

INNER FUNCTIONS AND DIVISION IN DOUGLAS ALGEBRAS

Keiji Izuchi and Yuko Izuchi

1. Introduction. Let H^∞ be the space of boundary functions for bounded analytic functions in the open unit disk D , and C be the space of continuous functions on ∂D . It is well known [14] that $H^\infty + C$ is an essentially supremum norm closed subalgebra of L^∞ . Guillory and Sarason began the study of division in $H^\infty + C$, and they showed the following.

THEOREM A [6]. *There is a positive integer N with the following property: if I is an inner function and f is a function in $H^\infty + C$ such that $|f| \leq |I|$ on $M(H^\infty + C)$, the maximal ideal space of $H^\infty + C$, then $f^N \bar{I} = f^N / I \in H^\infty + C$.*

Guillory and Sarason posed the problem of finding the best value of N . We solve this problem here. It has been shown [1; 7] that if I is a finite product of interpolating Blaschke products, then we may take $N=1$. As a consequence, we have the following.

THEOREM B [7]. *If I is an inner function which is not a finite product of interpolating Blaschke products, then we can not take $N=1$.*

In §2, we show that one may take $N=2$ in Theorem A. The technique used to prove this is almost the same as that used in [6]. For a function f in $H^\infty + C$, put $Z(f) = \{x \in M(H^\infty + C); f(x) = 0\}$. The condition $|f| \leq |I|$ on $M(H^\infty + C)$ implies $Z(I) \subset Z(f)$. The following question is given in [7, p. 5]: If I is an inner function and $f \in H^\infty + C$ with $Z(I) \subset Z(f)$, does there exist a positive integer K , depending on I and f , such that $f^K \bar{I} \in H^\infty + C$? We shall give its negative answer (Theorem 2). Also we shall show that Theorem B can not be extended to general Douglas algebras (Theorem 3). This is a negative answer to another question in [7, p. 5].

In §3, we shall study the factorization of Blaschke products. Let b be an inner function such that $\text{Ord}_b(x)$, the zero's order of b at $x \in Z(b)$, is uniformly bounded. By [12], b is a finite product of interpolating Blaschke products. By [10], there are interpolating Blaschke products $\{b_k\}_{k=1}^n$ and a finite Blaschke product b_0 such that $b = \prod_{k=0}^n b_k$ and $Z(b) = Z(b_1) \supset Z(b_2) \supset \cdots \supset Z(b_n)$ if and only if $Z(b)$ is an interpolation set for H^∞ , that is, if $H^\infty|_{Z(b)}$ coincides with the space of continuous functions on $Z(b)$. It is known that there exists a finite product of interpolating Blaschke products q such that $Z(q)$ is not an interpolation set for H^∞ (see [11]). We shall show that if I is an inner function such that $\text{Ord}_I(x) = k$ for every $x \in Z(I)$, then there are interpolating Blaschke products $\{b_n\}_{n=1}^k$ and a finite Blaschke product b_0 such that $I = \prod_{n=0}^k b_n$ and $Z(I) = Z(b_n)$ for $1 \leq n \leq k$ (Theorem 4).

Received November 27, 1985. Revision received February 14, 1986.
Michigan Math. J. 33 (1986).

2. Division by inner functions. For a function f in L^∞ , put $\text{dist}(f, H^\infty + C) = \inf\{\|f+h\|; h \in H^\infty + C\}$ and $\|f\|_1 = \int_{\partial D} |f| d\theta/2\pi$.

THEOREM 1. *Let I be an inner function. If f is a function in $H^\infty + C$ such that $|f| \leq |I|$ on $M(H^\infty + C)$, then $f^{k+1}\bar{I}^k \in H^\infty + C$ for every $k = 1, 2, \dots$*

To prove Theorem 1, we need a lemma.

LEMMA 1 [6, Theorem, p. 176]. *Let I be an inner function and $f \in H^\infty + C$. Then $f\bar{I}^n \in H^\infty + C$ for every positive integer n if and only if*

$$f = 0 \text{ on } \{x \in M(H^\infty + C); |I(x)| < 1\}.$$

Proof of Theorem 1. As in the proof of Theorem [6, pp. 177-179], for a small positive number ϵ there is a system Γ of simple closed rectifiable curves in \bar{D} with the following properties:

- (a) the curves in the system Γ have mutually disjoint interiors;
- (b) there is an absolute constant K (independent of I and ϵ) between 0 and 1 such that $\{z \in D; |I(z)| < \epsilon\}$ is contained in the interior of Γ and $|I| < \epsilon^K$ on Γ ;
- (c) arc length measure on $\Gamma \cap D$ is a Carleson measure such that $\int_\Gamma |h(z)| |dz| \leq K_1 \epsilon^{-2} \|h\|_1$ for every $h \in H^\infty$, where K_1 is another absolute constant.

