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Inner and Outer j-Radii of Convex Bodies in Finite-Dimensional Normed Spaces*

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Abstract. This paper is concerned with the various inner and outer radii of a convex body C in a d-dimensional normed space. The inner j-radius $r_j(C)$ is the radius of a largest j-ball contained in C, and the outer j-radius $R_j(C)$ measures how well C can be approximated, in a minimax sense, by a $(d-j)$-flat. In particular, $r_d(C)$ and $R_d(C)$ are the usual inradius and circumradius of C, while $2r_1(C)$ and $2R_1(C)$ are C's diameter and width.

Motivation for the computation of polytope radii has arisen from problems in computer science and mathematical programming. The radii of polytopes are studied in [GK1] and [GK2] from the viewpoint of the theory of computational complexity. This present paper establishes the basic geometric and algebraic properties of radii that are needed in that study.

Introduction

Throughout this paper, $\mathbb{M}$ denotes a *Minkowski space*—a normed finite-dimensional vector space over the real field $\mathbb{R}$. The dimension of $\mathbb{M}$ is denoted by d and the norm by $\| \ \|$. The *unit ball* and *unit sphere* of $\mathbb{M}$ are the sets $\mathbb{B} = \{x\colon \|x\| \le 1\}$ and $\mathbb{S} = \{x\colon \|x\| = 1\}$, respectively. As the terms are used here, a *body* in $\mathbb{M}$ is a d-dimensional compact convex set, and a *polytope* is a body that is the convex

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hull of a finite set of points or, equivalently, is the intersection of a finite collection of closed half-spaces.

Prefixes often indicate dimension. For example, the j-dimensional linear (resp. affine) subspaces of $\mathbb{M}$ are called *j-subspaces* (resp. *j-flats*), and a *j-ball of radius* ρ in $\mathbb{M}$ is a set of the form

$$(q + \rho\mathbb{B}) \cap F = \{x \in F: \|x - q\| \leq \rho\}$$

for some j-flat F in $\mathbb{M}$ and point $q \in F$.

For $1 \leq j \leq d$, the *inner j-radius* of a body $C \subset \mathbb{M}$ is the supremum $r_j(C)$ of the radii of the j-balls contained in C. By a routine compactness argument, this supremum is attained as a maximum. Hence there exist a j-flat $F_0 \subset \mathbb{M}$ and a point $q_0 \in F_0$ such that

$$(q_0 + r_j(C)\mathbb{B}) \cap \mathbb{F}_0 \subset C;$$

and $r_j(C)$ is the largest number that has this property. The number $r_j(X)$ can of course be defined in the same way for an arbitrary compact $X \subset \mathbb{M}$, but our interest here is confined to the case in which X is a body.

For $1 \leq j \leq d$, the *outer j-radius* of a body $C \subset \mathbb{M}$ is a number $R_j(C)$ that measures how well C can be approximated, in a minimax sense, by a $(d - j)$-flat. Specifically, $R_j(C)$ is the infimum of the positive numbers ρ such that $\mathbb{M}$ contains a $(d - j)$-flat F for which $C \subset F + \rho\mathbb{B}$. By a routine compactness argument, this infimum is attained as a minimum. Hence $\mathbb{M}$ contains a $(d - j)$-flat F_0 such that, for each $c \in C$, there exists $x \in F_0$ with $\|c - x\| \leq R_j(C)$; and $R_j(C)$ is the smallest number that has this property. The number $R_j(X)$ is defined in the same way for an arbitrary compact $X \subset \mathbb{M}$, and it is clear that $R_j(X) = R_j(\operatorname{conv} X)$.) (For definitions of some other notions of inner and outer radii, equivalent to these when $\mathbb{M}$ is Euclidean but not in general, see Section 2.)

The numbers $r_d(C)$ and $R_d(C)$ are respectively the radius of a largest d-ball contained in C and of a smallest d-ball containing C. They are called, respectively, the *inradius* and the *circumradius* of C, and the center of each such ball is called an *incenter* (resp. a *circumcenter*) of C. (When the unit ball $\mathbb{B}$ is rotund, each body has a unique circumcenter but may have many incenters. When there are segments in the unit sphere $\mathbb{S}$, circumcenters are also in general not unique.) The number $2r_1(C)$ is the *diameter* of C—the maximum distance that is realized between two points of C, and $2R_1(C)$ is the *width* of C—the smallest of the distances between pairs of parallel supporting hyperplanes of C. For each body C in a d-dimensional Minkowski space $\mathbb{M}$, it is true that

$$r_1 \geq r_2 \geq \cdots \geq r_d, \qquad R_1 \leq R_2 \leq \cdots \leq R_d,$$

$$r_1 \geq R_1, \qquad r_d \leq R_d.$$

For bodies C that are *symmetric* (meaning here that C is a translate of a body K for which $K = -K$), it is also true that $r_1(C) = R_d(C)$ and $r_d(C) = R_1(C)$ (see (1.3)).

Theorems involving inradius, circumradius, diameter, and width have long been standard fare among workers in the geometry of convex bodies [Bl], [BF], [Eg1], [DGK], and approximation problems involving all the outer j-radii have been studied in [Ec] in connection with problems of Gallai type about common transversals for collections of convex bodies. Workers in approximation theory have studied the outer j-radii in various function spaces [T1], [Br], but have usually focused instead on the closely related numbers involving approximation by $(d-j)$-subspaces rather than $(d-j)$-flats [Ko], [T2], [Lo], [Si], [Pi]. (The numbers are the same for bodies that are symmetric about the origin, but for our purposes asymmetric bodies are also of great interest.) Approximation theorists have used the terms "diameter" and "width," but we prefer "radius" because this is consistent with standard usage in the geometry of convex bodies and also seems to be a more accurate description of the numbers that interest us.

Certain pairs of inner and outer radii—particularly the pairs (r_1, R_1) and (r_d, R_d)—work together in some applications. It is in any case natural to consider the inner and outer radii together, because they are dual to each other in the sense that if $C = -C$ and C^0 is the polar of C (situated in $\mathbb{M}$'s conjugate space $\mathbb{M}^*$), then $r_j(C)R_j(C^0) = 1$ (see (1.2)).

Approximation theorists have computed the radii of particular symmetric bodies of special interest, and have estimated the radii when precise computation proved to be too difficult (see [Pi] and references therein). This paper is part of the preparation for a study [GK1], [GK2] that has a different focus, related to the fact that motivation for the computation of certain polytope radii has arisen recently from problems in computer graphics, pattern recognition, robotics, the sensitivity analysis of linear programming, and nonlinear global optimization. That study is concerned with the intrinsic complexity of computing (or approximating or bounding) the various inner and outer radii of polytopes in finite-dimensional spaces with ℓ_p norms or polytopal norms. These fundamental problems in computational convexity are approached in [GK1] and [GK2] from the viewpoint of the theory of computational complexity. However, the analysis there depends on a large number of basic geometric properties of radii along with a few algebraic properties, and these are of interest that extends beyond questions of computational complexity. The purpose of this paper is to present those geometric and algebraic properties.

Our section headings are as follows: 1. Basic geometric properties of radii; 2. Additional geometric properties; 3. Rationality of certain polytope radii; 4. Difficulties in computing radii.

In [GK1], essential use is made of all of the results in Sections 1 and 3 and of a key result in Section 4. The material in Section 2 is not used in [GK1], but is included in order to complete the picture of basic geometric properties of radii.

1. Basic Geometric Properties of Radii

With each Minkowski space $\mathbb{M}$ there is an associated *conjugate space* $\mathbb{M}^*$. The points of $\mathbb{M}^*$ are the linear functionals on $\mathbb{M}$, and the norm of a functional is the maximum of its values on the unit ball $\mathbb{B}$ of $\mathbb{M}$. The norm, unit ball, and unit sphere of $\mathbb{M}^*$ are denoted by $\| \ \|^*$, $\mathbb{B}^*$, and $\mathbb{S}^*$, respectively. As is customary, the norms in both $\mathbb{M}$ and $\mathbb{M}^*$ are denoted by $\| \ \|$ when there is no danger of confusion. The usual bilinear form on $\mathbb{M} \times \mathbb{M}^*$ is denoted by $\langle \, , \, \rangle$, so that, for $x \in \mathbb{M}$ and $y \in M^*$, $\langle x, y \rangle$ denotes the value of the functional y at the point x.

