \in \mathbb{C}^r$ such that $\gamma = \tilde{R}\alpha$. This system of $m$ linear equations for $\alpha$ is uniquely solvable iff $m = r$ and $B_0\tilde{R}$ is nonsingular. In that case,

$$\gamma = \tilde{R}(B_0\tilde{R})^{-1}\beta.$$

The estimates (2.13) and (2.14) can be shown using the results of Lemma 2.2 for $\lambda = 0$.

We now consider (2.2). Since in this case all initial conditions need to be posed at $t = 1$ the only possible form of $M$ is limited[6] to $M = 0$. This case is not interesting in its own right, but in the case of a matrix with more complex spectral properties (see Assumption A.3.2) the classical existence and uniqueness results and bounds for the solution of (2.2) will be incorporated into the analysis.

**Eigenvalues with positive real parts**

**LEMMA 2.5** *Let $\gamma \geq 0$ and in $J$ from (2.4) assume $\sigma > 0$. Then for*

$$u(t) := \int_t^\delta \left(\frac{t}{\tau}\right)^J \tau^{\gamma-1}\,d\tau, \quad t \in [0,\delta]$$

*the following estimates hold:*

$$|u(t)| \leq \begin{cases} \text{const.}\left(\frac{t}{\delta}\right)^\sigma \left(1+\left|\ln\left(\frac{t}{\delta}\right)\right|^{n-1}\right)\delta^\gamma, & \sigma < \gamma, \\ \text{const.}\, t^\sigma \left(1+\left|\ln\left(\frac{t}{\delta}\right)\right|^{n}\right), & \sigma = \gamma, \\ \text{const.}\, t^\gamma, & \sigma > \gamma. \end{cases}$$

**Proof:** We treat separately the cases $\sigma < \gamma$, $\sigma = \gamma$ and $\sigma > \gamma$.

(i) $\sigma < \gamma$:

$$\begin{aligned} |u(t)| &\leq \text{const.} \int_t^\delta \left(\frac{t}{\tau}\right)^\sigma \left(1+\left|\ln\left(\frac{t}{\tau}\right)\right|^{n-1}\right)\tau^{\gamma-1}\,d\tau \\ &\leq \text{const.}\, t^\sigma \left(1+\left|\ln\left(\frac{t}{\delta}\right)\right|^{n-1}\right)\int_t^\delta \tau^{\gamma-\sigma-1}\,d\tau. \end{aligned}$$

(ii) $\sigma = \gamma$:

$$|u(t)| \leq \text{const.}\, t^\sigma \left(1+\left|\ln\left(\frac{t}{\delta}\right)\right|^{n-1}\right)\int_t^\delta \tau^{-1}\,d\tau.$$

(iii) $\sigma > \gamma$: In this case there exists an $\varepsilon > 0$ such that $\sigma = \gamma + 2\varepsilon$ and hence the result follows from

$$\begin{aligned} |u(t)| &\leq \text{const.} \int_t^\delta \left(\frac{t}{\tau}\right)^{\gamma+\varepsilon} \left[\left(\frac{t}{\tau}\right)^\varepsilon \left(1+\left|\ln\left(\frac{t}{\tau}\right)\right|^{n-1}\right)\right]\tau^{\gamma-1}\,d\tau \\ &\leq \text{const.} \int_t^\delta \left(\frac{t}{\tau}\right)^{\gamma+\varepsilon} \tau^{\gamma-1}\,d\tau. \end{aligned}$$

We now use the above result to prove the following lemma.

[6] To see this consider the structure of the solution $v$ from (2.11) for $n = 2$: $v_1(t) = c_1 + c_2 \ln(t) + v_{p,1}(t)$, $v_2(t) = c_2 + v_{p,2}(t)$ with $v_1(1) = c_1 + v_{p,1}(1)$, $v_2(1) = c_2 + v_{p,2}(1)$. Clearly, $c_2 = 0$ is required for $v \in C[0,1]$ which is equivalent to the initial condition $v_2(1) = v_{p,2}(1)$, but the value of the particular solution $v_{p,2}(1)$ is unknown, in general.

**LEMMA 2.6** *Let all eigenvalues of $M$ have positive real parts. Then for every $f \in C^p[0,1]$, $p \geq 0$, and every $c \in \mathbb{C}^n$, there exists a solution $y \in C[0,1]$ of* (2.2a)*. This solution has the form*

$$y(t) = t^M c + t^M \int_1^t \tau^{-M} f(\tau)\, d\tau =: y_h(t) + y_p(t). \tag{2.15}$$

