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Initial Value Problems for the Heat Convection Equations in Exterior Domains

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1. Introduction.

We consider the initial value problem (IVP) of the heat convection equation (HCE) of Boussinesq type in an exterior domain $\Omega=K^c\subset\mathbf{R}^m$ ($m=2$ or 3), where K is a compact set with a smooth boundary $\Gamma=\partial K\in C^2$. We denote $\hat{\Omega}=\Omega\times(0,T)$. Then the problem (IVP) for (HCE) is as follows:

$$\begin{cases} u_t+(u\cdot\nabla)u=-(\nabla p)/\rho+\{1-\alpha(\theta-\Theta_0)\}g+\nu\Delta u & \text{in } \hat{\Omega}\,, \\ \qquad \operatorname{div}u=0 & \text{in } \hat{\Omega}\,, \\ \theta_t+(u\cdot\nabla)\theta=\kappa\Delta\theta & \text{in } \hat{\Omega}\,, \end{cases} \tag{1}$$

$$u|_\Gamma=0\,,\quad \theta|_\Gamma=\Theta_0>0\,,\quad \lim_{|x|\to\infty}u(x,t)=0\,,\quad \lim_{|x|\to\infty}\theta(x,t)=0 \text{ for } t\in(0,T)\,, \tag{2}$$

$$u|_{t=0}=a\,,\qquad \theta|_{t=0}=h\,. \tag{3}$$

Here, $u=u(x,t)$ is the velocity vector, $p=p(x,t)$ is the pressure and $\theta=\theta(x,t)$ is the temperature; ν, κ, α, ρ and $g=g(x)$ are the kinematic viscosity, the thermal conductivity, the coefficient of volume expansion, the density at $\theta=\Theta_0$ and the gravitational vector, respectively.

Hishida [2] and Hishida-Yamada [3] studied the exterior problem for (HCE) and proved the global existence of the strong solution of (IVP) when K is a ball, while the second author of our present paper has recently shown in [6] (which is her Master thesis) the existence of a weak solution of (IVP) for (HCE) in the case that K is a compact set with a smooth boundary of class C^2. In Ōeda-Matsuda [10], we announced the existence result ($m=2$ or 3) together with the uniqueness of a weak solution for the 2-dimensional problem. In the present paper, we will give details of proofs of the results announced in [10], and furthermore, show the uniqueness theorem of a weak solution for the 3-dimensional problem. As the equations (1) tell us, (HCE) is the system which

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consists of the Navier-Stokes equation with the buoyancy term and the heat equation with the convection term. In the Navier-Stokes equation, a sufficient condition for the existence of a strong solution of the initial-boundary value problem is given as a smallness condition on data. On the other hand, it is known that a weak solution exists for a large data. Concerning the Navier-Stokes equation, Serrin [11], for example, studied and reviewed the existence, uniqueness and the regularity of the solution when the domain Ω is a bounded or unbounded one in $\mathbf{R}^m$ $(m \geq 2)$. Our existence and uniqueness results on the weak solution obtained in this paper correspond to those for the Navier-Stokes equation.

Now, to prove the existence of the solution, we employ "the extending domain method". In other words, we can expect that problems on domains $\Omega_n \equiv \Omega \cap B_n$ (B_n being balls with radii n and center O) will approximate the problem on the domain Ω as $n \to \infty$. As for "the extending domain method", Ladyzhenskaya [4] referred briefly in relation to the Navier-Stokes equation, but skipped details. We are inspired by [4] and we will give the proof by this method in detail for (HCE). Moreover, we will use some inequalities with absolute constants (independent of domains) to estimate functions.

2. Preliminaries.

We make assumptions (A1)~(A3):

(A1) $\partial\Omega = \Gamma = \partial K \in C^2$.

(A2) $g(x)$ is a bounded and continuous vector function in $\mathbf{R}^m \backslash \operatorname{int} K$ and $g \in L^2(\Omega)$.

(A3) $K \subset B = B(O, d)$ (a ball with radius d and center O).

We use an auxiliary function $\bar{\theta}$ which is introduced as a solution of a problem:

$$\Delta\theta = 0 \ (\text{in } \Omega) \qquad \text{with } \theta(\Gamma) = \Theta_0 \text{ and } \lim_{|x| \to \infty} \theta(x) = 0 .$$

Then, we see $\theta = \bar{\theta} \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and $\|\nabla\bar{\theta}\|_\infty = \operatorname{ess.sup}_{x \in \Omega} |\nabla\bar{\theta}(x)| < \infty$.

Now, we make a change of variables:

$$u = \hat{u}, \quad \theta = \hat{\theta} + \bar{\theta}, \quad (x, y, z) = d(x^*, y^*, z^*) \ \ ((x, y) = d(x^*, y^*) \text{ in } \mathbf{R}^2),$$

$$t = d^2 t^*/\nu, \quad \hat{u} = \nu u^*/d, \quad \hat{\theta} = \nu\Theta_0\theta^*/\kappa, \quad p = \rho\nu^2 p^*/d^2 .$$

By these relations, we have new variables u^*, θ^* and so on, but we abbreviate asterisks for simplicity and use the same letters.

Then, equations (1), (2) and (3) are transformed to the following homogeneous boundary value problem:

$$\begin{cases} u_t + (u \cdot \nabla)u = -\nabla p + \Delta u - R\theta + d^3 g/\nu^2 - R(\bar{\theta} - P^{-1}) & \text{in } \hat{\Omega}, \\ \operatorname{div} u = 0 & \text{in } \hat{\Omega}, \\ \theta_t + (u \cdot \nabla)\theta = P^{-1}\Delta\theta - (u \cdot \nabla)\bar{\theta} & \text{in } \hat{\Omega}, \end{cases} \tag{4}$$

$$u|_{\Gamma}=0\,,\quad \theta|_{\Gamma}=0\,,\quad \lim_{|x|\to\infty} u(x,t)=0\,,\quad \lim_{|x|\to\infty}\theta(x,t)=0 \text{ for } t\in(0,T)\,, \tag{5}$$

$$u|_{t=0}=a\,,\qquad \theta|_{t=0}=h\,, \tag{6}$$

where $R=ag\Theta_0 d^3/\kappa\nu$, $P=\nu/\kappa$ and d is a radius of B.

Here, we define several function spaces:

$W^{k,p}(\Omega)=\{u;\, D^\alpha u\in L^p(\Omega),\, |\alpha|\le k\}$,

$W_0^{k,p}(\Omega)=$ the completion of $C_0^k(\Omega)$ in $W^{k,p}(\Omega)$,

$D_\sigma(\Omega)=\{\varphi\in C_0^\infty(\Omega);\, \operatorname{div}\varphi=0\}$,

$H_\sigma(\Omega)$ (resp. $H_\sigma^1(\Omega)$) $=$ the completion of $D_\sigma(\Omega)$ in $L^2(\Omega)$ (resp. $W^{1,2}(\Omega)$),

$\hat D_\sigma(\hat\Omega)=\{\varphi\in C_0^\infty(\hat\Omega');\, \operatorname{div}\varphi=0\}$ $\quad(\hat\Omega'=\Omega\times[0,T])$,

$\hat H_\sigma(\hat\Omega)$ (resp. $\hat H_\sigma^1(\hat\Omega)$) $=$ the completion of $\hat D_\sigma(\hat\Omega)$ in $L^2(\hat\Omega)$ (resp. $W^{1,2}(\hat\Omega)$),

where $\|u\|_{W^{1,2}(\hat\Omega)}=\{\int_0^T(\|u(t)\|^2_{L^2(\Omega)}+\|\nabla u(t)\|^2_{L^2(\Omega)})dt\}^{1/2}$;

furthermore

$D(\Omega)=\{\varphi\in C_0^\infty(\Omega\cup\Gamma);\, \varphi(\Gamma)=0\}$,

$H_0^1(\Omega)=$ the completion of $D(\Omega)$ in $W^{1,2}(\Omega)$,

$\hat D(\hat\Omega)=\{\varphi\in C_0^\infty(\widehat{\Omega\cup\Gamma}');\, \varphi(\Gamma)=0\}$ $\quad(\widehat{\Omega\cup\Gamma}'=(\Omega\cup\Gamma)\times[0,T])$,

$\hat H^1(\hat\Omega)=$ the completion of $\hat D(\hat\Omega)$ in $W^{1,2}(\hat\Omega)$,

and moreover

$\hat{\mathscr D}_\sigma(\hat\Omega)=\{\varphi\in\hat D_\sigma(\hat\Omega);\, \varphi=0 \text{ at } t=T\}$, $\quad\hat{\mathscr D}(\hat\Omega)=\{\psi\in\hat D(\hat\Omega);\, \psi=0 \text{ at } t=T\}$,

$\mathscr U(\hat\Omega)=\{\varphi\in\hat H_\sigma^1(\hat\Omega));\, \operatorname{ess.sup}_{0\le t\le T}\|\varphi(t)\|_{L^2(\Omega)}<\infty\}$,

$\mathscr T(\hat\Omega)=\{\psi\in\hat H^1(\hat\Omega));\, \operatorname{ess.sup}_{0\le t\le T}\|\psi(t)\|_{L^2(\Omega)}<\infty\}$.