For $X \subset \mathbb{M}$, the *polar* of X is the set $X^0 \subset \mathbb{M}^*$ given by

$$X^0 = \{ y \in \mathbb{M}^* \colon \langle x, y \rangle \leq 1 \text{ for all } x \in X \}.$$

Polars are defined in the same way for subsets of $\mathbb{M}^*$, and by identifying $(\mathbb{M}^*)^*$ with $\mathbb{M}$ in the usual way, these polars are regarded as subsets of $\mathbb{M}$. We use the well-known fact that if the origin is interior to a body $C \subset \mathbb{M}$, then the polar C^0 is a body in $\mathbb{M}^*$, with $(C^0)^0 = C$; if, in addition, C is a polytope, then so is C^0.

Some of the results collected in this section are old and some are new. However, the proofs are all easy so it seems simpler and more useful to include them all here rather than attempt to sort out the details in the literature. These results are all used in [GK1].

(1.1) A Property of Symmetric Bodies. *If the body $C \subset \mathbb{M}$ is symmetric about the origin* (*so that* $-C = C$), *and* $1 \leq j \leq d$, *then*:

(a) *C contains a j-ball of radius $r_j(C)$ centered at the origin*;
(b) *there is a $(d-j)$-subspace F of $\mathbb{M}$ such that $C \subset F + R_j(C)B$.*

Proof. By the definition of $r_j(C)$, there exist a point $q \in C$ and a j-subspace E of $\mathbb{M}$ such that C contains the j-ball $G = q + r_j(C)(\mathbb{B} \cap \mathbb{E})$. By symmetry, C also contains the j-ball $-G$, and then by convexity C contains the j-ball $K = r_j(C)(\mathbb{B} \cap \mathbb{E})$.

By the definition of $R_j(C)$, there exist a $(d-j)$-subspace F and a point q such that

$$C \subset (F + q) + R_j(C)\mathbb{B},$$

whence, by symmetry,

$$C \subset -(F + q) - R_j(C)\mathbb{B} = (F - q) + R_j(C)\mathbb{B},$$

and, by convexity,

$$C \subset \tfrac{1}{2}((F + q) + R_j(C)\mathbb{B}) + \tfrac{1}{2}((F - q) + R_j(C)\mathbb{B}) = F + R_j(C)\mathbb{B}. \qquad \square$$

With respect to radii, the most widely useful property of symmetric bodies is the one stated next.

(1.2) Duality between Inner and Outer Radii of Symmetric Bodies. *If the body $C \subset \mathbb{M}$ is symmetric about the origin, and $1 \leq j \leq d$, then*

$$r_j(C)R_j(C^0) = 1 \quad \textit{and} \quad R_j(C)r_j(C^0) = 1.$$

Proof. By (1.1) there is a j-dimensional linear subspace E of $\mathbb{M}$ such that

$$r_j(C)(\mathbb{B} \cap E) = K \subset C.$$

Using well-known properties of polarity and convexity, it then follows that $C^0 \subset K^0$ and, since $K = (r_j(C)\mathbb{B}) \cap E$, that

$$K^0 = \text{cl conv}((r_j(C)\mathbb{B})^0 \cup E^0) = (r_j(C)\mathbb{B})^0 + E^0.$$

However,

$$(r_j(C)\mathbb{B})^0 = (r_j(C))^{-1}\mathbb{B}^0 = (r_j(C))^{-1}\mathbb{B}^*$$

and E^0 is a $(d-j)$-subspace of $\mathbb{M}^*$, so it follows that $R_j(C^0) \leq (r_j(C))^{-1}$. The construction can be reversed to show that $r_j(C) \geq (R_j(C^0))^{-1}$, thus establishing the first statement of (1.2). The second statement is merely the dual equivalent of the first. □

(1.3) Equality of Certain Radii of Symmetric Bodies. *For each body C,*

$$r_1(C) \leq R_d(C) \quad \textit{and} \quad r_d(C) \leq R_1(C),$$

with equality in each case when C is symmetric.

Proof. It follows immediately from the triangle inequality that $r_1(C) \leq R_d(C)$. To see that $r_d(C) \leq R_1(C)$, use the fact that a d-ball of radius ρ is of width 2ρ.

Now let us assume that $C = -C$. By (1.1) there exists a segment T centered at 0 such that $T \subset C$ and T is of length $2r_1(C)$. Of course, $T \subset R_d(C)\mathbb{B}$, and it follows that $r_1(C) \leq R_d(C)$. Since the sphere $R_d(C)\mathbb{S}$ must include a point c of C, and since $-c \in C$ by symmetry, it is clear that $r_1(C) \geq R_d(C)$. Hence $r_1(C) = R_d(C)$. The second equality in (1.3) follows from this, with the aid of (1.2). □

For a body $C \subset \mathbb{M}$, and for unit vectors $s \in \mathbb{S}$ and $s^* \in \mathbb{S}^*$, the *s-length* $l_s(C)$ and the *s^*-breadth* $b_{s^*}(C)$ are defined as follows:

$l_s(C)$ is the length of the longest segment in C that is parallel to the line $\mathbb{R}s$;

$$b_{s^*}(C) = \max_{c \in C}\langle c, s^*\rangle - \min_{c \in C}\langle c, s^*\rangle.$$

(1.4) Lengths and Breadths under Symmetrization. *If C is a body in $\mathbb{M}$, and $K = \frac{1}{2}(C - C)$, then, for each $s \in \mathbb{S}$ and $s^* \in \mathbb{S}^*$, it is true that*

$$l_s(C) = l_s(K) \quad \textit{and} \quad b_{s^*}(C) = b_{s^*}(K).$$

Proof. The length $l_s(C)$ is merely the maximum of $\|x\|$ for points $x \in \mathbb{R}s$ of the form $c - c'$ with $c, c' \in C$. Hence the fact that $l_s(C) = l_s(K)$ follows from the observation that

$$\begin{aligned} C - C &= (\tfrac{1}{2}C + \tfrac{1}{2}C) - (\tfrac{1}{2}C + \tfrac{1}{2}C) \\ &= (\tfrac{1}{2}C - \tfrac{1}{2}C) - (\tfrac{1}{2}C - \tfrac{1}{2}C) = K - K. \end{aligned}$$

To establish the equality for breadths, note that

$$\begin{aligned} b_{s^*}(C) &= \max_{c, c' \in C} \langle c - c', s^* \rangle = \max_{q \in C - C} \langle q, s^* \rangle = 2 \max_{r \in (C - C)/2} \langle r, s^* \rangle \\ &= \max_{r \in (C-C)/2} \langle r, s^* \rangle - \min_{r \in (C-C)/2} \langle r, s^* \rangle = b_{s^*}(K). \end{aligned}$$ □

The body $C - C$ is generally called the *difference body* of C.

For each body C, determining any specific radius of C requires solving an optimization problem of special structure. The following description leads to polynomial-time algorithms under some circumstances (see [GK1] and [GK2]).

(1.5) Width and Diameter as Length and Breadth. *The following equalities hold for each body $C \subset \mathbb{M}$:*

$$2r_1(C) = \max_{s \in \mathbb{S}} l_s(C) = \max_{s^* \in \mathbb{S}^*} b_{s^*}(C);$$

$$2R_1(C) = \min_{s \in \mathbb{S}} l_s(C) = \min_{s^* \in \mathbb{S}^*} b_{s^*}(C).$$

Proof. Let us denote the minimum and maximum of the above lengths (resp. breadths) by $\lambda_{\min}$ and $\lambda_{\max}$ (resp. $\beta_{\min}$ and $\beta_{\max}$). It follows almost immediately from the relevant definitions that $2r_1(C) = \lambda_{\max}$ and $2R_1(C) = \beta_{\min}$. The proof will be completed by showing that $\lambda_{\max} = \beta_{\max}$ and $\lambda_{\min} = \beta_{\min}$. In proving these equalities we may assume, in view of (1.4), that $C = -C$.

In showing that $\lambda_{\max} = \beta_{\max}$, we assume further (for notational simplicity, and without loss of generality) that $\lambda_{\max} = 2$, from which it follows that $C \subset \mathbb{B}$ and that the boundary bd(C) includes two points q and $-q$ of $\mathbb{S}$. By the usual support theorem, there exists $s^* \in \mathbb{S}^*$ such that $\langle q, s^* \rangle = 1$. It is then apparent that $\beta_{\max} = (\text{width of } \mathbb{B}) = 2$.

