

# Input Output Analysis of Power Control in Wireless Networks

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**Abstract**—In a wireless communication network different users share a common resource. An objective of radio resource management is to assign the resources in an effective way between the users. Power control is an important component in this setting that has been extensively studied over the last two decades.

In many real networks there are inherent time delays due to filtering of signals and control signaling. Time delays can affect stability and convergence properties of the power control algorithms. We therefore consider power control laws of higher order to include models with delays and delay compensation.

The main contribution of this paper is to exploit more structure of the interference feedback to prove less conservative conditions for system stability. Our primary tool to reduce conservativeness is to use scalings in an input output framework for stability analysis.

## I. INTRODUCTION

Power control in wireless networks aims to assign powers to a set of users in a distributed fashion giving each user a Quality of Service (QoS). Users transmitting on the same channel interfere with each other, which degrades their QoS. This interference feedback has to be taken into account when analyzing system stability. In this paper we focus on the Signal-to-Interference Ratio (SIR) as a measure of QoS. This is a reasonable assumption since most other relevant measures of QoS, such as bit error rate (BER) or frame success rate, are monotone functions of the SIR. The SIR is modeled as

$$\bar{\gamma}_i = \frac{\bar{\delta}_i \bar{g}_{ii} \bar{p}_i}{\sum_{i \neq j} \bar{g}_{ij} \bar{p}_j + (1 - \bar{\delta}_i) \bar{g}_{ii} \bar{p}_i + \bar{\sigma}_i^2} \triangleq \frac{\bar{\delta}_i \bar{g}_{ii} \bar{p}_i}{\bar{R}_i(\bar{p})},$$

where  $\bar{p} = [\bar{p}_1 \dots \bar{p}_n]^T$  and  $\bar{p}_i$  is the power of user  $i$ ,  $\bar{\delta}_i \in [0, 1]$  is a constant modeling auto-interference,  $\bar{g}_{ij} > 0$  is the channel gain from user  $j$  to user  $i$  and  $\bar{\sigma}_i^2 > 0$  is the receiver noise.  $\bar{R}_i(\bar{p})$  is sometimes referred to as the *effective interference*.

Consider the following iterative update equation of the powers

$$\bar{p}[t+1] = \bar{I}(\bar{p}[t]). \quad (1)$$

An example is the Distributed Power Control (DPC) algorithm, proposed by Foschini and Miljanic [8], which balances the SIRs of the users. The next power update is given by

$$\bar{p}_i[t+1] = \bar{I}_i(\bar{p}[t]) \triangleq \frac{\bar{\gamma}_i^T}{\bar{\gamma}_i} \bar{p}_i[t] = \frac{\bar{\gamma}_i^T \bar{R}_i(\bar{p}[t])}{\bar{\delta}_i \bar{g}_{ii}}, \quad (2)$$

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where  $\bar{\gamma}_i^T$  is the SIR-target. This algorithm is suitable for speech applications where it is important to keep a SIR-level throughout time. Convergence for a class of algorithms denoted *standard*, including the DPC algorithm, was shown by Yates in [22].

For applications with data traffic more bandwidth is desired for each user. Contrary to traditional applications such as speech, it is more important to have an overall high bandwidth, rather than maintaining a stable high SIR-level. Hence it can be allowed that the QoS-level drops heavily over short periods of time, given that it is higher during others. Sung and Leung [19] proposed an opportunistic power control law that takes advantage of these characteristics. Their opportunistic algorithm is defined as

$$\bar{p}_i[t+1] = \bar{I}_i(\bar{p}[t]) \triangleq \frac{\bar{\zeta}_i}{\bar{R}_i(\bar{p}[t])}, \quad (3)$$

where  $\bar{\zeta}_i$  is called the Signal-to-Interference-Product (SIP) of user  $i$ . This implies that when a user has good interference conditions it transmits at a high power, and when it measures high interference it transmits at a low power. Furthermore they proposed a framework in [19] that generalizes Yates work to also include opportunistic algorithms. A key concept is *two-sided scalability*. Given the distance function

$$d(\bar{p}_1, \bar{p}_2) = \max_i \left\{ \max \left\{ \frac{\bar{p}_{1,i}}{\bar{p}_{2,i}}, \frac{\bar{p}_{2,i}}{\bar{p}_{1,i}} \right\} \right\}, \quad (4)$$

two-sided scalability for a function  $\bar{I}(\bar{p})$  is equivalent (see Appendix) to the condition

$$d(\bar{I}(\bar{p}_1), \bar{I}(\bar{p}_2)) < d(\bar{p}_1, \bar{p}_2),$$

which is used to derive Lipschitz constants in this paper.

## II. HIGHER ORDER CONTROL LAWS

Real communication networks can suffer from so called “power rushes”, where the transmission powers of the users significantly increase leading to system instability. It has been shown that delays can cause instability and many systems have inherent delays due to measuring and transmission of control commands. Gunnarsson [11] showed that using a Smith predictor it is possible to compensate for delay. The introduction of delays and delay compensation motivates the use of higher order power control laws.

