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INNER PRODUCT MODULES OVER B^*-ALGEBRAS

BY

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ABSTRACT. This paper is an investigation of right modules over a B^*-algebra B which posses a B-valued "inner product" respecting the module action. Elementary properties of these objects, including their normability and a characterization of the bounded module maps between two such, are established at the beginning of the exposition. The case in which B is a W^*-algebra is of especial interest, since in this setting one finds an abundance of inner product modules which satisfy an analog of the self-duality property of Hilbert space. It is shown that such self-dual modules have important properties in common with both Hilbert spaces and W^*-algebras. The extension of an inner product module over B by a B^*-algebra A containing B as a *-subalgebra is treated briefly. An application of some of the theory described above to the representation and analysis of completely positive maps is given.

1. **Introduction and conventions.** In this paper we investigate right modules over a B^*-algebra B which possess a B-valued "inner product" respecting the module action. These objects, which we call *pre-Hilbert B-modules*, are defined in the same way as I. Kaplansky's "C^*-modules" [4], but without the restriction that B be commutative. Our definition of a pre-Hilbert B-module also coincides with that of a "right B-rigged space" as recently introduced by M. A. Rieffel [7] except for his requirement that the range of the inner product generate a dense subalgebra of B. Fields of inner product modules have been studied by A. Takahashi [10]; for a discussion of some of this work we refer the reader to §8 of [2]. Pre-Hilbert B-modules and related objects appear to be useful in a variety of ways. The application which we will give concerns the representation and analysis of completely positive maps of U^*-algebras into B^*-algebras.

Our exposition begins with a section setting forth the elementary properties of pre-Hilbert B-modules. We show that these can be normed in a natural way, with norm and B-valued inner product related by an analog of the Cauchy-Schwarz inequality. For a B^*-algebra A containing B as a *-subalgebra, we give a characterization in terms of B- and A-valued inner products of bounded B-module maps

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from a pre-Hilbert B-module into a pre-Hilbert A-module. In §3, we investigate the case in which B is a W^*-algebra and show that in this setting the B-valued inner product on a pre-Hilbert B-module X lifts to a B-valued inner product on the right B-module X' of bounded module maps of X into B. The inner product module X' so obtained turns out to be *self-dual* in the sense that each bounded module map of X' into B arises by taking inner products with a fixed element of X'. (It is in some sense more difficult to produce self-dual inner product modules than to produce self-dual inner product spaces, since completeness of a pre-Hilbert B-module in its natural norm is in general insufficient for self-duality.) Such self-dual modules have properties in common with both Hilbert spaces and W^*-algebras. We show that they are all conjugate spaces and that the algebra of bounded module maps of such a module into itself is a W^*-algebra. A polar decomposition theorem for self-dual modules over a W^*-algebra is established and used to obtain an orthogonal direct sum decomposition for such modules. §4 treats the extension of a pre-Hilbert B-module by a B^*-algebra A containing B as a *-subalgebra. In §5 we show that a completely positive map from a *-algebra into a B^*-algebra B gives rise to a pre-Hilbert B-module in much the same way that a positive linear functional on a *-algebra gives rise to a pre-Hilbert space. The standard method of representing positive linear functionals via inner product spaces thus generalizes to a method of representing completely positive maps via inner product modules. Following W. B. Arveson's treatment of completely positive maps into the algebra of bounded operators on a Hilbert space [1], we use this representation scheme to characterize the order structure of the set of completely positive maps from a U^*-algebra with 1 into an arbitrary W^*-algebra. §6 is an appendix in which we establish an elementary but useful result on the positivity of matrices with entries in a B^*-algebra.

We make the following conventions. All algebras and linear spaces considered here are over the complex field C. An algebra with involution $a \longrightarrow a^*$ will be a **-algebra*. A map between *-algebras which respects their involutions will be called a **-map*. It is not assumed that all algebras herein possess a multipilcative identity; we will say "A has 1" if the algebra A has a multiplicative identity 1 and call A an "algebra with 1". If A is an algebra without 1, we let A^1 denote the algebra obtained by adjoining 1 to A. The identity operator on a linear space X will be denoted by I or I_X, depending on whether any possibility of ambiguity exists. The algebra of bounded linear operators on a normed linear space X will be denoted by $B(X)$ and we will write X^* for the conjugate space of X. We will denote the action of an algebra A on a right A-module X by $(x, a) \longrightarrow x \cdot a$; all such modules treated below will be assumed to have a vector space structure "compatible" with that of A in the sense that $\lambda(x \cdot a) = (\lambda x) \cdot a = x \cdot (\lambda a)\ \forall x \in X,\ a \in A,\ \lambda \in C$.

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2. **Elementary properties of pre-Hilbert and Hilbert B-modules.** Let B be a B^*-algebra.

2.1 **Definition.** A *pre-Hilbert B-module* is a right B-module X equipped with a conjugate-bilinear map $\langle \cdot, \cdot \rangle$: $X \times X \rightarrow B$ satisfying:

(i) $\langle x, x\rangle \geq 0$ $\forall x \in X$;

(ii) $\langle x, x\rangle = 0$ only if $x = 0$;

(iii) $\langle x, y\rangle = \langle y, x\rangle^*$ $\forall x, y \in X$;

(iv) $\langle x \cdot b, y\rangle = \langle x, y\rangle b$ $\forall x, y \in X$, $b \in B$.

The map $\langle \cdot, \cdot \rangle$ will be called a *B-valued inner product* on X.

Examples of such objects are numerous. If J is a right ideal of B, then J becomes a pre-Hilbert B-module when we define $\langle \cdot, \cdot \rangle$ by $\langle x, y\rangle = y^*x$ for $x, y \in J$. More generally, if $\{J_\alpha\}$ is a collection of right ideals of B, then the space X of all tuples $\{x_\alpha\}$ with $x_\alpha \in J_\alpha$ $\forall \alpha$ and $\Sigma_\alpha \|x_\alpha\|^2 < \infty$ becomes a right B-module when we define $\{x_\alpha\} \cdot b = \{x_\alpha b\}$ for $\{x_\alpha\} \in X$, $b \in B$, and a pre-Hilbert B-module when we set $\langle \{x_\alpha\}, \{y_\alpha\}\rangle = \Sigma_\alpha y_\alpha^* x_\alpha$ for $\{x_\alpha\}, \{y_\alpha\} \in X$. One checks easily that if H is a Hilbert space, then the algebraic tensor product $H \otimes B$, which is naturally a right B-module, admits a B-valued inner product $\langle \cdot, \cdot \rangle$ defined on elementary tensors by

$$\langle \xi \otimes a, \eta \otimes b\rangle = (\xi, \eta) b^* a.$$

We will see in §5 that pre-Hilbert B-modules can be constructed from completely positive maps of *-algebras into B in much the same way that pre-Hilbert spaces can be constructed from positive linear functionals on *-algebras.

Notice that if B has 1 and X is a pre-Hilbert B-module, then X is automatically unital, i.e. $x \cdot 1 = x$ $\forall x \in X$; this is because $\langle x \cdot 1, y\rangle = \langle x, y\rangle 1 = \langle x, y\rangle$ $\forall x, y \in X$. If B does not have 1, we can make X into a right module over the B^*-algebra B^1 in the obvious way. X is then clearly a pre-Hilbert B^1-module. The presence or absence of 1 in B will thus be of little importance in much of what follows. We also note in passing that $\langle x, y \cdot b\rangle = b^*\langle x, y\rangle$ $\forall x, y \in X$, $b \in B$; this follows from (iii) and (iv) of 2.1.

2.2 **Remark.** Suppose Y is a right B-module equipped with a conjugate-bilinear map $[\cdot, \cdot]$: $Y \times Y \rightarrow B$ satisfying (i), (iii), and (iv) of 2.1. Let $N = \{x \in Y: [x, x] = 0\}$. For each positive linear functional f on B, the map $(x, y) \rightarrow f([x, y])$ is a pseudo inner product (positive semidefinite hermitian conjugate-bilinear form) on Y, and it follows that $N_f = \{x \in Y: f([x, x]) = 0\}$ is a linear subspace of Y. N, being the intersection of all such N_f's, is thus a linear subspace of Y. We see from (iii) and (iv) that $N \cdot B \subseteq N$, so N is a submodule of Y. Let $X = Y/N$, so X is naturally a right B-module. The map $\langle \cdot, \cdot \rangle$: $X \times X \rightarrow B$ given by $\langle x + N, y + N\rangle = [x, y]$ is a (well-defined) B-valued inner product on X.

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For a pre-Hilbert B-module X, define $\|\cdot\|_X$ on X by $\|x\|_X = \|\langle x, x\rangle\|^{1/2}$.

2.3 **Proposition.** *$\|\cdot\|_X$ is a norm on X and satisfies:*
(i) $\|x \cdot b\|_X \leq \|x\|_X \|b\| \ \forall x \in X, b \in B$;
(ii) $\langle y, x\rangle\langle x, y\rangle \leq \|y\|_X^2 \langle x, x\rangle \ \forall x, y \in X$;
(iii) $\|\langle x, y\rangle\| \leq \|x\|_X \|y\|_X \ \forall x, y \in X$.

Proof. For each positive linear functional f on B, the map $(x, y) \to f(\langle x, y\rangle)$ is a pseudo inner product on X, whence it follows that $x \to f(\langle x, x\rangle)^{1/2}$ is a pseudonorm on X. We have

$$\|x\|_X = \|\langle x, x\rangle\|^{1/2} = \sup\{f(\langle x, x\rangle)^{1/2} : f \text{ a state of } B\}$$

for each $x \in X$; this exhibits $\|\cdot\|_X$ as the pointwise supremum of a collection of pseudonorms on X, so $\|\cdot\|_X$ is a pseudonorm, and hence, in light of (ii) of 2.1, a norm on X.

Item (i) of the proposition is established by a direct computation. For $x \in X$, $b \in B$, we have $\|x \cdot b\|_X^2 = \|\langle x \cdot b, x \cdot b\rangle\| = \|b^*\langle x, x\rangle b\| \leq \|b\|^2 \|\langle x, x\rangle\| = \|x\|_X^2 \|b\|^2$.

For (ii), take $x, y \in X$ and f a positive linear functional on B. Using the Cauchy-Schwarz inequality for the pseudo inner product $f(\langle\cdot, \cdot\rangle)$ on X, we compute

$$\begin{aligned} f(\langle y, x\rangle \langle x, y\rangle) &= f(\langle y \cdot \langle x, y\rangle, x\rangle) \\ &\leq f(\langle y \cdot \langle x, y\rangle, y \cdot \langle x, y\rangle\rangle)^{1/2} f(\langle x, x\rangle)^{1/2} \\ &= f(\langle y, x\rangle\langle y, y\rangle\langle x, y\rangle)^{1/2} f(\langle x, x\rangle)^{1/2} \\ &\leq \|\langle y, y\rangle\|^{1/2} f(\langle y, x\rangle\langle x, y\rangle)^{1/2} f(\langle x, x\rangle)^{1/2} \end{aligned}$$

so $f(\langle y, x\rangle\langle x, y\rangle) \leq \|y\|_X^2 f(\langle x, x\rangle)$. Since this holds for every positive linear functional f on B, (ii) follows.

Item (iii) is an immediate consequence of (ii).

We remark that 2.3 is also proved in §2 of [7].

2.4 **Definition.** A pre-Hilbert B-module X which is complete with respect to $\|\cdot\|_X$ will be called a *Hilbert B-module*.

2.5 **Remark.** If X is a pre-Hilbert B-module, $\tilde{X}$ its completion with respect $\|\cdot\|_X$, it follows easily from 2.3 that the module action of B on X and the B-valued inner product on X extend to $\tilde{X}$ in such a way as to make $\tilde{X}$ a Hilbert B-module.

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We now introduce a natural B-module analogue of the algebra of bounded operators on a Hilbert space. For a pre-Hilbert B-module X, we let $\mathfrak{A}(X)$ denote the set of operators $T \in B(X)$ for which there is an operator $T^* \in B(X)$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle\ \forall x, y \in X$. That is, $\mathfrak{A}(X)$ is the set of bounded operators on X which possess bounded adjoints with respect to the B-valued inner product. It is easy to see that for $T \in \mathfrak{A}(X)$, the adjoint T^* is unique and belongs to $\mathfrak{A}(X)$, so $\mathfrak{A}(X)$ is a $*$-algebra with involution $T \to T^*$. Without risk of confusion, we denote the operator norm on $B(X)$ by $\|\cdot\|_X$. A routine computation establishes that $\|T^*T\|_X = \|T\|_X^2\ \forall T \in \mathfrak{A}(X)$. If X is a Hilbert B-module, it is straightforward to show that $\mathfrak{A}(X)$ is closed in $B(X)$, so in this case $\mathfrak{A}(X)$ is a B^*-algebra.