*If the matrix $B_1$ is nonsingular, then there exists a unique solution of* (2.2)*. This solution is given by* (2.15) *with $c = B_1^{-1}\beta$, and the following estimates hold:*

$$|y(t)| \;\leq\; \begin{cases} \text{const.}\, t^{\sigma_+}(1+|\ln(t)|^{n_{\max}-1})(|B_1^{-1}\beta| + \|f\|), & \sigma_+ < 1, \\ \text{const.}\, t(1+|\ln(t)|^{n_{\max}})(|B_1^{-1}\beta| + \|f\|), & \sigma_+ = 1, \\ \text{const.}\, t(|B_1^{-1}\beta| + \|f\|), & \sigma_+ > 1, \end{cases} \tag{2.16}$$

$$|y'(t)| \;\leq\; \begin{cases} \text{const.}\, t^{\sigma_+-1}(1+|\ln(t)|^{n_{\max}-1})(|B_1^{-1}\beta| + \|f\|), & \sigma_+ < 1, \\ \text{const.}\, (1+|\ln(t)|^{n_{\max}})(|B_1^{-1}\beta| + \|f\|), & \sigma_+ = 1, \\ \text{const.}\, (|B_1^{-1}\beta| + \|f\|), & \sigma_+ > 1, \end{cases} \tag{2.17}$$

*where $\sigma_+$ is the smallest of the positive real parts of the eigenvalues of $M$ and $n_{\max}$ is the dimension of the largest Jordan block in the Jordan canonical form $J$ associated with $M$. This solution satisfies $y \in C[0,1] \cap C^{p+1}(0,1]$. Moreover, if $p < \sigma_+ \leq p+1$, then $y \in C^p[0,1] \cap C^{p+1}(0,1]$ and if $\sigma_+ > p+1$, then $y \in C^{p+1}[0,1]$.*

**Proof:** Clearly, $y_h(t) \in C[0,1]$, see (2.7). Also, $y_p(t) \in C[\varepsilon,1]$, for any $\varepsilon > 0$, and the limit $\lim_{\varepsilon\to 0} y_p(\varepsilon)$ exists according to Lemma 2.5, where $\delta = \gamma = 1$. Therefore, $y \in C[0,1]$. The estimate (2.16) follows immediately from

$$|t^M| \leq \text{const.}\, t^{\sigma_+}(1+|\ln(t)|^{n_{\max}-1})$$

and Lemma 2.5, $\gamma = \delta = 1$. The estimate (2.17) follows by substituting $y(t)$ into (2.2a) and using (2.16).

We now discuss the smoothness properties of $y$. It is clear from

$$y_h^{(r)}(t) = \left(\frac{d^r}{dt^r} t^M\right) c = M^r t^{M-rI} c, \quad r = 1,\dots,p+1,$$

that $p < \sigma_+ \leq p+1$ is sufficient for $y_h \in C^p[0,1] \cap C^{p+1}(0,1]$ and $\sigma_+ > p+1$ for $y_h \in C^{p+1}[0,1]$. We now consider $y_p(t)$. The substitution of $y_p(t)$ into (2.2a) yields

$$y_p'(t) = M t^{M-I} \int_1^t \tau^{-M} f(\tau)\, d\tau + f(t) = M \int_1^t \left(\frac{t}{\tau}\right)^{M-I} \tau^{-1} f(\tau)\, d\tau + f(t)$$

and it follows immediately from Lemma 2.5 that $y_p \in C^1[0,1]$ if $f \in C[0,1]$ and $\sigma_+ > 1$. We use integration by parts to rewrite the above representation for $y_p'$,

$$\begin{aligned} y_p'(t) \;&=\; M(I-M)^{-1} f(t) - M t^{M-I}(I-M)^{-1} f(1) \\ &\quad - M t^{M-I} \int_1^t \tau^{I-M}(I-M)^{-1} f'(\tau)\, d\tau + f(t), \end{aligned}$$

and from the differentiation thereof we have[7]

$$y_p''(t) = M t^{M-2I}\left(f(1) + \int_1^t \tau^{I-M} f'(\tau)\, d\tau\right) + f'(t).$$

[7]Note that $s^{I-M}(I-M)^{-1} = (I-M)^{-1} s^{I-M}$.

Consequently, if $\sigma_+ > 2$ and $f \in C^1[0,1]$, then $y_p \in C^2[0,1]$. Continuing this process we obtain analogous expressions for higher derivatives of $y$,

$$|y_p^{(p+1)}(t)| = Mt^{M-(p+1)I}\left(\int_1^t \tau^{pI-M} f^{(p)}(\tau)\,d\tau + \sum_{k=0}^{p-1}\prod_{l=p}^{k+2}(M-lI)f^{(k)}(1)\right) + f^{(p)}(t).$$

For $\sigma_+ > p+1$ and $f \in C^p[0,1]$, $y \in C^{p+1}[0,1]$ and the result follows.

**Remark:** From the estimate (2.16) it is clear that it is not possible to prescribe initial conditions at $t = 0$ to define a unique solution in the case where all the real parts of eigenvalues of $M$ are positive. In this case, for every solution of (2.2a) $y(0) = 0$ holds.

## General systems

The above discussion of the structure of smooth solutions suggests to associate different spectral properties of $M$ with (2.1) and (2.2), respectively. For the subsequent investigations we therefore make the following assumptions.

A.3.1 In (2.1) we assume all real parts of the eigenvalues of $M$ to be nonpositive[8].

