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INJECTIVE COVERS OVER COMMUTATIVE NOETHERIAN RINGS WITH GLOBAL DIMENSION AT MOST TWO

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Abstract. In [3], Del Valle, Enochs and Martínez studied flat envelopes over rings and they showed that over rings as in the title these are very well behaved. If we replace flat with injective and envelope with the dual notion of a cover we then have the injective covers. In this article we show that these injective covers over the commutative noetherian rings with global dimension at most 2 have properties analogous to those of the flat envelopes over these rings.

1. Introduction

R will denote a commutative ring with identity. Let $\mathcal{X}$ be a class which is closed under isomorphism, direct summands, and finite direct sums. An $\mathcal{X}$-*envelope* of a module M is a linear map $\phi : M \to X$ with $X \in \mathcal{X}$ such that the following two conditions hold;

(1) $\mathrm{Hom}_R(X, X') \to \mathrm{Hom}_R(M, X') \to 0$ is exact for any $X' \in \mathcal{X}$.
(2) Any $f : X \to X$ with $f \circ \phi = \phi$ is an automorphism of X.

If $\phi : M \to X$ satisfies (1), and perhaps not (2), ϕ is called an $\mathcal{X}$-*preenvelope* of M. Dually, an $\mathcal{X}$-*precover* of M is a linear map $\psi : X \to M$ with $X \in \mathcal{X}$ such that $\mathrm{Hom}_R(X', X) \to \mathrm{Hom}_R(X', M) \to 0$ is exact for any $X' \in \mathcal{X}$ and if an $\mathcal{X}$-precover $\psi : X \to M$ of M satisfies the condition that any $f : X \to X$ with $\psi \circ f = \psi$ is an automorphism of X, then $\psi : X \to M$ is called an $\mathcal{X}$-*cover* of M.

Note that an $\mathcal{X}$-envelope (an $\mathcal{X}$-cover, respectively) of a module is unique up to isomorphism, if it exists. By convention, (pre)envelopes and (pre)covers are named according to the name of the class $\mathcal{X}$. For

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example, an injective cover is an $\mathcal{X}$-cover where $\mathcal{X}$ is the class of injective modules.

We recall that for a class $\mathcal{X}$ of R-modules, $\mathcal{X}^{\perp}$ consists of all R-modules K such that $\mathrm{Ext}^1_R(X, K) = 0$ for all $X \in \mathcal{X}$ and ${}^{\perp}\mathcal{X}$ consists of all R-modules G such that $\mathrm{Ext}^1_R(G, X) = 0$ for all $X \in \mathcal{X}$ (See [7, p.29]).

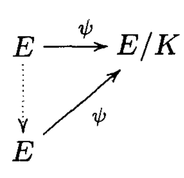
Wakamatsu's lemma ([7, Lemma 2.1.1] or [6]) says that if $\psi : X \to M$ is an $\mathcal{X}$-cover of an R-module M and if $\mathcal{X}$ is closed under extensions, then $\mathrm{Ker}\psi \in \mathcal{X}^{\perp}$. Conversely, if $\psi : X \to M$ is a surjection with $X \in \mathcal{X}$ and $\mathrm{Ker}\psi \in \mathcal{X}^{\perp}$, then for any $X' \in \mathcal{X}$, $\mathrm{Hom}_R(X', X) \to \mathrm{Hom}_R(X', M) \to \mathrm{Ext}^1_R(X', \mathrm{Ker}\psi) = 0$ is exact. So $\psi : X \to M$ is an $\mathcal{X}$-precover. Such a precover is called a *special $\mathcal{X}$-precover* of M.

In studying injective covers, the modules C such that $\mathrm{Hom}_R(E, C) = 0$ and $\mathrm{Ext}^1_R(E, C) = 0$ for all injective modules E play an important role (because of Wakamatsu's lemma). If the ring is $k[[x, y]]$ with k a field, this class contains all direct summands of products of modules of finite length (Theorem 2.9). One objective of this paper is to study modules C with $\mathrm{Hom}_R(E, C) = 0$ and $\mathrm{Ext}^1_R(E, C) = 0$ for all injective modules E over a noetherian ring R with global dimension at most 2.