The algebra $\mathfrak{A}(X)$ consists entirely of module maps, i.e. if $T \in \mathfrak{A}(X)$, then $T(x \cdot b) = (Tx) \cdot b\ \forall x \in X,\ b \in B$. To see this, take $y \in X$ and observe that $\langle T(x \cdot b), y\rangle = \langle x \cdot b, T^*y\rangle = \langle x, T^*y\rangle b = \langle (Tx) \cdot b, y\rangle$. This is enough to show that $T(x \cdot b) = (Tx) \cdot b$. One might guess by analogy with Hilbert space that every module map in $B(X)$ belongs to $\mathfrak{A}(X)$ when X is complete. This is not the case, however, as the following example shows. Suppose that J is a closed right ideal of a B^*-algebra B with 1 such that no element of J^* acts as a left multiplicative identity on J. (For instance, B could be the algebra of complex valued continuous functions on the unit interval, J the ideal of functions in B which vanish at 0.) Let X be the right B-module $J \times B$ with B-valued inner product defined by $\langle (a_1, b_1), (a_2, b_2)\rangle = a_2^*a_1 + b_2^*b_1$ for $a_1, a_2 \in J$, $b_1, b_2 \in B$. For $(a, b) \in X$ we have $\|(a, b)\|_X = \|a^*a + b^*b\|^{1/2}$, so

$$\max\{\|a\|, \|b\|\} \le \|(a, b)\|_X \le (\|a\|^2 + \|b\|^2)^{1/2},$$

whence it follows that X is complete with respect to $\|\cdot\|_X$. Define $T \in B(X)$ by $T(a, b) = (0, a)$ for $(a, b) \in X$. T is clearly a module map, but we claim that $T \notin \mathfrak{A}(X)$. For suppose that T has an adjoint T^* and let $T^*(0, 1) = (\alpha, \beta)$. For any $(a, b) \in X$ we have $a = \langle T(a, b), (0, 1)\rangle = \langle (a, b), (\alpha, \beta)\rangle = \alpha^*a + \beta^*b$. From this we see that $\beta = 0$ and $\alpha^*a = a\ \forall a \in J$. But $\alpha^* \in J^*$, and this contradicts our assumption about J. Hence $T \notin \mathfrak{A}(X)$. We remark that although $\mathfrak{A}(X)$ need not contain all bounded module maps of X into itself, it always contains nontrivial operators if X is nontrivial. For instance, we may take $x, y \in X$ and define $x \otimes y \in B(X)$ by $x \otimes y(w) = x \cdot \langle w, y\rangle$ for $w \in X$. It is easy to see that $x \otimes y \in \mathfrak{A}(X)$ with $(x \otimes y)^* = y \otimes x$.

2.6 **Proposition.** *For $T \in \mathfrak{A}(X)$, we have $\langle Tx, Tx\rangle \le \|T\|_X^2\langle x, x\rangle\ \forall x \in X$.*

Proof. Take $x \in X$ and let f be a positive linear functional on B. Repeated application of the Cauchy-Schwarz inequality for the pseudo inner product $f(\langle \cdot, \cdot\rangle)$ on X yields

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$$\begin{aligned} f(\langle Tx, Tx\rangle) &= f(\langle T^*Tx, x\rangle) \\ &\le f(\langle T^*Tx, T^*Tx\rangle)^{1/2} f(\langle x, x\rangle)^{1/2} \\ &\le f(\langle (T^*T)^2x, (T^*T)^2x\rangle)^{1/4} f(\langle x, x\rangle)^{1/2+1/4} \\ &\ \ \vdots \\ &\le f(\langle (T^*T)^{2^n}x, (T^*T)^{2^n}x\rangle)^{2^{-n}} f(\langle x, x\rangle)^{1/2+\cdots+2^{-n}} \\ &\le (\|f\|\,\|x\|^2)^{2^{-n}} \|T\|_X^2 f(\langle x, x\rangle)^{1/2+\cdots+2^{-n}} \end{aligned}$$

for $n = 1, 2, \cdots$, and in the limit we have

$$f(\langle Tx, Tx\rangle) \le \|T\|_X^2 f(\langle x, x\rangle),$$

as desired.

For the balance of this section, A will be a B^*-algebra, B a closed $*$-subalgebra of A, X a pre-Hilbert B-module, and Y a pre-Hilbert A-module. Denote the B- and A-valued inner products on X and Y by $\langle\cdot,\cdot\rangle_B$ and $\langle\cdot,\cdot\rangle_A$, respectively. Notice that Y is a right B-module. We will give a characterization of the bounded B-module maps of X into Y in terms of the inner products $\langle\cdot,\cdot\rangle_B$ and $\langle\cdot,\cdot\rangle_A$. To avoid unnecessary complications, we assume that A has 1 and that $1 \in B$. (Otherwise, we could regard Y as a pre-Hilbert A^1-module and X as a pre-Hilbert B'-module, where B' is the subalgebra of A^1 generated by 1 and B.) We begin by dealing with maps of B into A.

2.7 **Proposition.** *Let $\tau\colon B \to A$ be a linear map such that for some real $K \ge 0$ we have $\tau(x)^*\tau(x) \le Kx^*x$ $\forall x \in B$. Then $\tau(x) = \tau(1)x$ $\forall x \in B$.*

Proof. For each $x \in B$, we have $x^*\tau(1)^*\tau(1)x \le \|\tau(1)\|^2x^*x \le Kx^*x$, and (since $(\tau(x) + \tau(1)x)^*(\tau(x) + \tau(1)x) \ge 0$)

$$-(x^*\tau(1)^*\tau(x) + \tau(x)^*\tau(1)x) \le \tau(x)^*\tau(x) + x^*\tau(1)^*\tau(1)x \le 2Kx^*x,$$

so

$$(\tau(x) - \tau(1)x)^*(\tau(x) - \tau(1)x) \le 2Kx^*x - (x^*\tau(1)^*\tau(x) + \tau(x)^*\tau(1)x) \le 4Kx^*x.$$

Define $\tau_0\colon B \to A$ by $\tau_0(x) = (2K^{1/2})^{-1}(\tau(x) - \tau(1)x)$, so $\tau_0(1) = 0$ and $\tau_0(x)^*\tau_0(x) \le x^*x$ $\forall x \in B$. We must show that $\tau_0 = 0$.

We may assume that $A = B(H)$ for some Hilbert space H, so B is a closed $*$-subalgebra of $B(H)$ with $I \in B$. For $T \in B$, $\xi \in H$, we have $\tau_0(T)^*\tau_0(T) \le T^*T$ and hence $\|\tau_0(T)\xi\| \le \|T\xi\|$. From this it follows routinely that τ_0 extends to a linear map—call the extension τ_0 also—from B'' (the strong operator closure of B in $B(H)$) into $B(H)$ with the property that $\tau_0(T)^*\tau_0(T) \le T^*T$ $\forall T \in B''$. For any projection $P \in B''$, we have $\tau_0(P)^*\tau_0(P) \le P$ and also $\tau_0(P)^*\tau_0(P) = \tau_0(I - P)^*\tau_0(I - P) \le I - P$, forcing $\tau_0(P) = 0$. Since B'' is a W^*-algebra, it is the

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closed linear span of its projections, so we must have $\tau_0 = 0$ and the proof is complete.

It should be mentioned that in the case $B = A$, 2.7 follows from a result of B. E. Johnson [3].

2.8 **Theorem.** *For a linear map $T\colon X \to Y$ the following are equivalent.*

(i) *T is bounded and $T(x \cdot b) = (Tx) \cdot b$ $\forall x \in X$, $b \in B$.*

(ii) *There is a real $K \geq 0$ such that $\langle Tx, Tx\rangle_A \leq K\langle x, x\rangle_B$ $\forall x \in X$.*

Proof. To see that (i) implies (ii), assume that $T(x \cdot b) = (Tx) \cdot b$ $\forall x \in X$, $b \in B$ and that $\|T\| \leq 1$. We will show that in this case, $\langle Tx, Tx\rangle_A \leq \langle x, x\rangle_B$ $\forall x \in X$. Take $x \in X$ and for $n = 1, 2, \cdots$ set $b_n = (\langle x, x\rangle_B + n^{-1})^{-1/2}$ and $x_n = x \cdot b_n$. We have $\langle x_n, x_n\rangle_B = (\langle x, x\rangle_B\langle x, x\rangle_B + n^{-1})^{-1} \leq 1$, so $\|x_n\|_X \leq 1$, so $\|Tx_n\|_Y \leq 1$, so $\langle Tx_n, Tx_n\rangle_A \leq 1$ for $n = 1, 2, \cdots$. But $\langle Tx_n, Tx_n\rangle_A = b_n\langle Tx, Tx\rangle_A b_n$, so $\langle Tx, Tx\rangle_A \leq b_n^{-2} = \langle x, x\rangle_B + n^{-1}$ for $n = 1, 2, \cdots$, and hence $\langle Tx, Tx\rangle_A \leq \langle x, x\rangle_B$.

For the other direction, we assume that $\langle Tx, Tx\rangle_A \leq \langle x, x\rangle_B$ $\forall x \in X$, so clearly T is bounded with $\|T\| \leq 1$. Take $x \in X$, $y \in Y$, and consider the map $\tau\colon B \to A$ given by $\tau(b) = \langle T(x \cdot b), y\rangle_A$ for $b \in B$. Appealing to 2.3, we have

$$\begin{aligned}\tau(b)^*\tau(b) &= \langle y, T(x \cdot b)\rangle_A \langle T(x \cdot b), y\rangle_A \\ &\leq \|y\|_Y^2\langle T(x \cdot b), T(x \cdot b)\rangle_A \leq \|y\|_Y^2\langle x \cdot b, x \cdot b\rangle_B \\ &= \|y\|_Y^2 b^*\langle x, x\rangle_B b \leq \|y\|_Y^2\|x\|_X^2 b^*b \qquad \forall b \in B\end{aligned}$$

and hence by 2.7, $\tau(b) = \tau(1)b$ $\forall b \in B$, i.e. $\langle T(x \cdot b), y\rangle_A = \langle Tx, y\rangle_A b = \langle (Tx) \cdot b, y\rangle_A$ $\forall b \in B$. As x and y were arbitrary, (i) holds and the proof is complete.

2.9 **Remark.** It follows from the proof of 2.8 that for a bounded B-module map $T\colon X \to Y$, $\|T\| = \inf\{K^{1/2}\colon \langle Tx, Tx\rangle_A \leq K\langle x, x\rangle_B\ \forall x \in X\}$.

3. **Self-duality and modules over W^*-algebras.** For a pre-Hilbert B-module X, we let X' denote the set of bounded B-module maps of X into B. By 2.8 (with $A = B = Y$), X' is precisely the set of linear maps $\tau\colon X \to B$ for which there is a real $K \geq 0$ such that $\tau(x)^*\tau(x) \leq K\langle x, x\rangle$ $\forall x \in X$. Each $x \in X$ gives rise to a map $\hat{x} \in X'$ defined by $\hat{x}(y) = \langle y, x\rangle$ for $y \in X$ (see 2.3). We will call X *self-dual* if $\hat{X} = X'$, i.e. if every map in X' arises by taking B-valued inner products with some fixed $x \in X$. For a trivial example, we note that if B has 1, then B is itself a self-dual Hilbert B-module. If X is self-dual, X must be complete. (Otherwise, look at maps in X' of the form $\hat{z}$ where z belongs to the completion of X but not to X itself.) The converse is false; completeness is not enough to insure self-duality. For example, let J be a closed right ideal of B with the property that no element of J^* acts as a left identity on J. Then the injection of J into B is a map in J' which is not of the form $\hat{x}$ for any $x \in J$.

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If we define scalar multiplication on X' by $(\lambda\tau)(x) = \bar{\lambda}\tau(x)$ for $\lambda \in C$, $\tau \in X'$, $x \in X$ (so that we have $(\lambda x)\hat{} = \lambda\hat{x}$ for $x \in X$, $\lambda \in C$) and add maps in X' pointwise, then X' becomes a linear space. X' becomes a right B-module if we set $(\tau \cdot b)(x) = b^*\tau(x)$ for $\tau \in X'$, $b \in B$, $x \in X$. The map $x \longrightarrow \hat{x}$ is then a one-to-one module map of X into X'. We shall frequently regard X as a submodule of X' by identifying X with $\hat{X}$.