A.3.2 In (2.2) we assume all real parts of the eigenvalues of $M$ to be nonnegative. If zero is an eigenvalue of $M$, then the associated invariant subspace is assumed to be the eigenspace of $M$.

From the results of Lemmas 2.3, 2.4, and 2.6 and classical results for the case $M = 0$ it follows that in (2.1) the smoothness requirement (2.1c) is equivalent to $\text{rank}(M) = n - \text{rank}(R)$ homogeneous initial conditions, $My(0) = 0$, the solution $y$ must satisfy. In (2.2) the smoothness requirement (2.2c) is satisfied by any solution of (2.2a). Clearly, both statements are correct for the special spectral properties of $M$ formulated in A.3.1 and A.3.2. We stress that these assumptions establish most general singular *initial value* problems of the form (2.1) and (2.2), where *all* conditions necessary and sufficient for a unique solution $y \in C[0,1]$ have to be prescribed at one point, either at $t = 0$ or at $t = 1$. Now the next results follow immediately.

**THEOREM 2.7** *Let the $r \times r$ matrix $B_0\tilde{R}$ be nonsingular. Then for every $f \in C^p[0,1]$, $p \geq 0$, and any vector $\beta \in \mathbb{C}^r$, there is a unique solution $y \in C^{p+1}[0,1]$ of (2.1). This solution has the form*

$$y(t) = \tilde{R}(B_0\tilde{R})^{-1}\beta + t\int_0^1 s^{-M} f(st)\,ds. \tag{2.18}$$

*Furthermore,*

$$|y(t)| \leq \text{const.}\, t\|f\| + |\tilde{R}(B_0\tilde{R})^{-1}\beta|, \tag{2.19}$$

$$|y'(t)| \leq \text{const.}\|f\|. \tag{2.20}$$

**THEOREM 2.8** *Let $B_1$ be nonsingular. Then for every $f \in C^p[0,1]$, $p \geq 0$, and any vector $\beta \in \mathbb{C}^n$, there is a unique solution $y \in C[0,1] \cap C^{p+1}(0,1]$ of (2.2). This solution is given by*

$$y(t) = t^M B_1^{-1}\beta + t^M\int_1^t \tau^{-M} f(\tau)\,d\tau, \tag{2.21}$$

[8] Again, if $\sigma = 0$, then $\lambda = 0$.

*and the following estimates hold:*

$$|y(t)| \leq \begin{cases} \text{const.}\,(t^{\sigma_+}(1+|\ln(t)|^{n_{\max}-1})+1)(|B_1^{-1}\beta|+\|f\|), & \sigma_+<1,\\ \text{const.}\,(t(1+|\ln(t)|^{n_{\max}})+1)(|B_1^{-1}\beta|+\|f\|), & \sigma_+=1,\\ \text{const.}\,(t+1)(|B_1^{-1}\beta|+\|f\|), & \sigma_+>1,\end{cases}$$

$$|y'(t)| \leq \begin{cases} \text{const.}\,(t^{\sigma_+-1}(1+|\ln(t)|^{n_{\max}-1})(|B_1^{-1}\beta|+\|f\|)+\|f\|), & \sigma_+<1,\\ \text{const.}\,(1+|\ln(t)|^{n_{\max}})(|B_1^{-1}\beta|+\|f\|), & \sigma_+=1,\\ \text{const.}\,(|B_1^{-1}\beta|+\|f\|), & \sigma_+>1.\end{cases}$$

*Moreover, if* $p < \sigma_+ \leq p+1$*, then* $y \in C^p[0,1] \cap C^{p+1}(0,1]$ *and if* $\sigma_+ > p+1$*, then* $y \in C^{p+1}[0,1]$.

## 2.2 Linear problems with variable coefficient matrix $M(t)$

Here we study initial value problems of the form

(2.22a) $$y'(t) = \frac{M(t)}{t}y(t) + f(t), \quad t \in (0,1],$$

(2.22b) $$B_0 y(0) = \beta,$$

(2.22c) $$My(0) = 0,$$

where $M := M(0)$, and

(2.23a) $$y'(t) = \frac{M(t)}{t}y(t) + f(t), \quad t \in (0,1],$$

(2.23b) $$B_1 y(1) = \beta.$$

We discuss two cases, $M \in C[0,1] \cap C^1(0,1]$ and $M \in C^1[0,1]$, where $M$ is chosen to have the form

(2.24) $$M(t) = M + t^\gamma \mathring{C}(t), \quad \gamma > 0, \quad \mathring{C} \in C[0,1].$$

It follows that in the latter case we can choose $\gamma = 1$.

### 2.2.1 Coefficient matrix $M(t) \in C^1(0,1]$

Consider (2.22) with $M(t)$ given by (2.24) and assume that $M(0)$ satisfies A.3.1.