Stability of similar systems with delays have also been studied in e.g. [4], [3], [1], [7], [16], [18] and [17]. An important difference is that with the model we study, time-delays are allowed in both the interference and the transmission power of each user, without any assumption on local

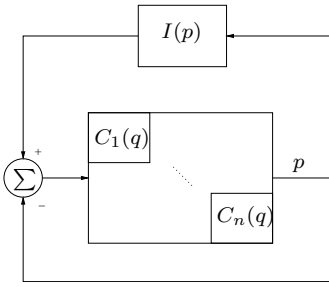


Fig. 1. Logarithmized system as block diagram.

replacement of delayed powers (delay compensation). Delays in the transmission powers of a user is present in many wireless networks with control signalling, e.g. when a base station controls the transmission power of a mobile user, see e.g. [6] and [20].

In [13] the framework of Sung and Leung was extended to include higher order control. Convergence and speed of convergence could be established. However the conditions were only sufficient and simulations indicated that they were conservative. In this paper we exploit more structure of the interference feedback to remove conservatism from the previous results. This implies that it is possible to prove stability for a wider class of control laws.

In section III we introduce logarithmic variables to rewrite the power control problem as a classical feedback interconnection. This model is then rewritten to input output form around an equilibrium point in section IV. Section V is devoted to input output analysis of the system and sufficient conditions for local stability are given. In section VI scalings are introduced to further improve the results of the previous section. We then introduce saturation of the transmission powers in the model to prove global stability in section VII. That is followed by section VIII on optimization of the Lipschitz constant to reduce conservativeness. An example illustrating the results is presented in section IX and the paper is concluded in section X.

The results of this paper generalizes our earlier conference contribution in [14] and technical report [12].

### III. SYSTEM MODEL IN LOGARITHMIC SCALE

The DPC and opportunistic algorithm can be interpreted as nonlinear feedback systems by introducing logarithmic variables. Let  $p_i = \ln(\bar{p}_i)$ ,  $\delta_i = \ln(\bar{\delta}_i)$ ,  $g_{ij} = \ln(\bar{g}_{ij})$ ,  $I_i(p) = \ln(\bar{I}_i(e^p)) = \ln(\bar{I}_i(e^p))$ ,  $R_i(p) = \ln(\bar{R}_i(e^p))$  and  $\gamma_i^T = \ln(\bar{\gamma}_i^T)$ . Then the fixed point iteration  $\bar{p}_i[t+1] = \bar{I}_i(\bar{p}[t])$  can be rewritten as

$$p_i[t+1] = I_i(p[t]) = p_i[t] + (I_i(p[t]) - p_i[t])$$

which can be represented in the block diagram in Figure 1 with  $C_i(q) = \frac{1}{q-1}$ , where  $q$  represents the time-shift operator, and with the interference function  $I(p) = [I_1(p) \dots I_n(p)]$ .

In implementations it is common that measurements are made on both the interference and signal part of the received

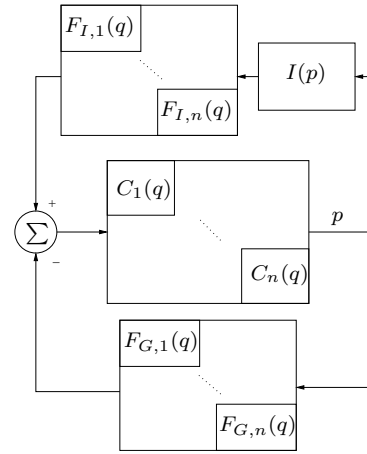


Fig. 2. Logarithmized system with filters as block diagram.

power, i.e. both the upper and lower loop in Figure 1 are implemented. In this paper we study a model that include linear filters both for the interference and signal part of the received power, as well as for the system itself, as shown in Figure 2. This extended system model, first proposed in [11], models a large variety of systems, where delays and higher order control laws can be included.

Define the linear filters as

$$C_i(q) = \frac{b_{i,0}q^{m_C-1} + b_{i,1}q^{m_C-2} + \dots + b_{i,m_C-1}}{(q-1)(q^{m_C-1} + a_{i,1}q^{m_C-2} + \dots + a_{i,m_C-1})} \quad (5)$$

$$F_{I,i}(q) = \frac{d_{i,0}q^{m_I} + d_{i,1}q^{m_I-1} + \dots + d_{i,m_I}}{q^{m_I} + c_{i,1}q^{m_I-1} + c_{i,2}q^{m_I-2} + \dots + c_{i,m_I}} \quad (6)$$

$$F_{G,i}(q) = \frac{f_{i,0}q^{m_G} + f_{i,1}q^{m_G-1} + \dots + f_{i,m_G}}{q^{m_G} + e_{i,1}q^{m_G-1} + e_{i,2}q^{m_G-2} + \dots + e_{i,m_G}} \quad (7)$$