It is natural to ask whether X' is a pre-Hilbert B-module, that is, whether $\langle \cdot, \cdot \rangle$ can be extended to a B-valued inner product on X'. It turns out that this *can* be done, at least when B is a W^*-algebra, but showing this requires some preparation. We begin by introducing some notation. Let f be a positive linear functional on B. We have already observed that $f(\langle \cdot, \cdot \rangle)$ is a pseudo inner product on X and that $N_f = \{x \in X: f(\langle x, x \rangle) = 0\}$ is a linear subspace of X. It follows that X/N_f is a pre-Hilbert space in the inner product $(\cdot, \cdot)_f$ defined by $(x + N_f, y + N_f)_f = f(\langle x, y \rangle)$ for $x, y \in X$. We let H_f denote the Hilbert space completion of X/N_f and write $\|\cdot\|_f$ for the norm on H_f gotten from its inner product.

Consider $\tau \in X'$. We have $\tau(x)^*\tau(x) \le \|\tau\|^2\langle x, x \rangle$ $\forall x \in X$ by 2.9, so if $x \in N_f$, then $f(\tau(x)^*\tau(x)) = 0 = f(\tau(x))$. This means that the map $x + N_f \longrightarrow f(\tau(x))$ is a well-defined linear functional on X/N_f. It is in fact bounded with norm not exceeding $\|\tau\| \, \|f\|^{1/2}$, since for $x \in X$ we have $|f(\tau(x))| \le \|f\|^{1/2} f(\tau(x)^*\tau(x))^{1/2} \le \|f\|^{1/2}\|\tau\| f(\langle x, x \rangle)^{1/2} = \|f\|^{1/2}\|\tau\| \, \|x + N_f\|_f$. From this, we see that there is a unique vector $\tau_f \in H_f$ such that $\|\tau_f\|_f \le \|\tau\| \, \|f\|^{1/2}$ and $(x + N_f, \tau_f)_f = f(\tau(x))$ $\forall x \in X$. Notice that $\hat{y}_f = y + N_f$ $\forall y \in X$.

Suppose that g is another positive linear functional on B with $g \le f$. We then have $N_f \subseteq N_g$ and the natural map $x + N_f \longrightarrow x + N_g$ of X/N_f into X/N_g is contractive and extends to a contractive map $V_{f,g}$ of H_f into H_g. For $x \in X$, we have $V_{f,g}(\hat{x}_f) = x + N_g = \hat{x}_g$. The next proposition says that every $\tau \in X'$ is similarly well-behaved with respect to the maps $V_{f,g}$.

3.1 **Proposition.** *Let X be a pre-Hilbert B-module, f and g positive linear functionals on B with $g \le f$. Then $V_{f,g}(\tau_f) = \tau_g$ $\forall \tau \in X'$.*

Proof. Take $\tau \in X'$. Since X/N_f is dense in H_f we can find a sequence $\{y_n + N_f\}$ in X/N_f such that $\|y_n + N_f - \tau_f\|_f \longrightarrow 0$. We have $V_{f,g}(\tau_f) = \lim_n V_{f,g}(y_n + N_f) = \lim_n (y_n + N_g)$. To see that $\tau_g = \lim_n (y_n + N_g)$, it suffices to show that $g(\langle x, y_n \rangle) \longrightarrow g(\tau(x))$ $\forall x \in X$. Take $x \in X$. We have

$$\begin{aligned}
&|g(\langle x, y_n \rangle - \tau(x))|^2 \\
&\quad \le \|g\| g(\langle x, y_n \rangle\langle y_n, x \rangle - \tau(x)\langle y_n, x \rangle - \langle x, y_n \rangle\tau(x)^* + \tau(x)\tau(x)^*) \\
&\quad \le \|f\| f(\langle x, y_n \rangle\langle y_n, x \rangle - \tau(x)\langle y_n, x \rangle - \langle x, y_n \rangle\tau(x)^* + \tau(x)\tau(x)^*)
\end{aligned}$$

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for $n = 1, 2, \cdots$. Observe that $f(\langle x, y_n\rangle \tau(x)^*) = f(\langle x \cdot \tau(x)^*, y_n\rangle) \to f(\tau(x \cdot \tau(x)^*))$ $= f(\tau(x)\tau(x)^*)$ by our choice of the sequence $\{y_n + N_f\}$. We will be done once we show that $f(\langle x, y_n\rangle\langle y_n, x\rangle - \tau(x)\langle y_n, x\rangle) \to 0$.

For each n we have

$$f(\langle x, y_n\rangle\langle y_n, x\rangle - \tau(x)\langle y_n, x\rangle) = f(\langle x \cdot \langle y_n, x\rangle, y_n\rangle - \tau(x \cdot \langle y_n, x\rangle))$$
$$= (x \cdot \langle y_n, x\rangle + N_f, y_n + N_f - \tau_f)_f.$$

Moreover, the sequence $\{x \cdot \langle y_n, x\rangle + N_f\}$ is $\|\cdot\|_f$-bounded. Indeed, we have

$$\|x \cdot \langle y_n, x\rangle + N_f\|_f^2 = f(\langle x \cdot \langle y_n, x\rangle, x \cdot \langle y_n, x\rangle\rangle) = f(\langle x, y_n\rangle\langle x, x\rangle\langle y_n, x\rangle)$$
$$\leq \|x\|_X^2 f(\langle x, y_n\rangle\langle y_n, x\rangle) \leq \|x\|_X^2 f(\|x\|_X^2\langle y_n, y_n\rangle) = \|x\|_X^4\|y_n + N_f\|_f^2$$

(the last inequality by virtue of 2.3), and $\{y_n + N_f\}$ is a bounded sequence. Since $\|y_n + N_f - \tau_f\|_f \to 0$, the proof is complete.

For the balance of this section, B will be a W^*-algebra unless it is explicity stated that this restriction on B is unnecessary. We will denote the predual of B by M, the set of normal positive linear functionals on B by P, and regard M as a subspace of B^*, the conjugate space of B, and P as a subset of M; M is then the linear span of P in B^*. For basic facts about W^*-algebras, we refer the reader to S. Sakai [8].

3.2 **Theorem.** *Let X be a pre-Hilbert B-module. The B-valued inner product $\langle\cdot, \cdot\rangle$ extends to $X' \times X'$ in such a way as to make X' into a self-dual Hilbert B-module. In particular, the extended inner product satisfies $\langle \hat{x}, \tau\rangle = \tau(x)$ $\forall x \in X$, $\tau \in X'$.*

Proof. Consider $\tau, \psi \in X'$. We proceed to define their inner product $\langle \tau, \psi\rangle \in B$. First, define $\Gamma: P \to C$ by $\Gamma(f) = (\tau_f, \psi_f)_f$ for $f \in P$. We wish to extend Γ to a linear functional on M.

Claim 1. If $\lambda_1, \cdots, \lambda_n \in C$, $f_1, \cdots, f_n \in P$ are such that $\sum_{j=1}^n \lambda_j f_j = 0$, then $\sum_{j=1}^n \lambda_j \Gamma(f_j) = 0$.

Proof of Claim. Let $f = \sum_{j=1}^n f_j$, so $f \in P$ and $f \geq f_j$ $(j = 1, \cdots, n)$. For $x, y \in X$, we have

$$\sum_{j=1}^n \lambda_j (V_{f,f_j}^* V_{f,f_j}(x + N_f), y + N_f)_f = \sum_{j=1}^n \lambda_j (x + N_{f_j}, y + N_{f_j})_{f_j} = \sum_{j=1}^n \lambda_j f_j(\langle x, y\rangle) = 0$$

by assumption, so $\sum_{j=1}^n \lambda_j V_{f,f_j}^* V_{f,f_j} = 0$. Now observe that

$$\sum_{j=1}^n \lambda_j \Gamma(f_j) = \sum_{j=1}^n \lambda_j (\tau_{f_j}, \psi_{f_j})_{f_j} = \sum_{j=1}^n \lambda_j (V_{f,f_j}\tau_f, V_{f,f_j}\psi_f)_{f_j}$$
$$= \sum_{j=1}^n \lambda_j (V_{f,f_j}^* V_{f,f_j}\tau_f, \psi_f)_f = 0,$$

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the second equality holding by virtue of 3.1.

This is enough to show that Γ extends to a linear functional (call it Γ also) on M, the linear span of P.

Claim 2. Γ is bounded.

Proof of claim. Take $g \in M$. By 1.14.3 of [8] we may write $g = f_1 - f_2 + i(f_3 - f_4)$ with $f_1, f_2, f_3, f_4 \in P$ and $\sum_{j=1}^4 \|f_j\| \leq 2\|g\|$. We then have

$$|\Gamma(g)| \leq \sum_{j=1}^{4} |(\tau_{f_j}, \psi_{f_j})_{f_j}| \leq \sum_{j=1}^{4} \|\tau_{f_j}\|_{f_j} \|\psi_{f_j}\|_{f_j}$$
$$\leq \sum_{j=1}^{4} \|f_j\| \, \|\tau\| \, \|\psi\| \leq 2\|\tau\| \, \|\psi\| \, \|g\|.$$

This proves the claim.

Now B is isometric to M^* under the natural duality, so there is a unique element $\langle \tau, \psi\rangle \in B$ such that $\Gamma(g) = g(\langle \tau, \psi\rangle)$ $\forall g \in M$ and in particular $(\tau_f, \psi_f)_f = f(\langle \tau, \psi\rangle)$ $\forall f \in P$. That the map $\langle \cdot, \cdot\rangle$: $X' \times X' \rightarrow B$ defined in this way is conjugate-bilinear follows from the linearity of the maps $\tau \rightarrow \tau_f$ of X' into H_f for $f \in P$. We now show that $\langle \cdot, \cdot \rangle$ satisfies properties (i)–(iv) of 2.1.

For (i), we have $f(\langle \tau, \tau\rangle) = (\tau_f, \tau_f)_f \geq 0$ $\forall \tau \in X'$, $f \in P$. This is enough to show that $\langle \tau, \tau\rangle \geq 0$ $\forall \tau \in X'$.

For (ii), suppose $\tau \in X'$ and $\langle \tau, \tau\rangle = 0$. Then $\tau_f = 0$ $\forall f \in P$, so $f(\tau(x)) = 0$ $\forall f \in P$, $x \in X$. This is enough to show that $\tau = 0$.

For (iii), take $\tau, \psi \in X'$. For any $f \in P$ we have $f(\langle \tau, \psi\rangle) = (\tau_f, \psi_f)_f = \overline{(\psi_f, \tau_f)_f} = \overline{f(\langle \psi, \tau\rangle)} = f(\langle \psi, \tau\rangle^*)$, which shows that $\langle \tau, \psi\rangle = \langle \psi, \tau\rangle^*$.

For (iv), consider $\tau, \psi \in X'$, $b \in B$, and $f \in P$. Define a functional f_b on B by $f_b(a) = f(ab)$. Then $f_b \in M$ and we may write $f_b = \sum_{j=1}^4 \lambda_j f_j$ with each $f_j \in P$ and each $\lambda_j \in C$. Let $g = f + \sum_{j=1}^4 f_j$, so $g \in P$ and $g \geq f, f_1, f_2, f_3, f_4$. We have

$$f(\langle \tau, \psi\rangle b) = \sum_{j=1}^{4} \lambda_j f_j(\langle \tau, \psi\rangle) = \sum_{j=1}^{4} \lambda_j (\tau_{f_j}, \psi_{f_j})_{f_j}$$
$$= \sum_{j=1}^{4} \lambda_j (\tau_{f_j}, V_{g, f_j} \psi_g)_{f_j}$$

by 3.1. For any $x \in X$, on the other hand,

$$\sum_{j=1}^{4} \lambda_j (\tau_{f_j}, V_{g, f_j}(x + N_g))_{f_j} = \sum_{j=1}^{4} \lambda_{f_j} (\tau_{f_j}, x + N_{f_j})_{f_j}$$
$$= \sum_{j=1}^{4} \lambda_j f_j(\tau(x)^*) = f_b(\tau(x)^*)$$
$$= \overline{f(b^*\tau(x))} = \overline{f((\tau \cdot b)(x))} = ((\tau \cdot b)_f, x + N_f)_f$$
$$= ((\tau \cdot b)_f, V_{g, f}(x + N_g))_f.$$

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Since X/N_g is dense in H_g, we must have

$$f(\langle \tau, \psi\rangle b) = \sum_{j=1}^{4} \lambda_j (\tau_{f_j}, V_{g,f_j}\psi_g)_{f_j}$$
$$= ((\tau \cdot b)_f, V_{g,f}\psi_g)_f = ((\tau \cdot b)_f, \psi_f)_f = f(\langle \tau \cdot b, \psi\rangle).$$

This holds $\forall f \in P$, so $\langle \tau, \psi\rangle b = \langle \tau \cdot b, \psi\rangle$ as desired.