2. Rings with global dimension at most 2

From results in [2, Proposition 8.1], we know that if $\phi : E \to M$ is an injective cover with kernel K, then $\mathrm{Hom}_R(\bar{E}, K) = 0$ and $\mathrm{Ext}^1_R(\bar{E}, K) = 0$ for all injective modules $\bar{E}$. Conversely, given such a module K, if $K \subset E$ is an injective envelope, then the natural map $\psi : E \to E/K$ is an injective precover. Moreover,

LEMMA 2.1. *The diagram*



can only be completed to a commutative diagram by id_E.

Proof. If f and g both complete the diagram

$$\begin{array}{ccc} E & \xrightarrow{\psi} & E/K \\ {\scriptstyle f}\vdots{\scriptstyle g} & \nearrow_{\psi} & \\ E & & \end{array}$$

then the map $f - g : E \to E$ has its image in K. But $\mathrm{Hom}(E, K) = 0$, so $f - g = 0$. Since $f = \mathrm{id}_E$ is one possibility, this is the only one. $\square$

DEFINITION 2.2. An $\mathcal{X}$-cover $\psi : X \to M$ for some class $\mathcal{X}$ will be called a *rigid $\mathcal{X}$-cover* if $\psi \circ f = \psi$ for $f : X \to X$ implies $f = \mathrm{id}_X$. A *rigid $\mathcal{X}$-envelope* is defined in an analogous manner.

LEMMA 2.3. *Let K be an R-module with $Hom_R(\bar{E}, K) = 0$ and $Ext^1_R(\bar{E}, K) = 0$ for all injective modules $\bar{E}$ and let $K \subset E$ be an injective envelope of K. Then E/K has no nonzero injective submodules.*

Proof. Let $\bar{E}$ be an injective submodule of E/K. Then $E/K = \bar{E} \oplus N$ for some N. Let $\psi : F \to N$ be an injective cover of N. Then $\sigma : F \oplus \bar{E} \to N \oplus \bar{E}$ is also an injective cover. Since the natural map $E \to E/K$ is a rigid injective cover by Lemma 2.1, so $F \oplus \bar{E} \cong E$. Moreover, $\mathrm{Ker}\sigma = \mathrm{Ker}\psi$, and so $K \subset F$. Since $F \cap \bar{E} = 0$, $K \cap \bar{E} = 0$. But $K \subset E$ is an injective envelope, and thus $\bar{E} = 0$. $\square$

DEFINITION 2.4. ([7]) Let $\mathcal{X}$ be a class of R-modules and let M be an R-module. Then an element $\xi \in \mathrm{Ext}^1_R(L, M)$ where $L \in \mathcal{X}$ is said to *generate* $\mathrm{Ext}^1_R(\mathcal{X}, M)$ (or ξ is a *generator* of $\mathrm{Ext}^1_R(\mathcal{X}, M)$) if for any $\bar{L} \in \mathcal{X}$ and $\bar{\xi} \in \mathrm{Ext}^1_R(\bar{L}, M)$, there is a linear map $f : \bar{L} \to L$ such that $\mathrm{Ext}^1_R(f, M)(\xi) = \bar{\xi}$.

Diagrammatically this says we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \bar{G} & \longrightarrow & \bar{L} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & M & \longrightarrow & G & \longrightarrow & L & \longrightarrow & 0, \end{array}$$

where the rows represent the extensions $\bar{\xi}$ and ξ. Moreover, $\xi \in \mathrm{Ext}^1_R(L, M)$ is called a *minimal generator* if it is a generator and if any $f \in \mathrm{Hom}_R(L, L)$ such that $\mathrm{Ext}^1_R(f, M)(\xi) = \xi$ is necessarily an automorphism of L. Then we see that if $\xi \in \mathrm{Ext}^1_R(L, M)$ and $\bar{\xi} \in \mathrm{Ext}^1_R(\bar{L}, M)$ are both minimal generators of $\mathrm{Ext}^1_R(\mathcal{X}, M)$, then any $f \in \mathrm{Hom}_R(\bar{L}, L)$ such that $\mathrm{Ext}^1_R(f, M)(\xi) = \bar{\xi}$ is an isomorphism.