REMARK 1. (i) If $\theta\in\hat H^1(\hat\Omega)$, then we have $\theta|_{\partial\Omega}=0$.

(ii) We note $H_\sigma(\Omega)\times L^2(\Omega)=H_\sigma(\Omega)\times\mathbf{O}+\mathbf{O}\times L^2(\Omega)$. We identify $u\in H_\sigma(\Omega)$ with ${}^t(u,0)\in H_\sigma(\Omega)\times L^2(\Omega)$, and moreover, $\theta\in L^2(\Omega)$ with ${}^t(0,\theta)\in H_\sigma(\Omega)\times L^2(\Omega)$, if necessary.

Later we use Friedrichs' lemma, so we state it here (see Remark 2 below):

LEMMA 2.1 (Friedrichs). *Suppose G is a bounded domain in $\mathbf{R}^m$ and its boundary ∂G is of class C^2. Let $\{w_k\}_{k=1}^\infty$ be an orthonormal basis of $L^2(G)$. Then for an arbitrary positive number ε, we can take a number N_ε such that the following inequality holds for all $u\in W^{1,p}(G)$:*

$$\|u\|_{L^2(G)}\le\left(\sum_{k=1}^{N_\varepsilon}(u,w_k)^2\right)^{1/2}+\varepsilon\|u\|_{W^{1,p}(G)}\,, \tag{7}$$

where $p>2m/(m+2)$ $(m\ge2)$, $p\ge1$ $(m=1)$ and N_ε is independent of u.

REMARK 2. (7) valids for all $u \in W^{1,p}(G)$ not only for $W_0^{1,p}(G)$. (See Lemma 2.4 of Chap. II in [5] (p. 72–73) which is proven by Theorem 2.1 and (2.5) (p. 61 of [5]), since $\partial G \in C^2$.)

LEMMA 2.2 (Chap. I of [4]). *Let $\Omega \subset \mathbf{R}^3$. Then*

$$\|u\|_{L^q(\Omega)} \leq c_3^\alpha \|\nabla u\|_{L^2(\Omega)}^\alpha \|u\|_{L^2(\Omega)}^{1-\alpha} \qquad \text{for} \quad u \in W_0^{1,2}(\Omega)\,, \tag{8}$$

where $2 \leq q \leq 6$, $\alpha = 3/2 - 3/q$ and $c_3 = (48)^{1/6}$.

3. Results.

For the simplicity of the representation, we prepare some notations. We put

$$I(s, G; u, \theta) \equiv \int_0^s \{(u, \varphi_\tau)_G + (\theta, \psi_\tau)_G + ((u \cdot \nabla)\varphi, u)_G + ((u \cdot \nabla)\psi, \theta)_G$$
$$+ (u, \Delta\varphi)_G + P^{-1}(\theta, \Delta\psi)_G - ((u \cdot \nabla)\bar{\theta}, \psi)_G - (R\theta, \varphi)_G + (f, \varphi)_G\} d\tau\,, \tag{9}$$

$$E(s, G; u, \theta) \equiv \|u(s)\|_G^2 + \|\theta(s)\|_G^2 + 2\int_0^s \|\nabla u(\tau)\|_G^2 d\tau + \frac{2}{P}\int_0^s \|\nabla\theta(\tau)\|_G^2 d\tau\,, \tag{10}$$

where $(\cdot, \cdot)_G = (\cdot, \cdot)_{L^2(G)}$, $\|\cdot\|_G = \|\cdot\|_{L^2(G)}$ and $f = d^3 g/v^2 - R(\bar{\theta} - P^{-1})$.

Before we state results, we give the definition of a weak solution.

DEFINITION 3.1. $U = {}^t(u(x, t), \theta(x, t))$ defined in $\hat{\Omega}$ is called a weak solution of (IVP) if (i) and (ii) hold:

(i) ${}^t(u, \theta) \in \mathscr{U}(\hat{\Omega}) \times \mathscr{T}(\hat{\Omega})$.

(ii) For all ${}^t(\varphi, \psi) \in \hat{\mathscr{D}}_\sigma(\hat{\Omega}) \times \hat{\mathscr{D}}(\hat{\Omega})$, the equality

$$I(T, \Omega; u, \theta) = -(a, \varphi(0))_\Omega - (h - \bar{\theta}, \psi(0))_\Omega \tag{11}$$

is satisfied.

Then, we have the existence theorem.

THEOREM 3.2. *Suppose the space dimension m is 2 or 3 and let assumptions* (A1)–(A3) *be satisfied. Then for any ${}^t(a, h) \in H_\sigma(\Omega) \times L^2(\Omega)$, there exists a weak solution of* (IVP) *and the following* (i), (ii) *and* (iii) *hold*:

(i) *${}^t(u(t), \theta(t)) = {}^t((u(\cdot, t), \theta(\cdot, t))$ is defined for all $t \in [0, T]$ and $(u(t), \varphi(t))_\Omega + (\theta(t), \psi(t))_\Omega$ is continuous on $[0, T]$ for every ${}^t(\varphi, \psi) \in \hat{D}_\sigma(\hat{\Omega}) \times \hat{D}(\Omega)$.*

(ii) *The following energy inequality holds for $t \in [0, T]$:*

$$E(t, \Omega; u, \theta) \leq \exp(cT)(\|a\|_\Omega^2 + \|h - \bar{\theta}\|_\Omega^2 + F(T))\,, \tag{12}$$

where $c = \|\nabla\bar{\theta}\|_\infty + \|R\|_\infty + 1$, $\|w\|_\infty = \text{ess.sup}_{x \in \Omega}|w(x)|$, $F(T) = \int_0^T \|f(\tau)\|_\Omega^2 d\tau$.

(iii) *For every ${}^t(\varphi, \psi) \in \hat{D}_\sigma(\hat{\Omega}) \times \hat{D}(\hat{\Omega})$, the next equality holds for all $t \in [0, T]$:*

$$(u(t), \varphi(t))_\Omega + (\theta(t), \psi(t))_\Omega = (a, \varphi(0))_\Omega + (h - \bar{\theta}, \psi(0))_\Omega + I(t, \Omega; u, \theta)\,. \tag{13}$$

Moreover, we have uniqueness theorems:

THEOREM 3.3. *Suppose the space dimension m is* 2. *If* $g \in L^2(\Omega) \cap L^{4/3}(\Omega)$, *then the weak solution of* (IVP) *for* (HCE) *is unique.*

THEOREM 3.4. *Let the space dimension m be* 3 *and assume* $g \in L^2(\Omega) \cap L^{4/3}(\Omega)$. *Then the weak solution* ${}^t(u, \theta)$ *of* (IVP) *is unique if*

$$u \in L^s(0, T; L^r(\Omega)) \quad and \quad \theta \in L^s(0, T; L^r(\Omega)) \tag{14}$$

hold for some $r > 3$, $s = 2r/(r-3)$.

4. Proof of theorems.

In order to prove the existence theorem, we employ "the extending domain method". We set $\Omega_n = B_n \cap \Omega$ ($B_n = B(O, n)$) and $\partial\Omega_n = \Gamma + \partial B_n$. We note that since K is included in a ball $B = B(O, d)$ by the assumption (A3), so $K \subset B_1$ after changing of variables. Then, we propose the following approximate problem (P_n) in Ω_n:

$$\begin{cases} u_t + (u \cdot \nabla)u = -\nabla p + \Delta u - R\theta + d^3 g/\nu^2 - R(\bar\theta - P^{-1}) & \text{in } \hat\Omega_n, \\ \operatorname{div} u = 0 & \text{in } \hat\Omega_n, \\ \theta_t + (u \cdot \nabla)\theta = P^{-1}\Delta\theta - (u \cdot \nabla)\bar\theta & \text{in } \hat\Omega_n, \end{cases} \tag{15}$$

$$u|_{\partial\Omega_n} = 0, \quad \theta|_{\partial\Omega_n} = 0, \qquad \text{for} \quad t \in (0, T), \tag{16}$$

$$u|_{t=0} = a_n, \qquad \theta|_{t=0} = h_n - \bar\theta_n, \tag{17}$$

where $a_n = \chi_{\Omega_n} a$, $h_n - \bar\theta_n = \chi_{\Omega_n}(h - \bar\theta)$ and χ_{Ω_n} is the characteristic function on the set Ω_n. We notice $a_n \in H_\sigma(\Omega_n)$ and $h_n - \bar\theta_n \in L^2(\Omega_n)$. Since $R = \alpha g \Theta_0 d^3/\kappa\nu$, $P = \nu/\kappa$ and d is an original radius of the ball B, therefore they are independent of n.

DEFINITION 4.1. $U = {}^t(u(x, t), \theta(x, t))$ defined in $\hat\Omega_n$, is called a weak solution of (P_n) if (i) and (ii) hold:

(i) ${}^t(u, \theta) \in \mathscr{U}(\hat\Omega_n) \times \mathscr{T}(\hat\Omega_n)$.