The B-valued inner product on X' which we have constructed is an extension of that on X (viewed as a submodule of X'). For $x, y \in X$ and $f \in P$, we have $f(\langle \hat{x}, \hat{y}\rangle) = (\hat{x}_f, \hat{y}_f) = (x + N_f, y + N_f)_f = f(\langle x, y\rangle)$, so $\langle \hat{x}, \hat{y}\rangle = \langle x, y\rangle$. Also, for $\tau \in X'$, $x \in X$, $f \in P$, we have $f(\langle \hat{x}, \tau\rangle) = (\hat{x}_f, \tau_f)_f = f(\tau(x))$, so $\langle \hat{x}, \tau\rangle = \tau(x)$.

It remains to show that X' is self-dual. Consider $\phi \in (X')'$. The restriction of ϕ to X belongs to X', so we can find a $\tau \in X'$ such that $\phi(\hat{X}) = \tau(x)$ $\forall x \in X$. Define $\phi_0 \in (X')'$ by $\phi_0(\psi) = \phi(\psi) - \langle \psi, \tau\rangle$ for $\psi \in X'$. We have $\phi_0(\hat{x}) = \{0\}$ and wish to show that $\phi_0 = 0$. Take $\psi \in X'$ and $f \in P$. We can find a sequence $\{y_n + N_f\}$ in X/N_f converging to ψ_f. Letting $K \geq 0$ be such that $\phi_0(\sigma)^*\phi_0(\sigma) \leq K\langle \sigma, \sigma\rangle$ $\forall \sigma \in X'$, we have for $n = 1, 2, \cdots$

$$f(\phi_0(\psi)^*\phi_0(\psi)) = f(\phi_0(\psi - \hat{y}_n)^*\phi_0(\psi - \hat{y}_n)) \leq Kf(\langle \psi - \hat{y}_n, \psi - \hat{y}_n\rangle).$$

But

$$\begin{aligned} &f(\langle \psi - \hat{y}_n, \psi - \hat{y}_n\rangle) \\ &= (\psi_f, \psi_f)_f - (y_n + N_f, \psi_f)_f - (\psi_f, y_n + N_f)_f - (y_n + N_f, y_n + N_f)_f \\ &= \|\psi_f - (y_n + N_f)\|_f^2 \qquad (n = 1, 2, \ldots) \end{aligned}$$

so $f(\langle \psi - \hat{y}_n, \psi - \hat{y}_n\rangle) \to 0$, forcing $f(\phi_0(\psi)^*\phi_0(\psi)) = 0$. This holds $\forall f \in P$, so $\phi_0(\psi) = 0$ as desired and the proof is complete.

3.3 **Remark.** There are ostensibly two ways of norming X', namely as bounded operators from X into B on the one hand, and by $\|\cdot\|_{X'}$ on the other. In fact, these two norms are identical. Letting $\|\cdot\|$ denote the operator norm on X', we have, for $\tau \in X'$ and $x \in X$, $\tau(x)^*\tau(x) = \langle \tau, \hat{x}\rangle\langle \hat{x}, \tau\rangle \leq \|\tau\|_{X'}^2\langle x, x\rangle$ by 2.3, so $\|\tau\| \leq \|\tau\|_{X'}$ by 2.9. But we have seen that $\|\tau_f\|_f \leq \|\tau\|\,\|f\|^{1/2}$ $\forall f \in P$, so $\|\tau\|_{X'}^2 = \|\langle \tau, \tau\rangle\| = \sup\{\|\tau_f\|_f^2 : f \in P, \|f\| = 1\} \leq \|\tau\|^2$, forcing $\|\tau\|_{X'} = \|\tau\|$.

It follows from §2 of [12] that 3.2 also holds when B is a commutative AW^*-algebra. Whether 3.2 holds for modules over arbitrary AW^*- algebras is unknown at present, but at any rate we cannot expect it to hold in much greater generality than this, as the following example shows. Let B be the algebra of complex-valued continuous functions on the unit interval, X the ideal of functions in B which vanish at 0, thought of as a Hilbert B-module. One checks easily that X' may

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be identified (as a normed right B-module) with the algebra of bounded complex-valued continuous functions on the half-open interval $(0, 1]$. Once this identification is made, it is not hard to show that the presence of functions in X' which do not admit continuous extensions to the closed unit interval implies that the natural B-valued inner product on X cannot be extended to a B-valued inner product on X'.

One pleasant property of self-dual Hilbert B-modules is that every bounded module map between two such has an adjoint. The following proposition (which does not require that B be a W^*-algebra) is proved in much the same way as the corresponding fact about Hilbert spaces.

3.4 **Proposition.** *Let X be a self-dual Hilbert B-module, Y a pre-Hilbert B-module, and T: $X \to Y$ a bounded module map. Then there is a bounded module map T^*: $Y \to X$ such that $\langle x, T^*y\rangle = \langle Tx, y\rangle$ $\forall x \in X$, $y \in Y$.*

3.5 **Corollary.** *If X is a self-dual Hilbert B-module, every module map in $B(X)$ belongs to $\mathfrak{A}(X)$.*

If B is a W^*-algebra, bounded module maps between two pre-Hilbert B-modules extend uniquely to bounded module maps between the corresponding self-dual modules.

3.6 **Proposition.** *Let X and Y be pre-Hilbert B-modules and T: $X \to Y$ a bounded module map. Then T extends uniquely to a bounded module map $\widetilde{T}$: $X' \to Y'$.*

Proof. Define $T^\#$: $Y \to X'$ by $(T^\# y)(x) = \langle Tx, y\rangle$ for $y \in Y$, $x \in X$. Notice that $\|(T^\# y)(x)\| \le \|T\|\,\|x\|\,\|y\|$, so by 3.3 $T^\#$ is bounded with $\|T^\# y\|_{X'} \le \|T\|\,\|y\|_Y$ $\forall y \in Y$. We also have $(T^\#(y \cdot b))(x) = \langle Tx, y \cdot b\rangle = b^*\langle Tx, y\rangle = ((T^\# y) \cdot b)(x)$ $\forall b \in B$, so $T^\#$ is a bounded module map. Define $\widetilde{T}$: $X' \to Y'$ by $(\widetilde{T}\tau)(y) = \langle T^\# y, \tau\rangle$ for $y \in Y$, $\tau \in X'$. Since $\widetilde{T}$ is just $(T^\#)^\#$, $\widetilde{T}$ is a bounded module map also. It is immediate that $(\widetilde{T}\hat{x})(y) = (Tx)\hat{}\,(y)$ $\forall x \in X$, $y \in Y$, so $\widetilde{T}$ is an extension of T.

To prove that $\widetilde{T}$ is unique in the desired sense, it suffices to show that if V: $X' \to Y'$ is a bounded module map with $V(\hat{X}) = \{0\}$, then $V = 0$. Indeed, let V^*: $Y' \to X'$ be the adjoint of V guaranteed by 3.4. For $\psi \in Y'$, $x \in X$, we have $(V^*\psi)(x) = \langle \hat{x}, V^*\psi\rangle = \langle V\hat{x}, \psi\rangle = 0$, so $V^* = 0$, so $V = 0$. This completes the proof.

If X is a pre-Hilbert B-module, the preceding proposition says in particular that every $T \in \mathfrak{A}(X)$ extends uniquely to a module map $\widetilde{T} \in B(X')$. By 3.5, $\widetilde{T} \in \mathfrak{A}(X')$. The map $T \to \widetilde{T}$ of $\mathfrak{A}(X)$ into $\mathfrak{A}(X')$ is clearly linear. For T, $U \in \mathfrak{A}(X)$ the operators $\widetilde{T}\widetilde{U}$ and $(\widetilde{T})^*$ are extensions of TU and T^*, respectively, so we must have $(TU)^\sim = \widetilde{T}\widetilde{U}$ and $(T^*)^\sim = (\widetilde{T})^*$, i.e. $T \to \widetilde{T}$ is a $*$-homomorphism.

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Since $\tilde{T} = 0$ implies $T = 0$, this map is a $*$-isomorphism. We record this information as a corollary to 3.6.

3.7 **Corollary.** *Let X be a pre-Hilbert B-module. Each $T \in \mathfrak{A}(X)$ extends to a unique $\tilde{T} \in \mathfrak{A}(X')$. The map $T \to \tilde{T}$ is a $*$-isomorphism of $\mathfrak{A}(X)$ into $\mathfrak{A}(X')$.*

We now proceed to investigate some of the special properties of self-dual modules over W^*-algebras. First, we will show that if X is a self-dual Hilbert B-module (over a W^*-algebra B), then X and $\mathfrak{A}(X)$ are conjugate spaces, so in particular $\mathfrak{A}(X)$ is a W^*-algebra. To this end, we introduce some notation. Let Y be the linear space X with "twisted" scalar multiplication (i.e. $\lambda \cdot x = \bar{\lambda} x$ for $\lambda \in C$, $x \in Y$), and consider the algebraic tensor product $M \otimes Y$, M as usual being the pre-dual of B. We norm $M \otimes Y$ with the greatest cross-norm. For $x \in X$, we define a linear functional $\check{x}$ on $M \otimes Y$ by

$$\check{x}\left(\sum_{j=1}^{n} f_j \otimes y_j\right) = \sum_{j=1}^{n} f_j(\langle x, y_j\rangle)$$

for $f_1, \cdots, f_n \in M$, $y_1, \cdots, y_n \in Y$. The functional $\check{x}$ is well defined and in fact bounded with $\|\check{x}\| \le \|x\|_X$, since

$$\left|\check{x}\left(\sum_{j=1}^{n} f_j \otimes y_j\right)\right| \le \|x\|_X \sum_{j=1}^{n} \|f_j\| \, \|y_j\|_X$$

$\forall f_1, \cdots, f_n \in M$, $y_1, \cdots, y_n \in Y$, which by definition of the greatest cross-norm yields the desired inequality. We actually have $\|\check{x}\| = \|x\|_X$. Indeed, let $\{g_n\}$ be a sequence of functionals of norm 1 in M such that $|g_n(\langle x, x\rangle)| \to \|x\|_X^2$. Each tensor $g_n \otimes x \in M \otimes Y$ has norm $\|g_n\| \, \|x\|_X = \|x\|_X$, and $|\check{x}(g_n \otimes x)| \to \|x\|_X^2$, so $\|x\|_X \le \|\check{x}\|$ and hence $\|\check{x}\| = \|x\|_X$. The map $x \to \check{x}$ is thus a linear isometry of X into $(M \otimes Y)^*$.

3.8 **Proposition.** *Let X be a self-dual Hilbert B-module. Then X is a conjugate space.*

Proof. It will suffice to show that $\check{X}$ is weak*-closed in $(M \otimes Y)^*$, since X will then be isometric with the conjugate space of a quotient space of $M \otimes Y$. Let $\{\check{x}_\alpha\}$ be a net in $\check{X}$ converging weak* to some $F \in (M \otimes Y)^*$. For $y \in X$, define a linear functional ψ_y on M by $\psi_y(g) = F(g \otimes y)$ for $g \in M$. The functional ψ_y is clearly bounded with norm not exceeding $\|F\| \, \|y\|_X$, and we conclude that there is a unique element $\tau(y) \in B$ with $\|\tau(y)\| \le \|F\| \, \|y\|_X$ and $F(g \otimes y) = g(\tau(y)^*)$ $\forall g \in M$.

The map τ is clearly linear and we have just seen that it is bounded. We claim that it is a module map (and therefore belongs to X'). Indeed, take $y \in X$,

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$b \in B$, $f \in M$ and define $g \in M$ by $g(a) = f(b^*a)$ for $a \in B$. We have

$$f(\tau(y \cdot b)^*) = F(f \otimes (y \cdot b)) = \lim_\alpha \check{x}_\alpha(f \otimes (y \cdot b)) = \lim_\alpha f(\langle x_\alpha, y \cdot b\rangle)$$

$$= \lim_\alpha g(\langle x_\alpha, y\rangle) = F(g \otimes y) = g(\tau(y)^*) = f(b^*\tau(y)^*).$$

This holds for every $f \in M$, so $\tau(y \cdot b) = \tau(y)b$ as claimed.

Since X is self-dual, we can find an $x_0 \in X$ such that $\tau(y) = \langle y, x_0\rangle$ $\forall y \in X$. It follows that $F = \check{x}_0$ and hence that X is weak*-closed in $(M \otimes Y)^*$. This completes the proof.

3.9 **Remark.** We let $\mathcal{T}$ denote the weak*-topology which X has by virtue of being a conjugate space in the manner demonstrated above. Closed, norm-bounded convex subsets of X are $\mathcal{T}$-compact. A bounded net $\{x_\alpha\}$ in X converges with respect to $\mathcal{T}$ to $x \in X$ if and only if $f(\langle x_\alpha, y\rangle) \to f(\langle x, y\rangle)$ $\forall f \in M$, $y \in X$.