**THEOREM 2.9** *If $B_0\tilde{R}$ is nonsingular, then for every $f \in C[0,1]$ and $\mathring{C} \in C[0,1]$, there exists a unique, continuous solution of (2.22). This solution satisfies $y \in C^1(0,1]$.*

**Proof:** Let $t \in [0,\delta]$. Then, according to Theorem 2.7, any continuous solution of the initial value problem (2.22) satisfies

(2.25) $$y(t) = (\mathcal{K}\mathcal{C}y)(t) + \psi(t),$$

where

$$(\mathcal{K}\mathcal{C}y)(t) = t^\gamma \int_0^1 s^{-M} s^{\gamma-1} \mathring{C}(st) y(st)\, ds$$

and

$$\psi(t) = \tilde{R}(B_0\tilde{R})^{-1}\beta + t\int_0^1 s^{-M} f(st)\, ds.$$

Hence $y$ is a fixed point of the operator $(\mathcal{KC}_\psi y)(t) := (\mathcal{KC}y)(t) + \psi(t)$, where $\mathcal{KC} : C[0,\delta] \to C[0,\delta]$ is a bounded linear operator with

$$\|\mathcal{KC}y\|_\delta \le D\,\delta^\gamma \|y\|_\delta, \quad D = \text{const.},$$

see Lemma 2.2. Therefore, the operator $\mathcal{KC}_\psi$ is contracting for $\delta$ sufficiently small, $\delta < (\frac{1}{D})^{\frac{1}{\gamma}}$. Now the Banach Fixed Point Theorem implies that there exists a unique solution $y \in C[0,\delta]$ of (2.25). This solution satisfies the initial condition (2.22b) and can be continued uniquely to $t = 1$. Finally, we substitute $y$ into (2.22a) and see that the structure of its first derivative is

$$y'(t) = t^{\gamma-1}\xi(t) + f(t), \quad \xi \in C[0,1].$$

Hence, $y \in C^1(0,1]$, and this completes the proof.

We now estimate $y(t)$ for $t \in [0,\delta]$, where $\delta$ is chosen such that $\|\mathcal{KC}\|_\delta = L < 1$. Using Lemma 2.2 we derive a bound for $\psi$,

$$\|\psi\|_\delta \le |\tilde{R}(B_0\tilde{R})^{-1}\beta| + \text{const.}\,\|f\|_\delta,$$

and by the Banach Lemma we conclude

$$\|y\|_\delta = \|(\mathcal{I} - \mathcal{KC})^{-1}\psi\|_\delta \le \frac{1}{1-L}\|\psi\|_\delta \le \text{const.}\,(|\tilde{R}(B_0\tilde{R})^{-1}\beta| + \|f\|_\delta).$$

Moreover, we can describe the local behavior of $y$ by deriving an estimate for $y(t)$ for $t$ close to zero. This is done by applying Lemma 2.2 and using the above bounds for $\|\psi\|_\delta$ and $\|y\|_\delta$ in (2.25),

$$|y(t)| \le \text{const.}\,t^\gamma(\|f\|_\delta + |\tilde{R}(B_0\tilde{R})^{-1}\beta|) + |\tilde{R}(B_0\tilde{R})^{-1}\beta|. \tag{2.26}$$

We now turn to (2.23) and assume that $M(0)$ satisfies A.3.2. For a nonsingular matrix $B_1$, the classical theory yields the existence of a unique solution $z(t)$ of (2.23), $z \in C[\delta,1]$, $0 < \delta \le 1$. Define $z(\delta) =: \omega$. In the following theorem we answer the question whether such a solution can be extended to $t = 0$.

**THEOREM 2.10** *If $B_1$ is nonsingular, then for any $f \in C[0,1]$ and $\overset{\circ}{C} \in C[0,1]$, there exists a unique, continuous solution $y$ of (2.23). This solution satisfies $y \in C^1(0,1]$.*

**Proof:** Consider (2.23a) for $t \in (0,\delta]$. Then, according to Theorem 2.8, any solution of (2.23a) subject to $y(\delta) = \omega$ satisfies

$$y(t) = (\mathcal{KC}y)(t) + \psi(t), \tag{2.27}$$

where

$$(\mathcal{KC}y)(t) = \int_\delta^t \left(\frac{t}{\tau}\right)^M \tau^{\gamma-1}\overset{\circ}{C}(\tau)y(\tau)\,d\tau$$

and

$$\psi(t) = \left(\frac{t}{\delta}\right)^M \omega + \int_\delta^t \left(\frac{t}{\tau}\right)^M f(\tau)\,d\tau.$$

Again, $y$ is a fixed point of the operator $(\mathcal{KC}_\psi y)(t) := (\mathcal{KC}y)(t) + \psi(t)$, where the integral operator $\mathcal{KC} : C[0,\delta] \to C[0,\delta]$ is linear and bounded. We now use Lemma 2.5 to estimate $\mathcal{KC}y$. For $\sigma = \gamma$ we obtain

$$\|\mathcal{KC}y\|_\delta \le D\,\delta^{\gamma/2}\|y\|_\delta, \quad D = \text{const.}$$

on noting that $|t^{\gamma/2}\left(1+\left|\ln\left(\frac{t}{\delta}\right)\right|^n\right)|$ is bounded uniformly in $t \in [0,\delta]$. For the cases $\sigma < \gamma$, $\sigma > \gamma$, and for the contribution associated with $\lambda = 0$ we have

$$\|\mathcal{K}\mathcal{C}y\|_\delta \le D\,\delta^\gamma \|y\|_\delta, \quad D = \text{const.}$$

and therefore $\mathcal{K}\mathcal{C}_\psi$ is contracting for a sufficiently small $\delta$.