(ii) For all ${}^t(\varphi, \psi) \in \hat{\mathscr{D}}_\sigma(\hat\Omega_n) \times \hat{\mathscr{D}}(\hat\Omega_n)$, the equality

$$I(T, \Omega_n; u, \theta) = -(a_n, \varphi(0))_{\Omega_n} - (h_n - \bar\theta_n, \psi(0))_{\Omega_n} \tag{18}$$

is satisfied.

We begin with the existence theorem for a weak solution in the interior domain Ω_n.

LEMMA 4.2. *There exists a weak solution* ${}^t(v_n, \Theta_n)$ *of the problem* (P_n) *and the following facts hold*:

(i) ${}^t(v_n(t), \Theta_n(t))$ *is weakly continuous on* $[0, T]$ *in* $H_\sigma(\Omega_n) \times L^2(\Omega_n)$.

(ii) *The energy inequality holds*:

$$E(t, \Omega_n; v_n, \Theta_n) \le C_T \equiv \exp(cT)(\|a\|^2_{L^2(\Omega)} + \|h - \bar\theta\|^2_{L^2(\Omega)} + F(T)), \tag{19}$$

where $c=\|\nabla\bar{\theta}\|_\infty+\|R\|_\infty+1$, *hence* C_T *does not depend on n and t.*

(iii) *For every* ${}^t(\varphi,\psi)\in\hat{D}_\sigma(\hat{\Omega}_n)\times\hat{D}(\hat{\Omega}_n)$ *and* $t\in[0,T]$ *the next equality holds*:

$$(v_n(t),\varphi(t))_{\Omega_n}+(\Theta_n(t),\psi(t))_{\Omega_n}=(a_n,\varphi(0))_{\Omega_n}+(h_n-\bar{\theta}_n,\psi(0))_{\Omega_n}+I(t,\Omega_n;v_n,\Theta_n)\,. \quad (20)$$

PROOF OF LEMMA 4.2 (see [6], [9]). Let n be fixed. We employ Galerkin's method. Let $\{\varphi_j\}$ (resp. $\{\psi_j\}$) be a basis of $H^1_\sigma(\Omega_n)$ (resp. $H^1_0(\Omega_n)$) and an orthonormal sequence of $L^2(\Omega_n)$ (resp. $L^2(\Omega_n)$). First we note that $H^1_\sigma(\Omega_n)$ is dense in $L^2(\Omega_2)$ and $H^1_0(\Omega_n)$ is dense in $L^2(\Omega_n)$. Therefore, for $a_n\in L^2(\Omega_n)$ and $b_n\equiv h_n-\bar{\theta}_n\in L^2(\Omega_n)$, we have

$$a_n=\sum_{j=1}^{\infty}\alpha_{nj}\varphi_j \quad\text{in } L^2(\Omega_n)\,,\qquad b_n=\sum_{j=1}^{\infty}\beta_{nj}\psi_j \quad\text{in } L^2(\Omega_n)\,. \quad (21)$$

We put

$$u^{(k)}(t)=\sum_{j=1}^{k}\alpha_j^{(k)}(t)\varphi_j\,,\qquad \theta^{(k)}(t)=\sum_{j=1}^{k}\beta_j^{(k)}(t)\psi_j\,. \quad (22)$$

Then we consider the following equations:

$$\frac{d}{dt}(u^{(k)}(t),\varphi_j)-((u^{(k)}(t)\cdot\nabla)\varphi_j,u^{(k)}(t))-(\nabla u^{(k)}(t),\nabla\varphi_j)+(R\theta^{(k)}(t),\theta_j)=(f_n,\varphi_j)\,, \quad (23)$$

$$\frac{d}{dt}(\theta^{(k)}(t),\psi_j)-((u^{(k)}(t)\cdot\nabla)\psi_j,\theta^{(k)}(t))+\frac{1}{P}(\nabla\theta^{(k)}(t),\nabla\psi_j)+((u^{(k)}(t)\cdot\nabla)\bar{\theta},\psi_j)=0\,, \quad (24)$$

with the initial condition

$$u^{(k)}(0)=a_n^{(k)}=\sum_{j=1}^{k}\alpha_{nj}\varphi_j\,,\qquad \theta^{(k)}(0)=b_n^{(k)}=\sum_{j=1}^{k}\beta_{nj}\psi_j\,, \quad (25)$$

where $(\cdot,\cdot)$ stands for $(\cdot,\cdot)_{L^2(\Omega_n)}$. Multiplying (23) (resp. (24)) by $\alpha_j^{(k)}(t)$ (resp. $\beta_j^{(k)}(t)$) and summing up with respect to j and noting $((u^{(k)}\cdot\nabla)u^{(k)},u^{(k)})=0$, $((u^{(k)}\cdot\nabla)\theta^{(k)},\theta^{(k)})=0$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u^{(k)}(t)\|^2_{\Omega_n}+\frac{1}{2}\frac{d}{dt}\|\theta^{(k)}(t)\|^2_{\Omega_n}+\|\nabla u^{(k)}(t)\|^2_{\Omega_n}+\frac{1}{P}\|\nabla\theta^{(k)}(t)\|^2_{\Omega_n}\\
&\le|((u^{(k)}(t)\cdot\nabla)\bar{\theta},\theta^{(k)}(t))_{\Omega_n}|+|(R\theta^{(k)}(t),u^{(k)}(t))_{\Omega_n}|+|(f_n,u^{(k)}(t))_{\Omega_n}|\\
&\le(\|\nabla\bar{\theta}\|_\infty+\|R\|_\infty)\|u^{(k)}(t)\|_{\Omega_n}\|\theta^{(k)}(t)\|_{\Omega_n}+\|f_n\|_{\Omega_n}\|u^{(k)}(t)\|_{\Omega_n}\\
&\le\tfrac{1}{2}(\|\nabla\bar{\theta}\|_\infty+\|R\|_\infty+1)(\|u^{(k)}(t)\|^2_{\Omega_n}+\|\theta^{(k)}(t)\|^2_{\Omega_n})+\tfrac{1}{2}\|f_n\|^2_{\Omega_n}\,.
\end{aligned} \quad (26)$$

From (26), we have an a priori estimate:

$$\|u^{(k)}(t)\|^2_{\Omega_n}+\|\theta^{(k)}(t)\|^2_{\Omega_n}\le\exp(ct)(\|a_n^{(k)}\|^2_{\Omega_n}+\|b_n^{(k)}\|^2_{\Omega_n}+\int_0^t\exp(-c\tau)\|f_n\|^2_{\Omega_n}d\tau)$$

$$\le\exp(ct)(\|a_n\|^2_{\Omega_n}+\|b_n\|^2_{\Omega_n}+\int_0^t\|f_n\|^2_{\Omega_n}d\tau)$$

$$\le\exp(cT)(\|a\|^2_{\Omega}+\|h-\bar\theta\|^2_{\Omega}+\int_0^T\|f\|^2_{\Omega}d\tau)\,, \tag{27}$$

where $c=\|\nabla\bar\theta\|_\infty+\|R\|_\infty+1$.

Moreover by means of (26) and (27) we have

$$\|u^{(k)}(t)\|^2_{\Omega_n}+\|\theta^{(k)}(t)\|^2_{\Omega_n}+2\int_0^t\|\nabla u^{(k)}(\tau)\|^2_{\Omega_n}d\tau+\frac{2}{P}\int_0^t\|\nabla\theta^{(k)}(\tau)\|^2_{\Omega_n}d\tau$$

$$\le c\int_0^t(\|u^{(k)}(\tau)\|^2_{\Omega_n}+\|\theta^{(k)}(\tau)\|^2_{\Omega_n})d\tau+\int_0^t\|f_n\|^2_{\Omega_n}d\tau+\|a_n^{(k)}\|^2_{\Omega_n}+\|b_n^{(k)}\|^2_{\Omega_n}$$

$$\le c\int_0^t e^{c\tau}\left(\|a_n\|^2_{\Omega_n}+\|b_n\|^2_{\Omega_n}+\int_0^\tau\|f_n\|^2_{\Omega_n}ds\right)d\tau+\|a_n\|^2_{\Omega_n}+\|b_n\|^2_{\Omega_n}+\int_0^t\|f_n\|^2_{\Omega_n}d\tau$$

$$\le\left(c\frac{1}{c}(e^{ct}-1)+1\right)\left(\|a_n\|^2_{\Omega_n}+\|b_n\|^2_{\Omega_n}+\int_0^T\|f\|^2_{\Omega_n}d\tau\right)$$

$$\le\exp(cT)\left(\|a\|^2_{\Omega}+\|h-\bar\theta\|^2_{\Omega}+\int_0^T\|f\|^2_{\Omega}dt\right)\equiv C_T\,, \tag{28}$$

where C_T is independent of k, n and t. Thus we have a k-dimensional energy inequality:

$$E(t,\Omega_n;\,u^{(k)}(t),\,\theta^{(k)}(t))\le C_T\,. \tag{29}$$

Fix n, then, by virtue of (29), we find that

$\{u^{(k)}\}$ is a bounded sequence in $L^2(0,T;H^1_\sigma(\Omega_n))$ and $L^\infty(0,T;L^2(\Omega_n))$,

$\{\theta^{(k)}\}$ is a bounded sequence in $L^2(0,T;H^1_0(\Omega_n))$ and $L^\infty(0,T;L^2(\Omega_n))$,

therefore, there exist subsequences $\{u^{(k)}\}$, $\{\theta^{(k)}\}$ (we use same symbols) and the limits v_n and Θ_n such that

$u^{(k)}\to v_n$ weakly in $L^2(0,T;H^1_\sigma(\Omega_n))$, weakly* in $L^\infty(0,T;L^2(\Omega_n))$,

$\theta^{(k)}\to\Theta_n$ weakly in $L^2(0,T;H^1_0(\Omega_n))$, weakly* in $L^\infty(0,T;L^2(\Omega_n))$.