Take a positive integer N such that $NK > 3$. We shall first prove that

$$(1) \quad (f\bar{I})^m I^N \in H^\infty + C \quad \text{for every } m = 1, 2, \dots$$

To see this, take m arbitrarily. Note $|f|^m \leq |I|^m$ on $M(H^\infty + C)$ and $\epsilon^m \leq |I|^m$ on Γ by (b). Using the corona theorem [5, p. 323], we have $|f|^m/|I|^m < 2$ on $\Gamma \cap \{z \in D; |z| > \delta\}$ for a positive number δ with $0 < \delta < 1$. Hence

$$(2) \quad \frac{|z|^n |f(z)|^m}{|I(z)|^m} < 2 \quad \text{on } \Gamma \text{ for sufficiently large } n.$$

By almost the same argument as the one in [6, pp. 177-179],

$$\begin{aligned} \text{dist}((f\bar{I})^m I^N, H^\infty + C) &= \inf_n \text{dist}((f\bar{I})^m I^N, \bar{z}^n H^\infty) \\ &= \inf_n \sup \left\{ \left| \frac{1}{2\pi} \int_\Gamma \frac{z^n f(z)^m I(z)^N h(z)}{I(z)^m} dz \right|; h \in H^\infty, \|h\|_1 = 1 \right\} \\ &\leq \sup \left\{ \frac{2}{2\pi} \int_\Gamma |I(z)|^N |h(z)| |dz|; h \in H^\infty, \|h\|_1 = 1 \right\} \text{ by (2)} \\ &\leq \sup \left\{ \frac{\epsilon^{NK}}{\pi} \int_\Gamma |h(z)| |dz|; h \in H^\infty, \|h\|_1 = 1 \right\} \text{ by (b)} \\ &\leq K_1 \epsilon^{NK-2}/\pi \text{ by (c)} \\ &\leq K_1 \epsilon/\pi \rightarrow 0 \quad (\epsilon \rightarrow 0). \end{aligned}$$

Thus we obtain (1).

We denote by B the Douglas algebra generated by H^∞ and $f\bar{I}$. By (1), we have

$$(3) \quad I^N B \subset H^\infty + C.$$

By the Chang–Marshall theorem [2; 13], for a fixed positive integer k there are sequences of H^∞ functions $\{h_n\}$ and $\{j_n\}$, with j_n inner, such that $\|f^k \bar{I}^k - \bar{j}_n h_n\| \rightarrow 0$ ($n \rightarrow \infty$) and $\bar{j}_n \in B$. By (3), $I^N \bar{j}_n^m \in H^\infty + C$ for every $m, n = 1, 2, \dots$. By Lemma 1, $I = 0$ on $\{x \in M(H^\infty + C); |j_n(x)| < 1\}$. By our assumption,

$$f = 0 \text{ on } \{x \in M(H^\infty + C); |j_n(x)| < 1\}.$$

Again by Lemma 1, $f \bar{j}_n \in H^\infty + C$ for every n . Hence

$$\begin{aligned} \text{dist}(f^{k+1} \bar{I}^k, H^\infty + C) &\leq \|f^{k+1} \bar{I}^k - f \bar{j}_n h_n\| \\ &\leq \|f\| \|f^k \bar{I}^k - \bar{j}_n h_n\| \rightarrow 0 \quad (n \rightarrow \infty). \end{aligned}$$

Thus we obtain $f^{k+1} \bar{I}^k \in H^\infty + C$ for $k = 1, 2, \dots$. □

THEOREM 2. *There are inner functions I and J such that $Z(I) \subset Z(J)$ and $J^N \bar{I} \notin H^\infty + C$ for every positive integer N .*

To prove the above theorem, we need some notations and lemmas. For a subset F of L^∞ , we denote by $[F]$ the closed subalgebra generated by F . We put $X = M(L^\infty)$, the maximal ideal space of L^∞ . The Shilov boundary of H^∞ may be identified with X [8, p. 174]. We denote by π the fiber projection from $M(H^\infty + C)$ onto ∂D ; $\pi(x) = z(x)$ for $x \in M(H^\infty + C)$. For $\lambda \in \partial D$, we put $X_\lambda = \{x \in X; \pi(x) = \lambda\}$ and $M_\lambda(H^\infty + C) = \{x \in M(H^\infty + C); \pi(x) = \lambda\}$.