The equality, $\lambda_{\min} = \beta_{\min}$, can be proved by an argument similar to the preceding one, after normalizing by assuming that $\beta_{\min} = 2$ and hence $\mathbb{B} \subset C$. Alternatively, with the aid of polarity and symmetry, the equality, $\lambda_{\min} = \beta_{\min}$, can be shown to be equivalent to the equality, $\lambda_{\max} = \beta_{\max}$. □

(1.6) Width and Diameter in Terms of Translates. *For each body $C \subset \mathbb{M}$, the diameter $2r_1(C)$ (resp. width $2R_1(C)$) is the largest number τ such that $(C + t) \cap C \neq \varnothing$ for some (resp. for each) $t \in \mathbb{M}$ with $\|t\| \leq \tau$.*

Proof. This is an immediate consequence of (1.5)'s description of width and diameter in terms of lengths. □

A boundary point q of a body C is *extreme* if it is not an inner point of any segment in C, *exposed* if C is supported at q by a hyperplane that intersects C only at q, and *smooth* if the hyperplane supporting C at q is unique. A body is *rotund* (resp. *smooth*) if each of its boundary points is extreme (resp. smooth). A pair of points of a set X is *diametral* in X if the distance between the points is equal to the diameter of X.

(1.7) Diameter Attained at Extreme Points. *Suppose that C is a body in the Minkowski space $\mathbb{M}$. Then there is a diametral pair $\{v, w\}$ that consists of extreme points of C, and if $C = -C$ it may be chosen so that $w = -v$. When the unit ball $\mathbb{B}$ of $\mathbb{M}$ is rotund, each diametral pair in C consists of exposed points of C, and if $C = -C$, then each point of a diametral pair is the negative of the other.*

Proof. For each point $c \in C$, define the function φ_c on C as follows:

$$\varphi_c(x) = \|x - c\| \qquad (x \in C).$$

The function φ_c is convex and continuous, and its domain is convex and compact. From this it follows that the maximum of φ on C is attained at an extreme point of C [Ba]. If $\mathbb{M}$'s unit ball $\mathbb{B}$ is rotund, then each boundary point of $\mathbb{B}$ is an exposed point of $\mathbb{B}$, and this implies that each point of C at which φ_c attains a maximum is an exposed point of C.

Now let t and u be points of C such that the distance $\|t - u\|$ is equal to the diameter $\delta = 2r_1(C)$ of C, and let w be an extreme point of C at which the function φ_t attains a maximum. Of course, $\|t - w\| = \delta$. Then let v be an extreme point of C at which the function φ_w attains a maximum, so that $\|v - w\| = \delta$.

Now suppose that $C = -C$, whence $-v$ and $-w$ are also extreme points of C. Since

$$\delta = \|v - w\| \leq \|v\| + \|w\|,$$

it is true that $\|v\| \geq \delta/2$ or $\|w\| \geq \delta/2$. Hence at least one of $\{v, -v\}$ and $\{w, -w\}$ is a pair of antipodal extreme points of C whose distance is equal to C's diameter. When the ball is rotund, these inequalities can be strengthened to show that the diameter of C is attained *only* by pairs of antipodal exposed points. □

(1.8) Width of Symmetric Bodies Attained at Smooth Points. *Suppose that C is a body such that $C = -C$ and such that each extreme point of the polar C^0 is exposed. Then C admits parallel supporting hyperplanes H_- and H_+ such that:*

(a) *the distance between H_- and H_+ is equal to the width of C;*
(b) *there are antipodal smooth points $-q$ and q of C such that $-q \in H_-$ and $q \in H_+$.*

If the ball $\mathbb{B}$ is smooth, every pair of hyperplanes that satisfies (a) *must satisfy* (b) *as well.*

Proof. By (1.7) there is an extreme point v of C^0 such that $\|v\|^* = r_1(C^0)$, and by hypothesis v is in fact an exposed point of C^0. Hence there is a boundary point q of C such that $\langle q, v\rangle = 1$ and $\langle q, y\rangle < 1$ for all $y \in C^0 \backslash \{v\}$. This implies that q is a smooth point of C, and then the desired conclusion follows with the aid of (1.5) and (1.2). □

In (1.8) the requirement concerning C's polar cannot be abandoned. To see this, let the unit ball B and the body C be concentric plane lenses with common apices, and let C be fatter than B. Then the nonsmooth apices are the only antipodal points of C that belong to parallel supporting lines $H_\pm$ satisfying condition (a). (To be more specific, suppose that $0 < \gamma < \beta$, and let the unit ball B and the body C in $\mathbb{R}^2$ be defined as follows:

$$B = \{(\xi, \eta) : \beta|\xi| \le 1 - \xi^2 - \eta^2\}, \qquad C = \{(\xi, \eta) : \gamma|\xi| \le 1 - \xi^2 - \eta^2\}.)$$

(1.9) Width of Polytopes. *Suppose that P is a polytope, and that H_- and H_+ are parallel supporting hyperplanes of P whose distance is equal to the width of P. Let $P_- = P \cap H_-$ and $P_+ = P \cap H_+$. Then*

$$\dim P_- + \dim P_+ \ge d - 1,$$

with $\dim P_- = \dim P_+ = d - 1$ *when P is symmetric.*

Proof. Let $Q = \frac{1}{2}(P - P)$. Then Q is a polytope and is equal to P when P is symmetric about the origin. Since the polar Q^0 is also a polytope, all of its extreme points are exposed, and hence (1.8) yields the existence of smooth points $q_\pm$ of Q such that $q_\pm \in H_\pm$. Since Q is a polytope, the smooth points of its boundary are precisely the points belonging to the relative interiors of facets. That settles the case in which $P = Q$. To complete the discussion in the general case, use the fact that each face of Q is of the form $\frac{1}{2}F - \frac{1}{2}G$, where F and G are faces of P. □

In order to determine the width of a polytope algorithmically, it seems to be essential to relate a direction of minimum breadth (or the position of the associated parallel supporting hyperplanes) to the facial structure of the polytope. For symmetric polytopes, the condition given in (1.9) is adequate (see [GK1]). The following result contains additional information about the relative positions of P_- and P_+, but even this does not seem to provide enough information to be able to construct an efficient algorithm in the nonsymmetric case.

(1.10) More on Widths of Polytopes. *With notation as in* (1.9), *let H denote the hyperplane that is parallel to and equidistant from H_- and H_+. Suppose that $q \in H$ and $s \in \mathbb{S}$ are such that $q \pm R_1(P)s \in H_\pm$. Then the sets $P^\pm = P_\pm \mp R_1(P)s$ are subsets of H that are not strictly separated in H. When the ball $\mathbb{B}$ is smooth, the sets $P^\pm = P_\pm \mp R_1(P)s$ are not even weakly separated in H, and hence*

$$\dim P_- + \dim P_+ \ge d - 1.$$

Proof. We assume without loss of generality that $q = 0$, whence the hyperplane H is a $(d-1)$-subspace of the d-dimensional Minkowski space $\mathbb{M}$. Each $(d-2)$-flat F in H is the boundary (relative to H) of two closed $(d-1)$-half-spaces in H. To say that the subsets $P^{\pm} := P_{\pm} \mp R_1(P)s$ of H are *weakly separated* in H is to say that F can be chosen so that one of the closed half-spaces (say F^-) contains P^- and the other (F^+) contains P^+. For *strict separation*, the separating F can be chosen disjoint from P^- and P^+. In either case, it may be assumed without loss of generality (by suitably translating P and F, while leaving H unchanged) that F is actually a $(d-2)$-subspace. Finally, we assume without loss of generality that $R_1(P) = 1$, so that $\pm s \in H_{\pm}$.

Now suppose, for the moment, that the space $\mathbb{M}$ is two-dimensional, whence P is a polygon, H_-, H, and H_+ are parallel lines, and $F = \{0\}$. A simple geometric argument shows that if the sets P^- and P^+ are strictly separated by F, and also if the ball $\mathbb{B}$ is smooth and the sets are weakly separated by F, then a suitable small rotation of H about F produces a pair of parallel supporting lines of P such that each of the lines intersects the interior of $\mathbb{B}$. Then the distance between the lines is less than 2, and the minimizing property of the pair $\{H_-, H_+\}$ is contradicted.

To handle the general case, simply proceed as in the two-dimensional case, rotating H about F. Alternatively, the proof can be completed by considering the images of the sets P, H, $\mathbb{B}$, and F under the natural mapping of the space $\mathbb{M}$ onto its quotient space $\mathbb{M}/F$, and applying the two-dimensional result to these images. □

Easy examples show that in ruling out the possibility that the sets P^- and P^+ are weakly separated, the assumption that $\mathbb{B}$ is smooth cannot be abandoned.