where we assume that  $F_{I,i}(q)$  and  $F_{G,i}(q)$  are stable and with  $F_{I,i}(1) = F_{G,i}(1) \neq 0$ . The pole at  $q = 1$  in  $C_i(q)$  implies that the system has integral action. Assume first that we are using the pure DPC-algorithm, with the parameters  $\gamma_i^T$ ,  $\delta_i$  and  $g_{ii}$ . With the filters introduced as in Figure 2, we have

$$C_i(q) = \frac{1}{q-1}, \quad I_i(p) = \hat{\gamma}_i^T + R_i(p) - \hat{\delta}_i - \hat{g}_{ii},$$

where  $\hat{\gamma}_i^T = \frac{1}{F_{I,i}(1)}\gamma_i^T$ ,  $\hat{\delta}_i = \frac{F_{G,i}(1)}{F_{I,i}(1)}\delta_i$  and  $\hat{g}_{ii} = \frac{F_{G,i}(1)}{F_{I,i}(1)}g_{ii}$ . The somewhat cumbersome relation between the constants is due to the physical implementation, where the received signal,  $\delta_i + g_{ii} + p_i$ , and the received interference,  $R_i(p)$ , are filtered separately and then compared with the (unfiltered) target SIR. Usually  $F_{I,i}(1) = F_{G,i}(1) = 1$ . This way of modelling the system does not affect the steady state and stability properties of the system. However the transient is different.

When we study the opportunistic algorithm with the SIP-

target  $\hat{\zeta}_i$ , we have

$$C_i(q) = \frac{1}{q-1}, \quad I_i(p) = \hat{\zeta}_i - R_i(p),$$

where  $\hat{\zeta}_i = \frac{1}{F_{I,i}(1)}\zeta_i$ .

#### IV. INPUT OUTPUT MODEL

In this section we will derive an input output model of the higher order control loop. The model will be on the form of Figure 3 where the output  $z$  corresponds to the deviation of the power level from its equilibrium values, while  $\delta r$  is a disturbance input that models for example the effect of initial conditions, variations in gain or measurement errors of the interference. The blocks  $H$  and  $\Phi$  denotes the linear and nonlinear parts of the model, respectively.

Before deriving the exact model we will introduce some notation and preliminary results.

##### A. Notation

Consider first the finite dimensional vector spaces

- (i)  $R_\infty^n = (R^n, |\cdot|_\infty)$ , where  $|x|_\infty = \max_{1 \leq k \leq n} |x_k|$
- (ii)  $R_2^n = (R^n, |\cdot|_2)$ , where  $|x|_2 = \left( \sum_{k=1}^n x_k^2 \right)^{1/2}$ .

Now consider the following function spaces.

- (i)  $l_\infty^n = \{z : N \rightarrow R_\infty^n : \|z\|_\infty < \infty\}$
- (ii)  $l_{2,\infty}^n = \{z : N \rightarrow R_\infty^n : \|z\|_{2,\infty} < \infty\}$
- (iii)  $l_2^n = \{z : N \rightarrow R_2^n : \|z\|_2 < \infty\}$ ,

where the norms are defined as  $\|z\|_\infty = \sup_k |z[k]|_\infty$ ,  $\|z\|_{2,\infty} = \left( \sum_{k=0}^\infty |z[k]|_\infty^2 \right)^{1/2}$  and  $\|z\|_2 = \left( \sum_{k=0}^\infty |z[k]|_2^2 \right)^{1/2}$ . The spatial dimension will often be suppressed.

In the next few definitions we consider a nonlinear operator  $F : X \rightarrow X$  such that  $F(0) = 0$  and  $X$  is a normed vector space. The *gain* of  $F$  is defined as

$$\|F\|_{X \rightarrow X} \triangleq \sup_{z \in X; z \neq 0} \frac{\|F(z)\|_X}{\|z\|_X},$$

where  $\|\cdot\|_X$  denotes the norm on  $X$ . A stronger assumption is Lipschitz continuity. The *global Lipschitz constant* of the operator  $F$  is defined as

$$L[F; X] \triangleq \sup_{z_1, z_2 \in X, z_1 \neq z_2} \frac{\|F(z_1) - F(z_2)\|_X}{\|z_1 - z_2\|_X}.$$

Notice that  $\|F\|_{X \rightarrow X} \leq L[F; X]$ . For us it will be interesting to consider the Lipschitz constant on a subset  $B_X$  of  $X$  defined by how large deviations around the equilibrium we consider. Define

$$L[F; B_X] \triangleq \sup_{z_1, z_2 \in B_X, z_1 \neq z_2} \frac{\|F(z_1) - F(z_2)\|_X}{\|z_1 - z_2\|_X}.$$

For linear operators the gain and Lipschitz constants coincide. The  $l_1$ -norm of a linear system  $H_i$  is defined as

$$\|H_i\|_1 \triangleq \sum_{k=0}^{\infty} |h_i[k]|,$$

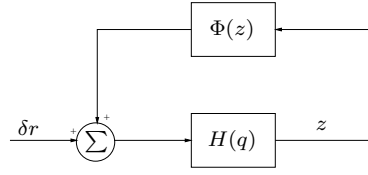


Fig. 3. The system on input-output form.

