INITIAL VALUE PROBLEMS
IN DISCRETE FRACTIONAL CALCULUS

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(Communicated by Jane M. Hawkins)

ABSTRACT. This paper is devoted to the study of discrete fractional calculus; the particular goal is to define and solve well-defined discrete fractional difference equations. For this purpose we first carefully develop the commutativity properties of the fractional sum and the fractional difference operators. Then a ν-th ($0 < \nu \leq 1$) order fractional difference equation is defined. A nonlinear problem with an initial condition is solved and the corresponding linear problem with constant coefficients is solved as an example. Further, the half-order linear problem with constant coefficients is solved with a method of undetermined coefficients and with a transform method.

1. INTRODUCTION

The purpose and contribution of this paper is to introduce a well-defined ν-th ($0 < \nu \leq 1$) order fractional difference equation, produce a method of solution for the general nonlinear problem, and exhibit two more methods of solution for the linear problem of half-order with constant coefficients. Fractional calculus has a long history and there seems to be new and recent interest in the study of fractional calculus and fractional differential equations; we provide two well-cited monographs here, [9] and [10].

The authors are trained with a perspective in differential equations; moreover, the kernel of the Riemann–Liouville fractional integral

$$\frac{(t-s)^{\nu-1}}{\Gamma(\nu)}$$

is a clear analogue of the Cauchy function for ordinary differential equations. Hence, the authors are heavily influenced by the approach taken by Miller and Ross [8] who study the linear ν-th order fractional differential equation as an analogue of the linear n-th order ordinary differential equation.

To the authors' knowledge, very little progress has been made to develop the theory of the analogous fractional finite difference equation. Miller and Ross [7] produced an early paper; the authors [1] have developed and applied a transform method. The authors [2] also developed and applied a transform method for fractional q-calculus problems. An appropriate bibliography for the fractional q-calculus is provided in [2].

We start with basic definitions and results so that this paper is self-contained. Let $\nu > 0$. Let $\sigma(s) = s + 1$. The ν-th fractional sum of f is defined by

$$(1.1) \qquad \Delta^{-\nu} f(t; a) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - \sigma(s))^{(\nu-1)} f(s).$$

Note that f is defined for $s = a \bmod (1)$ and $\Delta^{-\nu} f$ is defined for $t = a + \nu \bmod (1)$; in particular, $\Delta^{-\nu}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\nu}$, where $\mathbb{N}_t = \{t, t+1, t+2, \ldots\}$. We point out that we employ throughout the notation, $\sigma(s)$, because eventually progress will be made to develop the theory of the fractional calculus on time scales [4]. We remind the reader that $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t+\nu+1)}$, we shall suppress the dependence on a in $\Delta^{-\nu} f(t; a)$ since domains will be clear by the context, and finally we point out that Miller and Ross [7] have argued that $\lim_{\nu \to 0^+} \Delta_a^{-\nu} f(t) = f(t)$.

The following two results (the commutative property of the fractional sum operator and the power rule) and their proofs can be found in a paper by the authors [1].

Theorem 1.1. *Let f be a real-valued function defined on $\mathbb{N}_a$ and let $\mu, \nu > 0$. Then the following equalities hold:*

$$\Delta^{-\nu}[\Delta^{-\mu} f(t)] = \Delta^{-(\mu+\nu)} f(t) = \Delta^{-\mu}[\Delta^{-\nu} f(t)].$$

Lemma 1.1. *Let $\mu \neq -1$ and assume $\mu + \nu + 1$ is not a nonpositive integer. Then*

$$\Delta^{-\nu} t^{(\mu)} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} t^{(\mu+\nu)}.$$

The μ-th fractional difference is defined as

$$\Delta^\mu u(t) = \Delta^{m-\nu} u(t) = \Delta^m (\Delta^{-\nu} u(t)),$$

where $\mu > 0$ and $m - 1 < \mu < m$, where m denotes a positive integer, and $-\nu = \mu - m$. With this standard definition for the fractional derivative, it is a straightforward calculation to show that Lemma 1.1 is valid for ν real, $\mu > -1$.