(1.11) Helly's Theorem for the Circumradius. *If X is a bounded subset of $\mathbb{M}$ and $Y = \operatorname{cl}\operatorname{conv} X$, then*

$$R_d(Y) = \sup_{x_0,\ldots,x_d \in X} R_d(\operatorname{conv}\{x_0, \ldots, x_d\}).$$

Proof. Denote the supremum by σ. Then obviously $R_d(X) \geq \sigma$. The collection $\mathscr{B}$ of balls $\{y + \tau\mathbb{B}: y \in Y,\ \tau > \sigma\}$ is such that each $d+1$ members of $\mathscr{B}$ have a common point, hence by Helly's theorem [H] $\mathscr{B}$ has the finite intersection property, and then by compactness has nonempty intersection. Each point in the intersection is the center of a ball of radius σ that contains Y. □

(1.12) Helly's Theorem for the Inradius. *Suppose that Y is a finite subset of $\mathbb{M}^*$, and that, for each $y \in Y$, β_y is a real number and H_y is the closed half-space in $\mathbb{M}$ given by $H_y = \{x \in \mathbb{M}: \langle x, y\rangle \leq \beta_y\}$. Suppose also that the intersection $X = \bigcap_{y \in Y} H_y$ is a body. Then*

$$r_d(X) = \min_{y_0,\ldots,y_d \in Y} r_d\left(\bigcap_{i=0}^{d} H_{y_i}\right).$$

Proof. Denote the minimum by μ and let

$$H'_y = \{x \in \mathbb{M}: \langle x, y \rangle \leq \beta_y - \mu \|y\|^*\}.$$

Then it is true for each choice of $y_0, \ldots, y_d \in Y$ that $\bigcap_{i=0}^{d} H'_{y_i} \neq \emptyset$, and hence, by Helly's theorem, $\bigcap_{y \in Y} H'_y \neq \emptyset$. Each point in this intersection is the center of a ball of radius μ that is contained in X. □

For some "Helly tests" concerning the widths of bodies, see [GL].

We turn next to two results that are of a computational rather than geometric character. They are placed here because, like most of the other results in this section, they apply to all Minkowski spaces. In contrast, the computational results of [GK1] are restricted to spaces in which the norm is an ℓ_p norm or the unit ball is a polytope.

(1.13) Circumradius as Minimum of a Convex Function. *Suppose that W is a bounded subset of M. For each $w \in W$ and $x \in \mathbb{M}$, let $\varphi_w(x) = \|x - w\|$ and then set*

$$\Phi(x) = \sup_{w \in W} \varphi_w(x).$$

The function Φ is a convex contraction whose global minimum is the circumradius of the set W.

Proof. It is obvious that each function φ_w is real-valued, is convex, and is a *contraction* in the sense that $|\varphi_w(x_1) - \varphi_w(x_2)| \leq \|x_1 - x_2\|$ for all $x_1, x_2 \in \mathbb{M}$. Since the set W is bounded, the function Φ is real-valued and hence Φ is also a convex contraction and, in particular, is continuous. Since $\Phi(x) \to \infty$ as $\|x\| \to \infty$, the function Φ does indeed attain a global minimum. It is immediate from the definition of circumradius that the minimum value of Φ is the circumradius of the set W. □

(1.14) Inradius as the Solution of a Linear Program. *Suppose that Y is a subset of $\mathbb{M}^*$, and that, for each $y \in Y$, b_y is a real number and H_y is the closed half-space in $\mathbb{M}$ given by*

$$H_y = \{x \in \mathbb{M}: \langle x, y \rangle \leq \beta_y\}.$$

Let $C = \bigcap_{y \in Y} H_y$, and suppose that C has nonempty interior. Then the inradius of C is the solution of the following optimization problem:

$$\sup \xi$$
$$\langle x, y \rangle + \xi \|y\|^* \leq \beta_y \qquad \textit{for} \quad y \in Y.$$

Proof. Let $q \in C$. Then the largest ball that is centered at q and contained in C has radius

$$\xi_0 = \inf_{y \in Y} \frac{1}{\|y\|^*} (\beta_y - \langle q, y \rangle).$$

This follows from the fact that

$$\frac{1}{\|y\|^*}(\beta_y - \langle q, y\rangle)$$

is the distance of q from the hyperplane

$$\{x \colon \langle x, y\rangle = \beta_y\}.$$

Thus $r_d(C) \geq \xi$ if and only if the system of inequalities

$$\langle x, y\rangle + \xi\|y\|^* \leq \beta_y \qquad \text{for} \quad y \in Y$$

is feasible. The desired conclusion follows immediately from this. □

2. Additional Geometric Properties

The results in this section complete our survey of basic geometric properties of radii. They are stated without proof because they are not used in [GK1].

By (1.3) it is true for each body C in an arbitrary d-dimensional Minkowski space that

$$1 \leq \frac{R_d(C)}{r_1(C)} \quad \text{and} \quad 1 \leq \frac{R_1(C)}{r_d(C)},$$

with equality when C is symmetric. Upper bounds on these quotients are also of interest, and are supplied by the following results of [Bo], [Le], and [Eg2].

(2.1) Ratios of Radii. *For each body C in a d-dimensional Minkowski space,*

$$\frac{R_d(C)}{r_1(C)} \leq \frac{2d}{d+1}$$

and

$$\frac{R_1(C)}{r_d(C)} \leq \frac{d+1}{2}.$$

For each d, the stated bounds are attained for some choice of a d-dimensional $\mathbb{M}$ and a body $C \subset \mathbb{M}$.

The upper bounds can be considerably reduced in the case of Euclidean spaces, as follows from old theorems of [J] and [St]. (See [DGK] for references to later proofs of (2.1) and (2.2).)

(2.2) Ratios of Radii in Euclidean Spaces. *For each body C in the d-dimensional Euclidean space R_2^d it is true that*

$$\frac{R_d(C)}{r_1(C)} \leq \left(\frac{2d}{d+1}\right)^{1/2},$$

$$\frac{R_1(C)}{r_d(C)} \leq \sqrt{d} \qquad \textit{for odd } d,$$

and

$$\frac{R_1(C)}{r_d(C)} \leq \frac{d+1}{\sqrt{d+2}} \qquad \textit{for even } d.$$

In each case equality is attained when C is a regular d-simplex.

For j-radii with $1 < j < d$, the best results on ratios are those of [Pe]:

(2.3) Ratios of Radii in Euclidean Spaces. *For each body C in the d-dimensional Euclidean space R_2^d, and for each j with $1 \leq j \leq d$,*

$$\frac{R_{d+1-j}(C)}{r_j(C)} \leq j + 1.$$

Also,

$$\frac{R_2(C)}{r_2(C)} \leq 2.151$$

for each body C in Euclidean 3-space.

Some important aspects of the behavior of radii in Euclidean spaces do not carry over to non-Euclidean Minkowski spaces. The following summarizes results of [Kl], [Ga], and [KMZ].

(2.4) Circumspheres, Inspheres, and Inner Products. *For each Minkowski space $\mathbb{M}$ of dimension d, the following three conditions are equivalent:*

(a) *$d = 2$ or $\mathbb{M}$ is Euclidean;*
(b) *each polytope P in $\mathbb{M}$ is contained in a ball of radius $R_d(P)$ whose center belongs to P;*
(c) *each polytope P in $\mathbb{M}$ contains a ball G of radius $r_d(P)$ such that G's center belongs to the convex hull of $G \cap \mathrm{bd}(P)$.*

Though they are not used in [GK1], the facts stated in (2.4) do have computational consequences. Suppose, for example, that a polytope $P \subset \mathbb{M}$ is given and it is desired to find how well P can be approximated, in a minimax sense, by a single point $q \in P$. When $d \leq 2$ or the space $\mathbb{M}$ is Euclidean, the constraint $q \in P$ can be ignored and the problem can be treated as an unconstrained minimization problem. However, the constraint $q \in P$ cannot be ignored when $d \geq 3$ and $\mathbb{M}$ is not Euclidean. In this case, there arises a constrained minimization problem in which the minimax deviation from P may be greater than would be the case if q were required merely to lie in $\mathbb{M}$.