For a Douglas algebra B , the pseudo-hyperbolic distance between x and y in $M(B)$, the maximal ideal space of B , is

$$\rho_B(x, y) = \sup\{|f(y)|; f \in B, \|f\| \leq 1, f(x) = 0\}.$$

If $x, y \in D$, then we have $\rho_{H^\infty}(x, y) = |y - x|/|1 - \bar{x}y|$. For $x \in M(B)$, $P_B(x) = \{y \in M(B); \rho_B(y, x) < 1\}$ is called the Gleason part of B containing x . If $P_B(x) \neq \{x\}$, $P_B(x)$ is called nontrivial. We abbreviate $\rho = \rho_{H^\infty}$ and $P = P_{H^\infty}$. For $x \in M(B)$, we denote by μ_x the unique representing measure on X for a point x , and denote by $\text{supp } \mu_x$ the closed support set of μ_x .

LEMMA 2. *Let B be a Douglas algebra. If $x \in M(B)$, then $P(x) \subset M(B)$ and $P_B(x) = P(x)$.*

Proof. By the Chang–Marshall theorem,

$$B = [H^\infty, \{\bar{I}; I \text{ is an inner function with } \bar{I} \in B\}],$$

and

$$M(B) = \{y \in M(H^\infty); |I(y)| = 1 \text{ for every inner function } I \text{ with } \bar{I} \in B\}.$$

Hence $M(B) = \{y \in M(H^\infty); B \mid \text{supp } \mu_y = H^\infty \mid \text{supp } \mu_y\}$. Let $x \in M(B)$. By [4, p. 143], $\text{supp } \mu_x = \text{supp } \mu_y$ for every $y \in P(x)$. Thus $P(x) \subset M(B)$. Since $H^\infty \subset B$, $P_B(x) \subset P(x)$. To see $P_B(x) = P(x)$, suppose $P_B(x) \subsetneq P(x)$. Then there are $y_0 \in P(x) \setminus P_B(x)$ and a sequence $\{f_n\}$ in B such that

$$f_n(x) = 0, \quad \|f_n\| = 1, \quad \text{and} \quad f_n(y_0) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Since $\text{supp } \mu_x$ is a weak peak set for H^∞ [8, p. 207], there is a sequence $\{g_n\}$ in H^∞ [4, p. 58] such that

$$g_n(x) = 0, \quad \|g_n\| = 1 \quad \text{and} \quad g_n(y_0) \rightarrow 1 \quad \text{as } n \rightarrow \infty.$$

Thus $y_0 \notin P(x)$. This contradiction leads to $P_B(x) = P(x)$.

LEMMA 3 [16]. *Let I be an inner function. Then there is an interpolating Blaschke product b with $[H^\infty, \bar{b}] = [H^\infty, \bar{I}]$.*

For a positive singular measure μ with respect to $d\theta/2\pi$,

$$S(\mu)(z) = \exp\left(-\int_{\partial D} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta)\right) \quad \text{for } z \in D$$

is called a singular inner function.

LEMMA 4. *Let $S(\mu)$ be a singular inner function and let I be a product of finitely many interpolating Blaschke products. Then $I\bar{S}(\mu) \notin H^\infty + C$.*

Proof. Let λ be a point in the support set of μ . Then there is a sequence $\{z_n\}$ in D such that $z_n \rightarrow \lambda$ and $S(\mu)(z_n) \rightarrow 0$ as $n \rightarrow \infty$. We may assume that $\{z_n\}$ is an interpolating sequence. Let x_0 be a cluster point of $\{z_n\}$ in $M(H^\infty)$. Then $x_0 \notin D$. By the work of Hoffman [9], $P(x_0)$ is nontrivial, $S(\mu) = 0$ on $P(x_0)$, and $I \neq 0$ on $P(x_0)$. To see $I\bar{S}(\mu) \notin H^\infty + C$, suppose $I\bar{S}(\mu) \in H^\infty + C$. Put $h = I\bar{S}(\mu)$. Then $I = hS(\mu)$. Hence $0 \neq I|_{P(x_0)} = hS(\mu)|_{P(x_0)} = 0$. This is a contradiction. \square

Proof of Theorem 2. Put $\lambda_n = e^{\pi i/n} \in \partial D$. Then $\lambda_n \rightarrow 1$. Let $\{a_n\}_{n=1}^\infty$ be a sequence of positive numbers with $\sum_{n=1}^\infty na_n < \infty$. Let δ_{λ_n} be the unit point mass at λ_n . Put

$$S_1 = S\left(\sum_{n=1}^\infty na_n \delta_{\lambda_n}\right) \quad \text{and} \quad S_2 = S\left(\sum_{n=1}^\infty a_n \delta_{\lambda_n}\right).$$