The plan of this paper is the following. In Section 2, we shall state and prove the commutative type properties of the fractional sum and difference operators. We shall also introduce and develop properties for a characteristic function, $F(t, \nu, \alpha)$, which plays a role analogous to that of the exponential function for finite difference equations. In Section 3, we introduce the ν-th $(0 < \nu \leq 1)$ order fractional difference equation with an initial condition; employing the commutativity properties of Section 2, we shall construct an equivalent summation equation. Further, we shall solve an initial value problem for a nonlinear equation. We formally produce a series solution of a linear equation with constant coefficients. Finally, we shall focus on the half-order linear equation with constant coefficients and provide two more methods of solution: a method of undetermined coefficients and a transform method.

2. Preliminaries

Theorem 2.1. *For any $\nu > 0$, the following equality holds:*

$$(2.1) \qquad \Delta^{-\nu} \Delta f(t) = \Delta \Delta^{-\nu} f(t) - \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)} f(a),$$

where f is defined on $\mathbb{N}_a$.

Proof. First recall the summation by parts formula [6]:

$$\Delta_s((t-s)^{(\nu-1)}f(s)) = (t-\sigma(s))^{(\nu-1)}\Delta_s f(s) - (\nu-1)(t-\sigma(s))^{(\nu-2)}f(s).$$

Sum by parts to obtain

$$\frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-\sigma(s))^{(\nu-1)}\Delta_s f(s)$$

$$= \frac{\nu-1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-\sigma(s))^{(\nu-2)}f(s) + \frac{(t-s)^{(\nu-1)}f(s)}{\Gamma(\nu)}\Big|_a^{t+1-\nu}$$

$$= \frac{\nu-1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-\sigma(s))^{(\nu-2)}f(s) + \frac{(\nu-1)^{(\nu-1)}f(t+1-\nu)}{\Gamma(\nu)} - \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)}f(a)$$

$$= \frac{1}{\Gamma(\nu-1)}\sum_{s=a}^{t-(\nu-1)}(t-\sigma(s))^{(\nu-2)}f(s) - \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)}f(a).$$

Since $\Delta\Delta^{-\nu}f(t) = \frac{1}{\Gamma(\nu-1)}\sum_{s=a}^{t-(\nu-1)}(t-\sigma(s))^{(\nu-2)}f(s)$, the desired equality follows. $\square$

Remark 2.1. Replace ν by $\nu+1$ in (2.1) and employ Theorem 1.1 to obtain

$$\Delta^{-\nu-1}\Delta f(t) = \Delta^{-\nu}f(t) - \frac{(t-a)^{(\nu)}}{\Gamma(\nu+1)}f(a).$$

This implies

$$(2.2) \qquad \Delta^{-\nu}f(t) = \Delta^{-\nu-1}\Delta f(t) + \frac{(t-a)^{(\nu)}}{\Gamma(\nu+1)}f(a).$$

Remark 2.2. Let $p-1 < \nu < p$, where p is a positive integer. Theorem 2.1 implies that

$$\Delta\Delta^\nu f(t) = \Delta\Delta^p(\Delta^{-(p-\nu)}f(t)) = \Delta^{p+1}(\Delta^{-(p-\nu)}f(t))$$

$$= \Delta^p(\Delta\Delta^{-(p-\nu)}f(t)) = \Delta^p[\Delta^{-(p-\nu)}\Delta f(t) + \frac{(t-a)^{(p-\nu-1)}}{\Gamma(p-\nu)}f(a)]$$

$$= \Delta^p\Delta^{-(p-\nu)}\Delta f(t) + \Delta^p\frac{(t-a)^{(p-\nu-1)}}{\Gamma(p-\nu)}f(a)$$

$$= \Delta^\nu\Delta f(t) + \frac{(t-a)^{(-\nu-1)}}{\Gamma(-\nu)}f(a).$$

So we conclude that (2.1) is valid for any real number ν.