Estimates for the solution and its first derivative, analogous to those in Theorem 2.8, hold. We omit them here to avoid unnecessary repetitions.
Due to the unsatisfactory smoothness properties of the solution $y$, the case $M \in C^1(0,1]$ or $\gamma < 1$ is not going to be considered any further. However, the above considerations can now be utilized to cover the case $M \in C^1[0,1]$ with no additional effort.

### 2.2.2 Coefficient matrix $M(t) \in C^1[0,1]$

In this section we study initial value problems (2.22) and (2.23), where $M(t)$ is given by (2.24) with $\gamma = 1$. The existence of continuous and unique solutions is obvious from the considerations in the previous section. Therefore, in the following two theorems we discuss in more detail only the smoothness properties of $y$.

**THEOREM 2.11** *If $B_0\tilde{R}$ is nonsingular, then for every $f, \overset{\circ}{C} \in C^p[0,1]$, $p \ge 0$, there exists a unique solution $y$ of* (2.22). *This solution satisfies $y \in C^{p+1}[0,1]$.*

**Proof:** In (2.25) we set $\gamma = 1$ and conclude that any continuous solution of (2.22) has the form

$$y(t) = t\eta(t) + \tilde{R}(B_0\tilde{R})^{-1}\beta, \quad \eta(t) \in C[0,1].$$

We now substitute $y$ into (2.22a) and conclude $y \in C^1[0,1]$. The result follows by successively applying the above argument in Theorem 2.7.

**THEOREM 2.12** *If $B_1$ is nonsingular, then for any $f, \overset{\circ}{C} \in C^p[0,1]$, $p \ge 0$, there exists a unique solution $y$ of* (2.23). *Moreover, if $p < \sigma_+ \le p+1$, then $y \in C^p[0,1] \cap C^{p+1}(0,1]$ and if $\sigma_+ > p+1$, then $y \in C^{p+1}[0,1]$.*

**Proof:** The smoothness results can be shown using techniques developed in Lemma 2.6.

## 3 Analytic results for nonlinear systems of first order

In this section we discuss nonlinear problems of the form

(3.1a) $$y'(t) = \frac{M(t)}{t}y(t) + f(t,y(t)), \quad t \in (0,1],$$

(3.1b) $$B_0 y(0) = \beta,$$

(3.1c) $$My(0) = 0,$$

and

(3.2a) $$y'(t) = \frac{M(t)}{t}y(t) + f(t,y(t)), \quad t \in (0,1],$$

(3.2b) $$B_1 y(1) = \beta,$$

where $f(t,y)$ is assumed to be continuous and Lipschitz-continuous with respect to $y$ on a suitably defined domain.
We restrict our attention to the case, where $M \in C^1[0,1]$ is of the form (2.24) with $\gamma = 1$. Here, we assume that all quantities (except for the eigenvalues of $M$) are real.

**THEOREM 3.1** *Let $f \in C^p([0,1] \times \mathbb{R}^n)$, $\mathring{C} \in C^p[0,1]$, $p \geq 0$, and let the matrix $B_0\tilde{R}$ be nonsingular. Assume $f(t,y)$ to be Lipschitz-continuous with respect to $y$ on $[0,1] \times \mathbb{R}^n$. Then there exists a unique solution $y$ of (3.1) and $y \in C^{p+1}[0,1]$.*

**Proof:** We first prove the result on the subinterval $[0,\delta]$. Then standard arguments yield the extension to the whole interval, see Section 2.2.
Clearly, solving (3.1) on $[0,\delta]$ is equivalent to finding a fixed point of the nonlinear operator $(\mathcal{KF}_\gamma y)(t) := (\mathcal{KF}y)(t) + \gamma$, or equivalently, solving the nonlinear integral equation

$$y(t) = (\mathcal{KF}y)(t) + \gamma, \quad t \in [0,\delta], \tag{3.3}$$

where

$$(\mathcal{KF}y)(t) = t\int_0^1 s^{-M}\left(\mathring{C}(st)y(st) + f(st, y(st))\right) ds$$

and

$$\gamma = \tilde{R}(B_0\tilde{R})^{-1}\beta.$$

From Lemma 2.2 we conclude that for a sufficiently small $\delta$ the operator $\mathcal{KF}_\gamma$ is contracting (with constant $L < 1$) on $C[0,\delta]$. Consequently, there exists a unique solution $y$ of (3.3) and $y \in C[0,\delta]$. The smoothness of higher derivatives of $y$, $y \in C^{p+1}[0,1]$, can be shown in a manner indicated in Theorem 2.11.