An elaboration of the technique employed in the proof of 3.8 can be used to show that $\mathfrak{A}(X)$ is a conjugate space under the circumstances which we are considering. Let Y be as above, and norm $X \otimes Y \otimes M$ with the greatest cross-norm. For $T \in \mathfrak{A}(X)$, define a linear functional $\check{T}$ on $X \otimes Y \otimes M$ by

$$\check{T}\left(\sum_{j=1}^{n} x_j \otimes y_j \otimes g_j\right) = \sum_{j=1}^{n} g_j(\langle Tx_j, y_j\rangle)$$

for $x_j, y_j \in X$, $g_j \in M$ $(j = 1, \cdots, n)$. $\check{T}$ is well defined and it is easy to see that $\check{T} \in (X \otimes Y \otimes M)^*$ with $\|\check{T}\| = \|T\|_X$. The map $T \to \check{T}$ is thus a linear isometry of $\mathfrak{A}(X)$ into $(X \otimes Y \otimes M)^*$.

3.10 **Proposition.** *Let X be a self-dual Hilbert B-module. Then $\mathfrak{A}(X)$ is a W^*-algebra.*

Proof. It suffices to show that $\mathfrak{A}(X)$ is a conjugate space, and for this in turn it suffices to show that $\mathfrak{A}(X)\check{}$ is weak*-closed in $(X \otimes Y \otimes M)^*$. Let $\{T_\alpha\}$ be a net in $\mathfrak{A}(X)$ with $\{\check{T}_\alpha\}$ converging weak* to some $\Phi \in (X \otimes Y \otimes M)^*$. For $x, y \in X$, define $\tau_{x,y}: M \to C$ by $\tau_{x,y}(g) = \Phi(x \otimes y \otimes g)$ for $g \in M$. The functional $\tau_{x,y}$ is clearly linear and bounded with norm not greater than $\|\Phi\|\,\|x\|_X\|y\|_X$, so there is a unique element $\tau_x(y) \in B$ with $\|\tau_x(y)\| \le \|\Phi\|\,\|x\|_X\|y\|_X$ such that $\Phi(x \otimes y \otimes g) = g(\tau_x(y))$ $\forall g \in M$.

Claim. For $x, y \in X$, $b \in B$, we have $\tau_{x\cdot b}(y) = \tau_x(y)b$ and $\tau_x(y \cdot b) = b^*\tau_x(y)$.

Proof of claim. We establish only the first equality; the second is proved similarly. Take $f \in M$ and define $g \in M$ by $g(a) = f(ab)$ for $a \in B$. We have $f(\tau_{x\cdot b}(y)) = \Phi((x \cdot b) \otimes y \otimes f) = \lim_\alpha \check{T}_\alpha((x \cdot b) \otimes y \otimes f) = \lim_\alpha f(\langle T_\alpha(x \cdot b), y\rangle) = \lim_\alpha g(\langle T_\alpha x, y\rangle) = \Phi(x \otimes y \otimes g) = g(\tau_x(y)) = f(\tau_x(y)b)$. This holds $\forall f \in M$, so $\tau_{x\cdot b}(y) = \tau_x(y)b$.

For any $y \in X$, the map $x \to \tau_x(y)$ is thus a bounded module map of X into

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B. Since X is self-dual, we can find a unique $Uy \in X$ such that $\tau_x(y) = \langle x, Uy\rangle$ $\forall x \in X$. U is clearly linear, and in fact a module map since for $x, y \in X$, $b \in B$ we have $\langle x, U(y \cdot b)\rangle = \tau_x(y \cdot b) = b^*\tau_x(y) = \langle x, (Uy) \cdot b\rangle$. Moreover, for any $y \in X$, we have $\|Uy\|_X^2 = \|\langle Uy, Uy\rangle\| = \|\tau_{Uy}(y)\| \le \|\Phi\| \, \|Uy\|_X \|y\|_X$, whence $\|Uy\|_X \le \|\Phi\| \, \|y\|_X$. U, being a bounded module map, belongs to $\check{\mathfrak{A}}(X)$ by 3.5. Let $T = U^*$. It is immediate that $\Phi = \tilde{T}$, which completes the proof of the proposition.

Our next result gives a "polar decomposition" for elements of a self-dual module over a W^*-algebra. Its proof mimics that of 1.12.1 in [8].

3.11 **Proposition.** *Let X be a self-dual Hilbert B-module. Each $x \in X$ can be written $x = u \cdot \langle x, x\rangle^{1/2}$, where $u \in X$ is such that $\langle u, u\rangle$ is the range projection of $\langle x, x\rangle^{1/2}$. This decomposition is unique in the sense that if $x = v \cdot b$ where $b \ge 0$ and $\langle v, v\rangle$ is the range projection of b, then $v = u$ and $b = \langle x, x\rangle^{1/2}$.*

Proof. Take $x \in X$ and for $n = 1, 2, \cdots$ set $b_n = (\langle x, x\rangle + n^{-1})^{1/2}$ and $x_n = x \cdot b_n^{-1}$. We have $\langle x_n, x_n\rangle = \langle x, x\rangle(\langle x, x\rangle + n^{-1})^{-1}$, so $\|x_n\|_X \le 1$ for $n = 1, 2, \cdots$. Let y be a $\mathcal{J}$-accumulation point of the sequence $\{x_n\}$ (see 3.9). Since $\|b_n - \langle x, x\rangle^{1/2}\| \to 0$ and $x_n \cdot b_n = x$ $(n = 1, 2, \cdots)$, we conclude that $x = y \cdot \langle x, x\rangle^{1/2}$. Let p be the range projection of $\langle x, x\rangle^{1/2}$. We have $p\langle x, x\rangle^{1/2} = \langle x, x\rangle^{1/2}p = \langle x, x\rangle^{1/2}$, so $x = y \cdot p\langle x, x\rangle^{1/2}$ and $\langle x, x\rangle = \langle x, x\rangle^{1/2}p\langle y, y\rangle\lambda\langle x, x\rangle^{1/2}$. Hence $\langle x, x\rangle^{1/2}(p - p\langle y, y\rangle p)\langle x, x\rangle^{1/2} = 0$. Since $\|y\|_X \le 1$, we have $p - p\langle y, y\rangle p \ge 0$, so $\langle x, x\rangle^{1/2}(p - p\langle y, y\rangle p)^{1/2} = 0$. This forces $p(p - p\langle y, y\rangle p)^{1/2} = 0$ and hence $p = p\langle y, y\rangle p$. Now let $u = y \cdot p$. We have $u \cdot \langle x, x\rangle^{1/2} = y \cdot p\langle x, x\rangle^{1/2} = x$ and $\langle u, u\rangle = p\langle y, y\rangle p = p$ as desired.

To prove the uniqueness of the decomposition, suppose $x = v \cdot b$, where $b \ge 0$ and $\langle v, v\rangle$ is the range projection of b. Then $\langle x, x\rangle = b\langle v, v\rangle b = b^2$, so $b = \langle x, x\rangle^{1/2}$, and $\langle v, v\rangle = p$. We have $\langle v - v \cdot p, v - v \cdot p\rangle = p - p - p + p = 0$, so $v = v \cdot p$ and likewise $u = u \cdot p$. Also, $\langle x, u\rangle = \langle x, x\rangle^{1/2} = \langle v, u\rangle\langle x, x\rangle^{1/2}$, i.e. $(p - \langle v, u\rangle)\langle x, x\rangle^{1/2} = 0$. This forces $(p - \langle v, u\rangle)p = p - \langle v \cdot p, u\rangle = p - \langle v, u\rangle = 0$. Hence $\langle u - v, u - v\rangle = p - p - p + p = 0$, so $u = v$ and the proof is complete.

Our next project is to obtain a "direct sum" decomposition for self-dual Hilbert B-modules over a W^*-algebra B. The summands here will be right ideals of B of the form pB, where $p \in B$ is a projection, viewed as (self-dual) Hilbert B-modules with B-valued inner product $\langle pa, pb\rangle = b^*pa$ for $a, b \in B$. First, we must develop a notion of "direct sum" appropriate to such a decomposition.

Let I be an index set, and $\{X_\alpha : \alpha \in I\}$ a collection of pre-Hilbert B-modules indexed by I. Let $\mathcal{F}$ denote the set of finite subsets of I, directed upwards by inclusion. For I-tuples $x = \{x_\alpha\}$, $y = \{y_\alpha\}$ $(x_\alpha, y_\alpha \in X_\alpha \ \forall \alpha \in I)$ and $S \in \mathcal{F}$, we set $\langle x, y\rangle_S = \Sigma\{\langle x_\alpha, y_\alpha\rangle : \alpha \in S\}$. Let X denote the set of I-tuples $x = \{x_\alpha\}$ such that $\sup\{\|\langle x, x\rangle_S\| : S \in \mathcal{F}\} < \infty$. Notice that for $x \in X$, the net $\{\langle x, x\rangle_S : S \in \mathcal{F}\}$

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is norm-bounded and increasing; we let $\langle x, x\rangle$ denote its least upper bound. Take $x, y \in X$ and consider the net $\{\langle x, y\rangle_S: S \in \mathfrak{F}\}$. We claim that this net is norm-bounded and ultraweakly convergent. For each state f of B and each $S \in \mathfrak{F}$, we have

$$\begin{aligned}|f(\langle x, y\rangle_S)| &\le \sum\{|f(\langle x_\alpha, y_\alpha\rangle)|: \alpha \in S\}\\ &\le \sum\{f(\langle x_\alpha, x_\alpha\rangle)^{1/2} f(\langle y_\alpha, y_\alpha\rangle)^{1/2}: \alpha \in S\}\\ &\le \Big(\sum\{f(\langle x_\alpha, x_\alpha\rangle): \alpha \in S\}\Big)^{1/2}\Big(\sum\{f(\langle y_\alpha, y_\alpha\rangle): \alpha \in S\}\Big)^{1/2}\\ &= f(\langle x, x\rangle_S)^{1/2} f(\langle y, y\rangle_S)^{1/2}\\ &\le \|\langle x, x\rangle_S\|^{1/2}\|\langle y, y\rangle_S\|^{1/2} \le \|\langle x, x\rangle\|^{1/2}\|\langle y, y\rangle\|^{1/2}.\end{aligned}$$

This is enough to show that $\|\langle x, y\rangle_S\| \le 2\|\langle x, x\rangle\|^{1/2}\|\langle y, y\rangle\|^{1/2}$ $\forall S \in \mathfrak{F}$, i.e. the net in question is norm-bounded. To see that it converges ultraweakly, it therefore suffices to show that the net $\{f(\langle x, y\rangle_S): S \in \mathfrak{F}\}$ is Cauchy $\forall f \in P$. Take $f \in P$ and consider $S, S_1, S_2 \in \mathfrak{F}$ with $S \subseteq S_1 \cap S_2$. We have

$$\begin{aligned}|f(\langle x, y\rangle_{S_1} - \langle x, y\rangle_{S_2})| &= |f(\langle x, y\rangle_{S_1\setminus S_2} - \langle x, y\rangle_{S_2\setminus S_1})|\\ &\le |f(\langle x, y\rangle_{S_1\setminus S_2})| + |f(\langle x, y\rangle_{S_2\setminus S_1})|\\ &\le f(\langle x, x\rangle_{S_1\setminus S_2})^{1/2} f(\langle y, y\rangle_{S_1\setminus S_2})^{1/2} + f(\langle x, x\rangle_{S_2\setminus S_1})^{1/2} f(\langle y, y\rangle_{S_2\setminus S_1})^{1/2}\\ &\le f(\langle x, x\rangle_{S_1\setminus S})^{1/2} f(\langle y, y\rangle_{S_1\setminus S})^{1/2} + f(\langle x, x\rangle_{S_2\setminus S})^{1/2} f(\langle y, y\rangle_{S_2\setminus S})^{1/2}.\end{aligned}$$

But the last quantity may be made as small as desired by choosing S sufficiently large (since f is normal), so we are done. We let $\langle x, y\rangle$ denote the ultraweak limit of the net $\{\langle x, y\rangle_S: S \in \mathfrak{F}\}$. It is now clear that S is a right B-module under coordinatewise operations and that $\langle\cdot, \cdot\rangle$ defined as above is a B-valued inner product on X. We call the pre-Hilbert B-module X the *ultraweak direct sum* of the modules X_α and write $X = \mathrm{UDS}\{X_\alpha: \alpha \in I\}$. It is routine to show that X is self-dual if and only if each X_α is.