Theorem 2.2. *For any real number ν and any positive integer p, the following equality holds:*

$$(2.3) \qquad \Delta^{-\nu}\Delta^p f(t) = \Delta^p\Delta^{-\nu}f(t) - \sum_{k=0}^{p-1}\frac{(t-a)^{(\nu-p+k)}}{\Gamma(\nu+k-p+1)}\Delta^k f(a),$$

where f is defined on $\mathbb{N}_a$.

Proof. We replace f by Δf in (2.1):

$$\Delta^{-\nu}\Delta^2 f(t) = \Delta\Delta^{-\nu}\Delta f(t) - \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)}\Delta f(a)$$

$$= \Delta[\Delta\Delta^{-\nu}f(t) - \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)}f(a)] - \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)}\Delta f(a)$$

$$= \Delta^2\Delta^{-\nu}f(t) - \frac{(t-a)^{(\nu-2)}}{\Gamma(\nu-1)}f(a) - \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)}\Delta f(a)$$

$$= \Delta^2\Delta^{-\nu}f(t) - \sum_{k=0}^{1}\frac{(t-a)^{(\nu-2+k)}}{\Gamma(\nu+k-1)}\Delta^k f(a).$$

Repeated iterations give the desired result. $\qquad\square$

Remark 2.3. Again replace ν by $\nu+p$ in (2.3) and employ Theorem 1.1 to obtain

$$(2.4) \qquad \Delta^{-\nu}f(t) = \Delta^{-\nu-p}\Delta^p f(t) + \sum_{k=0}^{p-1}\frac{(t-a)^{(\nu+k)}}{\Gamma(\nu+k+1)}\Delta^k f(a).$$

Theorem 2.3. *Let p be a positive integer and let $\nu > p$. Then*

$$(2.5) \qquad \Delta^p[\Delta^{-\nu}f(t)] = \Delta^{-(\nu-p)}f(t).$$

Proof. By the definition of the fractional sum,

$$\Delta^{-\nu}f(t) = \frac{1}{\Gamma(\nu)}\sum_{s=a}^{t-\nu}(t-\sigma(s))^{(\nu-1)}f(s),$$

we see that

$$\Delta^{p-1}\Delta^{-\nu}f(t) = \frac{1}{\Gamma(\nu-p+1)}\sum_{s=a}^{t-(\nu-p+1)}(t-\sigma(s))^{(\nu-p)}f(s) = \Delta^{-(\nu-p)-1}f(t),$$

since $\nu > p$. Apply the difference operator to each side of the above equation to obtain

$$\Delta^p[\Delta^{-\nu}f(t)] = \Delta[\Delta^{p-1-\nu}f(t)].$$

Apply (2.1) with ν replaced by $\nu-p+1$ to obtain

$$\Delta^p[\Delta^{-\nu}f(t)] = \Delta^{p-1-\nu}[\Delta f(t)] + \frac{(t-a)^{(\nu-p)}}{\Gamma(\nu+1-p)}f(a).$$

Apply (2.2) with ν replaced by $\nu-p$ and (2.5) is proved. $\qquad\square$

Recall [6] that for linear difference equations with constant coefficients, the family of functions $(1+\alpha)^t$ plays the same role that the family of functions $e^{\alpha t}$ plays for linear ordinary differential equations with constant coefficients. Miller and Ross [8] employ the family of functions $D^{-\nu}e^{\alpha t}$ in a similar role in their study of linear fractional differential equations with constant coefficients. We develop fundamental properties for a family of functions

$$F(t,\nu,\alpha) = \Delta^{-\nu}(1+\alpha)^t,$$

where ν is any real number so that $\Gamma(\nu)$ is defined. Technically we should write

$$F(t,\nu,\alpha;a) = \Delta^{-\nu}((1+\alpha)^t;a),$$

but we continue the convention to suppress notational dependence on a.