In addition to the inner and outer radii studied here, there are some closely related notions that deserve to be mentioned. To define them, let us consider, for a given Minkowski space $\mathbb{M}$ and an integer j with $1 \leq j \leq d$, the various ways of expressing M as the direct sum of a j-subspace and a $(d-j)$-subspace—that is,

$$\mathbb{M} = E \oplus F \qquad (\dim E = j, \dim F = d - j). \tag{1}$$

For each $\rho > 0$, the intersection

$$E \cap (F + \rho\mathbb{B}) \tag{2}$$

is the image of the ball $\rho\mathbb{B}$ under the linear projection that carries $\mathbb{M}$ onto E and has F as nullspace. Let us call each set of the form (2) a *j-proball of radius ρ*, and for each body $C \subset \mathbb{M}$ define $s_j(C)$ as the largest radius ρ such that C contains a translate of a j-proball of radius ρ. Define $S_j(C)$ as the smallest ρ such that, for some expression of the form (1) and for some translate F' of F,

$$C \subset F' + ((\rho\mathbb{B}) \cap E).$$

It is evident that

$$s_1 \geq s_2 \geq \cdots \geq s_d, \qquad S_1 \leq S_2 \leq \cdots \leq S_d,$$
$$s_1 \geq S_1, \qquad s_d \leq S_d,$$

and for symmetric bodies, $s_1(C) = S_d(C)$ and $s_d(C) = S_1(C)$.

Theorem (2.5) below can be proved by using the Blaschke–Kakutani characterization of inner-product spaces [Bl], [Ka]. The proof of (2.6) is similar to that of (1.3).

(2.5) Relations Between Two Kinds of Radii. *Suppose that $\mathbb{M}$ is a d-dimensional Minkowski space and $1 \leq j \leq d$. Then, for each body $C \subset \mathbb{M}$,*

$$S_j(C) \leq r_j(C) \quad \text{and} \quad S_j(C) \geq R_j(C),$$

with equality when $j = 1$ or $j = d$. For $1 < j < d$, the following three conditions are equivalent:

(a) *$\mathbb{M}$ is Euclidean:*
(b) *$s_j(C) = r_j(C)$ for each body $C \subset \mathbb{M}$;*
(c) *$S_j(C) = R_j(C)$ for each body $C \subset \mathbb{M}$.*

(2.6) Duality Between Inner and Outer Radii of Symmetric Bodies. *If C is a body in a Minkowski space $\mathbb{M}$, and $-C = C$, then*

$$s_j(C)S_j(C^0) = 1 \quad \textit{and} \quad S_j(C)s_j(C^0) = 1.$$

3. Rationality of Certain Polytope Radii

Except for some assumptions of smoothness or rotundity, the results of Section 1 apply to arbitrary Minkowski spaces. In this section we are getting closer to computational questions and thus confine our attention to the spaces $\mathbb{R}_p^d$ for $1 \leq p \leq \infty$. In $\mathbb{R}_p^d$ each point x is given by its d-tuple $(\xi_1, \ldots, \xi_d)$ of real coordinates, with

$$\|x\|_p = \left(\sum_{i=1}^{d} |\xi_i|^p\right)^{1/p} \qquad \text{when} \quad 1 \leq p < \infty$$

and

$$\|x\|_\infty = \max\{|\xi_1|, \ldots, |\xi_d|\}.$$

The conjugate space of $\mathbb{R}_p^d$ is the space $\mathbb{R}_{\bar{p}}^d$, where $\bar{p}$ is defined by the condition that $1/p + 1/\bar{p} = 1$. (When there is no danger of confusion, subscripts on the norms are suppressed and the norms in both $\mathbb{R}_p^d$ and $\mathbb{R}_{\bar{p}}^d$ are denoted simply by $\|\ \|$.)

This section establishes the algebraic tractability of certain radii of rational polytopes in certain spaces $\mathbb{R}_p^d$. Attention is confined to the case in which p or $\bar{p}$ is a positive integer or ∞. Results are necessarily limited, for it is shown in [GHK] that if $p = 2$ and ρ is the inradius function for rational right triangles in $\mathbb{R}_p^2$, or p is an integer greater than 2 and ρ is the circumradius function for rational isosceles triangles in $\mathbb{R}_p^2$, then there is no *rationalizing polynomial* for ρ—that is, for no polynomial φ with rational coefficients can $\varphi(\rho(T))$ be rational for each triangle of the indicated sort. However, it turns out that there are a few (radius, exponent) pairs (ρ, p) for which the powers ρ^p or $\rho^{\bar{p}}$ must be rational, thus providing very simple rationalizing polynomials in those cases. (It is convenient to define $\xi^\infty = \xi$ for each $\xi > 0$.) The pairs that we know about are described in (3.1) below, where they are accompanied by explicit bounds on the magnitudes of the numerators and denominators in question. These bounds, expressed in terms of the size L of a rational presentation of the polytope P, are used in proofs [GK1] of the polynomial-time computability of certain powers of radii.

Some definitions are required in order to discuss input size for the the two principal ways of presenting a polytope. When κ is an integer, we define

$$\langle\kappa\rangle = 1 + \lceil\log(|\kappa| + 1)\rceil,$$

where the logarithm is to the base 2 and $\lceil\eta\rceil$ denotes the integer ceiling of the real number η. When γ and δ are integers and $\delta > 0$, we define

$$\left\langle \frac{\gamma}{\delta} \right\rangle = \langle \gamma \rangle + \langle \delta \rangle.$$

A *$\mathscr{V}$-presentation* of a polytope $P \subset \mathbb{R}^d$ consists of integers m and d with $m > d \geq 1$, and an m-tuple $v_1, \ldots, v_m$ of rational points of $\mathbb{R}^d$ such that

$$P = \operatorname{conv}\{v_1, \ldots, v_m\}.$$

When

$$v_i = \left(\frac{\gamma_{i1}}{\delta_{i1}}, \ldots, \frac{\gamma_{id}}{\delta_{id}} \right)$$

the *size* of this input is defined as

$$L = \langle P \rangle = \sum_{i=1}^{m} \sum_{j=1}^{d} \left\langle \frac{\gamma_{ij}}{\delta_{ij}} \right\rangle.$$

An *$\mathscr{H}$-presentation* of a polytope P consists of integers m and d with $m > d \geq 1$, a rational $m \times d$ matrix A, and a rational m-vector b such that

$$P = \{x \in \mathbb{R}^d \colon Ax \leq b\}.$$

When $A = [\alpha_{ij}]$ and $b = (\beta_1, \ldots, \beta_m)$, we define

$$\langle A \rangle = \sum_{i=1}^{m} \sum_{j=1}^{d} \langle \alpha_{ij} \rangle, \qquad \langle b \rangle = \sum_{i=1}^{m} \langle \beta_i \rangle,$$

and the *size* of the input is defined as

$$L = \langle P \rangle = \langle A \rangle + \langle b \rangle.$$

As is well known, a polytope P admits both a $\mathscr{V}$-presentation and an $\mathscr{H}$-presentation, However, since P may have many more vertices than facets (or vice versa), it may happen that the minimum size for one sort of presentation is much larger than the minimum size for the other sort. That is why the complexity results for $\mathscr{V}$-presentations differ from those for $\mathscr{H}$-presentations.

(3.1) Rationality of a Few Powers of Radii. *Suppose that P is a rational polytope in $\mathbb{R}_p^d$, given by means of a $\mathscr{V}$- or $\mathscr{H}$-presentation of size L. Under each of the additional assumptions listed on the left below, the power of the radius given in the middle can be expressed as the quotient of two positive integers neither of which* *exceeds the bound indicated on the right*:

Assumption	*Radius*	*Bound*
$p \in \mathbb{N} \cup \{\infty\}$	$r_1(P)^p$	2^{2pdL}
$\bar{p} \in \mathbb{N} \cup \{\infty\}$	$R_1(P)^{\bar{p}}$	$2^{8\bar{p}d^3L}$
$p \in \mathbb{N} \cup \{\infty\}$, *P symmetric*	$R_d(P)^p$	2^{2pdL}
$p \in \{1, 2, \infty\}$	$R_d(P)^p$	2^{16d^2L}
$\bar{p} \in \mathbb{N} \cup \{\infty\}$, *P symmetric*	$r_d(P)^{\bar{p}}$	$2^{2\bar{p}d^3L}$
$\bar{p} \in \{1, \infty\}$	$r_d(P)^{\bar{p}}$	2^{16d^4L}

Proof. The proof uses the following basic facts (see, e.g., [GLS]):

(i) If the polytope P is given by a $\mathscr{V}$- or an $\mathscr{H}$-presentation of size L, then each component of each vertex of P can be expressed as the quotient of an integer γ and a positive integer δ with $\langle\gamma\rangle, \langle\delta\rangle \le L - 1$. This implies in particular that $|\gamma|, \delta \le 2^{L-1}$.