3.12 **Theorem.** *Let X be a self-dual Hilbert B-module. There is a collection $\{p_\alpha: \alpha \in I\}$ of (not necessarily distinct) nonzero projections in B such that X and* $\mathrm{UDS}\{p_\alpha B: \alpha \in I\}$ *are isomorphic as Hilbert B-modules.*

Proof. Let $\{e_\alpha: \alpha \in I\}$ be a subset of X which is maximal with respect to the following properties: (i) $\langle e_\alpha, e_\alpha\rangle$ is a nonzero projection; (ii) $\langle e_\alpha, e_\beta\rangle = 0$ for $\alpha \neq \beta$. (Such a set clearly exists by virtue of 3.11 and Zorn's lemma.) Let $p_\alpha = \langle e_\alpha, e_\alpha\rangle$ for each $\alpha \in I$. (Notice that $\langle e_\alpha - e_\alpha \cdot p_\alpha, e_\alpha - e_\alpha \cdot p_\alpha\rangle = 0$, so e_α

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$= e_\alpha \cdot p_\alpha\ \forall \alpha \in I$.) For $S \in \mathfrak{F}$ (= set of finite subsets of I) and $x \in X$, one sees by imitating the proof of Bessel's inequality for Hilbert space that

$$\sum \{\langle e_\alpha, x\rangle\langle x, e_\alpha\rangle : \alpha \in S\} \leq \langle x, x\rangle.$$

Since $\langle x, e_\alpha\rangle = p_\alpha\langle x, e_\alpha\rangle\ \forall \alpha \in I$, this shows that the I-tuple $\{\langle x, e_\alpha\rangle : \alpha \in I\}$ belongs to $\mathrm{UDS}\{p_\alpha B : \alpha \in I\}$. We define $T: X \to \mathrm{UDS}\{p_\alpha B : \alpha \in I\}$ by $Tx = \{\langle x, e_\alpha\rangle\}$. It is clear that T is a module map. We wish to show that T is onto and that $\langle Tx, Tx\rangle = \langle x, x\rangle\ \forall x \in X$.

Consider $\{p_\alpha b_\alpha\} \in \mathrm{UDS}\{p_\alpha B\}$ and for each $S \in \mathfrak{F}$, set $y_S = \Sigma\{e_\alpha \cdot b_\alpha : \alpha \in S\}$. We have $\langle y_S, y_S\rangle = \langle\{p_\alpha b_\alpha\}, \{p_\alpha b_\alpha\}\rangle_S\ \forall S \in \mathfrak{F}$, so the net $\{y_S : S \in \mathfrak{F}\}$ is norm-bounded in X. Let y be a $\mathfrak{T}$-accumulation point of this net (see 3.9). For each $f \in M$ and $\alpha \in I$, $f(\langle y, e_\alpha\rangle)$ is an accumulation point of $\{f(\langle y_S, e_\alpha\rangle) : S \in \mathfrak{F}\}$. But for sufficiently large S, $\langle y_S, e_\alpha\rangle = \langle e_\alpha \cdot b_\alpha, e_\alpha\rangle = p_\alpha b_\alpha$, so $\langle y, e_\alpha\rangle = p_\alpha b_\alpha\ \forall \alpha \in I$, i.e. $Ty = \{p_\alpha b_\alpha\}$, showing that T is onto.

It follows routinely from 3.11 that if $x \in X$ and $\langle x, e_\alpha\rangle = 0\ \forall \alpha \in I$, then the range projection of $\langle x, x\rangle^{1/2}$ is orthogonal to each e_α and hence 0 by the maximality of $\{e_\alpha : \alpha \in I\}$. This means that T is one-to-one. Finally, take $x \in X$ and for each $S \in \mathfrak{F}$ set $x_S = \Sigma\{e_\alpha \cdot \langle x, e_\alpha\rangle : \alpha \in S\}$. We have seen that $\{\langle x, e_\alpha\rangle\} = \{p_\alpha\langle x, e_\alpha\rangle\} \in \mathrm{UDS}\{p_\alpha B\}$, so the net $\{x_S : S \in \mathfrak{F}\}$ is norm-bounded and any $\mathfrak{T}$-accumulation y thereof satisfies $\langle y, e_\alpha\rangle = \langle x, e_\alpha\rangle\ \forall \alpha \in I$. It follows that the net $\{x_S\}$ is $\mathfrak{T}$-convergent to x. For each $f \in M$ we have $f(\langle x, x\rangle) = \lim_S f(\langle x, x_S\rangle) = \lim_S f(\langle x_S, x_S\rangle) = \lim_S f(\langle Tx, Tx\rangle_S) = f(\langle Tx, Tx\rangle)$, so $\langle x, x\rangle = \langle Tx, Tx\rangle$ and the proof is complete.

4. **Extension of a module by a bigger algebra.** Let A be a B^*-algebra with 1, B a closed *-subalgebra of A with $1 \in B$, and X a pre-Hilbert B-module. In this section we construct an "extension" $X \odot A$ of X by A which is a pre-Hilbert A-module and show that under certain circumstances $(X \odot A)'$ is isometrically isomorphic to the right A-module of all bounded B-module maps of X into A. One consequence of this is that the set of bounded B-module maps of X into B^{**} can be made into a self-dual Hilbert B^{**}-module.

Consider the algebraic tensor product $X \otimes A$, which becomes a right A-module when we set $(x \otimes a) \cdot a_1 = x \otimes aa_1$, for $x \in X$, $a, a_1 \in A$. Define $[\cdot, \cdot]: X \otimes A \times X \otimes A \to A$ by

$$\left[\sum_{j=1}^{n} x_j \otimes a_j, \sum_{i=1}^{m} y_i \otimes \alpha_i\right] = \sum_{i,j} \alpha_i^* \langle x_j, y_i\rangle a_j.$$

It is immediate that $[\cdot, \cdot]$ is well defined and conjugate-bilinear, and that $[z, w] = [w, z]^*$ and $[z \cdot a, w] = [z, w]a\ \forall z, w \in X \otimes A$, $a \in A$. For $x_1, \cdots, x_n \in X$ and $b_1, \cdots, b_n \in B$ we have

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$$\sum_{i,j} b_i^* \langle x_j, x_i \rangle b_j = \left\langle \sum_{i=1}^n x_i \cdot b_i, \sum_{i=1}^n x_i \cdot b_i \right\rangle \geq 0,$$

so by 6.1 the matrix $[\langle x_j, x_i \rangle]$ in $B_{(n)}$, the B^*-algebra of $n \times n$ matrices with entries in B, is positive. Hence it is positive as an element of the larger B^*-algebra $A_{(n)}$ and by 6.1 again, $\Sigma_{i,j} a_i^* \langle x_j, x_i \rangle a_j \geq 0$ $\forall a_1, \cdots, a_n \in A$, i.e. $[z, z] \geq 0$ $\forall z \in X \otimes A$. If we let $N = \{z \in X \otimes A: [z, z] = 0\}$, then N is an A-submodule of $X \otimes A$ and $Y = (X \otimes A)/N$ is a pre-Hilbert A-module in a natural way (see 2.2). A direct computation shows that $(x \cdot b) \otimes 1 - x \otimes b \in N$ $\forall x \in X$, $b \in B$, so the map $x \longrightarrow x \otimes 1 + N$ is a B-module map of X into Y. Moreover, we have $\langle x \otimes 1 + N, y \otimes 1 + N \rangle = \langle x, y \rangle$ $\forall x, y \in X$ so we may regard X as a B-submodule of Y. We call Y the *extension of* X *by* A and write $Y = X \odot A$.

Let $M(X, A)$ denote the set of bounded B-module maps of X into A, made into a linear space by adding maps pointwise and "twisting" the natural scalar multiplication (i.e. $(\lambda\phi)(x) = \bar{\lambda}\phi(x)$ for $\lambda \in C$, $\phi \in M(X, A)$, $x \in X$). $M(X, A)$ becomes a right A-module when we define $\phi \cdot a$ for $\phi \in M(X, A)$ and $a \in A$ by $(\phi \cdot a)(x) = a^*\phi(x)$ $\forall x \in X$. Notice that each $\tau \in (X \odot A)'$ gives rise to a map $\tau_R \in M(X, A)$ by restriction to X; explicitly, $\tau_R(x) = \tau(x \otimes 1 + N)$ for $x \in X$. If $(X \odot A)'$ and $M(X, A)$ are normed as linear spaces of bounded linear maps, it is clear that the map $\tau \longrightarrow \tau_R$ is a contractive A-module map of $(X \odot A)'$ into $M(X, A)$. We shall see that under certain conditions (which obtain in reasonable generality), this map is an isometry of $(X \odot A)'$ onto $M(X, A)$.

We will need the following lemma.

4.1 **Lemma.** *Let* $\mathfrak{A}$ *be a* B^**-algebra with* 1, *and* S *a set of positive linear functionals on* $\mathfrak{A}$ *of norm not exceeding* 1 *such that* $\|a\| = \sup\{f(a): f \in S\}$ $\forall a \in \mathfrak{A}$ *with* $a \geq 0$. *Then if* $b \in \mathfrak{A}$ *is selfadjoint and* $f(b) \geq 0$ $\forall f \in S$, *we have* $b \geq 0$.

Proof. Let $[\lambda, \Lambda]$ be the smallest closed subinterval of the real line containing the spectrum of b. We must show that $\lambda \geq 0$. Since $\Lambda - b \geq 0$, we have

$$\Lambda - \lambda = \|\Lambda - b\| = \sup\{\Lambda\|f\| - f(b): f \in S\}$$
$$\leq \sup\{\Lambda - f(b): f \in S\} = \Lambda - \inf\{f(b): f \in S\},$$

so $\lambda \geq \inf\{f(b): f \in S\} \geq 0$, which is what we wanted.

4.2 **Theorem.** *With* A *and* B *as above, the following are equivalent*:

(i) *For each pre-Hilbert* B*-module* X, *the restriction map of* $(X \odot A)'$ *into* $M(X, A)$ *is an isometry onto*;

(ii) *for any subset* $\{c_{ij}: i, j = 1, \cdots, n\}$ *of* A *such that* $\Sigma_{i,j} b_i^* c_{ij} b_j \geq 0$ $\forall b_1, \cdots, b_n \in B$, *we have* $\Sigma_{i,j} a_i^* c_{ij} a_j \geq 0$ $\forall a_1, \cdots, a_n \in A$.

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Proof. We first show that (i) implies (ii). Suppose we have $c_{ij} \in A$ $(i, j = 1, \cdots, n)$ such that

$$\sum_{i,j} b_i^* c_{ij} b_j \geq 0 \quad \forall b_1, \cdots, b_n \in B. \tag{1}$$

Let X be the direct sum of n copies of B, made into a Hilbert B-module with B-valued inner product defined by

$$\langle (b_1, \cdots, b_n), (\beta_1, \cdots, \beta_n) \rangle = \sum_{j=1}^{n} \beta_j^* b_j$$

for $b_j, \beta_j \in B$ $(j = 1, \cdots, n)$. One checks that $X \odot A$ is just the direct sum of n copies of A (with A-valued inner product defined in like manner) via the identification $(b_1, \cdots, b_n) \otimes a + N \rightarrow (b_1 a, \cdots, b_n a)$. Now consider the B^*-algebra $\mathfrak{A}(X \odot A)$, which is easily seen to be $*$-isomorphic with the B^*-algebra $A_{(n)}$ of $n \times n$ matrices with entries in A. For $T \in \mathfrak{A}(X \odot A)$, $T \geq 0$, we have (using the assumption that the restriction map is an isometry)

$$\begin{aligned}
\|T\|_{X \odot A}^{1/2} &= \|T^{1/2}\|_{X \odot A} \\
&= \sup\{\|T^{1/2} y\|_{X \odot A}: y \in X \odot A,\ \|y\|_{X \odot A} \leq 1\} \\
&= \sup\{\|\langle x, T^{1/2} y\rangle\|: y \in X \odot A,\ x \in X,\ \|y\|_{X \odot A} \leq 1,\ \|x\|_X \leq 1\} \\
&= \sup\{\|\langle T^{1/2} x, y\rangle\|: y \in X \odot A,\ x \in X,\ \|y\|_{X \odot A} \leq 1,\ \|x\|_X \leq 1\} \\
&= \sup\{\|T^{1/2} x\|_{X \odot A}: x \in X,\ \|x\|_X \leq 1\} \\
&= \sup\{\|\langle T^{1/2} x, T^{1/2} x\rangle\|^{1/2}: x \in X,\ \|x\|_X \leq 1\} \\
&= \sup\{\|\langle T x, x\rangle\|^{1/2}: x \in X,\ \|x\|_X \leq 1\}
\end{aligned}$$

i.e. $\|T\|_{X \odot A} = \sup\{\|\langle Tx, x\rangle\|: x \in X, \|x\|_X \leq 1\}$. Let S be the family of functionals $U \rightarrow f(\langle Ux, x\rangle)$ on $\mathfrak{A}(X \odot A)$, where f is a state of A and $x \in X$, $\|x\|_X \leq 1$. For $T \in \mathfrak{A}(X \odot A)$, $T \geq 0$, we have just shown that $\|T\|_{X \odot A} = \sup\{g(T): g \in S\}$, so S satisfies the hypotheses of 4.1.