By Lemma 3, there is an interpolating Blaschke product b with zeros $\{z_n\}_{n=1}^\infty$ in D such that

$$(1) \quad [H^\infty, \bar{b}] = [H^\infty, \bar{S}_1].$$

Then $\pi(Z(b)) = \pi(Z(S_1)) = \pi(Z(S_2)) = \{1, \lambda_n; n = 1, 2, \dots\}$. We divide $\{z_n\}_{n=1}^\infty$ into countably many disjoint infinite subsets such that

$$(2) \quad \bigcup_{k=1}^\infty \{z_{k,n}; n = 1, 2, \dots\} = \{z_n\}_{n=1}^\infty,$$

and

$$(3) \quad z_{k,n} \rightarrow \lambda_k \quad (n \rightarrow \infty) \quad \text{for each } k.$$

Let b_k be an interpolating Blaschke product with zeros $\{z_{k,n}\}_{n=1}^\infty$. By (2) and (3), $b = \prod_{k=1}^\infty b_k$ and $[H^\infty, \bar{S}(a_k \delta_{\lambda_k})] = [H^\infty, \bar{S}(ka_k \delta_{\lambda_k})] = [H^\infty, \bar{b}_k]$. By Lemma 4, $b_k^m \bar{S}(a_k \delta_{\lambda_k}) \notin H^\infty + C$. By [14, p. 401], there is $\lambda \in \partial D$ such that $b_k^m \bar{S}(a_k \delta_{\lambda_k})|_{X_\lambda} \notin H^\infty|_{X_\lambda}$. Since $\bar{S}(a_k \delta_{\lambda_k})|_{X_\lambda}$ is constant whenever $\lambda \neq \lambda_k$, we have

$$(4) \quad b_k^m \bar{S}(a_k \delta_{\lambda_k})|_{X_{\lambda_k}} \notin H^\infty|_{X_{\lambda_k}} \quad \text{for every } m \text{ and } k.$$

Take a sequence of positive integers $\{m_k\}_{k=1}^\infty$ such that

$$\sum_{k=1}^\infty m_k \sum_{n=1}^\infty (1 - |z_{k,n}|) < \infty,$$

$$(5) \quad m_k \leq m_{k+1}, \quad \text{and} \quad m_k \rightarrow \infty \quad (k \rightarrow \infty).$$

Let ψ be a Blaschke product with zeros $\{z_{k,n}; k, n = 1, 2, \dots\}$ and corresponding multiplicities $\{m_k; k = 1, 2, \dots\}$, that is, $\psi = \prod_{k=1}^{\infty} b_k^{m_k}$. Put $I = \psi S_1$ and $J = \psi S_2$. We note that $\pi(Z(I)) = \pi(Z(J)) = \{1, \lambda_n; n = 1, 2, \dots\}$. We shall show that I and J satisfy our assertions.

To see $Z(I) \subset Z(J)$, let $x \in M(H^\infty + C)$ with $I(x) = 0$. If $\psi(x) = 0$, then evidently $J(x) = 0$. Suppose $S_1(x) = 0$. There occur two cases.

Case 1. Suppose $\pi(x) = \lambda_n$ for some n . Then $S(na_n \delta_{\lambda_n})(x) = 0$. Hence

$$S(a_n \delta_{\lambda_n})(x) = 0 \quad \text{and} \quad S_2(x) = 0.$$

Thus $J(x) = 0$.

Case 2. Suppose $\pi(x) = 1$. By (1), $|b(x)| < 1$. Then we have

$$\begin{aligned} |\psi(x)| &= \left| \left(\prod_{k=1}^{\infty} b_k^{m_k} \right)(x) \right| \\ &= \left| \left(\prod_{k=p}^{\infty} b_k^{m_k} \right)(x) \right| \quad \text{for every } p \text{ by (3)} \\ &\leq \left| \left(\prod_{k=p}^{\infty} b_k^{m_p} \right)(x) \right| \quad \text{by (5)} \\ &= |b(x)|^{m_p} \quad \text{by (3)} \\ &\rightarrow 0 \quad (p \rightarrow \infty) \quad \text{by (5)}. \end{aligned}$$

Hence $\psi(x) = 0$. Thus we get $Z(I) \subset Z(J)$.