(ii) If v and w are rational d-vectors such that each of their components can be expressed as the quotient of integers of size at most L, then each component of $v + w$ can be expressed as the quotient of two integers of absolute values at most 2^{2L-1}.

(iii) If $p \in \mathbb{N}$ and v is a rational d-vector whose components can be expressed as the quotient of two integers of absolute values at most 2^{2L-1}, then $\|v\|_p^p$ can be expressed as the quotient of two integers of absolute values at most $2pdL$.

Statement (i) is trivial for $\mathscr{V}$-presentations; actually, each component of each vertex of P is of size at most $L - 2md + 2$. For an $\mathscr{H}$-presentation we apply Cramer's rule to express each vertex of P as the quotient of determinants of integer matrices that are given by selecting d linearly independent rows of the matrix (A, b). The assertion then follows by means of some elementary calculations. Statements (ii) and (iii) are trivial.

Let us now turn to the proof of (3.1).

With the aid of (i)–(iii) the statement about r_1 becomes obvious.

To prove the statement about R_1 we might try to proceed as follows. With $Q = \frac{1}{2}(P - P)$, it follows from (1.4) that $R_1(P) = R_1(Q)$, and since Q is symmetric it follows from (1.2) that $R_1(Q) = r_1(Q^0)^{-1}$. A $\mathscr{V}$-presentation of P of size L yields a $\mathscr{V}$-presentation of Q in $\mathbb{R}_p^d$ of size less than $2m^2L$, and this in turn yields an $\mathscr{H}$-presentation for Q^0 in $\mathbb{R}_{\bar{p}}^d$ of size less than $2m^2dL$. (Note that we have to transform the rational presentation as an intersection of half-spaces to an integer presentation of this kind by multiplying by the appropriate common denominators.) Now a sightly different version of the desired conclusion follows from the result of the preceding paragraph.

For an $\mathscr{H}$-presented polytope P, the approach of the preceding paragraph does not yield a bound of the required type, for in general Q may have exponentially many vertices and exponentially many facets (see [RS]). Thus in general we cannot convert the rational presentations obtained into integral $\mathcal{V}$- or $\mathcal{H}$-presentations by multiplying by the common denominators, for this might produce integers that are exponential in the size of the input. However, we can deal with the width of $\mathcal{H}$-presented polytopes in the manner described in the next paragraph, and this also yields the desired bound for $\mathcal{V}$-polytopes.

By (1.9) there exist two sets $F_1 = \operatorname{conv}\{w_0, \ldots, w_k\}$, $F_2 = \operatorname{conv}\{w_{k+1}, \ldots, w_d\}$ in the boundary of P such that the set

$$\begin{aligned} H &= \operatorname{lin}\{w_1 - w_0, \ldots, w_k - w_0, w_{k-1} - w_d, \ldots, w_{d-1} - w_d\} \\ &= \operatorname{lin}((F_1 - w_0) \cup (F_2 - w_d)) \end{aligned}$$

is a $(d-1)$-subspace of $\mathbb{R}^d$ and the distance between the hyperplanes $w_0 + H$ and $w_d + H$ is precisely $2R_1(P)$. Suppose (without loss of generality) that $w_0 \neq 0$ and let y_0 be the solution of the linear system

$$\begin{aligned} \langle y, w_i - w_0 \rangle &= 0 \qquad (1 \le i \le k), \\ \langle y, w_i - w_d \rangle &= 0 \qquad (k+1 \le i \le d-1), \\ \langle y, w_0 \rangle &= 1. \end{aligned}$$

Since all vertices of P are rational with absolute values of numerators and denominators bounded by 2^{L-1}, it follows from Cramer's rule (after multiplying the equations by the respective common denominators) that y_0 is a rational vector with absolute values of numerators and denominators bounded by 2^{2d^2L}. However,

$$2R_1(P) = \frac{1}{\|y_0\|_{\bar{p}}} (1 - \langle w_d, y_0 \rangle),$$

so $R_1(P)^{\bar{p}}$ is a rational number with bounds as asserted.

When P is symmetric, it follows from (1.3) that $R_d(P) = r_1(P)$ and $r_d(P) = R_1(P)$. That completes the discussion of the cases in which P is symmetric.

For the asymmetric case, let us first dispose of the polytopal norms $\| \; \|_1$ and $\| \; \|_\infty$.

We begin with R_d. By (1.11) there are $d+1$ vertices $w_0, \ldots, w_d$ of P such that $R_d(P) = R_d(T)$, where T is the d-simplex $\operatorname{conv}\{w_0, \ldots, w_d\}$. Now let Y denote the set of all d-vectors of the form $(0, \ldots, 0, \pm 1, 0, \ldots, 0)$, and let Z be the set of all vectors of the form $(\pm 1, \ldots, \pm 1)$. Then $R_d(P)$ is the solution of the linear system

$$\begin{aligned} &\min \rho \\ &\rho + \langle c, y \rangle \ge \langle w_j, y \rangle \qquad (1 \le j \le d;\, y \in Y) \end{aligned}$$

for $P \subset \mathbb{R}^d_\infty$ and of the same problem with Y replaced by Z for $P \subset \mathbb{R}^d_1$. In both cases, the vertices of the associated feasible region are rational vectors with absolute values of numerators and denominators bounded by 2^{2dL}. Notice that in an optimal solution c is a circumcenter of P. That takes care of R_d.

In the following, we deal with the inradius in $\mathbb{R}_1^d$ and $\mathbb{R}_\infty^d$, starting with the case of a $\mathscr{V}$-polytope $P = \operatorname{conv}\{v_1, \ldots, v_m\}$ in $\mathbb{R}_1^d$. With Y as before, it is easy to see that $r_d(P)$ is the solution of the following linear program:

$$\max \rho$$

$$\sum_{j=1}^{m} \tau_j^{(y)} v_j - \rho y - c = 0 \qquad (y \in Y)$$

$$\sum_{j=1}^{m} \tau_j^{(y)} = 1 \qquad (y \in Y)$$

$$\tau_j^{(y)} \geq 0 \qquad (j = 1, \ldots, m;\ y \in Y).$$

Thus, by Cramer's rule and some calculations, $r_d(P)$ is of size at most 2^{16d^4L}. Notice that in an optimal solution c is the center of an insphere.

Next, let P be the $\mathscr{H}$-polytope, $P = \bigcap_{i=1}^m \{x \in \mathbb{R}^d\colon \langle x, a_i\rangle \leq \beta_i\}$, in $\mathbb{R}_1^d$ or $\mathbb{R}_\infty^d$. By (1.14) $r_d(P)$ is the solution of the linear program

$$\max \rho$$

$$\langle x, a_i\rangle + \rho \|a_i\|^* \leq \beta_i \qquad (i = 1, \ldots, m).$$

This characterization of $r_d(P)$ can be used to obtain the asserted bounds if we can show that $\|a_1\|^*, \ldots, \|a_m\|^*$ are rational and of suitable size, and that follows easily from the facts that $\|\ \|_1$ is conjugate to $\|\ \|_\infty$ and that $\|\ \|_\infty$ is conjugate to $\|\ \|_1$.

For polytopal norms there remains only the case of a $\mathscr{V}$-presented polytope $P \subset \mathbb{R}_\infty^d$. Let

$$P = \bigcap_{i=1}^{r} \{x \in \mathbb{R}_\infty^d\colon \langle y_i, x\rangle \leq 1\}$$

be the irredundant rational $\mathscr{H}$-representation of P that is naturally associated with the given $\mathscr{V}$-presentation. (This presentation is obtained by solving $\binom{m}{d}$ systems of linear equations and disregarding all half-spaces whose bounding hyperplanes are not determined by facets of P.) The y_i's are rational vectors with absolute values of numerators and denominators bounded by 2^{2d^2L}. By (1.12) there are $d+1$ of these half-spaces—say with indices $0, \ldots, d$—such that $r_d(P) = r_d(T)$, where

$$T = \bigcap_{i=0}^{d} \{x \in \mathbb{R}_\infty^d\colon \langle y_i, x\rangle \leq 1\}.$$

This is an $\mathscr{H}$-presentation, and our previous results can be applied.