where  $h_i[k]$  is the impulse response at time  $k$ . For a diagonal matrix  $H$ ,  $H(q) = \text{diag}(H_i : i = 1, \dots, n)$ , the induced norms from  $l_\infty$  and  $l_{2,\infty}$  become (see e.g. [5])

$$\|H\|_{l_\infty \rightarrow l_\infty} = \|H\|_1 \triangleq \max(\|H_1\|_1, \dots, \|H_n\|_1)$$

and

$$\|H\|_{l_{2,\infty} \rightarrow l_{2,\infty}} \leq \|H\|_{1,1} \triangleq \sum_{k=0}^{\infty} |h[k]|_1,$$

where we used the matrix norm  $|M|_1 = |M|_{R_\infty^n \rightarrow R_\infty^n} = \max_{1 \leq i \leq n} \sum_{j=1}^n |M_{ij}|$ . Clearly  $\|H\|_1 \leq \|H\|_{1,1}$ , and equality holds if  $H_i = H_j, \forall (i, j)$ .

##### B. Existence of equilibrium point

The model of the wireless network in Figure 2 can be rewritten. Define

$$H(q) = \text{diag}(H_i : i = 1, \dots, n), \quad (8)$$

where

$$H_i(q) = \frac{F_{I,i}(q)C_i(q)}{1 + C_i(q)F_{G,i}(q)}.$$

We assume that  $H_i$  has the following time domain representation in terms of its impulse response

$$(H_i u)[k] = \sum_{l=0}^k h_i[k-l]u[l], \quad k \geq 0.$$

Therefore the system can be represented as

$$p[t] = (H \circ I(p))[t] = \sum_{l=0}^t h[t-l]I(p[l]), \quad t \geq 0, \quad (9)$$

where  $\circ$  denotes the composition of operators.

The assumption that  $C_i(q)$  has integral action and that  $F_I$  and  $F_G$  have the same steady state gain implies that  $H_i(1) = \sum_{l=0}^{\infty} h_i[l] = 1, \forall i$ . Hence at equilibrium (9) reduces to

$$p^* = I(p^*), \quad (10)$$

from which we conclude that the equilibrium is the same as for the standard power update equation in (1).

Let  $L[I, B]$  denote the Lipschitz constant of the nonlinear function  $I$  over any compact set  $B \subset R_\infty^n$ .

*Proposition 1:* Assume  $I(p) : R_\infty^n \rightarrow R_\infty^n$  is two-sided scalable. Then  $I$  is Lipschitz on any compact subset  $B \subset R_\infty^n$  and  $L[I, B] < 1$ .

A proof can be found in the appendix.

It is possible to derive conditions for the iterations in (1) to remain within a compact set  $B = [p_{\min}, p_{\max}]$ , see e.g. [13]. In such cases  $I : B \rightarrow B$  and since  $L[I, B] < 1$ , the Banach fixed point theorem proves the existence and uniqueness of an equilibrium point.

### C. Input Output model

Define deviations in the powers around the equilibrium point as  $z = p - p^*$ . We now also introduce a disturbance term,  $\delta r$ , which can model for example disturbances in the measurements of the interference or the target SIR. We can then write the system as the operator equation

$$p = H \circ (\delta r + I(p)),$$

where  $H$  and  $I$  are interpreted as operators on a Banach space  $X$ . Furthermore

$$\begin{aligned} z = p - p^* &= H \circ (\delta r + I(p)) - H \circ (I(p^*)) \\ &= H \circ (I(p) - I(p^*) + \delta r) \\ &= H \circ (\Phi(z) + \delta r), \end{aligned}$$

where  $\Phi(z) \triangleq I(p^* + z) - I(p^*)$ . This system is illustrated in Figure 3.

## V. INPUT OUTPUT ANALYSIS

In this section we perform an input output analysis of the model that was derived in the previous section. We will show that the gains of the linear and nonlinear parts of the system do not satisfy the small gain theorem. Instead, we need to analyze the system in bounded regions of the signal space. This is not surprising in view of the fact that no guaranteed rate of convergence can be obtained for the power update law in (1) for general two-sided scalable functions. It is also interesting to notice that input output analysis in the standard  $l_2$  space seems infeasible, see Subsection V-C.