To finish the proof, let N be a positive integer. Take a positive integer k as $k > N$. Then

$$\begin{aligned} J^N \bar{I} | X_{\lambda_k} &= \psi^N S_2^N \bar{\psi} \bar{S}_1 | X_{\lambda_k} \\ &= a b_k^{(N-1)m_k} \bar{S}((k-N)a_k \delta_{\lambda_n}) | X_{\lambda_k} \quad \text{for some constant } a \text{ with } |a| = 1 \\ &\notin H^\infty | X_{\lambda_k} \quad \text{by (4)}. \end{aligned}$$

Hence $J^N \bar{I} \notin H^\infty + C$ for every positive integer N . □

For a Douglas algebra B and $f \in B$, we put $Z_B(f) = \{x \in M(B); f(x) = 0\}$. For a subset E of ∂D , we put $L_E^\infty = \{f \in L^\infty; f \text{ is continuous at each } x \in E\}$. By [3], $H^\infty + L_E^\infty$ is a Douglas algebra.

THEOREM 3 (cf. [7, Corollary 2]). *Let $B = H^\infty + L_{\partial D \setminus \{1\}}^\infty$. Then there is an inner function I satisfying the following conditions.*

- (i) *I vanishes identically on a nontrivial Gleason part of B .*
- (ii) *If $g \in B$ with $Z_B(I) \subset Z_B(g)$, then $g\bar{I} \in B$.*

To prove Theorem 3, we need some lemmas.

LEMMA 5 [9, Lemma 4.2]. *Let b be an interpolating Blaschke product with zeros $\{z_n\}_{n=1}^\infty$. Then there exist δ ($0 < \delta < 1$) and r ($0 < r < 1$) satisfying the following properties: The set $\{z \in D; |b(z)| < r\}$ is the union of pairwise disjoint domains $V_n, z_n \in V_n$; and $V_n \subset \{z \in D; \rho(z, z_n) < \delta\}$. If $|w| < r$, then $b_w(z) =$*

$(b(z) - w)/(1 - \bar{w}b(z))$ is an interpolating Blaschke product having one zero in each V_n .

LEMMA 6 [1; 7]. Let B be a Douglas algebra and I be an interpolating Blaschke product. If $g \in B$ satisfies $Z_B(I) \subset Z_B(g)$, then $g\bar{I} \in B$.

LEMMA 7 [5, p. 314]. Let $\{z_n\}$ and $\{w_n\}$ be interpolating sequences in D . If $\inf\{\rho(z_n, w_m); n, m = 1, 2, \dots\} > 0$, then $\{z_n\} \cup \{w_m\}$ is interpolating.

Proof of Theorem 3. Put $\lambda_n = e^{\pi i/n} \in \partial D$. It is well known that there is an interpolating Blaschke product b with zeros $\{z_n\}_{n=1}^\infty$ in D such that $\pi(Z(b)) = \{1, \lambda_n; n = 1, 2, \dots\}$. We divide $\{z_n\}$ into countably many disjoint infinite subsets as

$$(1) \quad \bigcup_{k=1}^\infty \{z_{k,n}; n = 1, 2, \dots\} = \{z_n\}_{n=1}^\infty,$$

and

$$(2) \quad z_{k,n} \rightarrow \lambda_k \quad (n \rightarrow \infty) \quad \text{for each } k.$$

We may assume

$$(3) \quad \sum_{k=1}^\infty k \sum_{n=1}^\infty (1 - |z_{k,n}|) < \infty.$$

By Lemma 5, there exist δ and r ($0 < \delta < 1$, $0 < r < 1$) and there is a sequence of pairwise disjoint domains $\{V_{k,n}; k, n = 1, 2, \dots\}$ such that $z_{k,n} \in V_{k,n}$,

$$(4) \quad V_{k,n} \subset \{z \in D; \rho(z, z_{k,n}) < \delta\},$$

and

$$(5) \quad \text{if } |w| < r, \text{ then } b_w(z) = (b(z) - w)/(1 - \bar{w}b(z)) \text{ is an interpolating Blaschke product having one zero in each } V_{k,n}.$$