If $\xi \in \mathrm{Ext}^1_R(L, M)$ is a generator of $\mathrm{Ext}^1_R(\mathcal{X}, M)$ and $\mathrm{Ext}^1_R(f, M)(\bar{\xi}) = \xi$ for $\bar{\xi} \in \mathrm{Ext}^1_R(\bar{L}, M)$ and $\bar{L} \in \mathcal{X}$, then $\bar{\xi}$ is also a generator.

THEOREM 2.5. *If R is a noetherian ring, then every R-module M has a generator of $\mathrm{Ext}^1_R(\mathcal{E}, M)$, where $\mathcal{E}$ is the class of injective R-modules.*

Proof. We will only sketch this argument (which is due to Naveed Zaman). Since R is noetherian, there is a representative set of indecomposable R-modules. Then for the given R-module M, we see that there is a family $0 \to M \to G_i \to E_i \to 0 (i \in I)$ of short exact sequences where E_i is indecomposable and injective such that any short exact sequence $0 \to M \to G \to E \to 0$ with E indecomposable and injective is isomorphic over M to one of the sequence $0 \to M \to G_i \to E_i \to 0$. Let $\bar{G}$ be the direct sum of the G_i amalgamated over M. Then we have an exact sequence $0 \to M \to \bar{G} \to \oplus E_i \to 0$. Now let $0 \to M \to H \to E \to 0$ be an exact sequence with E an arbitrary injective module. Then E is the direct sum of indecomposable injective modules. If E' is any indecomposable injective submodule of E, we have the short exact sequence $0 \to M \to H' \to E' \to 0$ which we derive from $0 \to M \to H \to E \to 0$ (H' is the preimage of E'). By construction there is a commutative diagram ;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & H' & \longrightarrow & E' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & \bar{G} & \longrightarrow & \oplus E_i & \longrightarrow & 0. \end{array}$$

But then using the fact that E is the direct sum of some set of such $E' \subset E$ we see that we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & H & \longrightarrow & E & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & \bar{G} & \longrightarrow & \oplus E_i & \longrightarrow & 0. \end{array}$$

Hence the exact sequence $0 \to M \to \bar{G} \to \oplus E_i \to 0$ is a generator of $\mathrm{Ext}^1_R(\mathcal{E}, M)$. □

PROPOSITION 2.6. *If $0 \to M \to K \to E \to 0$ is an exact sequence with E injective and $K \in \mathcal{E}^{\perp}$, where $\mathcal{E}$ is the class of injective R-modules, then $M \to K$ is an envelope for the class $\mathcal{E}^{\perp}$.*

Proof. Let $G \in \mathcal{E}^{\perp}$. Then $\mathrm{Hom}_R(K, G) \to \mathrm{Hom}_R(M, G) \to \mathrm{Ext}^1_R(E, G) = 0$ is exact. So $M \to K$ is an $\mathcal{E}^{\perp}$-preenvelope. Now if $0 \to M \to$

$\bar{K} \to \bar{E} \to 0$ is exact with $\bar{E} \in \mathcal{E}$, then $\mathrm{Hom}_R(\bar{K}, K) \to \mathrm{Hom}_R(M, K) \to \mathrm{Ext}^1_R(\bar{E}, K) = 0$ is exact, so the diagram

$$\begin{array}{ccc} M & \longrightarrow & \bar{K} \\ \| & & \vdots \\ M & \longrightarrow & K \end{array}$$

can be completed to a commutative diagram. This shows that $0 \to M \to K \to E \to 0$ is a generator of $\mathrm{Ext}^1_R(\mathcal{E}, M)$. Since $\mathcal{E}$ is closed under direct limits, $0 \to M \to K \to E \to 0$ is a minimal generator by [7, Theorem 2.2.2]. So the diagram

$$\begin{array}{ccc} M & \longrightarrow & K \\ \downarrow & \swarrow & \\ K & & \end{array}$$

can be completed only by automorphisms. Thus $M \to K$ is an $\mathcal{E}^{\perp}$-envelope. □

Note that if $0 \to M \to K \to K/M \to 0$ is an exact sequence with $M \to K$ an $\mathcal{E}^{\perp}$-envelope, then K/M is an injective R-module.