Theorem 2.4. *Assume that the following factorial functions are defined:*

(*i*) $F(t, \nu, \alpha) = \alpha F(t, \nu + 1, \alpha) + \frac{(t-a)^{(\nu)}}{\Gamma(\nu+1)}$.

(*ii*) $\Delta_t F(t, \nu + 1, \alpha) = F(t, \nu, \alpha)$, *where* Δ_t *denotes the forward difference operator in* t.

(*iii*) $\Delta_t^p F(t, \nu + p, \alpha) = F(t, \nu, \alpha)$, *for* $p = 0, 1, 2, \ldots$.

(*iv*) $\Delta^\mu F(t, \nu, \alpha) = F(t, \nu - \mu, \alpha)$, *where* $p - 1 < \mu \le p$.

(*v*) $\Delta^{-\mu} F(t, \nu, \alpha) = F(t, \nu + \mu, \alpha)$.

(*vi*) $F(t - a, \nu, \alpha) = (1 + \alpha)^{-a} F(t, \nu, \alpha)$.

Proof. (*i*) Let $f(t) = (1 + \alpha)^t$. Then the formula (2.2) in Remark 2.1 implies that

$$\Delta^{-\nu}(1 + \alpha)^t = \Delta^{-\nu-1}\Delta(1 + \alpha)^t + \frac{(t-a)^{(\nu)}}{\Gamma(\nu+1)} = \Delta^{-\nu-1}\alpha(1 + \alpha)^t + \frac{(t-a)^{(\nu)}}{\Gamma(\nu+1)}.$$

Hence we have $F(t, \nu, \alpha) = \alpha F(t, \nu + 1, \alpha) + \frac{(t-a)^{(\nu)}}{\Gamma(\nu+1)}$.

(*ii*) Employ (2.1) with $f(t) = (1 + \alpha)^t$, and

$$\alpha F(t, \nu, \alpha) = \Delta F(t, \nu, \alpha) - \frac{(t-a)^{(\nu-1)}}{\Gamma(\nu)}.$$

Now, (*ii*) follows from (*i*).

(*iii*) Employ (2.3) and (2.4) with $f(t) = (1 + \alpha)^t$ to obtain the following property for $F(t, \nu, \alpha)$:

$$\Delta_t^p F(t, \nu + p, \alpha) = F(t, \nu, \alpha), \text{ for } p = 0, 1, 2, \ldots.$$

(*iv*) Apply the definition of the fractional difference operator, Theorem 1.1 and (*iii*), respectively:

$$\begin{aligned}
\Delta^\mu F(t, \nu, \alpha) &= \Delta^p \Delta^{-(p-\mu)} F(t, \nu, \alpha) \\
&= \Delta^p \Delta^{-(p-\mu)} \Delta^{-\nu}(1 + \alpha)^t = \Delta^p \Delta^{-(p-\mu+\nu)}(1 + \alpha)^t \\
&= \Delta^p F(t, p - \mu + \nu, t) = F(t, \nu - \mu, t).
\end{aligned}$$

(*v*) This is a consequence of Theorem 1.1.

(*vi*) This is a simple consequence of the linearity of the fractional sum and difference operators. $\square$

3. Existence result for an initial value problem

In this section, we introduce a nonlinear fractional difference equation with an initial condition and obtain the existence and uniqueness of a solution.

Consider the following nonlinear fractional difference equation with an initial condition:

(3.1) $\qquad \Delta^\nu y(t) = f(t + \nu - 1, y(t + \nu - 1)), \quad t = 0, 1, 2, \ldots,$

(3.2) $\qquad \Delta^{\nu-1} y(t)|_{t=0} = a_0,$

where $\nu \in (0, 1]$, f is a real-valued function, and a_0 is a real number. Note that the solution, $y(t)$, if it exists, is defined on $\mathbb{N}_{\nu-1}$.