Let $\{w_n\}$ be a distinct sequence of complex numbers with $w_1 = 0$, $|w_n| < r$, and $w_n \rightarrow 0$ ($n \rightarrow \infty$). For i and k with $1 \leq i \leq k$, let $\zeta_{k,i,n}$ be the point in $V_{k,n}$ with $b_{w_i}(\zeta_{k,i,n}) = 0$. By (5), $\{\zeta_{k,i,n}\}_{n=1}^\infty$ is an interpolating sequence for each k and i with $1 \leq i \leq k$. Let i and j with $i \neq j$ and $1 \leq i, j \leq k$. Since $b_{w_i}(\zeta_{k,i,n}) = 0$, $b(\zeta_{k,i,n}) = w_i$. Hence for $n, m = 1, 2, \dots$,

$$\begin{aligned} \rho(\zeta_{k,i,n}, \zeta_{k,j,m}) &\geq |b_{w_j}(\zeta_{k,i,n})| \\ &= \left| \frac{w_i - w_j}{1 - \bar{w}_j w_i} \right| > 0 \quad \text{by (5)}. \end{aligned}$$

By Lemma 7, for each fixed k , $\{\zeta_{k,i,n}; 1 \leq i \leq k, n = 1, 2, \dots\}$ is an interpolating sequence. Let b_k be the interpolating Blaschke product with zeros $\{\zeta_{k,i,n}; 1 \leq i \leq k, n = 1, 2, \dots\}$. Since $\zeta_{k,i,n} \in V_{k,n}$, by (4) we have

$$\begin{aligned} \delta > \rho(\zeta_{k,i,n}, z_{k,n}) &= \left| \frac{z_{k,n} - \zeta_{k,i,n}}{1 - \bar{\zeta}_{k,i,n} z_{k,n}} \right| \\ &\geq \frac{|z_{k,n}| - |\zeta_{k,i,n}|}{1 - |\zeta_{k,i,n}| |z_{k,n}|}. \end{aligned}$$

By elementary calculations, $1 - |\zeta_{k,i,n}| \leq ((1 + \delta)/(1 - \delta))(1 - |z_{k,n}|)$. Then

$$\begin{aligned} & \sum_{k=1}^{\infty} \sum \{1 - |\zeta_{k,i,n}|; 1 \leq i \leq k, n = 1, 2, \dots\} \\ & \leq \frac{1 + \delta}{1 - \delta} \sum_{k=1}^{\infty} \sum \{1 - |z_{k,n}|; 1 \leq i \leq k, n = 1, 2, \dots\} \\ & = \frac{1 + \delta}{1 - \delta} \sum_{k=1}^{\infty} k \sum \{1 - |z_{k,n}|; n = 1, 2, \dots\} \\ & < \infty \quad \text{by (3)}. \end{aligned}$$

Hence $\prod_{k=1}^{\infty} b_k$ is a Blaschke product. Put $I = \prod_{k=1}^{\infty} b_k$ and we shall show that I is a desired inner function. By (2) and (4), $\pi(Z(I)) = \{1, \lambda_n; n = 1, 2, \dots\}$. Since $w_1 = 0$, $I\bar{b} \in H^\infty$, so $Z(b) \subset Z(I)$. Take a sequence $\{x_k\}_{k=1}^{\infty}$ in $Z(b)$ with $\pi(x_k) = \lambda_k$. Then $x_k \in M(B)$. Let x be a cluster point of $\{x_k\}_{k=1}^{\infty}$. Then $x \in Z(b)$ and $\pi(x) = 1$. By the work of Hoffman [9], $P(x)$ is a nontrivial Gleason part and $I = 0$ on $P(x)$.

To see (i), it is sufficient to prove $x \in M(B)$. Since

$$B = [H^\infty, \{\bar{J}; J \text{ is an inner function with } Z(J) \subset M_1(H^\infty + C)\}],$$

we have $|J(x)| = 1$ for every inner function J with $\bar{J} \in B$. Because if $|J(x)| < 1$ then $|J(x_k)| < 1$ for some k , so $\bar{J} \notin B$. Thus we obtain $x \in M(B)$, hence $P_B(x) = P(x) \subset M(B)$ by Lemma 2.

To see (ii), let $g \in B$ with $Z_B(g) \supset Z_B(I)$. Let $\lambda \in \partial D$ with $\lambda \neq 1$. If $\lambda \neq \lambda_n$ for every n , then $\bar{I} | X_\lambda$ is constant, so $g\bar{I} | X_\lambda \in H^\infty | X_\lambda$. Suppose that $\lambda = \lambda_n$ for some n . Then $I | X_{\lambda_n} = cb_n | X_{\lambda_n}$ for some constant c with $|c| = 1$. Put

$$B_n = \{f \in L^\infty; f | X_{\lambda_n} \in H^\infty | X_{\lambda_n}\}.$$

Then B_n is a Douglas algebra. Since $Z_{B_n}(g) \supset Z_{B_n}(I) = Z_{B_n}(b_n)$, by Lemma 6 we have $g\bar{b}_n \in B_n$. Hence $g\bar{I} | X_{\lambda_n} = cg\bar{b}_n | X_{\lambda_n} \in H^\infty | X_{\lambda_n}$. Thus we obtain $g\bar{I} \in B$, because $B = \{f \in L^\infty; f | X_\lambda \in H^\infty | X_\lambda \text{ for every } \lambda \text{ with } \lambda \neq 1\}$. □