COROLLARY 2.7. *If R is a noetherian ring with gl.dim.$R \leq 2$, then every R-module has a rigid $\mathcal{E}^{\perp}$-envelope.*

Proof. By Theorem 2.5 and [7, Theorem 2.2.2], every R-module M has a minimal generator for $\mathrm{Ext}^1_R(\mathcal{E}, M)$ where $\mathcal{E}$ is the class of injective R-modules. So if $0 \to M \to K \to E \to 0$ is a minimal generator of $\mathrm{Ext}^1_R(\mathcal{E}, M)$, then $K \in \mathcal{E}^{\perp}$ by [7, Proposition 2.2.1]. Thus $M \to K$ is an $\mathcal{E}^{\perp}$-envelope by the proposition above. Now if f, g both complete the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & K & \longrightarrow & K/M & \longrightarrow & 0 \\ & & \| & & \vdots \, f \; g & & & & \\ & & M & \longrightarrow & K & & & & \end{array}$$

then $f - g = 0$ by the argument in the proof of Lemma 2.1. So $f = g$. □

When we say that $(R, \mathcal{M}, k)$ is a local ring, we mean that R is a local ring, $\mathcal{M}$ is the unique maximal ideal of R and k is the residue field of R. For any module M, the injective envelope of M is denoted by $E(M)$.

EXAMPLE 2.8. If $(R, \mathcal{M}, k)$ is a complete local ring of *depthR* ≥ 2, then by [1], $\mathrm{Ext}^1_R(k, R) = 0$. But $\hat{R} \cong E(k)$, $\hat{k} \cong k$ and $\mathrm{Ext}^1_R(k, R) \cong$

$\mathrm{Ext}^1_R(\hat{R},\hat{k})$ (this uses the fact that the elements of $\mathrm{Ext}^1_R(M,N)$ can be identified with equivalence classes of short exact sequences $0 \to N \to U \to M \to 0$). So for $P \in \mathrm{Spec}R$, if $P \neq \mathcal{M}$, let $t \notin P$ and $t \in \mathcal{M}$. Then multiplication by t on $E(R/P)$ is an isomorphism and is zero on k. From this we can get $\mathrm{Hom}_R(E(R/P),k) = 0$ and $\mathrm{Ext}^1_R(E(R/P),k) = 0$. If $P = \mathcal{M}$, the Matlis duality gives $\mathrm{Hom}_R(E(k),k) = 0$ and $\mathrm{Ext}^1_R(E(k),k) = 0$. This is because $\mathrm{Hom}_R(k,\hat{R}) = \mathrm{Ext}^1_R(k,\hat{R}) = 0$. For example, if $R = k[[x,y]]$, then $k \cong \dfrac{k[[x,y]]}{(x,y)}$ has the above properties.

The next theorem shows that there is a plentiful supply of modules C over $R = k[[x,y]]$ with $\mathrm{Hom}_R(E,C) = 0$ and $\mathrm{Ext}^1_R(E,C) = 0$ whenever E is an injective R-module.

THEOREM 2.9. *If a module C over $R = k[[x,y]]$ with k a field is a direct summand of a product of modules of finite length, then C has the property that $Hom_R(E,C) = 0$ and $Ext^1_R(E,C) = 0$ for all injective R-modules E.*

Proof. If we use induction on the length of C, it is not hard to argue that any module C of finite length over R has the desired properties. And we easily see that the class of modules C with these properties is closed under products and summands. □

DEFINITION 2.10. ([2]) Let R be a noetherian ring and M an R-module. The complex $\cdots \to E_1 \to E_0 \to M \to 0$ with the requirement that $E_0 \to M$, $E_1 \to \mathrm{Ker}(E_0 \to M)$ and $E_{n+1} \to \mathrm{Ker}(E_n \to E_{n-1})$ for $n \geq 1$ are injective precovers is called an *injective resolvent* of M. If the maps $E_{n+1} \to \mathrm{Ker}(E_n \to E_{n-1})$ are furthermore injective covers, then we call our sequence a *minimal injective resolvent.*

Note that the minimal injective resolvent is unique up to isomorphism and finite sums of minimal resolvents are minimal resolvents.

For arbitrary class $\mathcal{X}$, it is not in general true that the product of $\mathcal{X}$-covers is an $\mathcal{X}$-cover (even if $\mathcal{X}$ is closed under products). The next result shows this is the case in our situation.