### A. Lipschitz Constants for the Interference Mapping

Consider the nonlinear operator  $\Phi$ , defined by  $\Phi(z) = I(p^* + z) - I(p^*)$ , where  $I$  is two-sided scalable. Clearly  $\Phi(0) = 0$ . Initially we will focus attention on the case when the system consists of DPC and opportunistic users, i.e. when some users use the power update in (2) while others use (3). This means that the interference function has the following form

$$\bar{I} = [\bar{I}_1(\bar{p}), \dots, \bar{I}_n(\bar{p})],$$

where  $\bar{I}_i(\bar{p})$  has either of the following forms

$$\bar{I}_i(\bar{p}) = \frac{\bar{\gamma}_i^T}{\bar{\delta}_i \bar{g}_{ii}} \bar{R}_i(\bar{p}) \quad \text{DPC}$$

$$\bar{I}_i(\bar{p}) = \frac{\bar{\zeta}_i}{\bar{R}_i(\bar{p})} \quad \text{Opportunistic,}$$

where

$$\bar{R}_i(\bar{p}) = \bar{F}^i \bar{p} + \bar{\sigma}_i^2$$

and where  $\bar{F}^i = [\bar{F}_{i1}, \dots, \bar{F}_{in}]$ , where

$$\bar{F}_{ij} = \begin{cases} (1 - \bar{\delta}_i) \bar{g}_{ii}, & i = j, \\ \bar{g}_{ij}, & i \neq j. \end{cases} \quad (11)$$

Since  $\Phi_i(z) = I_i(p^* + z) - I_i(p^*)$ , we have

$$\frac{\partial \Phi_i}{\partial z_j}(z) = \pm \frac{\partial}{\partial z_j} (\ln(\bar{R}_i(z)))$$

depending on whether it is a DPC or opportunistic user, respectively. This gives

$$\nabla \Phi(z) = \begin{bmatrix} \pm \left[ \frac{(1-\bar{\delta}_1) \bar{g}_{11} e^{p_1^* + z_1}}{\bar{\sigma}_1^2 + F^1 e^{p^* + z}} \quad \dots \quad \frac{\bar{g}_{1n} e^{p_n^* + z_n}}{\bar{\sigma}_1^2 + F^1 e^{p^* + z}} \right] \\ \vdots \\ \pm \left[ \frac{\bar{g}_{n1} e^{p_1^* + z_1}}{\bar{\sigma}_n^2 + F^n e^{p^* + z}} \quad \dots \quad \frac{(1-\bar{\delta}_n) \bar{g}_{nn} e^{p_n^* + z_n}}{\bar{\sigma}_n^2 + F^n e^{p^* + z}} \right] \end{bmatrix}, \quad (12)$$

where the plus sign corresponds to DPC users and the minus sign to opportunistic users. The results in this paper will be independent of whether the users are DPC, opportunistic or combinations thereof, since the differences in sign will be removed by norms.

*Proposition 2:* Let  $\Phi$  be defined by a combination of DPC and opportunistic users. For  $l_\infty$  and  $l_{2,\infty}$  we have that the gain and global Lipschitz constants satisfy

$$\begin{aligned} \|\Phi\|_{l_\infty \rightarrow l_\infty} &= L[\Phi; l_\infty] = 1 \\ \|\Phi\|_{l_{2,\infty} \rightarrow l_{2,\infty}} &= L[\Phi; l_{2,\infty}] = 1. \end{aligned}$$

*Proof:* A proof is given in the appendix.  $\blacksquare$

A consequence of the above result is that global versions of the small gain theorem cannot not be applied. This follows since the integral action in the controller implies that  $H(1) = 1$ , which in turn implies that  $\|H\|_1 \geq 1$ . Since in Proposition 2 it was shown that the global Lipschitz constant is equal to one for our considered signal spaces it follows that  $\|H\|_1 L[\Phi; l_\infty] \geq 1$  and the small gain condition of the loop gain is violated. Global stability can be proven if the condition on  $H(1)$  is relaxed to  $H(1) < 1$ . Then, however, the relation  $p^* = I(p^*)$  in (10) does not hold, which in the DPC case means that the target SIR is not reached in steady state. This fact motivates the study of local stability instead of global.

### B. Local Analysis

Let  $X$  be either of the spaces  $l_\infty^n$  or  $l_{2,\infty}^n$ . Consider the set  $B$ , defined componentwise by  $p_{\min,i} \leq p_i \leq p_{\max,i}, \forall i$ . The induced sets for the deviations around the equilibrium point,  $z$ , is then given by

$$B^* = \{z \in \mathbf{R}_\infty^n : p_{\min,i} - p_i^* \leq z_i \leq p_{\max,i} - p_i^*, \forall i\} \quad (13)$$

$$B_X^* = \{z \in X : z[k] \in B^*, \forall k\}. \quad (14)$$

*Proposition 3:* Any  $\Phi : X \rightarrow X$  defined by two-sided scalable interference functions is Lipschitz on the closed convex set  $B_X^*$  with

$$L[\Phi; B_{l_\infty}^*] = L[\Phi; B_{l_{2,\infty}}^*] = L[\Phi; B^*] < 1.$$

Assume furthermore that  $\Phi$  is continuously differentiable and that the Jacobian is Lipschitz on  $B^*$ . Then

$$L[\Phi; B^*] = \max_{z \in B^*} |\nabla \Phi(z)|_1.$$

For the DPC and opportunistic algorithms the Lipschitz constant is easily computed. It is given by

$$\max_{z \in B^*} |\nabla \Phi(z)|_1 = \max_i \frac{\bar{F}^i e^{p^* + z_{\max}}}{\bar{\sigma}_i^2 + \bar{F}^i e^{p^* + z_{\max}}},$$

where  $e^{p^* + z_{\max}} = [e^{p_1^* + z_{\max,1}}, \dots, e^{p_n^* + z_{\max,n}}]$  and  $z_{\max,i} = p_{\max,i} - p_i^*, \forall i$ .