There remains only the case of the circumradius $R_d(P)$ of an asymmetric presented polytope $P \subset \mathbb{R}_2^d$. Let V, c, and ρ denote P's vertex-set, circumcenter, and circumradius, and let S denote the unit sphere of the space. Since the sphere

$c + \rho S$ intersects V, ρ is the distance from c to a point of V. The points of V are all rational, and hence the number ρ^2 must be rational if the point c is rational. In showing that c is rational, the relevant geometric properties of Euclidean spaces are the following:

the set of points equidistant from two given points is a hyperplane;
the circumcenter of a polytope belongs to the affine hull of the polytope.

Among Minkowski spaces of dimension at least three, each of these properties actually characterizes Euclidean spaces (see [D] and [PK] for the first property and [Kl] and [Ga] for the second).

Now let $v_0, \dots, v_r$ be the points of the intersection $V \cap (c + \rho S)$, and write each as a row vector. Then the center $c = (\gamma_1, \dots, \gamma_d)$ is the unique solution of the linear system:

$$\langle v_j - v_{j-1}, c\rangle = \tfrac{1}{2}(\|v_j\|_2^2 - \|v_{j-1}\|_2^2) \qquad (j = 1, \dots, r),$$
$$\sum_{i=0}^{r} \tau_i v_i = c,$$
$$\sum_{i=0}^{r} \tau_i = 1.$$

This implies that c is a rational vector with the property that, for each component γ_i of c, both the numerator and denominator are of absolute value at most 2^{6d^2L}, whence ρ^2 is a rational number whose numerator and denominator are of absolute value less than 2^{16d^3L}. □

4. Difficulties in Computing Radii

Each radius computation requires finding the global optimum of a certain nonlinear function. It is standard practice, in seeking to optimize a function, to look first for necessary conditions for optimality, and then to attempt to isolate the true optimum among the "solutions" that satisfy the necessary condition. Difficulty may be expected when there are many such solutions. That is illustrated here by several examples involving radii.

For a symmetric polytope P centered at the origin in $\mathbb{M}$, it is true that

$$r_1(P) = R_d(P) = \max\{\|x\|: x \in P\},$$

and it is well known [Ba] that this global maximum is attained at a vertex of P. However, it may happen that only two vertices provide the global maximum for the norm function while all others provide strict local maxima that are not global maxima. When the polytope is $\mathscr{H}$-presented, this is a serious obstacle to determining the global maximum, for the possible number of vertices is not bounded by any polynomial in the size of the presentation. Indeed, it is proved in [BGKV] that, for each positive integer p, the problem of determining $\max\{\|x\|^p: x \in P\}$ in a finite-dimensional ℓ_p space is $\mathbb{NP}$-hard, even for the restricted situation in which P is a parallelotope centered at the origin.

When a polytope P is $\mathscr{V}$-presented in $\mathbb{R}_p^d$, the pth power of the diameter $2r_1(P)$ can be determined by the obvious $O(L^2)$ procedure: compute the pth powers of distances between pairs of vertices, and take the maximum of the numbers thus obtained. When $p = 2$, this obvious algorithm has been significantly improved for $d \leq 3$ [PS], [Y] but not for $d \geq 4$. As is noted in [PS], that appears to be related to the fact that if a polytope $P \subset \mathbb{R}_2^d$ has m vertices and k diametral pairs of vertices, then $k \leq m$ when $d = 2$ [Er1] and $k \leq 2m - 2$ when $d = 3$ [Gr], [He], [Str], while, for $d \geq 4$, k may be almost as large as $m^2/2$ [Er2]. In the next two paragraphs the construction of [Er2] is modified slightly to bring local maxima more clearly into the picture.

Suppose that $\mathbb{E}$ is a Euclidean space of dimension $d \geq 4$, expressed as the direct sum of two subspaces E_i of dimension $d_i \geq 2$. Let V_i be a finite subset of E_i's unit sphere S_i such that the diameter of V_i is less than $\sqrt{2}$ and the set $P_i = \operatorname{conv} V_i$ is of dimension d_i. Then the d-polytope $P = \operatorname{conv}(V_1 \cup V_2)$ has vertex-set $V_1 \cup V_2$ and its diameter $\sqrt{2}$ is attained as $\|v_1 - v_2\|$ for each $v_1 \in V_1$, $v_2 \in V_2$.

Now perturb the set $V_1 \cup V_2$ by replacing each $v \in V_1 \cup V_2$ with a multiple $\mu_v v$ for some $\mu_v \geq 1$, where the μ_v's are sufficiently small to preserve (for the perturbed V_i's) the conditions that the diameter of V_i is less than $\sqrt{2}$ and that S_i is intersected by each segment that joins two points of V_i. Then on the (perturbed) P_i, the norm attains a strict local maximum at each point of V_i, and for the (perturbed) d-polytope P, the distance function

$$\varphi(x, y) = \|x - y\| \qquad (x, y) \in P \times P,$$

attains a strict local maximum at each $(v_1, v_2) \in V_1 \times V_2$. The length-function $l_s(P)$ (defined on $\mathbb{E}$'s unit sphere $\mathbb{S}$) attains a strict local maximum at each point

$$q(v_1, v_2) = \frac{1}{\|v_1 - v_2\|}(v_1 - v_2) \in \mathbb{S}.$$

Of course the multipliers μ_v can be chosen so that the global maximum of the length-function is attained only at a single pair of antipodal points of $\mathbb{S}$.

The next construction is formulated as a theorem, because it plays an essential role in one of the $\mathbb{NP}$-completeness proofs of [GK1].

(4.1) The Width of Certain Simplices. *For an arbitrary fixed $p \in [1, \infty]$, let $\{e_0, \ldots, e_d\}$ and $\{\bar{e}_0, \ldots, \bar{e}_d\}$ denote the standard dual bases for the respective spaces $\mathbb{R}_p^{d+1}$ and $\mathbb{R}_{\bar{p}}^{d+1} = (\mathbb{R}_p^{d+1})^*$. Suppose that $\eta_0, \ldots, \eta_d$ are positive real numbers, and let*

$$y = \sum_{i=0}^{d} \eta_i \bar{e}_i \in \mathbb{R}_{\bar{p}}^{d+1}.$$

Consider the hyperplane

$$G = \left\{ x = \sum_{i=0}^{d} \xi_i e_i \colon \langle x, y \rangle = 1 \right\} \subset \mathbb{R}_p^{d+1},$$

and for each i let

$$v_i = \frac{1}{\eta_i} e_i \in G.$$

For each nonempty subset I of the index-set $N = \{0, \ldots, d\}$, let

$$F_I = \operatorname{conv}\{v_i \colon i \in I\} \subset G,$$

and when $p > 1$ define

$$\sigma_I = \sum_{i \in I} \eta_i^{\bar{p}}, \qquad q_I = \frac{1}{\sigma_I} \sum_{i \in I} \eta_i^{\bar{p}} v_i \in \operatorname{relint} F_I.$$

Then F_N is a d-simplex in G, and, for $0 \le k \le d$, the k-faces of F_N are precisely the sets F_I with $I \subset N$ and $|I| = k + 1$. The width of F_N relative to its affine hull G is the minimum, over all pairs (I, J) of complementary proper subsets of N, of the quantities $b_p(I, J)$ defined as follows:

$$b_1(I, J) = \frac{1}{\max_{i \in I} \eta_i} + \frac{1}{\max_{j \in J} \eta_j}; \qquad b_\infty(I, J) = \max\left\{ \frac{1}{\sum_{i \in I} \eta_i}, \frac{1}{\sum_{j \in J} \eta_j} \right\};$$

for $1 < p < \infty$,

$$b_p(I, J) = \left(\frac{1}{(\sum_{i \in I} \eta_i^{\bar{p}})^{p-1}} + \frac{1}{(\sum_{j \in J} \eta_j^{\bar{p}})^{p-1}} \right)^{1/p}.$$

When $p > 1$ it is true for all (I, J) that

$$b_p(I, J) = \|q_I - q_J\|_p.$$

Proof. To see that $b_p(I, J) = \|q_I - q_J\|_p$ when $p > 1$ just apply the formula for the ℓ_p norm, using the fact that $p + \bar{p} = p\bar{p}$ and that since I and J are complementary,

$$\|q_I - q_J\|_p = \|(\|q_I\|_p, \|q_J\|_p)\|_p.$$

In particular, for $1 < p < \infty$ we have

$$\|q_I - q_J\|_p^p = \frac{1}{\sigma_I^p} \sum_{i \in I} \eta_i^{(\bar{p}-1)p} + \frac{1}{\sigma_J^p} \sum_{j \in J} \eta_j^{(\bar{p}-1)p}$$
$$= \frac{1}{\sigma_I^p} \sigma_I + \frac{1}{\sigma_J^p} \sigma_J.$$