Furthermore, by using compactness argument, we can select subsequences (we use same symbols) such that (see [9], Theorem 2.2 of Chap. III in [12])

$u^{(k)}\to v_n$ strongly in $L^2(0,T;L^2(\Omega_n))$,

$\theta^{(k)}\to\Theta_n$ strongly in $L^2(0,T;L^2(\Omega_n))$.

Owing to these facts, we find that the limit ${}^t(v_n, \Theta_n)$ is a weak solution of (P_n) with the initial value ${}^t(a_n, b_n)$. Moreover we also see similarly that it satisfies (iii) in Lemma 4.2. Concerning the energy inequality (ii), by the lower semicontinuity of the norms of Hilbert spaces with respect to the weak convergence, we have

$$\begin{aligned}
&\|v_n(t)\|^2_{\Omega_n} + \|\Theta_n(t)\|^2_{\Omega_n} + 2\int_0^t \|\nabla v_n(\tau)\|^2_{\Omega_n} d\tau + \frac{2}{P}\int_0^t \|\nabla\Theta_n(\tau)\|^2_{\Omega_n} d\tau \\
&\le \liminf_{k\to\infty} \|u^{(k)}(t)\|^2_{\Omega_n} + \liminf_{k\to\infty} \|\theta^{(k)}(t)\|^2_{\Omega_n} + 2\liminf_{k\to\infty}\int_0^t \|\nabla u^{(k)}(\tau)\|^2_{\Omega_n} d\tau \\
&\quad + \liminf_{k\to\infty} \frac{2}{P}\int_0^t \|\nabla\theta^{(k)}(\tau)\|^2_{\Omega_n} d\tau \\
&\le \liminf_{k\to\infty}\left(\|u^{(k)}(t)\|^2_{\Omega_n} + \|\theta^{(k)}(t)\|^2_{\Omega_n} + 2\int_0^t \|\nabla u^{(k)}(\tau)\|^2_{\Omega_n} d\tau + \frac{2}{P}\int_0^t \|\nabla\theta^{(k)}(\tau)\|^2_{\Omega_n} d\tau\right) \\
&\le \exp(cT)\left(\|a\|^2_{\Omega} + \|h-\bar{\theta}\|^2_{\Omega} + \int_0^T \|f\|^2_{L^2(\Omega)} d\tau\right).
\end{aligned} \tag{30}$$

Thus we obtained the energy inequality (ii). To get (i), we show that

for any fixed j, the sequence $\{(u^{(k)}(t), \varphi_j)_{\Omega_n} + (\theta^{(k)}(t), \psi_j)_{\Omega_n}\}_{k=j}^{\infty}$
is uniformly bounded and equicontinuous on $[0, T]$. (31)

Once we show that (31) is right, then by means of Ascoli-Arzelà's theorem, for each j we find a uniformly convergent subsequence. Moreover, with the aid of diagonal argument, we can get a subsequence commonly with respect to j. Since $\{\varphi_j\}$ and $\{\psi_j\}$ are dense in $H_\sigma(\Omega_n)$ and $L^2(\Omega_n)$ respectively, for $\varphi \in H_\sigma(\Omega_n)$ and $\psi \in L^2(\Omega_n)$ the sequence $\{(u^{(k)}(t), \varphi)_{\Omega_n} + (\theta^{(k)}(t), \psi)_{\Omega_n}\}_{k=1}^{\infty}$ forms a uniform Cauchy sequence on $[0, T]$ except for an arbitrary $\varepsilon > 0$. This implies (i) holds for any $\varphi \in H_\sigma(\Omega_n)$, $\psi \in L^2(\Omega_n)$.

We return to the claim (31). Uniformly boundedness is an immediate consequence of (30). To show the equicontinuity, we integrate (23) and (24) on $[s, t]$. Indeed,

$$\begin{aligned}
&|(u^{(k)}(t), \varphi_j)_{\Omega_n} + (\theta^{(k)}(t), \psi_j)_{\Omega_n} - (u^{(k)}(s), \varphi_j)_{\Omega_n} - (\theta^{(k)}(s), \psi_j)_{\Omega_n}| \\
&\le \int_s^t \{C^2 |||\nabla\varphi_j||| \cdot \|u^{(k)}(\tau)\|^{1/2}_{L^2} \|\nabla u^{(k)}(\tau)\|^{3/2}_{L^2} \\
&\quad + C^2 |||\nabla\psi_j||| \cdot \|\theta^{(k)}(\tau)\|^{1/4}_{L^2} \|\nabla\theta^{(k)}(\tau)\|^{3/4}_{L^2} \|u^{(k)}(\tau)\|^{1/4}_{L^2} \|\nabla u^{(k)}(\tau)\|^{3/4}_{L^2} \\
&\quad + |||\nabla\varphi_j||| \cdot \|\nabla u^{(k)}(\tau)\|_{L^2} + (1/P)|||\nabla\psi_j||| \cdot \|\nabla\theta^{(k)}(\tau)\|_{L^2} \\
&\quad + C_T^{1/2} |||\psi_j||| \cdot \|\nabla\bar{\theta}\|_\infty + C_T^{1/2} \|R\|_\infty \cdot |||\varphi_j||| + \|f\|_{L^2} \cdot |||\varphi_j|||\} d\tau \\
&\le 2^{-3/4} C^2 C_T^{1/4+3/4} |||\nabla\varphi_j|||(t-s)^{1/4} + (4/P)^{-3/8} C^2 C_T^{1/4+3/8+3/8} |||\nabla\psi_j|||(t-s)^{1/4} \\
&\quad + (C_T/2)^{1/2} |||\nabla\varphi_j|||(t-s)^{1/2} + (2P)^{-1/2} C_T^{1/2} |||\nabla\psi_j|||(t-s)^{1/2} \\
&\quad + C_T^{1/2} |||\psi_j||| \cdot \|\nabla\bar{\theta}\|_\infty (t-s) + C_T^{1/2} \|R\|_\infty \cdot |||\varphi_j|||(t-s) + \|f\|_{L^2} \cdot |||\varphi_j|||(t-s),
\end{aligned}$$

where $|||w||| = \max_{0 \le t \le T} \|w(t)\|_{L^2(\Omega_n)}$ and C is a dimension constant. Hence we have the equicontinuity and the lemma is shown.

Here, we will give a series of lemmas which yields a candidate pair of functions for a weak solution of the exterior problem. First, we make functions defined on whole of the region Ω.

LEMMA 4.3. *Put $u_n(x, t) = v_n(x, t)$ if $x \in \Omega_n$ and $u_n(x, t) = 0$ if $x \in \Omega \setminus \Omega_n$; $\theta_n(x, t) = \Theta_n(x, t)$ if $x \in \Omega_n$ and $\theta_n(x, t) = 0$ if $x \in \Omega \setminus \Omega_n$, where v_n, Θ_n are those obtained in Lemma 4.2. Then we have*

(i) ${}^t(u_n, \theta_n) \in H^1_\sigma(\Omega) \times W^{1,2}_0(\Omega)$ *and* $u_n|_{\partial\Omega_n} = 0$, $\theta_n|_{\partial\Omega_n} = 0$.

(ii) $E(t, \Omega; u_n, \theta_n) \le C_T$.

(iii) *For every ${}^t(\varphi, \psi) \in \hat{D}_\sigma(\hat{\Omega}) \times \hat{D}(\hat{\Omega})$, there exists n_0 such that if $n \ge n_0$ then the next equality holds for $t \in [0, T]$:*

$$(u_n(t), \varphi(t))_\Omega + (\theta_n(t), \psi(t))_\Omega = (a_n, \varphi(0))_\Omega + (h_\pi - \bar{\theta}_n, \psi(0))_\Omega + I(t, \Omega; u_n, \theta_n).$$

Since $\operatorname{supp}\varphi$ and $\operatorname{supp}\psi$ are both compact sets in $\hat{\Omega}$, we can easily show Lemma 4.3.