Now let $T \in \mathfrak{A}(X \odot A)$ be the operator corresponding to the matrix $[c_{ij}] \in A_{(n)}$. We see from (1) that $g(T) \geq 0$ $\forall g \in S$. It also follows easily from (1) that $T = T^*$ (i.e. $c_{ij} = c_{ji}^*$ for $i, j = 1, \cdots, n$), and we conclude from 4.1 that $T \geq 0$. By 6.1, this means that $\sum_{i,j} a_i^* c_{ij} a_j \geq 0$ $\forall a_1, \cdots, a_n \in A$, which is what we wanted.

For the other direction, assume that (ii) holds and let X be a pre-Hilbert B-module. To establish (i), it will suffice to show that given $\phi \in M(X, A)$ with $\|\phi\| \leq 1$, we can extend ϕ to a unique $\tau \in (X \odot A)'$ with $\|\tau\| \leq 1$.

Consider τ_0: $X \otimes A \rightarrow A$ defined by $\tau_0(\sum_{i=1}^n x_i \otimes a_i) = \sum_{i=1}^n \phi(x_i) a_i$. τ_0 is

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clearly an A-module map. Moreover, for $b_1, \cdots, b_n \in B$, $x_1, \cdots, x_n \in X$ we have

$$\sum_{i,j} b_i^* \phi(x_i)^* \phi(x_j) b_j = \sum_{i,j} \phi(x_i \cdot b_i)^* \phi(x_j \cdot b_j) = \left(\phi\left(\sum_{i=1}^n x_i \cdot b_i\right)\right)^* \phi\left(\sum_{i=1}^n x_i \cdot b_i\right)$$

$$\leq \left\langle \sum_{i=1}^n x_i \cdot b_i, \sum_{i=1}^n x_i \cdot b_i \right\rangle = \sum_{i,j} b_i^* \langle x_j, x_i \rangle b_j,$$

the inequality holding by virtue of 2.8 and our assumption that $\|\phi\| \leq 1$. By (ii), we must have

$$\sum_{i,j} a_i^* \phi(x_i)^* \phi(x_j) a_j \leq \sum_{i,j} a_i^* \langle x_j, x_i \rangle a_j \quad \forall a_1, \cdots, a_n \in A, \ x_1, \cdots, x_n \in X,$$

i.e.

$$\tau_0(z)^* \tau_0(z) \leq [z, z] \quad \forall z \in X \otimes A.$$

This shows that the map $\tau\colon X \odot A \to A$ given by $\tau(\Sigma_{i=1}^n x_i \otimes a_i + N) = \Sigma_{i=1}^n \phi(x_i) a_i$ is well defined and satisfies $\tau(y)^* \tau(y) \leq \langle y, y \rangle$ $\forall y \in X \odot A$ (so $\|\tau\| \leq 1$). Hence $\tau \in (X \odot A)'$. Notice that $\tau(x \otimes 1 + N) = \phi(x)$ $\forall x \in X$, so τ is an extension of ϕ. This completes the proof.

We mention two situations in which the pair (A, B) (where A has 1 and $1 \in B$) satisfies (ii) of 4.2. If A is commutative, it follows from a result of M. Takesaki [11] that the pure states of $A_{(n)}$ all have the form $[c_{ij}] \to \Sigma_{i,j} \bar{\lambda}_i \lambda_j \pi(c_{ij})$, where π is a multiplicative linear functional on A and $\lambda_1, \cdots, \lambda_n \in C$ are such that $\Sigma_{i=1}^n |\lambda_i|^2 = 1$. From this it is immediate that (ii) holds whenever A is commutative. We claim that (ii) also holds whenever A is a W^*-algebra and B is ultraweakly dense in A. In this situation, balls about 0 in B of finite radius are dense in the corresponding balls of A with respect to the strong*-topology of A (see 1.8 of [8]). Moreover, the involution on A is strong*-continuous and multiplication is jointly strong*-continuous on norm-bounded subsets of A. Hence if $c_{ij} \in A$ $(i, j = 1, \cdots, n)$ and $\Sigma_{i,j} b_i^* c_{ij} b_j \geq 0$ $\forall b_1, \cdots, b_n \in B$, then $\Sigma_{i,j} a_i^* c_{ij} a_j \geq 0$ $\forall a_1, \cdots, a_n \in A$.

We remark in passing that it is not difficult to find pairs (A, B) for which (ii) fails. For example, let A be the algebra of 2×2 complex matrices and B the subalgebra of A consisting of complex multiples of the identity matrix. If we let

$$c_{11} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad c_{12} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad c_{21} = c_{12}^*,$$

$$c_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a_2 = a_1^*,$$

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then $\Sigma_{i,j} \lambda_i \bar{\lambda}_j c_{ij} \geq 0$ $\forall \lambda_1, \lambda_2 \in C$, but $\Sigma_{i,j} a_i^* c_{ij} a_j = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

If B is an arbitrary B^*-algebra with 1, and X a pre-Hilbert B-module, it appears that we cannot in general expect to be able to extend the B-valued inner product on X to a B-valued inner product on X' as in 3.2. We can, however, obtain a reasonably satisfactory general substitute for 3.2 by considering bounded B-module maps of X into B^{**}, the second conjugate space of B. By 3.2, $(X \odot B^{**})'$ is a self-dual Hilbert B^{**}-module with a B^{**}-valued inner product extending that of $X \odot B^{**}$. Since B^{**} is a W^*-algebra containing B as an ultraweakly dense subalgebra, (ii) of 4.2 holds for the pair (B^{**}, B) and we may therefore transfer the inner product on $(X \odot B^{**})'$ over to $M(X, B^{**})$. X may be regarded as a B-submodule of $M(X, B^{**})$ in an obvious way, and it is clear that the B^{**}-valued inner product which we have put on $M(X, B^{**})$ extends the B-valued inner product on X. We thus obtain the following corollary as a special case of 4.2.

4.3 **Corollary.** *Let B be a B^*-algebra with 1 and X a pre-Hilbert B-module. Then the B-valued inner product on X can be extended to a B^{**}-valued inner product on $M(X, B^{**})$ in such a way as to make the latter into a self-dual B^{**}-module.*

5. **Representation of completely positive maps.** Let B be a B^*-algebra, A a $*$-algebra, and ϕ: $A \to B$ a linear map. We call ϕ *positive* if $\phi(a^*a) \geq 0$ $\forall a \in A$. For $n = 1, 2, \cdots$, ϕ induces a map ϕ_n from the algebra $A_{(n)}$ of $n \times n$ matrices with entries in A (made into a $*$-algebra by setting $[a_{ij}]^* = [a_{ji}^*]$ $\forall$ matrices $[a_{ij}] \in A_{(n)}$) into the corresponding B^*-algebra $B_{(n)}$ defined by $\phi_n([a_{ij}]) = [\phi(a_{ij})]$; we say that ϕ is *completely positive* if each of the induced maps ϕ_n is positive. It should be noted that positivity does not in general imply complete positivity. For example, the map from the algebra of 2×2 complex matrices onto itself which sends each matrix to its transpose is positive but not completely positive (see [1].)

5.1 **Remark.** A linear map ϕ: $A \to B$ is completely positive if and only if $\Sigma_{i,j} b_i^* \phi(a_i^* a_j) b_j \geq 0$ $\forall a_1, \cdots, a_n \in A, b_1, \cdots, b_n \in B$. To see this, observe that the matrices in $A_{(n)}$ of the form M^*M $(M \in A_{(n)})$ are precisely those which can be written as the sum of n or fewer matrices of the form $[a_i^* a_j]$ $(a_1, \cdots, a_n \in A)$. The remark now follows from 6.1.

Let ϕ: $A \to B$ be completely positive and suppose in addition that $\phi(a^*) = \phi(a)^*$ $\forall a \in A$. (This additional assumption is frequently superfluous, for instance if A has 1.) The map ϕ gives rise to a pre-Hilbert B-module as follows. Consider the algebraic tensor product $A \otimes B$, which becomes a right B-module when we set $(a \otimes b) \cdot \beta = a \otimes b\beta$ for $b, \beta \in B$, $a \in A$. Define $[\cdot, \cdot]$: $A \otimes B \times A \otimes B \to B$ by

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$$\left[\sum_{j=1}^{n} a_j \otimes b_j, \sum_{i=1}^{m} \alpha_i \otimes \beta_i\right] = \sum_{i,j} \beta_i^* \phi(\alpha_i^* a_j) b_j$$

for $a_1, \cdots, a_n, \alpha_1, \cdots, \alpha_m \in A$, $b_1, \cdots, b_n, \beta_1, \cdots, \beta_m \in B$. $[\cdot, \cdot]$ is clearly well defined and conjugate-bilinear. We have $[x, x] \geq 0$ $\forall x \in A \otimes B$ (since ϕ is completely positive), $[x, y] = [y, x]^*$ $\forall x, y \in A \otimes B$ (since ϕ is a *-map), and $[x \cdot b, y] = [x, y]b$ $\forall x, y \in A \otimes B$, $b \in B$ (by inspection). By 2.2, the set $N = \{x \in A \otimes B: [x, x] = 0\}$ is a submodule of $A \otimes B$ and $x_0 = (A \otimes B)/N$ is a pre-Hilbert B-module with B-valued inner product $\langle x + N, y + N\rangle = [x, y]$ for $x, y \in A \otimes B$.

The construction of X_0 is a generalization of the process whereby a hermitian positive linear functional on A gives rise to a pre-Hilbert space. It should be compared with a similar construction carried out by W. F. Stinespring [9].

Following T. W. Palmer [5], we call an element v of the *-algebra A *quasi-unitary* if $vv^* = v^*v = v + v^*$ and say that A is a *U^*-algebra* if it is the linear span of its quasi-unitary elements. All Banach *-algebras are U^*-algebras [6]. Notice that if A has 1, then $u \in A$ is unitary (i.e. $u^*u = uu^* = 1$) if and only if $1 - u$ is quasi-unitary, so in this case A is a U^*-algebra if and only if it is spanned by its unitaries.

Let X be a Hilbert B-module. Given a *-homomorphism $\pi: A \to \mathfrak{A}(X)$ (henceforth called a **-representation of A on X*) and an element $e \in X$, we may define a linear map $\phi: A \to B$ by $\phi(a) = \langle \pi(a)e, e\rangle$ for $a \in A$. Using 5.1, an easy computation shows that ϕ is completely positive. The following theorem says that if A is a U^*-algebra with 1, then all completely positive maps of A into B arise in this manner. Its proof is modeled on that of a result of W. F. Stinespring [9].

5.2 **Theorem.** *Let A be a U^*algebra with* 1, *B a B^*-algebra with* 1, *and $\phi: A \to B$ a completely positive map. There is a Hilbert B-module X, a *-representation π of A on X, and an element $e \in X$ such that $\phi(a) = \langle \pi(a)e, e\rangle$ $\forall a \in A$ and the set $\{\pi(a)(e \cdot b): a \in A, b \in B\}$ spans a dense subspace of X.*

Proof. First observe that ϕ is automatically a *-map. (For each positive linear functional f on B, the map $a \to f(\phi(a))$ is a positive linear functional on A. Since A has 1, each such functional is hermitian and we have $f(\phi(a^*)) = \overline{f(\phi(a))} = f(\phi(a)^*)$ for every $a \in A$ and every positive linear functional f on B. This shows that $\phi(a^*) = \phi(a)^*$ $\forall a \in A$.) Notice also that $A \otimes B$ becomes a left A-module when we define $\alpha \cdot (a \otimes b) = \alpha a \otimes b$ for $\alpha, a \in A$, $b \in B$. If $[\cdot, \cdot]$ and N are defined as in the construction of the pre-Hilbert B-module X_0 at the beginning of this section, then N is an A-submodule of $A \otimes B$. Indeed, if $u \in A$ is unitary, a direct computation shows that $[u \cdot x, u \cdot x] = [x, x]$ $\forall x \in A \otimes B$, so in particular $u \cdot N \subseteq N$. Since A is spanned by its unitaries, we have $A \cdot N \subseteq N$.