Finally, we estimate $y$ and $y'$. From

$$\|\mathcal{KF}_\gamma y\|_\delta - \|\mathcal{KF}_\gamma 0\|_\delta \leq \|\mathcal{KF}_\gamma y - \mathcal{KF}_\gamma 0\|_\delta \leq L\|y\|_\delta$$

we obtain

$$\|y\|_\delta \leq \frac{1}{1-L}\left(|\gamma| + D\max_{\tau\in[0,\delta]}|f(\tau,0)|\right) =: r.$$

We can now estimate $f$ on the bounded domain

$$U := [0,\delta] \times \{y \in \mathbb{R}^n : |y| \leq r\}$$

and define

$$F_\delta := \max_{(t,y)\in U}|f(t,y)|.$$

Using this bound and Lemma 2.2 we obtain for $t \in [0,\delta]$

$$|y(t)| \leq \text{const.}\, t(F_\delta + |\tilde{R}(B_0\tilde{R})^{-1}\beta|) + |\tilde{R}(B_0\tilde{R})^{-1}\beta|, \tag{3.4}$$

$$|y'(t)| \leq \text{const.}\,(F_\delta + |\tilde{R}(B_0\tilde{R})^{-1}\beta|). \tag{3.5}$$

For weaker smoothness assumptions on $f$, $f \in C^p([0,1] \times \{y \in \mathbb{R}^n : |y| \leq r^*\})$ with $r^* \geq r$, the arguments of Theorem 3.1 can be used to show the existence, uniqueness and smoothness of $y$ on $[0,\delta]$. In this case, though, the classical theory does not answer the question if this solution can be extended to the whole interval $[0,1]$.

**THEOREM 3.2** *Let $f \in C^p([0,1] \times \mathbb{R}^n)$, $\overset{\circ}{C} \in C^p[0,1]$, $p \geq 0$, and let the matrix $B_1$ be nonsingular. Moreover, assume $f(t,y)$ to be Lipschitz-continuous with respect to $y$ on $[0,1] \times \mathbb{R}^n$. Then there exists a unique solution $y$ of (3.2). This solution satisfies $y \in C[0,1] \cap C^{p+1}(0,1]$. If $p < \sigma_+ \leq p+1$, then $y \in C^p[0,1] \cap C^{p+1}(0,1]$, and if $\sigma_+ > p+1$, then $y \in C^{p+1}[0,1]$.*

**Proof:** The existence result follows by representing the solution in a way indicated in Theorem 2.10[9] and applying techniques from Theorem 3.1 with Lemma 2.5 instead of Lemma 2.2. The smoothness results can be shown using techniques from Theorem 2.12.

We note that for the solution of (3.1), $y \in C^1[0,1]$, the value of $y'(0)$ can be calculated by using the local Taylor expansion at $t = 0$,

$$y'(0) = (I - M(0))^{-1}(M'(0)y(0) + f(0, y(0))). \tag{3.6}$$

Note that $M(0)$ has only nonpositive eigenvalues and hence, $I - M(0)$ is nonsingular. The above representation does not hold for the solution of (3.2) in general.

# 4 Analytic results for systems of second order

In this section we briefly discuss second order problems of the form (1.2). No conditions for $y'(0)$ are incorporated into the initial condition (1.2b), since we will show that $y'(0)$ is uniquely determined by the smoothness requirement (1.2c). In order to keep the presentation short, we have decided not to discuss the case where $f$ depends on $y'$ here. This analysis does not result in gaining more insight, but requires a lot of new notation related to a redefinition of the involved Banach spaces, cf. [18]. This restriction is also well justified by applications. For the same reasons terminal value problems at $t = 1$ are not considered in this section, all results can be found in [11].
We use the linear transformation $z(t) = (z_1(t), z_2(t))^T := (y(t), ty'(t))^T$ to rewrite the second order system to the first order form and obtain

$$z'(t) = \frac{M(t)}{t} z(t) + t\overset{\circ}{f}(t, z(t)), \quad t \in (0,1], \tag{4.1}$$

where

$$M(t) = \begin{pmatrix} 0 & I \\ A_0(t) & I + A_1(t) \end{pmatrix}, \quad \overset{\circ}{f}(t, z(t)) = \begin{pmatrix} 0 \\ f\,(t, z_1(t)) \end{pmatrix}. \tag{4.2}$$

Clearly, techniques and results from Sections 2 and 3 can now be utilized in the investigation of (1.2). Notation is used accordingly. Especially, we again associate the assumption A.3.1 with the matrix $M := M(0)$ from (4.2). In this section $I_1$ and $I_2$ denote the $n$ upper and $n$ lower rows of the $2n$-dimensional identity matrix, respectively.

### Linear problems with constant coefficient matrices $A_0$ and $A_1$

As a first step in the analysis of second order systems we study the linear problem

$$y''(t) = \frac{A_1}{t} y'(t) + \frac{A_0}{t^2} y(t) + f(t), \quad t \in (0,1], \tag{4.3a}$$

$$B_0 y(0) = \beta, \tag{4.3b}$$

$$A_0 y(0) = 0, \;\; y'(0) = 0. \tag{4.3c}$$

[9]Here, by classical theory, a solution on $[\delta, 1]$ exists.