*Proof:* A proof can be found in the appendix. ■

For one of our main analysis results we need to consider the maximum interior ball in  $B^*$  and  $B_X^*$ , respectively, which are defined as

$$B^*(\gamma) = \{z \in \mathbf{R}_\infty : |z|_\infty \leq \gamma\},$$

$$B_X^*(\gamma) = \{z \in X : z[k] \in B^*(\gamma), \forall k\},$$

where  $\gamma = \min_i \{\min\{p_i^* - p_{\min,i}, p_{\max,i} - p_i^*\}\}$  and the corresponding Lipschitz bounds

$$L[\Phi; B_{l_\infty}^*(\gamma)] = L[\Phi; B_{l_2, \infty}^*(\gamma)] = L[\Phi; B^*(\gamma)].$$

*Theorem 1:* If  $\|H\|_1 < \frac{1}{L[\Phi; B^*(\gamma)]}$ , then there exists a unique power distribution  $z \in B_{l_\infty}^*(\gamma)$  for all

$$\|\delta r\|_\infty \leq \frac{\gamma(1 - \|H\|_1 L[\Phi; B^*(\gamma)])}{\|H\|_1}. \quad (15)$$

If it in addition holds that  $\|\delta r\|_{2, \infty} < \infty$  and

$$\|H\|_{l_2, \infty \rightarrow l_2, \infty} < \frac{1}{L[\Phi; B^*(\gamma)]},$$

then  $z[k] \rightarrow 0$  as  $k \rightarrow \infty$ .

*Proof:* A proof is given in the appendix. ■

### C. Failure of $l_2$ -based analysis

It is often advantageous to analyze the system in a Hilbert space such as  $l_2^n$ . Then the inner product structure and Fourier domain tools may be used to capture phase information and frequency domain interpretations. Note also that the gain  $\|H\|_{l_2^n \rightarrow l_2^n} = \sup_{\omega \in \mathbf{R}} \max_i |H_i(j\omega)|$  is less than or equal to  $\|H\|_1$ . Despite these potential advantages it turns out that the interference nonlinearity has a structure that appears to be unsuitable for  $l_2$ -analysis. Our first negative result shows that its gain and Lipschitz constant grows with the number of users for the cases where the interference nonlinearity is defined from the DPC or opportunistic algorithms.

*Proposition 4:*  $\|\Phi\|_{l_2 \rightarrow l_2} = L[\Phi; l_2] = \sqrt{n}$ . See [12] for a proof.

Our second negative observation shows that for the same algorithms, the interference nonlinearity violates the definition of incremental positivity in [21]. This implies that powerful characterizations of memoryless nonlinearities from the input output theory cannot be used, see e.g. [21], [15]

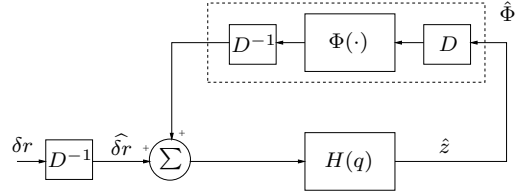


Fig. 4. The system on input-output form with scale factors.

and the references therein. Consider first the DPC case. The proof of our claim follows because if

$$z_1 = \begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad z_2 = \begin{bmatrix} z_{21} \\ z_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and  $\bar{\delta}_i = 1, \bar{\sigma}_i^2 > 0, i = 1, 2$ . Then one can show that

$$(z_1 - z_2)^T (\Phi(z_1) - \Phi(z_2)) = \ln \left( \frac{\bar{\sigma}_1^2 + \bar{g}_{12} e^{p_2^*}}{\bar{\sigma}_1^2 + \bar{g}_{12} e^{p_2^*}} \right) - \ln \left( \frac{\bar{\sigma}_2^2 + g_{21} e^{p_1^*}}{\bar{\sigma}_2^2 + \bar{g}_{21} e^{p_1^*}} \right) < 0,$$

which implies that the interference nonlinearity cannot be incrementally positive. Now consider the opportunistic case, and the choice of signals

$$z_1 = \begin{bmatrix} z_{11} \\ z_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad z_2 = \begin{bmatrix} z_{21} \\ z_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and  $\bar{\delta}_i = 1, \bar{\sigma}_i^2 > 0, i = 1, 2$ . Then one can show that

$$(z_1 - z_2)^T (\Phi(z_1) - \Phi(z_2)) = \ln \left( \frac{\bar{\sigma}_1^2 + \bar{g}_{12} e^{p_2^*} e^{-1}}{\bar{\sigma}_1^2 + \bar{g}_{12} e^{p_2^*}} \right) + \ln \left( \frac{\bar{\sigma}_2^2 + g_{21} e^{p_1^*}}{\bar{\sigma}_2^2 + \bar{g}_{21} e^{p_1^*}} \right) < 0,$$

and hence the same holds for the opportunistic case.