LEMMA 4.4. *Let ${}^t(\varphi, \psi) \in \hat{D}_\sigma(\hat{\Omega}) \times \hat{D}(\hat{\Omega})$ be any element and let us fix it. Then, if N is the least number satisfying $\operatorname{supp}\varphi \subset \Omega_N$ and $\operatorname{supp}\psi \subset \Omega_N$, then the system of functions $\{(u_n(t), \varphi(t))_\Omega + (\theta_n(t), \psi(t))_\Omega\}_N^\infty$ is uniformly bounded and equicontinuous on $[0, T]$.*

PROOF OF LEMMA 4.4. Recall the constant C_T (which appears in (ii) of Lemma 4.2) is independent of n, t, then we get the uniform boundedness by the energy inequality. Next, we show the equicontinuity. To this end, with the aid of the equality in (iii) of Lemma 4.3 and by means of Hölder's and Sobolev's inequalities, we calculate as follows:

$$\begin{aligned}
&|(u_n(t), \varphi(t))_\Omega + (\theta_n(t), \psi(t))_\Omega - (u_n(s), \varphi(s))_\Omega - (\theta_n(s), \psi(s))_\Omega| \\
&= |I(t, \Omega; u_n, \theta_n) - I(s, \Omega; u_n, \theta_n)| \\
&\le \int_s^t \{C_T^{1/2}|||\varphi_\tau||| + C_T^{1/2}|||\psi_\tau||| + C^2|||\nabla\varphi||| \cdot \|u_n(\tau)\|_{L^2}^{1/2}\|\nabla u_n(\tau)\|_{L^2}^{3/2} \\
&\quad + C^2|||\nabla\psi||| \cdot \|\theta_n(\tau)\|_{L^2}^{1/4}\|\nabla\theta_n(\tau)\|_{L^2}^{3/4}\|u_n(\tau)\|_{L^2}^{1/4}\|\nabla u_n(\tau)\|_{L^2}^{3/4} + |||\nabla\varphi||| \cdot \|\nabla u_n(\tau)\|_{L^2} \\
&\quad + (1/P)|||\nabla\psi||| \cdot \|\nabla\theta_n(\tau)\|_{L^2} + C_T^{1/2}|||\psi||| \cdot \|\nabla\bar{\theta}\|_\infty + C_T^{1/2}\|R\|_\infty|||\varphi||| + \|f\|_{L^2} \cdot |||\varphi|||\}d\tau \\
&\le C_T^{1/2}|||\varphi_\tau|||(t-s) + C_T^{1/2}|||\psi_\tau|||(t-s) + 2^{-3/4}C^2C_T^{1/4+3/4}|||\nabla\varphi|||(t-s)^{1/4} \\
&\quad + (4/P)^{-3/8}C^2C_T^{1/4+3/8+3/8}|||\nabla\psi|||(t-s)^{1/4} + (C_T/2)^{1/2}|||\nabla\varphi|||(t-s)^{1/2} \\
&\quad + (2P)^{-1/2}C_T^{1/2}|||\nabla\psi|||(t-s)^{1/2} + C_T^{1/2}|||\psi||| \cdot \|\nabla\bar{\theta}\|_\infty(t-s) \\
&\quad + C_T^{1/2}\|R\|_\infty|||\varphi|||(t-s) + \|f\|_{L^2} \cdot |||\varphi|||(t-s),
\end{aligned}$$

where $|||w||| \equiv \max_{0 \le t \le T} \|w(t)\|_{L^2(\Omega)}$ and C depends only on the space dimension. Thus we have shown the equicontinuity.

LEMMA 4.5. *There exists a subsequence* $\{{}^t(u_{n_k}(t), \theta_{n_k}(t)\}$ *of* $\{{}^t(u_n(t), \theta_n(t)\}$ *such that the following* (i), (ii) *and* (iii) *hold*:

(i) *There exists* ${}^t(u(t), \theta(t)) \in H_\sigma(\Omega) \times L^2(\Omega)$ *such that* ${}^t(u_{n_k}(t), \theta_{n_k}(t))$ *converges weakly to* ${}^t(u(t), \theta(t))$ *uniformly on* $[0, T]$. *Consequently,* ${}^t(u_{n_k} \theta_{n_k})$ *converges to* ${}^t(u, \theta)$ *weakly in* $L^2(\hat{\Omega}) \times L^2(\hat{\Omega})$.

(ii) ${}^t(\nabla u_{n_k}, \nabla\theta_{n_k})$ *converges to* ${}^t(\nabla u, \nabla\theta)$ *weakly in* $L^2(\hat{\Omega}) \times L^2(\hat{\Omega})$. *Hence* ${}^t(u, \theta) \in \hat{H}^1_\sigma(\hat{\Omega}) \times \hat{H}^1(\hat{\Omega})$.

(iii) *If we rechoose a subsequence* $\{{}^t(u_{n_k}, \theta_{n_k})\}$, *if necessary, then* ${}^t(u_{n_k}, \theta_{n_k})$ *converges strongly to* ${}^t(u, \theta)$ *in the sense of* $L^2(\hat{\Omega}') \times L^2(\hat{\Omega}')$. *Here* Ω' *is an arbitrary bounded domain satisfying* $\Omega' \subset \Omega$.

PROOF OF LEMMA 4.5. In order to prove (i) and (ii), we use Ascoli-Arzelà's theorem, the diagonal argument and Riesz's representation theorem (see [1], [6], [9]). To prove (iii), we need Lemma 2.1 (the extended Friedrichs' lemma). Let $\{w_k\} \times \{z_k\}$ be an orthonormal basis of $H_\sigma(\Omega') \times L^2(\Omega')$ and we take $p=2$. Then, thanks to the Friedrichs' lemma, for any given $\varepsilon > 0$ there is N_ε such that the following estimates hold (see Remark 2):

$$\|u\|_{L^2(\Omega')} \le \left(\sum_{i=1}^{N_\varepsilon} (u, w_i)^2_{L^2(\Omega')} \right)^{1/2} + \varepsilon \|u\|_{W^{1,2}(\Omega')}, \qquad \text{for} \quad u \in W^{1,2}(\Omega'),$$

$$\|\theta\|_{L^2(\Omega')} \le \left(\sum_{i=1}^{N_\varepsilon} (\theta, z_i)^2_{L^2(\Omega')} \right)^{1/2} + \varepsilon \|\theta\|_{W^{1,2}(\Omega')}, \qquad \text{for} \quad \theta \in W^{1,2}(\Omega').$$

Therefore we have ($\|\cdot\|_{\Omega'}$ stands for $\|\cdot\|_{L^2(\Omega')}$)

$$\begin{aligned}
&\int_0^T \{\|u_{n_k}(t) - u_{n_l}(t)\|^2_{\Omega'} + \|\theta_{n_k}(t) - \theta_{n_l}(t)\|^2_{\Omega'}\} dt \\
&\le \int_0^T \Big\{ 2 \sum_{i=1}^{N_\varepsilon} (u_{n_k}(t) - u_{n_l}(t), w_i)^2_{\Omega'} + 2\varepsilon^2 (\|u_{n_k}(t) - u_{n_l}(t)\|^2_{\Omega'} + \|\nabla(u_{n_k}(t) - u_{n_l}(t))\|^2_{\Omega'}) \\
&\quad + 2 \sum_{i=1}^{N_\varepsilon} (\theta_{n_k}(t) - \theta_{n_l}(t), z_i)^2_{\Omega'} + 2\varepsilon^2 (\|\theta_{n_k}(t) - \theta_{n_l}(t)\|^2_{\Omega'} + \|\nabla(\theta_{n_k}(t) - \theta_{n_l}(t))\|^2_{\Omega'}) \Big\} d\tau \\
&\le \int_0^T \Big\{ 2 \sum_{i=1}^{N_\varepsilon} (u_{n_k}(t) - u_{n_l}(t), w_i)^2_{\Omega'} + 2\varepsilon^2 (2C_T + 2C_T + \|\nabla(u_{n_k}(t) - u_{n_l}(t))\|^2_{\Omega'}) \\
&\quad + 2 \sum_{i=1}^{N_\varepsilon} (\theta_{n_k}(t) - \theta_{n_l}(t), z_i)^2_{\Omega'} + 2\varepsilon^2 (2C_T + 2C_T + \|\nabla(\theta_{n_k}(t) - \theta_{n_l}(t))\|^2_{\Omega'}) \Big\} d\tau \\
&\le \int_0^T \Big\{ 2 \sum_{i=1}^{N_\varepsilon} (u_{n_k}(t) - u_{n_l}(t), w_i)^2_{\Omega'} + 2 \sum_{i=1}^{N_\varepsilon} (\theta_{n_k}(t) - \theta_{n_l}(t), z_i)^2_{\Omega'} \Big\} d\tau \\
&\quad + 2\varepsilon^2 (8C_T T + 2C_T + 2C_T P) \to 4\varepsilon^2 (4C_T T + C_T + C_T P) \qquad \text{as} \quad n_k, n_l \to \infty, \qquad (32)
\end{aligned}$$

here we used (i) of the lemma and the energy inequality (ii) of Lemma 4.3. Since ε is arbitrary, we established the proof of strong convergence in $L^2(\hat{\Omega}')$.