**THEOREM 4.1** *Let $m = r$ and assume that the $r \times r$ matrix $B_0 I_1 \tilde{R}$ is nonsingular. Then for every $f \in C^p[0,1]$, $p \geq 0$, there exists a unique continuous*[10] *solution of* (4.3). *This solution has the form*

$$y(t) = I_1 \tilde{R}(B_0 I_1 \tilde{R})^{-1}\beta + t^2 I_1 \int_0^1 s s^{-M} \mathring{f}(st)\, ds, \tag{4.4}$$

$$y'(t) = t I_2 \int_0^1 s s^{-M} \mathring{f}(st)\, ds. \tag{4.5}$$

*Moreover, $y \in C^{p+2}[0,1]$, and the following estimates hold:*

$$|y(t)| \leq \text{const.}\, t^2 \|f\| + |\tilde{R}(B_0 I_1 \tilde{R})^{-1}\beta|, \tag{4.6}$$
$$|y'(t)| \leq \text{const.}\, t\|f\|, \tag{4.7}$$
$$|y''(t)| \leq \text{const.}\, \|f\|. \tag{4.8}$$

**Proof:** The result follows in a straightforward manner from the discussion of the related first order problem analogous to (4.1). The representation of the solution can be obtained from Theorem 2.7, and the condition $A_0 y(0) = 0$ from $Mz(0) = 0$. $y'(0) = 0$ is a consequence of (4.5). The smoothness result is derived by substituting (4.4) and (4.5) into (4.3a) and using Theorem 2.7.

## Linear problems with variable coefficient matrices $A_0(t)$ and $A_1(t)$

We consider the problem

$$y''(t) = \frac{A_1(t)}{t} y'(t) + \frac{A_0(t)}{t^2} y(t) + f(t), \quad t \in (0,1], \tag{4.9a}$$
$$B_0 y(0) = \beta, \tag{4.9b}$$
$$A_0 y(0) = 0, \;\; y'(0) = 0, \tag{4.9c}$$

where $A_0 := A_0(0)$. Motivated by the smoothness statements from Theorems 2.9 and 2.11, we assume $A_0, A_1 \in C^1[0,1]$ and write both matrices in the form

$$A_i(t) = A_i(0) + tC_i(t), \quad C_i \in C[0,1], \quad i = 1,2. \tag{4.10}$$

However, in the next theorem we need a finer structure of $A_0$ than of $A_1$ and therefore we additionally assume

$$A_0(t) = A_0(0) + tC_0(t) = A_0(0) + tA_0'(0) + t^2 D_0(t), \quad D_0 \in C[0,1]. \tag{4.11}$$

**THEOREM 4.2** *Let the matrix $B_0 I_1 \tilde{R}$ be nonsingular. Then for every $f$, $C_1 \in C^p[0,1]$ and $C_0 \in C^{p+1}[0,1]$, $p \geq 0$, there exists a unique solution $y$ of* (4.9) *iff $A_0'(0)y(0) = 0$. This solution satisfies $y \in C^{p+2}[0,1]$. Moreover, there exists a $\delta > 0$, such that for every $t \in [0,\delta]$ the following estimates hold:*

$$|y(t)| \leq \text{const.}\, t^2(|\tilde{R}(B_0 I_1 \tilde{R})^{-1}\beta| + \|f\|_\delta) + |\tilde{R}(B_0 I_1 \tilde{R})^{-1}\beta|, \tag{4.12}$$
$$|y'(t)| \leq \text{const.}\, t(|\tilde{R}(B_0 I_1 \tilde{R})^{-1}\beta| + \|f\|_\delta), \tag{4.13}$$
$$|y''(t)| \leq \text{const.}\, (|\tilde{R}(B_0 I_1 \tilde{R})^{-1}\beta| + \|f\|_\delta). \tag{4.14}$$

[10] Conditions (4.3c) are necessary and sufficient for the solution of (4.3a) to be continuous.

**Proof:** Again, consider the equivalent first order problem corresponding to (4.9). Define $\gamma := (\gamma_1, 0)^T \in \mathcal{R}(R)$, $\gamma_1 := I_1\tilde{R}(B_0 I_1 \tilde{R})^{-1}\beta$[11]. It follows from Theorem 2.11 that for sufficiently small $\delta$ there exists a unique solution $z \in C^{p+1}[0,\delta]$ of (4.1). This solution has the form $z(t) = \gamma + t\zeta(t)$ and satisfies $Mz(0) = 0$. Substituting $z$ into (4.1) yields

$$z'(t) = \frac{1}{t}Mz(t) + \overset{\circ}{C}(t)\gamma + t(\overset{\circ}{C}(t)\zeta(t) + \overset{\circ}{f}(t)), \tag{4.15}$$