## VI. INPUT OUTPUT ANALYSIS USING SCALINGS

In this section we will see how the results of the previous section can be improved. We insert multipliers into the loop, which gives the transformed, but equivalent system

$$\hat{H}(q) \triangleq D^{-1}H(q)D = H(q)$$

$$\hat{\Phi}(\hat{z}) \triangleq D^{-1}\Phi(D\hat{z})$$

$$\hat{\delta r} \triangleq D^{-1}\delta r, \quad \hat{z} \triangleq D^{-1}z$$

for any  $D \in \mathcal{D} = \{D = \text{diag}(d_1, \dots, d_n) : d_k > 0\}$ . See Figure 4. An interpretation of this is that we study stability of an equivalent, transformed system where the norm is changed. The advantage is that structure of the interconnection matrix can be taken into account in the computation of the local Lipschitz constant. In the following we assume that  $\Phi$  is continuously differentiable. Let  $\hat{x} = D^{-1}x, \hat{y} = D^{-1}y$ . Note that  $\nabla(M\Phi(x)) = M\nabla\Phi(x), M \in \mathbf{R}^{n \times n}$ , and  $\nabla(\Phi(Mx)) = \nabla\Phi(Mx)M$ , where  $\nabla\Phi(x)$  is the Jacobian of  $\Phi$ . We have

$$\begin{aligned}
& |\hat{\Phi}(\hat{x}) - \hat{\Phi}(\hat{y})|_\infty \\
&= \left| \int_0^1 \nabla \hat{\Phi}(\hat{y} + \theta(\hat{x} - \hat{y})) (\hat{x} - \hat{y}) d\theta \right|_\infty \\
&= \left| \int_0^1 D^{-1} \nabla \Phi(D\hat{y} + D\theta(\hat{x} - \hat{y})) D(\hat{x} - \hat{y}) d\theta \right|_\infty \\
&\leq \int_0^1 |D^{-1} \nabla \Phi(y + \theta(x - y)) D|_1 d\theta |\hat{x} - \hat{y}|_\infty \\
&\leq \underbrace{\max_{z \in B^*} |D^{-1} \nabla \Phi(z) D|_1}_{\triangleq K_D} |\hat{x} - \hat{y}|_\infty.
\end{aligned}$$

where  $B^*$  is given in (13).

By a suitable choice of scalings we may obtain lower Lipschitz gain and thus less restrictive stability criteria. The following result shows that  $K_D$  is the Lipschitz constant of the scaled nonlinearity  $\hat{\Phi}$ .

*Proposition 5:* Assume that the Jacobian of  $\Phi$  is Lipschitz on  $B^*$ . Then the scaled nonlinearity  $\hat{\Phi} : R_\infty^n \rightarrow R_\infty^n$  is Lipschitz on  $D^{-1}B^* \subset R_\infty^n$  with

$$L[\hat{\Phi}; D^{-1}B^*] = \max_{z \in B^*} |D^{-1} \nabla \Phi(z) D|_1 = K_D.$$

*Proof:* A proof is given in the appendix. ■

Let  $d_i$  be our scalings. Define

$$\hat{\gamma} = \min_i \left\{ \min \left\{ \frac{1}{d_i} (p_i^* - p_{\min, i}), \frac{1}{d_i} (p_{\max, i} - p_i^*) \right\} \right\},$$

and the sets

$$C(\hat{\gamma}) = \{z \in \mathbf{R}_\infty : -d_i \hat{\gamma} \leq z_i \leq d_i \hat{\gamma}, \forall i\}$$

$$C_X(\hat{\gamma}) = \{z \in X : z[k] \in C(\hat{\gamma}), \forall k\}$$

$$C_{\delta, X}(\hat{\gamma}) = \left\{ \delta r \in X : |\delta r_i[k]| \leq \frac{\hat{\gamma} d_i (1 - \|H\|_1 K_D)}{\|H\|_1}, \forall i, k \right\}$$

In complete analogy to Proposition 3 it follows that the Lipschitz gain of the scaled nonlinearity  $\hat{\Phi}$  over the set  $C_X(\hat{\gamma})$  is computed according to the next proposition.