In the last part of this section, we shall give some comments. Let B be a Douglas algebra with $B \supset H^\infty + C$. An inner function I is called B -interpolating if there is an interpolating Blaschke product b such that $|b| = |I|$ on $M(B)$. Then we have the following.

PROPOSITION. *Let I be an inner function. If I is B -interpolating, then I satisfies the following condition.*

(#) *If J is an inner function with $Z_B(I) \subset Z_B(J)$, then $|J| \leq |I|$ on $M(B)$.*

Proof. Let b be an interpolating Blaschke product with $|b| = |I|$ on $M(B)$. Then $Z_B(b) = Z_B(I)$. Let J be an inner function with $Z_B(I) \subset Z_B(J)$. By Lemma 6, $J\bar{b} \in B$. Put $h = J\bar{b}$; then h is unimodular and $J = bh$. Thus $|J| \leq |b||h| \leq |b| = |I|$ on $M(B)$. □

Theorem 3 says that the converse of this proposition is not true for $B = H^\infty + L_{\partial D \setminus \{1\}}^\infty$. To see this, let I be an inner function in Theorem 3. By (i) of Theorem 3, I is not B -interpolating. By (ii) of Theorem 3, I satisfies (#).

We do not know whether the converse of the proposition is true or not for $B = H^\infty + C$. We note that by [10, Theorem 1], if I is an $(H^\infty + C)$ -interpolating inner function then $I = b_0 b_1$, where b_0 is a finite Blaschke product and b_1 is an interpolating Blaschke product.

3. Factorization of Blaschke products. Let $h \in H^\infty$ and $x \in M(H^\infty + C)$ with $h(x) = 0$. If $P(x)$ is a nontrivial Gleason part, $P(x)$ carries the structure of an analytic disk, so we can define $\text{Ord}_h(x)$, the order of zero of h at x . If $P(x)$ is trivial, we define $\text{Ord}_h(x) = \infty$.

THEOREM 4. *Let I be an inner function with $\text{Ord}_I(x) = k$ for every $x \in Z(I)$. Then there are interpolating Blaschke products $\{b_n\}_{n=1}^k$ and a finite Blaschke product b_0 such that $I = \prod_{n=0}^k b_n$ and $Z(I) = Z(b_n)$ for $1 \leq n \leq k$.*

To see our theorem, we need two lemmas.

LEMMA 8. *Let b be an inner function, let $\{x_j\}_{j=1}^\infty$ be a sequence in $Z(b)$ such that $\text{Ord}_b(x_j) = n$ for every j , and let x_0 be a cluster point of $\{x_j\}_{j=1}^\infty$ in $M(H^\infty + C)$. If $\{y_j\}_{j=1}^\infty$ is a sequence in $Z(b)$ such that $\{x_j\}_{j=1}^\infty \cap \{y_j\}_{j=1}^\infty = \emptyset$, $\text{Ord}_b(y_j) = m$ for every j , and $\rho(x_j, y_j) \rightarrow 0$ as $j \rightarrow \infty$, then $\text{Ord}_b(x_0) \geq n + m$.*

Proof. It is enough to prove this for the case $\text{Ord}_b(x_0) < \infty$. Let $k = \text{Ord}_b(x_0)$. By [9, Theorem 5.3], there are interpolating Blaschke products b_1, \dots, b_k and an inner function b_0 such that $b = \prod_{j=0}^k b_j$ and $b_j(x_0) = 0$ for $1 \leq j \leq k$. Since x_0 is a cluster point of $\{x_j\}_{j=1}^\infty$, we may assume that $b_0(x_j) \neq 0$ for any j . Moreover, since $\rho(x_j, y_j) \rightarrow 0$ as $j \rightarrow \infty$, we may assume that $b_0(y_j) \neq 0$ for any j .