First, we construct a summation equation that is equivalent to the IVP (3.1)–(3.2). Apply the $\Delta^{-\nu}$ operator to each side of (3.1) to obtain

(3.3) $\qquad \Delta^{-\nu}\Delta^\nu y(t) = \Delta^{-\nu} f(t + \nu - 1, y(t + \nu - 1)), \quad t = \nu, \nu + 1, \ldots.$

Apply Theorem 1.1 to the right-hand side of (3.3) to obtain

$$\Delta^{-\nu}\Delta^{\nu}y(t) = \Delta^{-\nu}\Delta\Delta^{-(1-\nu)}y(t) = \Delta\Delta^{-\nu}\Delta^{-(1-\nu)}y(t) - \frac{t^{(\nu-1)}}{\Gamma(\nu)}y(\nu-1)$$
$$= \Delta^{-\nu}f(t+\nu-1, y(t+\nu-1)).$$

So, for $t \in \mathbb{N}_{\nu-1}$ we have

$$(3.4) \qquad y(t) = \frac{t^{(\nu-1)}}{\Gamma(\nu)}a_0 + \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{(\nu-1)}f(s+\nu-1, y(s+\nu-1)).$$

The recursive iteration to this sum equation implies that (3.4) represents the unique solution of the IVP (3.1)–(3.2).

We close with three examples in which techniques are exhibited to solve discrete fractional equations with constant coefficients.

Example 3.1. Consider

$$(3.5) \qquad\qquad \Delta^{\nu}y(t) = \lambda y(t+\nu-1), \quad t = 0, 1, 2, \ldots,$$

$$(3.6) \qquad\qquad \Delta^{\nu-1}y(t)|_{t=0} = a_0.$$

Note that the solution is defined on $\mathbb{N}_{\nu-1}$ and $\Delta^{\nu-1}y(t)|_{t=0} = y(\nu-1)$ since $(-\nu)^{(-\nu)} = \Gamma(1-\nu)$. So the IVP (3.5)–(3.6) is equivalent to the IVP

$$\Delta^{\nu}y(t) = \lambda y(t+\nu-1), \quad t = 0, 1, 2, \ldots,$$
$$y(\nu-1) = a_0.$$

By (3.4), the solution of the IVP (3.5)–(3.6) is a solution of the summation equation

$$y(t) = \frac{t^{(\nu-1)}}{\Gamma(\nu)}a_0 + \frac{\lambda}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{(\nu-1)}y(s+\nu-1).$$

We employ the method of successive approximations. Set

$$y_0(t) = \frac{t^{(\nu-1)}}{\Gamma(\nu)}a_0,$$

$$y_m(t) = y_0(t) + \frac{\lambda}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{(\nu-1)}y_{m-1}(s+\nu-1)$$
$$= y_0(t) + \lambda\Delta^{-\nu}y_{m-1}(t+\nu-1),$$
$$m = 1, 2, \ldots.$$

Apply the power rule (Lemma 1.1) to show that

$$y_1(t) = y_0(t) + \lambda\Delta^{-\nu}y_0(t+\nu-1) = a_0\left(\frac{t^{(\nu-1)}}{\Gamma(\nu)} + \lambda\frac{(t+\nu-1)^{(2\nu-1)}}{\Gamma(2\nu)}\right).$$

With repeated applications of the power rule it follows inductively that

$$y_m(t) = a_0\sum_{i=0}^{m}\frac{\lambda^i}{\Gamma((i+1)\nu)}(t+(i-1)(\nu-1))^{(i\nu+\nu-1)},$$
$$m = 0, 1, 2, \ldots.$$

Formally, take the limit $m \to \infty$ to obtain

$$(3.7) \qquad y(t) = a_0 \sum_{i=0}^{\infty} \frac{\lambda^i}{\Gamma((i+1)\nu)} (t + (i-1)(\nu-1))^{(i\nu+\nu-1)}.$$

One immediate observation can be made. Set $\nu = 1$. Then $y(t) = a_0 \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} t^{(i)}$. Since the IVP (3.5)–(3.6) with $\nu = 1$ has the unique solution $a_0(1+\lambda)^t$, we obtain $(1+\lambda)^t = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} t^{(i)}$. This equality appears as a special case of [3, Lemma 4.4] for a time scale $\mathbb{T} = \mathbb{Z}$, the set of integers.