THEOREM 2.11. *Let R be a noetherian ring with $gl.dim.R \leq 2$.*

(1) *If $\phi_i : E_i \to M_i$ are injective covers for $i \in I$, then $\prod \phi_i : \prod E_i \to \prod M_i$ is also an injective cover.*
(2) *If $\phi_i : E_i \to M_i$ are injective covers for $i = 1, 2, \cdots, n$, then so is $\oplus_{i=1}^n \phi_i : \oplus_{i=1}^n E_i \to \oplus_{i=1}^n M_i$.*

Proof. (1) Since $\prod E_i$ is injective, $\prod \phi_i : \prod E_i \to \prod M_i$ is an injective precover by [7, Theorem 1.2.9]. From [2, Proposition 8.1], we see that if R is noetherian and $gl.dim.R \leq 2$, then the minimal resolvent of any R-module M is of the form

$$0 \longrightarrow 0 \longrightarrow 0 \longrightarrow E \xrightarrow{\phi} M \longrightarrow 0\,.$$

But this means that $\mathrm{Hom}_R(E', \mathrm{Ker}\phi) = 0$ for any injective module E'. To get $\prod \phi_i : \prod E_i \to \prod M_i$ is an injective cover, we need only show that if $C_i = \mathrm{Ker}\phi_i$ for each $i \in I$, then $\prod C_i$ has no nonzero injective submodules. Now if $\bar{E}$ is an injective submodule of $\prod C_i$, then $\mathrm{Hom}_R(\bar{E}, \prod C_i) \cong \prod \mathrm{Hom}_R(\bar{E}, C_i) = 0$. So $\bar{E} = 0$.

(2) This follows from [2, Proposition 4.1] when we take $E_m = 0$ for $m > n$. □

THEOREM 2.12. *If $(R, \mathcal{M}, k)$ is a Gorenstein local ring of Krull dimension ≥ 2, then $\mathrm{Ext}^1_R(E(k), R) = 0$.*

Proof. From [1], we know that if $0 \to R \to E^0(R) \to \cdots \to E^h(R) \to \cdots$ is a minimal injective resolution of R, then for each $h \geq 0$,

$$E^h(R) = \oplus_{\mathrm{ht}P=h} E(R/P).$$

To get $\mathrm{Ext}^1_R(E(k), R) = 0$ we only need to argue that $\mathrm{Hom}_R(E(k), E(R/P)) = 0$ when $\mathrm{ht}P = 1$ by [5]. If $\gamma \in \mathcal{M}$ and $\gamma \notin P$, then the multiplication $E(R/P) \to E(R/P)$ by γ is an isomorphism. But for $x \in E(R/P)$, $\gamma^n x = 0$ for some $n \geq 1$. If $f : E(k) \to E(R/P)$ is linear, then $0 = f(\gamma^n x) = \gamma^n f(x)$. This implies $f(x) = 0$ by the above. So we get $\mathrm{Ext}^1_R(E(k), R) = 0$. □

3. An equivalence of module categories

In this section we assume R is a commutative noetherian ring with global dimension at most 2. We find an equivalence between the category of modules C we have been considering and another category of modules.

Let $\mathcal{C}$ be the category of R-modules C such that $\mathrm{Hom}_R(E, C) = 0$ and $\mathrm{Ext}^1_R(E, C) = 0$ for all injective modules E and let $\mathcal{D}$ be the category of R-modules D such that D is isomorphic to a quotient of an injective R-module (or equivalently the injective cover $E \to D$ is surjective) and D has no nonzero injective submodules.

We can define a functor $G : \mathcal{D} \to \mathcal{C}$ as follows: for each $D \in \mathrm{obj}(\mathcal{D})$, define $G(D) =$ the kernel of the injective cover of D. Then $\mathrm{Hom}_R(E, G(D)) = 0$ and $\mathrm{Ext}^1_R(E, G(D)) = 0$ for all injective R-modules

E. So $G(D) \in \mathrm{obj}(\mathcal{C})$. For the given $\mathcal{D}$-morphism $D_1 \to D_2$ with the injective covers $E_1 \to D_1$ and $E_2 \to D_2$, we can complete uniquely to the following commutative diagram

$$\begin{array}{ccc} E_1 & \longrightarrow & D_1 \\ \downarrow & & \downarrow \\ E_2 & \longrightarrow & D_2. \end{array}$$

So from this way we get a well-defined functor $G : \mathcal{D} \to \mathcal{C}$. Also, it is easy to see that G is an additive functor.