PROOF OF THEOREM 3.2. Let ${}^t(u, \theta)$ be the limit element which is obtained by Lemma 4.5. We only show that the equality (which is in Definition 3.1)

$$\int_0^T \{(u, \varphi_t)+(\theta, \psi_t)+((u\cdot\nabla)\varphi, u)+((u\cdot\nabla)\psi, \theta)+(u, \Delta\varphi)+P^{-1}(\theta, \Delta\psi)$$
$$-((u\cdot\nabla)\bar{\theta}, \psi)-(R\theta, \varphi)+(f, \varphi)\}dt = -(a, \varphi(0))-(h-\bar{\theta}, \psi(0))$$

holds for all ${}^t(\varphi, \psi)\in\hat{\mathscr{D}}_\sigma(\hat{\Omega})\times\hat{\mathscr{D}}(\hat{\Omega})$. Here $(\cdot, \cdot)$ means $(\cdot, \cdot)_{L^2(\Omega)}$.

To this end, we investigate the convergence of the two nonlinear terms and we skip the remaining part. First we notice that for a given test function ${}^t(\varphi, \psi)\in\hat{\mathscr{D}}_\sigma(\hat{\Omega})\times\hat{\mathscr{D}}(\hat{\Omega})$, we can take a bounded domain Ω' and a number n_0 satisfying $\operatorname{supp}\varphi$, $\operatorname{supp}\psi\subset\Omega'\times[0, T]$ and $\Omega'\subset\Omega_{n_0}\subset\Omega_n$ for all $n\geq n_0$.

If $\Omega\subset\mathbf{R}^3$, we use $\|w\|_{L^6(\Omega)}\leq c_3\|\nabla w\|_{L^2(\Omega)}$, where $c_3=(48)^{1/6}$. Owing to (iii) of Lemma 4.5, we have (C is a dimension constant)

$$\int_0^T |((u_{n_k}\cdot\nabla)\varphi, u_{n_k})_\Omega-((u\cdot\nabla)\varphi, u)_\Omega|dt$$
$$\leq\int_0^T C\{\|u_{n_k}-u\|_{L^2(\Omega')}\|u_{n_k}\|_{L^6(\Omega)}\|\nabla\varphi\|_{L^3(\Omega')}+\|u\|_{L^6(\Omega)}\|u_{n_k}-u\|_{L^2(\Omega')}\|\nabla\varphi\|_{L^3(\Omega')}\}dt$$
$$\leq c_3C_T^{1/2}C\max_{0\leq t\leq T}\|\nabla\varphi(t)\|_{L^3(\Omega')}\left(\int_0^T\|u_{n_k}(t)-u(t)\|^2_{L^2(\Omega')}dt\right)^{1/2}\to 0 \qquad (\text{as } n_k\to\infty).$$

Moreover, we see

$$\int_0^T |((u_{n_k}\cdot\nabla)\psi, \theta_{n_k})_\Omega-((u\cdot\nabla)\psi, \theta)_\Omega|dt$$
$$\leq\int_0^T C\{\|\theta_{n_k}-\theta\|_{L^2(\Omega')}\|u_{n_k}\|_{L^6(\Omega)}\|\nabla\psi\|_{L^3(\Omega')}+\|\theta\|_{L^6(\Omega)}\|u_{n_k}-u\|_{L^2(\Omega')}\|\nabla\psi\|_{L^3(\Omega')}\}dt$$
$$\leq c_3C_T^{1/2}C\max_{0\leq t\leq T}\|\nabla\psi(t)\|_{L^3(\Omega')}\left\{\left(\frac{1}{2}\right)^{1/2}\left(\int_0^T\|\theta_{n_k}(t)-\theta(t)\|^2_{L^2(\Omega')}dt\right)^{1/2}\right.$$
$$\left.+\left(\frac{P}{2}\right)^{1/2}\left(\int_0^T\|u_{n_k}(t)-u(t)\|^2_{L^2(\Omega')}dt\right)^{1/2}\right\}\to 0 \qquad (\text{as } n_k\to\infty).$$

If $\Omega\subset\mathbf{R}^2$, we use $\|w\|_{L^4(\Omega)}\leq c_2\|w\|^{1/2}_{L^2(\Omega)}\|\nabla w\|^{1/2}_{L^2(\Omega)}$, where $c_2=2^{1/4}$:

$$\int_0^T |((u_{n_k}\cdot\nabla)\varphi, u_{n_k})_\Omega-((u\cdot\nabla)\varphi, u)_\Omega|dt$$

$$\le \int_0^T C\{\|u_{n_k}-u\|_{L^2(\Omega')}\|u_{n_k}\|_{L^4(\Omega)}\|\nabla\varphi\|_{L^4(\Omega')}+\|u\|_{L^4(\Omega)}\|u_{n_k}-u\|_{L^2(\Omega')}\|\nabla\varphi\|_{L^4(\Omega')}\}dt$$

$$\le c_2C_T^{1/4}C\max_{0\le t\le T}\|\nabla\varphi(t)\|_{L^4(\Omega')}\left(\int_0^T\|u_{n_k}(t)-u(t)\|^2_{L^2(\Omega')}dt\right)^{1/2}\left(\int_0^T\|\nabla u_{n_k}\|_{L^2(\Omega)}\right)^{1/2}$$

$$+c_2C_T^{1/4}C\max_{0\le t\le T}\|\nabla\varphi(t)\|_{L^4(\Omega')}\left(\int_0^T\|u_{n_k}(t)-u(t)\|^2_{L^2(\Omega')}dt\right)^{1/2}\left(\int_0^T\|\nabla u\|_{L^2(\Omega)}\right)^{1/2}$$

$$\le 2c_2C_T^{1/4}C_T^{1/2}\left(\frac{T}{2}\right)^{1/2}C\max_{0\le t\le T}\|\nabla\varphi(t)\|_{L^4(\Omega')}\left(\int_0^T\|u_{n_k}(t)-u(t)\|^2_{L^2(\Omega')}dt\right)^{1/2}\to 0$$

$$(\text{as } n_k\to\infty).$$

Moreover, we see

$$\int_0^T |((u_{n_k}\cdot\nabla)\psi, \theta_{n_k})_\Omega-((u\cdot\nabla)\psi, \theta)_\Omega|dt$$

$$\le \int_0^T C\{\|\theta_{n_k}-\theta\|_{L^2(\Omega')}\|u_{n_k}\|_{L^4(\Omega)}\|\nabla\psi\|_{L^4(\Omega')}+\|\theta\|_{L^4(\Omega)}\|u_{n_k}-u\|_{L^2(\Omega')}\|\nabla\psi\|_{L^3(\Omega')}\}dt$$

$$\le c_2C_T^{1/4}C\max_{0\le t\le T}\|\nabla\psi(t)\|_{L^4(\Omega')}\left(\int_0^T\|\theta_{n_k}(t)-\theta(t)\|^2_{L^2(\Omega')}dt\right)^{1/2}\left(\int_0^T\|\nabla u_{n_k}\|_{L^2(\Omega)}\right)^{1/2}$$

$$+c_2C_T^{1/4}C\max_{0\le t\le T}\|\nabla\psi(t)\|_{L^4(\Omega')}\left(\int_0^T\|u_{n_k}(t)-u(t)\|^2_{L^2(\Omega')}dt\right)^{1/2}\left(\int_0^T\|\nabla\theta\|_{L^2(\Omega)}\right)^{1/2}$$

$$\le c_3C_T^{1/4}C_T^{1/2}C\max_{0\le t\le T}\|\nabla\psi(t)\|_{L^4(\Omega')}\left\{\left(\frac{1}{2}\right)^{1/2}\left(\int_0^T\|\theta_{n_k}(t)-\theta(t)\|^2_{L^2(\Omega')}dt\right)^{1/2}\right.$$

$$\left.+\left(\frac{P}{2}\right)^{1/2}\left(\int_0^T\|u_{n_k}(t)-u(t)\|^2_{L^2(\Omega')}dt\right)^{1/2}\right\}\to 0 \qquad (\text{as } n_k\to\infty).$$

Hence we have established Theorem 3.2.