For each j , $Z(b_j)$ is an interpolation set for H^∞ [8, p. 205]; thus there exists $\epsilon_j > 0$ such that $\rho(x, y) \geq \epsilon_j$ for any $x, y \in Z(b_j)$. Let $\epsilon = \min_{1 \leq j \leq k} \epsilon_j$. Without loss of generality we may assume that $\rho(x_j, y_j) \leq \epsilon/2$ for all j . By assumption, b has a zero of order n at x_j . Since $b_0(x_j) \neq 0$, there exist n factors of b other than b_0 which vanish at x_j . None of these factors can vanish at y_j , for $x, y \in Z(b_\ell)$ forces $\rho(x, y) \geq \epsilon_\ell$ and $\rho(x_j, y_j) \leq \epsilon/2 \leq \epsilon_\ell/2$. Hence m factors of b distinct from these and b_0 must vanish at y_j . Therefore $k \geq n + m$. \square

LEMMA 9 [10]. *If I is an inner function such that $Z(I)$ is an interpolation set for H^∞ , then there is an interpolating Blaschke product b such that $I\bar{b} \in H^\infty$ and $Z(I) = Z(b)$.*

Proof of Theorem 4. Let I be an inner function with $\text{Ord}_I(x) = k$ for every $x \in Z(I)$. By [12], there are interpolating Blaschke products $\{\psi_1, \psi_2, \dots, \psi_n\}$ and a finite Blaschke product ψ_0 such that $I = \prod_{i=0}^n \psi_i$. By Lemma 8, there is $\epsilon > 0$ such that $\rho(x, y) \geq \epsilon$ for every $x, y \in Z(I)$ with $x \neq y$. By Varopoulos [15], $Z(I) = \bigcup_{i=1}^n Z(\psi_i)$ is an interpolation set for H^∞ . By Lemma 9, there is an interpolating Blaschke product b_1 such that $I\bar{b}_1 \in H^\infty$ and $Z(b_1) = Z(I)$. Applying Lemma 9 k times, there are interpolating Blaschke products $\{b_1, b_2, \dots, b_k\}$ such that $I\bar{b}_1\bar{b}_2 \cdots \bar{b}_j \in H^\infty$ and $Z(b_j) = Z(I)$ for $1 \leq j \leq k$. Since $Z(I \prod_{j=1}^k \bar{b}_j) = \emptyset$, $b_0 = I \prod_{j=1}^k \bar{b}_j$ is a finite Blaschke product. This completes the proof. \square

The authors would like to thank the referee for shortening the proof of Lemma 8.

REFERENCES

1. S. Axler and P. Gorkin, *Divisibility in Douglas algebras*, Michigan Math. J. 31 (1984), 89–94.
2. S.-Y. Chang, *A characterization of Douglas subalgebras*, Acta Math. 137 (1976), 81–89.
3. A. M. Davie, T. W. Gamelin, and J. Garnett, *Distance estimates and pointwise bounded density*, Trans. Amer. Math. Soc. 175 (1973), 37–68.
4. T. W. Gamelin, *Uniform algebras*, Prentice-Hall, Englewood Cliffs, N.J., 1969.
5. J. Garnett, *Bounded analytic functions*, Academic Press, New York, 1981.
6. C. Guillory and D. Sarason, *Division in $H^\infty + C$* , Michigan Math. J. 28 (1981), 173–181.
7. C. Guillory, K. Izuchi, and D. Sarason, *Interpolating Blaschke products and division in Douglas algebras*, Proc. Roy. Irish Acad. Sect. A 84A (1984), 1–7.
8. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N.J., 1962.
9. ———, *Bounded analytic functions and Gleason parts*, Ann. of Math. (2) 86 (1967), 74–111.
10. K. Izuchi, *Zero sets of interpolating Blaschke products*, Pacific J. Math. 119 (1985), 337–342.
11. K. Izuchi and Y. Izuchi, *Annihilating measures for Douglas algebras*, Yokohama Math. J. 32 (1984), 135–151.
12. A. Kerr-Lawson, *Some lemmas on interpolating Blaschke products and a correction*, Canad. J. Math. 21 (1969), 531–534.
13. D. Marshall, *Subalgebras of L^∞ containing H^∞* , Acta Math. 137 (1976), 91–98.
14. D. Sarason, *Functions of vanishing mean oscillation*, Trans. Amer. Math. Soc. 207 (1975), 391–405.
15. N. Th. Varopoulos, *Sur la réunion de deux ensembles d'interpolation d'une algèbre uniforme*, C. R. Acad. Sci. Paris Sér. A-B 272 (1971), A950–A952.
16. R. Younis, *Division in Douglas algebras and some applications*, Arch. der Math. 45 (1985), 555–560.

Kanagawa University
Yokohama 221, Japan

and

Kanagawa Dental College
Yokosuka 238, Japan