For each (I, J), the d-simplex F_N is supported in G by a unique pair of parallel $(d-1)$-flats H_I and H_J that contain F_I and F_J respectively. The existence and uniqueness of the $(d-1)$-flats H_I, H_J follows from dimensional and linear algebraic considerations independent of p, and (1.9) then implies that the width of F_N in G is equal to the minimum over all (I, J) of the distance $\delta_p(H_I, H_J)$ between H_I and H_J with respect to the ℓ_p norm in $\mathbb{R}^{d+1}$. When $p > 1$ the points q_I and q_J are defined, and since they belong to H_I and H_J respectively it follows that $\delta_p(H_I, H_J) \leq \|q_I - q_J\|_p$. Hence for $p > 1$ it remains only to show that

$$\delta_p(H_I, H_J) \geq \|q_I - q_J\|_p.$$

To establish the desired inequality for $1 < p < \infty$, we shall produce a linear functional

$$\varphi_{I,J} \in (\mathbb{R}_p^{d+1})^* = \mathbb{R}_{\bar{p}}^{d+1}$$

and two numbers α and β such that

$$F_I \subset \varphi_{I,J}^{-1}(\alpha), \qquad F_J \subset \varphi_{I,J}^{-1}(\beta),$$

and

$$\frac{|\alpha - \beta|}{\|\varphi_{I,J}\|_{\bar{p}}} \geq \|q_I - q_J\|_p.$$

This implies that the distance between the parallel d-flats $\varphi_{I,J}^{-1}(\alpha)$ and $\varphi_{I,J}^{-1}(\beta)$ in $\mathbb{R}_p^{d+1}$ is at least $\|q_I - q_J\|_p$, and from this it follows that the same is true of the intersections of these hyperplanes with G. However, these intersections are precisely the $(d-1)$-flats H_I and H_J.

For $1 < p < \infty$ set

$$\varphi_{I,J} = \frac{1}{\sigma_I^{p-1}} \sum_{i \in I} \eta_i \bar{e}_i - \frac{1}{\sigma_J^{p-1}} \sum_{j \in J} \eta_j \bar{e}_j,$$

$$\alpha = \varphi_{I,J} F_I = \frac{1}{\sigma_I^{p-1}}, \qquad \beta = \varphi_{I,J} F_J = \frac{1}{\sigma_J^{p-1}}.$$

Since

$$\|\varphi_{I,J}\|_{\bar{p}} = \left(\frac{1}{\sigma_I^{(p-1)\bar{p}}}\,\sigma_I + \frac{1}{\sigma_J^{(p-1)\bar{p}}}\,\sigma_J\right)^{1/\bar{p}},$$

it follows that

$$\frac{|\alpha - \beta|}{\|\varphi_{I,J}\|_{\bar{p}}} = \frac{1/\sigma_I^{p-1} + 1/\sigma_J^{p-1}}{(1/\sigma_I^{p-1} + 1/\sigma_J^{p-1})^{1/\bar{p}}} = \left(\frac{1}{\sigma_I^{p-1}} + \frac{1}{\sigma_J^{p-1}}\right)^{1-1/\bar{p}} = \|q_I - q_J\|_p.$$

Hence the proof is complete except for $p = 1$ and $p = \infty$.

Now note that if η_I denotes a point of $\mathbb{R}^{|I|}$ whose coordinates are the η_i's for $i \in I$, and η_J is similarly defined, then the quantity $b_p(I, J)$ is given for all p by the following formula that involves the $\ell_{\bar{p}}$ norm in $\mathbb{R}^{|I|}$ and $\mathbb{R}^{|J|}$ and the ℓ_p norm in $\mathbb{R}^2$:

$$b_p(I, J) = \left\|\left(\frac{1}{\|\eta_I\|_{\bar{p}}}, \frac{1}{\|\eta_J\|_{\bar{p}}}\right)\right\|_p.$$

For $1 < p < \infty$ we have shown that

$$b_p(I, J) = \delta_p(H_I, H_J),$$

and hence by a routine continuity argument the latter equality continues to hold when $p = 1$ and when $p = \infty$. (Use the fact that as $p \to 1$ or $p \to \infty$, the convergence of $\|\ \|_p$ to $\|\ \|_1$ of $\|\ \|_\infty$ is uniform on compact subsets of $\mathbb{R}^{d+1}$.) A continuity argument also shows that $b_\infty(I, J) = \|q_I - q_J\|_\infty$. $\square$

By (4.1) the numbers of the form $\|q_I - q_J\|_p$ are exactly the ones realized as the distance between two supporting hyperplanes of the simplex F_N which satisfy the final necessary condition on dimension-sums provided by (1.9). There may be as many as $2^{|N|} - 2$ such numbers, and each is a candidate for being the width of the simplex. The list of candidates could in many cases be reduced by applying (1.10)'s "no weak-separation" condition. However, finding the width of the simplex does require deciding, at least implicitly, which number of the form $\|q_I - q_J\|_p$ is smallest, and that does not appear to be easy. Indeed, (4.1) is used in [GK1] to show that the problem of determining the width of a simplex is $\mathbb{NP}$-hard. There is no need to distinguish here between $\mathscr{V}$-presented and $\mathscr{H}$-presented simplices, because for a simplex either sort of presentation can be derived in polynomial time from the other. The essential difficulty seems to come from the fact that the number of complementary pairs (I, J) (or, equivalently, the number of facets of the difference body) increases exponentially with the dimension.

The set F_N of (4.1) is a d-simplex in $\mathbb{R}^{d+1}$, and its vertices have rational coordinates. Nevertheless, F_N is not a polytope in our special sense because it is not full-dimensional. Of course, it does have nonempty interior relative to its own affine hull G, and even though G depends on the choice of the numbers η_i it is true when $p = 2$ that G is always isometric to the space $\mathbb{R}_2^d$. However, even when all the η_i are equal and the simplex F_N is therefore regular, F_N may fail to be similar to any rational (equivalently, to any integral) d-simplex in $\mathbb{R}_2^d$. (See the paragraph following (4.2).) For use in our analysis of computational complexity, that difficulty is overcome in [GK1] by a simple technical device. However, the following related number-theoretic problem appears to be both difficult and interesting.

(4.2) Two Questions about Integral Presentations of Simplices. *For each sequence* $k = (\kappa_0, \ldots, \kappa_d)$ *of positive integers, let*

$$T(k) = \operatorname{conv}\{\kappa_0 e_0, \ldots, \kappa_d e_d\} \subset \mathbb{R}_2^{d+1}.$$

Then:

(a) *For which k is the d-simplex $T(k)$ isometric to an integral d-simplex in $\mathbb{R}_2^d$?*
(b) *For which k is the d-simplex $T(k)$ similar to an integral d-simplex in $\mathbb{R}^d$?*

A result of [Sc], [P], and [M] settles the case of (b) in which all the k_i are equal. They show that $\mathbb{R}^d$ contains a regular d-simplex with integral vertices if and only if one of the following three conditions is satisfied:

(i) d is even and of the form $m^2 - 1$;
(ii) d is of the form $4m - 1$;
(iii) d is of the form $4m + 1$ where $2m + 1$ is a sum of two squares.

For a polytope P in Euclidean 3-space $\mathbb{R}_2^3$, $R_2(P)$ is the radius of a smallest circular cylinder containing P. Difficulties in computing this may be related to the remaining examples in this section.

For R_2 in $\mathbb{R}_2^3$, there is no result of Helly type such as (1.11) and (1.12). Consider a regular $(2m + 1)$-gon situated in a plane in $\mathbb{R}_2^3$. Since half its width in its affine hull is smaller than its circumradius, and since that width is reduced by removing any vertex, none of the minimum containing cylinders for the $(2m + 1)$-gon is minimum for any proper subset of the vertices. This example can easily be modified to show that there is no Helly-type theorem for R_2 that applies to 3-polytopes in $\mathbb{R}_2^3$.

There is no upper bound on the number of vertices of a 3-polytope P in $\mathbb{R}_2^3$ such that there are at least two cylinders of radius $R_2(P)$, each of which contains all vertices of P. For example, consider two different cylinders of the same radius whose axes intersect and are perpendicular to each other. Let C be the curve consisting of all points that belong to the boundaries of both cylinders. Then there is a six-pointed subset V of C (the vertex-set of a octahedron) such that, for each finite set W with $V \subset W \subset C$, each of the cylinders is a minimum cylinder among those containing the polytope conv W.

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