PROOF OF THEOREM 3.3. Suppose $\Omega\subset\mathbf{R}^2$. Put $V_1=H^1_\sigma(\Omega)$, $V_2=H^1_0(\Omega)$ and let V_1' (resp. V_2') be the dual space of V_1 (resp. V_2). We note that if ${}^t(u,\theta)$ is a weak solution of (HCE), then their distribution derivative (with respect to time variable t) u' and θ' belong to the spaces $L^2(0, T; V_1')$ and $L^2(0, T; V_2')$ respectively. Indeed, using $\|w\|_4\le c_2\|w\|^{1/2}\|\nabla w\|^{1/2}\le c_2\|w\|_{V_i}$, we have the following estimates for $\varphi\in V_1$ and $\psi\in V_2$:

$$|((u\cdot\nabla)\varphi, u)|\le C\|u\|_4\|\nabla\varphi\|\cdot\|u\|_4\le c_2^2C\|u\|\cdot\|\nabla u\|\cdot\|\varphi\|_{V_1}$$

$$|(\nabla u, \nabla\varphi)|\le\|\nabla u\|\cdot\|\nabla\varphi\|\le\|\nabla u\|\cdot\|\varphi\|_{V_1}$$

$$|(R\theta, \varphi)|\le\|R\|_\infty\|\theta\|\cdot\|\varphi\|\le\|R\|_\infty\|\theta\|\cdot\|\varphi\|_{V_1}$$

$$|(f, \varphi)|\le\|f\|_{4/3}\|\varphi\|_4\le c_2\|f\|_{4/3}\|\varphi\|_{V_1}$$

$$|((u\cdot\nabla)\psi, \theta)|\le C\|u\|_4\|\nabla\psi\|\cdot\|\theta\|_4\le c_2^2C\|u\|^{1/2}\|\nabla u\|^{1/2}\|\psi\|_{V_2}\|\theta\|^{1/2}\|\nabla\theta\|^{1/2}$$

$$|(\nabla\theta, \nabla\psi)|\le\|\nabla\theta\|\cdot\|\nabla\psi\|\le\|\nabla\theta\|\cdot\|\psi\|_{V_2}$$

$$|((u\cdot\nabla)\bar\theta, \psi)|\le\|\nabla\bar\theta\|_\infty\|u\|\cdot\|\psi\|\le\|\nabla\bar\theta\|_\infty\|u\|\cdot\|\psi\|_{V_2},$$

here C is a dimension constant and we used $g\in L^2(\Omega)\cap L^{4/3}(\Omega)$. By the energy inequality, we see $-(u\cdot\nabla)u+\Delta u-R\theta+f\in L^2(0, T; V_1')$ and $-(u\cdot\nabla)\theta+(1/P)\Delta\theta-(u\cdot\nabla)\bar\theta\in L^2(0, T; V_2')$, so $u'\in L^2(0, T; V_1')$ and $\theta'\in L^2(0, T; V_2')$. Then we have (see (3.62) of Chap. III of [12])

$$d/dt\|u(t)\|^2=2\langle u'(t), u(t)\rangle \quad\text{and}\quad d/dt\|\theta(t)\|^2=2\langle\theta'(t), \theta(t)\rangle. \tag{33}$$

Here, let ${}^t(u_i, \theta_i)$ $(i=1, 2)$ are two weak solutions of (IVP) and put ${}^t(u, \theta)={}^t(u_1-u_2, \theta_1-\theta_2)$. Then we prepare several estimates:

$$\begin{aligned}|((u\cdot\nabla)u_1, u)|&\le C\|u\|_4\|\nabla u_1\|\cdot\|u\|_4\le C\cdot 2^{1/2}\|u\|\cdot\|\nabla u\|\cdot\|\nabla u_1\|\\ &\le\tfrac14\|\nabla u\|^2+2C^2\|u\|^2\|\nabla u_1\|^2\le\tfrac14\|\nabla u\|^2+\tfrac12(\|u\|^2+\|\theta\|^2)\cdot 4C^2\|\nabla u_1\|^2,\end{aligned} \tag{34}$$

$$\begin{aligned}|((u\cdot\nabla)\theta_1, \theta)|&\le C\|u\|_4\|\nabla\theta_1\|\cdot\|\theta\|_4\le C\cdot 2^{1/4}\|u\|^{1/2}\|\nabla u\|^{1/2}\|\nabla\theta_1\|2^{1/4}\|\theta\|^{1/2}\|\nabla\theta\|^{1/2}\\ &\le\tfrac12(2^{1/2}C^2\|u\|\cdot\|\nabla u\|\cdot\|\nabla\theta_1\|+2^{1/2}\|\nabla\theta_1\|\cdot\|\theta\|\cdot\|\nabla\theta\|)\\ &\le\frac12\left(\frac12\|\nabla u\|^2+\frac12 2C^4\|u\|^2\|\nabla\theta_1\|^2+\frac1P\|\nabla\theta\|^2+\frac P4\cdot 2\|\nabla\theta_1\|^2\|\theta\|^2\right)\\ &\le\frac14\|\nabla u\|^2+\frac1{2P}\|\nabla\theta\|^2+\frac12(\|u\|^2+\|\theta\|^2)\left(C^4\|\nabla\theta_1\|^2+\frac P2\|\nabla\theta_1\|^2\right).\end{aligned} \tag{35}$$

Now we can consider u_i (resp. θ_i) satisfies the equation (4) as an element of $L^2(0, T; V_1')$ (resp. $L^2(0, T; V_2')$). Subtracting each sides of corresponding two equations $(i=1, 2)$, taking the scalar product of the difference of equations with u, θ in the duality between V_i and V_i', integrating on $[0, t]$ and using $u(0)=0$, $\theta(0)=0$, then we get by above estimates (34) and (35)

$$\frac{1}{2}\|u(t)\|^2+\frac{1}{2}\|\theta(t)\|^2+\int_0^t\|\nabla u(\tau)\|^2d\tau+\frac{1}{P}\int_0^t\|\nabla\theta(\tau)\|^2d\tau$$

$$=-\int_0^t\{((u\cdot\nabla)u_1,u)+((u\cdot\nabla)\theta_1,\theta)+((u\cdot\nabla)\bar{\theta},\theta)+(R\theta,u)\}dt$$

$$\leq\int_0^t\left\{\frac{1}{2}(\|u\|^2+\|\theta\|^2)\left(4C^2\|\nabla u_1\|^2+C^4\|\nabla\theta_1\|^2+\frac{P}{2}\|\nabla\theta_1\|^2+\|\nabla\bar{\theta}\|_\infty+\|R\|_\infty\right)\right\}d\tau$$

$$+\frac{1}{2}\int_0^t\|\nabla u(\tau)\|^2d\tau+\frac{1}{2P}\int_0^t\|\nabla\theta(\tau)\|^2d\tau\,. \tag{36}$$

Since $4C^2\|\nabla u_1\|^2+C^4\|\nabla\theta_1\|^2+\frac{P}{2}\|\nabla\theta_1\|^2+\|\nabla\bar{\theta}\|_\infty+\|R\|_\infty$ belongs to $L^1(0,T)$, so by (36) and Gronwall's inequality, we obtain the uniqueness.

PROOF OF THEOREM 3.4. Let ${}^t(u,\theta)$ be a weak solution of (IVP) satisfying the condition (14):

$$u\in L^s(0,T;L^r(\Omega))\quad\text{and}\quad\theta\in L^s(0,T;L^r(\Omega))\qquad(r>3,\ s=2r/(r-3))\,.$$

From (14), we can show

$$u'\in L^2(0,T;V_1')\quad\text{and}\quad\theta'\in L^2(0,T;V_2')\,, \tag{37}$$

where $V_1=H_\sigma^1(\Omega)$ and $V_2=H_0^1(\Omega)$. To do this, we will verify

$$(u\cdot\nabla)u\in L^2(0,T;V_1')\quad\text{and}\quad(u\cdot\nabla)\theta\in L^2(0,T;V_2')\,. \tag{38}$$

Indeed, if we take $2\leq q\leq 6$ such that $1/q+1/r=1/2$, then, for $\varphi\in V_1$, we have

$$|((u\cdot\nabla)\varphi,u)|\leq C\|u\|_q\|\nabla\varphi\|_2\|u\|_r\qquad\text{for}\quad\varphi\in V_1\,, \tag{39}$$

where $\|\cdot\|_p=\|\cdot\|_{L^p(\Omega)}$ and C is a dimension constant. Here we use Lemma 2.2 with $\alpha=3/2-3/q$. Then we have

$$\|u\|_q\leq c_3^\alpha\|\nabla u\|^\alpha\|u\|^{1-\alpha}\,, \tag{40}$$

from which we have

$$|((u\cdot\nabla)\varphi,u)|\leq Cc_3^\alpha\|\nabla u\|^\alpha\|u\|^{1-\alpha}\|\nabla\varphi\|\cdot\|u\|_r\,. \tag{41}$$

Since $\|\nabla\varphi\|\leq\|\varphi\|_{V_1}$, therefore we find

$$\|(u\cdot\nabla)u\|_{V_1'}\leq Cc_3^\alpha\|\nabla u\|^\alpha\|u\|^{1-\alpha}\|u\|_r\,. \tag{42}$$

By means of the energy inequality, we obtain

$$\int_0^T \|(u\cdot\nabla)u\|_{V_1'}^2 dt \le \int_0^T C^2 c_3^{2\alpha}\|\nabla u\|^{2\alpha}\|u\|^{2(1-\alpha)}\|u\|_r^2 dt$$

$$\le C^2 c_3^{2\alpha} C_T^{1-\alpha}\int_0^T \|\nabla u\|^{2\alpha}\|u\|_r^2 dt \le C^2 c_3^{2\alpha} C_T^{1-\alpha}\left(\int_0^T \|\nabla u\|^2 dt\right)^{\alpha}\left(\int_0^T \|u\|_r^{2/(1-\alpha)}\right)^{1-\alpha}$$

$$\le C^2 c_3^{2\alpha} C_T^{1-\alpha}(C_T/2)^{\alpha}\|u\|_{r,s}^2 = C^2 c_3^{2\alpha} C_T (1/2)^{\alpha}\|u\|_{r,s}^2, \tag{43}$$

where $\|u\|_{r,s}=\|u\|_{L^s(0,T;L^r(\Omega))}$ and we used $\alpha=3/r$ and $s=2/(1-\alpha)$ because of $1/q=1/2-1/r$. Similarly we get

$$\int_0^T \|(u\cdot\nabla)\theta\|_{V_2'}^2 dt \le C^2 c_3^{2\alpha} C_T\left(\frac{1}{2}\right)^{\alpha}\|\theta\|_{r,s}^2 . \tag{44}$$