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For each $a \in A$, we may thus define a B-module map $\pi_0(a)$ of X_0 into itself by $\pi_0(a)(x + N) = a \cdot x + N$ for $x \in A \otimes B$. For any unitary $u \in A$, $\pi_0(u)$ is an isometry of X_0, so each $\pi_0(a)$ is a linear combination of isometries and therefore bounded. Since $[a \cdot x, y] = [x, a^* \cdot y]$ $\forall a \in A$, $x, y \in A \otimes B$, we have $\pi_0(a) \in \mathcal{A}(X_0)$ with $\pi_0(a^*) = \pi_0(a)^*$ $\forall a \in A$. Let X be the Hilbert B-module completion of X_0. Each $\pi_0(a)$ extends uniquely to an operator $\pi(a) \in \mathcal{A}(X)$. It is clear that π is a $*$-representation of A on X.

Finally, set $e = 1 \otimes 1 + N$. For $a \in A$, $b \in B$, we have $\pi(a)(e \cdot b) = a \otimes b + N$, so the linear span of the set $\{\pi(a)(e \cdot b)\colon a \in A,\ b \in B\}$ is precisely X_0, which is dense in X. We have $\langle \pi(a)e, e\rangle = [a \otimes 1, 1 \otimes 1] = \phi(a)$ $\forall a \in A$, which completes the proof.

Suppose in addition that $\phi(1) = 1$. Then $\langle e, e\rangle = 1$ and it follows that the operator $e \otimes e \in \mathcal{A}(X)$ is a projection. It is a routine matter to verify that the map $b \to (e \cdot b) \otimes e$ is a $*$-isomorphism of B onto the closed $*$-subalgebra $(e \otimes e)\mathcal{A}(X)(e \otimes e)$ of $\mathcal{A}(X)$. Notice that $(e \cdot \phi(a)) \otimes e = (e \otimes e)\pi(a)(e \otimes e)$ $\forall a \in A$. These observations yield the following corollary to 5.2.

5.3 **Corollary.** *Let A and B be as above, and $\phi\colon A \to B$ a completely positive map such that $\phi(1) = 1$. There is a B^*-algebra $\mathcal{A}$ containing B, a projection $p \in \mathcal{A}$ such that $B = p\mathcal{A}p$, and a $*$-homomorphism $\pi\colon A \to \mathcal{A}$ such that $\phi(a) = p\pi(a)p$ $\forall a \in A$.*

Let A be a U^*-algebra with 1, and B a W^*-algebra. Our goal is a description of the order structure of the set of completely positive maps from A into B similar to that given in 1.4.2 of [1] for the case $B = B(H)$, H a Hilbert space. Let $\phi\colon A \to B$ be a completely positive map. If X, π, and e are as in 5.2, we may define a $*$-representation $\tilde{\pi}$ of A on the self-dual Hilbert B-module X' by composing the $*$-isomorphism $T \to \tilde{T}$ of $\mathcal{A}(X)$ into $\mathcal{A}(X')$ (see 3.7) with π, i.e. we set $\tilde{\pi}(a) = \pi(a)^\sim \in \mathcal{A}(X')$ $\forall a \in A$. Suppose $\psi\colon A \to B$ is another completely positive map. We write $\psi \leq \phi$ if $\phi - \psi$ is completely positive and let $[0, \phi]$ denote the set of completely positive maps from A into B which are $\leq \phi$.

For $T \in \mathcal{A}(X')$, define $\phi_T\colon A \to B$ by $\phi_T(a) = \langle T\tilde{\pi}(a)\hat{e}, \hat{e}\rangle$ for $a \in A$. Notice that $\phi_I = \phi$ and that the map $T \to \phi_T$ is a linear map of $\mathcal{A}(X')$ into the space of linear transformations of A into B. The proof of the next proposition is much like that of 1.4.2 in [1].

5.4 **Proposition.** *The map $T \to \phi_T$ is an affine order isomorphism of $\{T \in \tilde{\pi}(A)'\colon 0 \leq T \leq I_{X'}\}$ onto $[0, \phi]$ (where $\tilde{\pi}(A)'$ denotes the commutant of $\tilde{\pi}(A)$ in $\mathcal{A}(X')$).*

Proof. First we show that $T \to \phi_T$ is one-to-one on $\tilde{\pi}(A)'$. Indeed, if $T \in$

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$\tilde{\pi}(A)'$ and $\phi_T = 0$, a direct computation shows that $\langle T(\pi(a_1)(e \cdot b_1))\hat{}$, $(\pi(a_2)(e \cdot b_2))\hat{}\rangle = 0$ $a_1, a_2 \in A$, $b_1, b_2 \in B$, so $\langle T(\hat{X}_0), \hat{X}_0\rangle = 0$, so $\langle T(\hat{X}), \hat{X}\rangle = 0$, so $T = 0$ by the uniqueness assertion of 3.7. Next, we claim that ϕ_T is completely positive if $T \in \tilde{\pi}(A)'$ and $T \geq 0$. For $a_1, \cdots, a_n \in A$, $b_1, \cdots, b_n \in B$, set $x = \sum_{j=1}^n \pi(a_j)(e \cdot b_j) \in X$. One checks that

$$\sum_{i,j} b_i^* \phi_T(a_i^* a_j) b_j = \langle T\hat{x}, \hat{x}\rangle = \langle T^{1/2}\hat{x}, T^{1/2}\hat{x}\rangle \geq 0$$

so ϕ_T is completely positive by 5.1. This is enough to show that $T \to \phi_T$ is an affine order isomorphism of $\{T \in \tilde{\pi}(A)' \colon 0 \leq T \leq I\}$ into $[0, \phi]$.

To show that this isomorphism is onto, take $\psi \in [0, \phi]$. From 5.2 we get a $*$-representation ρ of A on a Hilbert B-module Y and a $d \in Y$ such that $\psi(a) = \langle \rho(a)d, d\rangle$ $\forall a \in A$ and the set $\{\rho(a)(d \cdot b) \colon a \in A, b \in B\}$ spans a dense subspace Y_0 of Y. Since $\psi \leq \phi$, it follows routinely that there is a well-defined bounded module map $W \colon X_0 \to Y_0$ such that $W(\pi(a)(e \cdot b)) = \rho(a)(d \cdot b)$ $\forall a \in A$, $b \in B$ and $\langle Wx, Wx\rangle \leq \langle x, x\rangle$ $\forall x \in X_0$. W extends to a bounded module map $W \colon X \to Y$. A straightforward computation shows that the maps $W\pi(a)$ and $\rho(a)W$ agree on X_0 $\forall a \in A$, whence $W\pi(a) = \rho(a)W$ $\forall a \in A$. We appeal to 3.6 to get a bounded module map $\tilde{W} \colon X' \to Y'$ extending W. It is clear from the proof of 3.6 that $\langle \tilde{W}\tau, \tilde{W}\tau\rangle \leq \langle \tau, \tau\rangle$ $\forall \tau \in X'$. Let $\tilde{W}^* \colon Y' \to X'$ be the adjoint of $\tilde{W}$ given by 3.4 and set $T = \tilde{W}^*\tilde{W}$, so $T \in \mathfrak{A}(X')$ and $T = T^*$. For $\tau \in X'$, we have $\langle T\tau, \tau\rangle = \langle \tilde{W}\tau, \tilde{W}\tau\rangle$, so $0 \leq \langle T\tau, \tau\rangle \leq \langle \tau, \tau\rangle$. From this it follows (see the proof of 6.1) that $0 \leq T \leq I$.

Notice that for $a \in A$, the bounded module maps $\tilde{W}\tilde{\pi}(a)$ and $\tilde{\rho}(a)\tilde{W}$ of X' into Y' are both extensions of $W\pi(a) = \rho(a)W$, so by the uniqueness assertion of 3.6 we have $\tilde{W}\tilde{\pi}(a) = \tilde{\rho}(a)\tilde{W}$ $\forall a \in A$. It follows from this that $\tilde{\pi}(a)\tilde{W}^* = \tilde{W}^*\tilde{\rho}(a)$ $\forall a \in A$. Hence for any $a \in A$, we have $T\tilde{\pi}(a) = \tilde{W}^*\tilde{W}\tilde{\pi}(a) = \tilde{W}^*\tilde{\rho}(a)\tilde{W} = \tilde{\pi}(a)\tilde{W}^*\tilde{W} = \tilde{\pi}(a)T$, i.e. $T \in \tilde{\pi}(A)'$.

Finally, $\phi_T = \psi$, since for $a \in A$ we have $\phi_T(a) = \langle T\tilde{\pi}(a)\hat{e}, \hat{e}\rangle = \langle \tilde{W}\tilde{\pi}(a)\hat{e}, \tilde{W}\hat{e}\rangle = \langle W\pi(a)e, We\rangle = \langle \rho(a)d, d\rangle = \psi(a)$. This completes the proof.

With A and B as above and $b \in B$, $b \geq 0$, we denote the set of completely positive maps $\phi \colon A \to B$ such that $\phi(1) = b$ by $\Sigma(A, B, b)$. Notice that $\Sigma(A, B, b)$ is a convex subset of the space of linear maps from A into B. The following characterization of the set of extreme points of $\Sigma(A, B, b)$ follows from 5.4 in exactly the same way that 1.4.6 of [1] follows from 1.4.2.

5.4 **Theorem.** *Let A be a U^*-algebra with* 1, *B a W^*-algebra, and $\phi \in \Sigma(A, B, b)$ where $b \in B$, $b \geq 0$. Then (in the notation of* 5.2) *ϕ is an extreme point of $\Sigma(A, B, b)$ if and only if the map $T \to \langle T\hat{e}, \hat{e}\rangle$ of $\mathfrak{A}(X')$ into B is one-to-one on $\tilde{\pi}(A)'$.*

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6. **Appendix: Positivity of matrices over B^*-algebras.** Let B be a B^*-algebra and for $n = 1, 2, \cdots$, let $B_{(n)}$ denote the B^*-algebra of $n \times n$ matrices with entries in B. The following criterion for the positivity of a matrix in $B_{(n)}$ is used several times in this paper.

6.1 **Proposition.** *Let $c_{ij} \in B$ $(i, j = 1, \cdots, n)$. The matrix $[c_{ij}] \in B_{(n)}$ is ≥ 0 if and only if $\sum_{i,j} a_i^* c_{ij} a_j \geq 0$ $\forall a_1, \cdots, a_n \in B$.*

Proof. Without loss of generality, we may assume that B has 1. Let X be the direct sum of n copies of B, made into a Hilbert B-module with B-valued inner product defined by $\langle (b_1, \cdots, b_n), (\beta_1, \cdots, \beta_n) \rangle = \sum_{j=1}^n \beta_j^* b_j$ for $b_j, \beta_j \in B$ $(j = 1, \cdots, n)$. (That X is complete with respect to $\|\cdot\|_X$ follows from the fact that

$$\max\{\|b_j\|: j = 1, \cdots, n\} \leq \|(b_1, \cdots, b_n)\|_X \leq \left(\sum_{j=1}^n \|b_j\|^2\right)^{1/2}$$

$\forall (b_1, \cdots, b_n) \in X$.) For $j = 1, \cdots, n$, let e_j be the element of X with jth coordinate 1 and all other coordinates 0. It is routine to show that the map $T \to [\langle Te_j, e_i \rangle]$ is a $*$-isomorphism of $\mathfrak{A}(X)$ onto $B_{(n)}$.

Let T be the operator in $\mathfrak{A}(X)$ corresponding to the matrix $[c_{ij}] \in B_{(n)}$, so for $(b_1, \cdots, b_n) \in X$, the kth coordinate of $T(b_1, \cdots, b_n)$ is $\sum_{j=1}^n c_{kj} b_j$ $(k = 1, \cdots, n)$. It is clear that $\sum_{i,j} a_i^* c_{ij} a_j \geq 0$ $\forall a_1, \cdots, a_n \in B$ if and only if $\langle Tx, x \rangle \geq 0$ $\forall x \in X$. On the other hand, $[c_{ij}] \geq 0$ if and only if $T \geq 0$. Now certainly if $T \geq 0$, then $\langle Tx, x \rangle = \langle T^{1/2}x, T^{1/2}x \rangle \geq 0$ $\forall x \in X$. Conversely, suppose $\langle Tx, x \rangle \geq 0$ $\forall x \in X$. We may write $T = U + iV$ for selfadjoint $U, V \in \mathfrak{A}(X)$. Since $\langle Ux, x \rangle$ and $\langle Vx, x \rangle$ are selfadjoint $\forall x \in X$, it follows that $\langle Vx, x \rangle = 0$ $\forall x \in X$ and hence (exactly as for a bounded operator on a Hilbert space) $V = 0$, i.e. T is selfadjoint. We have $f(\langle Tx, x \rangle) \geq 0$ for each $x \in X$ and each positive linear functional f on B. It follows from 4.1, applied to the family S of functionals on $\mathfrak{A}(X)$ of the form $W \to f(\langle Wx, x \rangle)$ where $x \in X$ with $\|x\|_X = 1$ and f is a state of B, that $T \geq 0$. This completes the proof.

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