Example 3.2. Consider the following half-order difference equation with constant coefficients:

$$\Delta^{1/2} y(t) + a\Delta^0 y(t - 1/2) = 0, \quad t = 0, 1, 2, \ldots.$$

We shall employ a method of undetermined coefficients as motivated by Miller and Ross [8]. Assume that a solution has the form

$$y(t) = AF(t, 0, \lambda) + F(t, -1/2, \lambda).$$

Note the y is defined on $\mathbb{N}_{-1/2}$. Substitute $y(t)$ into the equation to obtain

$$(3.8) \quad \Delta^{1/2} y(t) + a\Delta^0 y(t - 1/2) = A\Delta^{1/2}(1+\lambda)^t + \Delta^{1/2}\Delta^{1/2}(1+\lambda)^t$$
$$+ aA(1+\lambda)^{-1/2}(1+\lambda)^t + a(1+\lambda)^{-1/2}\Delta^{1/2}(1+\lambda)^t.$$

Before we continue and simplify (3.8), we shall perform the following calculation for the sake of clarity. We shall show that $\Delta^{1/2}\Delta^{1/2}(1+\lambda)^t = \lambda(1+\lambda)^t$. Indeed, it follows from the definition of the fractional difference, Theorem 2.1 and the power rule that

$$\Delta^{1/2}\Delta^{1/2}(1+\lambda)^t = \Delta^{1/2}\Delta\Delta^{-1/2}(1+\lambda)^t$$

$$= \Delta^{1/2}\left(\Delta^{-1/2}\Delta(1+\lambda)^t + \frac{(t+1/2)^{(-1/2)}}{\Gamma(1/2)}(1+\lambda)^{-1/2}\right)$$

$$= \Delta\Delta^{-1/2}\Delta^{-1/2}\Delta(1+\lambda)^t + \Delta^{1/2}\left(\frac{(t+1/2)^{(-1/2)}}{\Gamma(1/2)}(1+\lambda)^{-1/2}\right)$$

$$= \lambda(1+\lambda)^t + \Delta\Delta^{-1/2}\left(\frac{(t+1/2)^{(-1/2)}}{\Gamma(1/2)}(1+\lambda)^{-1/2}\right)$$

$$= \lambda(1+\lambda)^t.$$

Now return to (3.8):

$$\Delta^{1/2} y(t) + a\Delta^0 y(t - 1/2) = \Delta^{1/2}(1+\lambda)^t \left(A + a(1+\lambda)^{-1/2}\right)$$

$$+ (1+\lambda)^t \left(\lambda + Aa(1+\lambda)^{-1/2}\right)$$

$$= (1+\lambda)^{-1/2} F(t, -1/2, \lambda) \left(A(1+\lambda)^{1/2} + a\right)$$

$$+ (1+\lambda)^{-1/2} F(t, 0, \lambda) \left(\lambda(1+\lambda)^{1/2} + Aa\right)$$

$$= (1+\lambda)^{-1/2} F(t, -1/2, \lambda) \left(A(1+\lambda)^{1/2} + a\right)$$

$$+ A(1+\lambda)^{-1/2} F(t, 0, \lambda) \left(A^{-1}\lambda(1+\lambda)^{1/2} + a\right).$$

Set $P(x) = \sqrt{x(1+x)} + a$. Set $\lambda = A^2$. Then (3.8) reduces to

$$\Delta^{1/2}y(t) + a\Delta^0 y(t - 1/2) = (1 + \lambda)^{-1/2} P(\lambda) \left(F(t, -1/2, \lambda) + \sqrt{\lambda} F(t, 0, \lambda) \right).$$

Thus, if λ is a root of $P(x) = 0$, then

$$y(t) = \sqrt{\lambda} F(t, 0, \lambda) + F(t, -1/2, \lambda)$$

is a solution of the discrete fractional equation for $t = -1/2, 1/2, 3/2, \ldots$. This completes the illustration of the method of a characteristic equation.