where $\overset{\circ}{C}(t)$ is defined in an obvious way. We conclude $y = z_1 \in C^{p+2}[0,\delta]$ on noting that $\overset{\circ}{C}(t)\gamma = (0, C_0(t)\gamma_1)^T \in C^{p+1}[0,\delta]$ and $\overset{\circ}{C}(t)\zeta(t) + \overset{\circ}{f}(t) \in C^p[0,\delta]$, see Theorem 2.11 and Theorem 4.1.
It is clear that in general $y'(0) \neq 0$, since from

$$z'(0) = (I - M)^{-1}\overset{\circ}{C}(0)\gamma$$

one only has[12]

$$y'(0) = -(A_0(0) + A_1(0))^{-1}C_0(0)y(0).$$

In order to show that $A_0'(0)y(0) = 0$ is sufficient for $y$ to satisfy $y'(0) = 0$ we rewrite $\overset{\circ}{C}(t)\gamma$,

$$\overset{\circ}{C}(t)\gamma = \begin{pmatrix} 0 & 0 \\ A_0'(0) & 0 \end{pmatrix}\gamma + t\begin{pmatrix} 0 & 0 \\ D_0(t) & \frac{C_1(t)}{t} \end{pmatrix}\gamma = t\overset{\circ}{D}(t)\gamma, \quad \overset{\circ}{D}(t) = \begin{pmatrix} 0 & 0 \\ D_0(t) & 0 \end{pmatrix},$$

and substitute the latter expression into (4.15),

$$z'(t) = \frac{1}{t}Mz(t) + t(\overset{\circ}{D}(t)\gamma + \overset{\circ}{C}(t)\zeta(t) + \overset{\circ}{f}(t)).$$

Consequently,

$$z(t) = \gamma + t^2\vartheta(t), \quad y(t) = \gamma_1 + t^2 I_1\vartheta(t), \quad y'(t) = tI_2\vartheta(t),$$

where

$$\vartheta(t) := \int_0^1 s s^{-M}\left(\overset{\circ}{D}(st)\gamma + \overset{\circ}{C}(st)\zeta(st) + \overset{\circ}{f}(st)\right) ds, \quad \vartheta \in C[0,\delta].$$

Clearly, the above $y$ can be uniquely continued to $t = 1$ and the result follows.

We stress that any continuous solution $z$ of (4.1) satisfies $M(0)z(0) = 0$ and the corresponding conditions expressed in terms of $y$ solving (4.9a) and (4.9b) read either

$$A_0(0)y(0) = 0, \quad (A_0(0) + A_1(0))y'(0) + C_0(0)y(0) = 0,$$

or

$$A_0(0)y(0) = 0, \quad y'(0) = 0,$$

if $y(0) \in \ker A_0'(0)$. To avoid technicalities we use the second set of conditions derived above while analyzing the nonlinear problem. This set of initial conditions is particularly relevant for applications, where typically $A_0$ is a constant matrix.

[11] Note that $A_0(0)\gamma_1 = 0$.
[12] $(A_0(0) + A_1(0))$ is nonsingular because $(I - M)$ is nonsingular.

**Nonlinear problems**

Consider the problem

(4.16a) $$y''(t) = \frac{A_1(t)}{t}y'(t) + \frac{A_0(t)}{t^2}y(t) + f(t, y(t)), \quad t \in (0, 1],$$

(4.16b) $$B_0 y(0) = \beta,$$

(4.16c) $$A_0 y(0) = 0, \quad y'(0) = 0.$$

The following result is a consequence of Theorem 4.2 applied to (4.16), where the Lipschitz-condition for $\overset{\circ}{f}$ is used.

**THEOREM 4.3** *Let $f \in C^p([0,1] \times \mathbb{R}^n), C_1 \in C^p[0,1], C_0 \in C^{p+1}[0,1]$*[13]*, $p \geq 0$, and let the matrix $B_0 I_1 \tilde{R}$ be nonsingular. Assume $f(t,y)$ to be Lipschitz-continuous with respect to $y$ on $[0,1] \times \mathbb{R}^n$. Then there exists a unique solution $y$ of (4.16) iff $A_0'(0)y(0) = 0$. This solution satisfies $y \in C^{p+2}[0,1]$.*

Finally, using Taylor's Theorem we can derive a representation for $y''(0)$, where $y \in C^2[0,1]$ is a solution of (4.16),

(4.17) $$\left(I - A_1(0) - \frac{A_0(0)}{2}\right) y''(0) = \frac{A_0''(0)}{2} y(0) + f(0, y(0)).$$

This is a system of $n$ linear equations for $y''(0)$ which is uniquely solvable iff the leading coefficient matrix is nonsingular.

When applying a numerical method to solve (3.1) or (4.16) one often needs to provide a discretization at $t = 0$ for the system (3.1a) or (4.16a), respectively. In most of the methods this involves an evaluation of the corresponding right-hand side at $t = 0$ which is not possible when the singularity is present. In such a case the local behavior of $y'(0)$ or $y''(0)$ described in (3.6) and (4.17), respectively, can be used to remedy the difficulty (see [9], [17]).

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