Therefore we showed (38). Moreover, thanks to $g\in L^2(\Omega)\cap L^{4/3}(\Omega)$, we find $f\in L^2(0, T; V_1')$ (we omit verifications about other terms). Thus we obtain (37). Hence we have

$$d/dt\|u(t)\|^2 = 2\langle u'(t), u(t)\rangle \quad\text{and}\quad d/dt\|\theta(t)\|^2 = 2\langle \theta'(t), \theta(t)\rangle . \tag{45}$$

Let ${}^t(u_1, \theta_1)$ and ${}^t(u_2, \theta_2)$ be two weak solutions of (IVP) with the same initial data satisfying the condition (14). Put ${}^t(u, \theta)={}^t(u_1-u_2, \theta_1-\theta_2)$. We can show

$$|((u\cdot\nabla)u_1, u)| \le \tfrac{1}{4}\|\nabla u\|^2 + (1-\alpha^2)(Cc_3^{\alpha})^{2/(1+\alpha)}\|u\|^2\|u_1\|_r^s, \tag{46}$$

$$|((u\cdot\nabla)\theta_1. \theta)| \le \tfrac{1}{4}\|\nabla u\|^2 + (1/2P)\|\nabla\theta\|^2 + (1-\alpha^2)(Cc_3^{\alpha})^{2/(1+\alpha)}(P/2)^{1/(1-\alpha)}\|u\|^2\|\theta_1\|_r^s . \tag{47}$$

In fact, we see by (40)

$$|((u\cdot\nabla)u_1, u)| = |((u\cdot\nabla)u, u_1)| \le C\|u\|_q\|\nabla u\|\cdot\|u_1\|_r$$
$$\le (Cc_3^{\alpha}\|\nabla u\|^{1+\alpha})(\|u\|^{1-\alpha}\|u_1\|_r) . \tag{48}$$

Here we take $p'=2/(1+\alpha)$, $q'=2/(1-\alpha)$ $(=s)$ and k such that $(k^{p'}/p')(Cc_3^{\alpha})^{p'}=1/4$. Then, for such a k, we find $(1/q'k^{p'})=(1-\alpha^2)(Cc_3^{\alpha})^{2/(1+\alpha)}$. Then we have from (48)

$$|((u\cdot\nabla)u_1, u)| \le \frac{k^{p'}}{p'}(Cc_3^{\alpha})^{p'}\|\nabla u\|^{(1+\alpha)p'} + \frac{1}{q'k^{p'}}\|u\|^{(1-\alpha)q'}\|u_1\|_r^{q'}$$
$$= \tfrac{1}{4}\|\nabla u\|^2 + (1-\alpha^2)(Cc_3^{\alpha})^{2/(1+\alpha)}\|u\|^2\|u_1\|_r^s . \tag{49}$$

Similarly we have

$$|((u\cdot\nabla)\theta_1, \theta)| \le C\|u\|_q\|\nabla\theta\|\cdot\|\theta_1\|_r \le (Cc_3^{\alpha}\|\nabla u\|^{\alpha}(2/P)^{1/2}\|\nabla\theta\|)((P/2)^{1/2}\|u\|^{1-\alpha}\|\theta_1\|_r)$$
$$\le \tfrac{1}{4}\|\nabla u\|^{2\alpha/(1+\alpha)}(2/P)^{1/(1+\alpha)}\|\nabla\theta\|^{2/(1+\alpha)} + (1-\alpha^2)(Cc_3^{\alpha})^{2/(1+\alpha)}(P/2)^{1/(1-\alpha)}\|u\|^2\|\theta_1\|_r^s . \tag{50}$$

Here, for $\lambda=(1+\alpha)/\alpha$ and $\mu=1+\alpha$, we find

$$\|\nabla u\|^{2\alpha/(1+\alpha)}(2/P)^{1/(1+\alpha)}\|\nabla\theta\|^{2/(1+\alpha)} \le \frac{1}{\lambda}\|\nabla u\|^{2\alpha\lambda/(1+\alpha)} + \frac{1}{\mu}\left(\frac{2}{P}\right)^{(1/(1+\alpha))\mu}\|\nabla\theta\|^{(2/(1+\alpha))\mu} . \tag{51}$$

By (50) and (51), we obtain (note $0<1/\lambda,\ 1/\mu<1$)

$$|((u\cdot\nabla)\theta_1,\theta)|\le\frac{1}{4}\|\nabla u\|^2+\frac{1}{2P}\|\nabla\theta\|^2+(1-\alpha^2)(Cc_3^{\alpha})^{2/(1+\alpha)}(P/2)^{1/(1-\alpha)}\|u\|^2\|\theta_1\|_r^s. \quad (52)$$

Now, using (49) and (52), putting $C_\alpha=(1-\alpha^2)(Cc_3^{\alpha})^{2/(1+\alpha)}$, then we get

$$\frac{1}{2}\|u(t)\|^2+\frac{1}{2}\|\theta(t)\|^2+\int_0^t\|\nabla u(\tau)\|^2d\tau+\frac{1}{P}\int_0^t\|\nabla\theta(\tau)\|^2d\tau$$

$$\le\frac{1}{2}\int_0^t\|\nabla u(\tau)\|^2d\tau+\frac{1}{2P}\int_0^t\|\nabla\theta(\tau)\|^2d\tau+\int_0^t\frac{1}{2}(\|u\|^2+\|\theta\|^2)(\|\nabla\bar\theta\|_\infty+\|R\|_\infty)d\tau$$

$$+\int_0^t\|u\|^2\left(C_\alpha\|u_1\|_r^s+C_\alpha\left(\frac{P}{2}\right)^{1/(1-\alpha)}\|\theta_1\|_r^s\right)d\tau$$

$$\le\frac{1}{2}\int_0^t\|\nabla u(\tau)\|^2d\tau+\frac{1}{2P}\int_0^t\|\nabla\theta(\tau)\|^2d\tau$$

$$+\int_0^t\frac{1}{2}(\|u\|^2+\|\theta\|^2)\left(2C_\alpha\|u_1\|_r^s+2C_\alpha\left(\frac{P}{2}\right)^{1/(1-\alpha)}\|\theta_1\|_r^s+\|\nabla\bar\theta\|_\infty+\|R\|_\infty\right)d\tau. \quad (53)$$

Since $2C_\alpha\|u_1\|_r^s+2C_\alpha(P/2)^{1/(1-\alpha)}\|\theta_1\|_r^s+\|\nabla\bar\theta\|_\infty+\|R\|_\infty\in L^1(0,T)$, by virtue of (53) and Gronwall's inequality, we have the uniqueness theorem.

References

[1] H. FUJITA and N. SAUER, On existence of weak solutions of Navier-Stokes equations in regions with moving boundaries, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **17** (1970), 403–420.

[2] T. HISHIDA, Asymptotic behavior and stability of solutions to the exterior convection problem, Nonlinear Anal. TMA **22** (1994), 895–925.

[3] T. HISHIDA and Y. YAMADA, Global solutions for the heat convection equations in an exterior domain, Tokyo J. Math. **15** (1992), 135–151.

[4] O. A. LADYZHENSKAYA, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach (1963).

[5] O. A. LADYZHENSKAYA, V. A. SOLONNIKOV and N. N. URALCEVA, *Linear and Quasilinear Equations of Parabolic Type*, Amer. Math. Soc. (1968).

[6] N. MATSUDA, On existence of weak solutions of the heat convection equations in exterior domains, Master Thesis, St. Paul's Univ. (1996) (in Japanese).

[7] H. MORIMOTO, Non-stationary Boussinesq equations, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **39** (1992), 61–75.

[8] K. ŌEDA, On the initial value problem for the heat convection equation of Boussinesq approximation in a time-dependent domain, Proc. Japan Acad. **64** (1988), 143–146.

[9] K. ŌEDA, Weak and strong solutions of the heat convection equations in regions with moving boundaries, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **36** (1989), 491–536.

[10] K. ŌEDA and N. MATSUDA, On existence and uniqueness of weak solutions of the heat convection equations in exterior domains, (preprint).

[11] J. SERRIN, The initial value problem for the Navier-Stokes equations, *Nonlinear Problems* (R. E. Langer, ed.), Univ. Wisconsin Press (1963), 69–98.
[12] R. TEMAM, *Navier-Stokes Equations*, North-Holland (1984).

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