For the third solution method, we shall employ the R-transform method to solve the half-order difference equation with initial condition (3.1)–(3.2). For the sake of self-containment, we provide basic definitions and properties.

The discrete transform (R-transform) is defined by

$$R_{t_0}(f(t))(s) = \sum_{t=t_0}^{\infty} \left(\frac{1}{s+1} \right)^{t+1} f(t),$$

where f is defined on $\mathbb{N}_{t_0}$. The R-transform is the Laplace transform on the time scale of integers [5] (see also [4]), and it is not intended to be the more commonly employed z-transform.

Lemma 3.1 ([1]). *For $0 < \nu < 1$ and the function f defined for $\nu - 1, \nu, \nu + 1, \ldots$,*

$$R_0(\Delta^\nu f(t))(s) = s^\nu R_{\nu-1}(f(t)) - f(\nu - 1).$$

Lemma 3.2. *Let λ be any real number. Then*
(i) $R_{-1/2} F(t, -1/2, \lambda) = R_{-1/2} \Delta^{1/2} (1 + \lambda)^t = \frac{\sqrt{s}}{s - \lambda}$,
(ii) $R_{-1/2} F(t, 0, \lambda) = R_{-1/2} (1 + \lambda)^t = \frac{1}{\sqrt{\lambda+1}(s-\lambda)}$.

Example 3.3. Apply the R_0-transform to each side of

$$\Delta^{1/2} y(t) + a\Delta^0 y(t - 1/2) = 0, \quad t = 0, 1, 2, \ldots,$$

where a is any nonpositive real number. Then by Lemma 3.1, it follows that

$$\sqrt{s} R_{-1/2} y(t) - y(-1/2) + \frac{1}{\sqrt{s+1}} R_{-1/2} y(t) = 0.$$

In particular,

$$(3.9) \qquad R_{-1/2} y(t) = \frac{a_0 \sqrt{s+1}}{\sqrt{s(s+1)} + a}.$$

Set $\alpha = \sqrt{a^2 + 1/2^2} - 1/2$ and $\beta = 1/2 + \sqrt{a^2 + 1/2^2}$. Then

$$\frac{a_0 \sqrt{s+1}}{\sqrt{s(s+1)} + a} = \frac{a_0}{\alpha + \beta} \left(\frac{(1+\alpha)\sqrt{s}}{s - \alpha} - \frac{(1-\beta)\sqrt{s}}{s + \beta} - \frac{a\sqrt{s+1}}{s - \alpha} + \frac{a\sqrt{s+1}}{s + \beta} \right).$$

Apply Lemma 3.2 to each side of (3.9) (where of course we use the calculation above for the right-hand side of (3.9)) to obtain

$$y(t) = \frac{a_0}{\alpha + \beta} \left((\alpha + 1) F(t, -1/2, \alpha) - a\sqrt{\alpha + 1} F(t, 0, \alpha) \right)$$
$$+ \frac{a_0}{\alpha + \beta} \left((\beta - 1) F(t, -1/2, -\beta) + a\sqrt{1 - \beta} F(t, 0, -\beta) \right).$$

Note that $x = \beta$ satisfies the equation $\sqrt{x(x-1)} + a = 0$. So this implies that $\lambda_1 = \beta - 1$ and $\lambda_2 = -\beta$ are the roots of the characteristic equation $\sqrt{x(1+x)} + a = 0$

that we obtained in Example 3.2. Employing these observations and the relation $\alpha + 1 = \beta$, we can write the solution in the following form:

$$y(t) = \frac{a_0 \beta}{2\beta - 1} \left(F(t, -1/2, \beta - 1) + \sqrt{\beta - 1} F(t, 0, \beta - 1) \right)$$
$$+ \frac{a_0(\beta - 1)}{2\beta - 1} \left(F(t, -1/2, -\beta) + \sqrt{-\beta} F(t, 0, -\beta) \right).$$

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