REMARK 3.1. (1) Since an injective cover of D is unique up to isomorphism, if we make a different choice of injective cover $E' \to D$ for each D and get another functor $G' : \mathcal{D} \to \mathcal{C}$, then it is not hard to see that G and G' would be naturally isomorphic functors.

(2) It is easy to see that D is an object $\mathcal{D}$ if and only if its injective cover $E \to D$ is surjective and if $\mathrm{Ker}(E \to D) \subset E$ is an injective envelope.

We know from Lemma 2.1 and Lemma 2.3 that for each $C \in \mathrm{obj}(\mathcal{C})$, $E(C)/C$ has no nonzero injective submodules and $E(C) \to E(C)/C$ is the injective cover. If $f : C_1 \to C_2$ is a $\mathcal{C}$-morphism, then there is a linear $\bar{f} : E(C_1) \to E(C_2)$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_1 & \longrightarrow & E(C_1) & \xrightarrow{\alpha} & E(C_1)/C_1 & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & C_2 & \longrightarrow & E(C_2) & \xrightarrow[\beta]{} & E(C_2)/C_2 & \longrightarrow & 0. \end{array}$$

So if $f \in \mathrm{Hom}_{\mathcal{C}}(G(D_1), G(D_2))$, then there exists an extension $\bar{f} : E(G(D_1)) \to E(G(D_2))$ such that $\bar{f}|_{G(D_1)} = f$. Thus there is a map $g : E(G(D_1))/G(D_1) \to E(G(D_2))/G(D_2)$ such that $g \circ \alpha = \beta \circ \bar{f}$. Hence we get the following proposition.

PROPOSITION 3.2. *$Hom_{\mathcal{D}}(D_1, D_2) \overset{\Delta}{\to} Hom_{\mathcal{C}}(G(D_1), G(D_2))$ is always bijective.*

Proof. We only need to ensure that Δ is injective. From the following diagram ;

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G(D_1) & \longrightarrow & E_1 & \longrightarrow & D_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G(D_2) & \longrightarrow & E_2 & \longrightarrow & D_2 & \longrightarrow & 0 \end{array}$$

Ker $\Delta = \{f \in \mathrm{Hom}_{\mathcal{D}}(D_1, D_2) |\, G(f) \cong 0\}$
$= \{f \in \mathrm{Hom}_{\mathcal{D}}(D_1, D_2) |\, G(f)$ can be factored through an injective module $E_1\}$ by [4, 13.2]
$= \{f \in \mathrm{Hom}_{\mathcal{D}}(D_1, D_2) |\, f$ can be factored through an injective module $E_2\}$ by [4, 13.8]
$= \{f \in \mathrm{Hom}_{\mathcal{D}}(D_1, D_2) |\, f \cong 0\}$
$= [0]$. □

REMARK 3.3. (1) We know that the functor G "preserves products", that is, $G(\prod D_i) \cong \prod G(D_i)$ naturally. It is probably not true that $G(\oplus D_i) \cong \oplus G(D_i)$ in general.

(2) If $gl.dim.R \leq 1$ (so then $gl.dim.R \leq 2$), then every injective cover is just of the form $E \hookrightarrow M$, where E is the largest injective submodule of M. So $\mathcal{D} = 0$ and $\mathcal{C} = 0$.

References

[1] H. Bass, *On the ubiquity of Gorenstein rings*, Math. Z. **82** (1963), 8–28.

[2] E. Enochs, *Injective and flat covers, envelopes, and resolvents*, Israel J. Math. **39** (1981), 189–209.

[3] E. Enochs, J. Martínez Hernández and A. Del Valle, *Coherent rings of finite weak global dimension*, Proc. Amer. Math. Soc. **126** (1998), 1611–1620.

[4] P. Hilton, *Homotopy theory and duality*, Gordon and Breach, New York, 1965.

[5] E. Matlis, *Injective modules over Noetherian rings*, Pacific J. Math. **8** (1958), 511–528.

[6] T. Wakamatsu, *Stable equivalence of self-injective algebras and a generalization of tilting modules*, J. Algebra **134** (1990), 298–325.

[7] J. Xu, *Flat Covers of Modules*, Lecture Notes in Math. **1634**, Springer-Verlag, Berlin, 1996.

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