*Proposition 6:* The scaled nonlinearity  $\hat{\Phi} : X \rightarrow X$  is Lipschitz on  $C_X(\hat{\gamma})$  with

$$\begin{aligned}
L[\hat{\Phi}; C_{l_\infty}(\hat{\gamma})] &= L[\hat{\Phi}; C_{l_{2, \infty}}(\hat{\gamma})] = L[\hat{\Phi}; C(\hat{\gamma})] \\
&\leq \max_{z \in B^*} |D^{-1} \nabla \Phi(z) D|_1 = K_D
\end{aligned}$$

*Proof:* We have that  $C(\hat{\gamma}) \subset D^{-1}B^*$  and hence  $L[\hat{\Phi}; C(\hat{\gamma})] \leq L[\hat{\Phi}; D^{-1}B^*] = K_D$ . The proof of the equalities follows the lines of Proposition 3. ■

The following central result gives improved sufficient conditions of stability and convergence under disturbances for the wide class of power control algorithms studied.

*Corollary 1:* If  $\|H\|_1 < \frac{1}{K_D}$  then there exists a unique power distribution  $z \in C_{l_\infty}(\hat{\gamma})$  for all  $\delta r \in C_{\delta, l_\infty}(\hat{\gamma})$ .

If it in addition holds that  $\|\delta r\|_{2, \infty} < \infty$  and  $\|H\|_{l_{2, \infty} \rightarrow l_{2, \infty}} < \frac{1}{K_D}$ , then  $z[k] \rightarrow 0$  as  $k \rightarrow \infty$ .

*Proof:* We study stability in the scaled signal space, where  $\hat{z} = D^{-1}z$ .  $z \in C_{l_\infty}(\hat{\gamma})$  implies that  $\hat{z} \in B_{l_\infty}^*(\hat{\gamma})$  and  $\delta r \in C_{\delta, l_\infty}(\hat{\gamma})$  implies that

$$\|\hat{\delta r}\|_\infty \leq \hat{\gamma} \frac{1 - \|H\|_1 K_D}{\|H\|_1}.$$

Theorem 1 then proves the statement. ■

Note that some conservativeness is introduced in Corollary 1 since the Lipschitz constant is taken over the set  $B^*$ , whereas stability is guaranteed in the subset  $C(\hat{\gamma})$ . By working on the set  $D^{-1}B^*$  there is no conservatism, but the bounds on the powers may be violated. In section VIII we will optimize over the scalings to obtain a lower Lipschitz constant.

## VII. GLOBAL ANALYSIS USING SATURATED POWERS

In this section we will consider a model with a saturation on the powers entering the interference nonlinearity. The modified system is shown in Figure 5. With the saturation introduced we prove global stability and show that some conservatism in the results of the original model in Figure 1 is removed for the saturated system.

A saturation operator was used already in the proof of Theorem 1. We then showed that the saturation operator is inactive for an appropriate choice of bound on the disturbances. In that procedure we used a maximum interior ball, which introduced conservatism. By instead starting from a saturated system an analogous stability analysis is developed for the modified system where there is less conservatism and global results can be established.

A drawback of using the saturated model is that the class of system architectures which can be included is different from the original model. In Section VII-A we discuss architectures that can be modelled and compare possible approaches to include saturation. In Section VII-B we apply the previous stability analysis to the saturated model.

### A. System Architecture and Modelling

Saturation of the powers models bounds on the maximal and minimal transmission power of the users. It is a realistic assumption that such limitations exist. However, including the saturation operator imposes structure on the location of information in the network. Parting from the original model in Figure 1 there are two natural ways to introduce saturation.

Consider first introducing saturation only to the transmission powers in the interference, i.e. the upper loop, see Figure 5. The filters  $F_I$  and  $F_G$  are given as before in (6), (7) and  $\tilde{C}(q) \triangleq \frac{q-1}{\beta} C(q)$ , with  $C(q)$  given in (5).

An architecture included in this model is a base station measuring interference and computing unsaturated internal states of the transmission powers. If the transmission gains between user  $i$  and receiver  $i$  is known for all users, by for example pilot transmissions in the network, this can be motivated.

Next consider including the saturation in the transmission powers in both the upper and lower loop, see Figure 6. This includes an architecture with a base station measuring both interference and transmission powers before computing and transmitting a feedback signal to the mobile user. This model implies that a marginally stable loop is saturated, which may cause instability problems. Analysis of this model is beyond the scope of this paper.

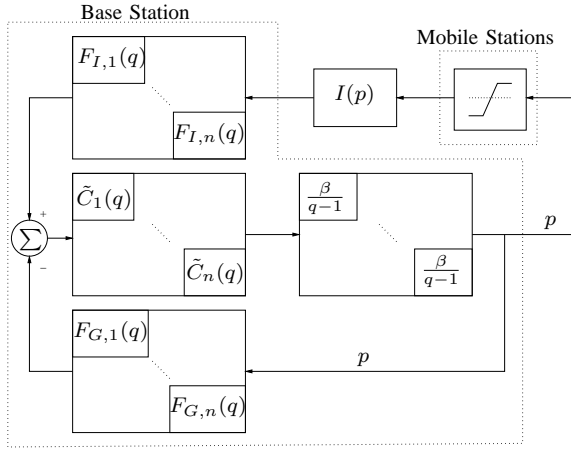


Fig. 5. System model with saturated powers and internal feedback.

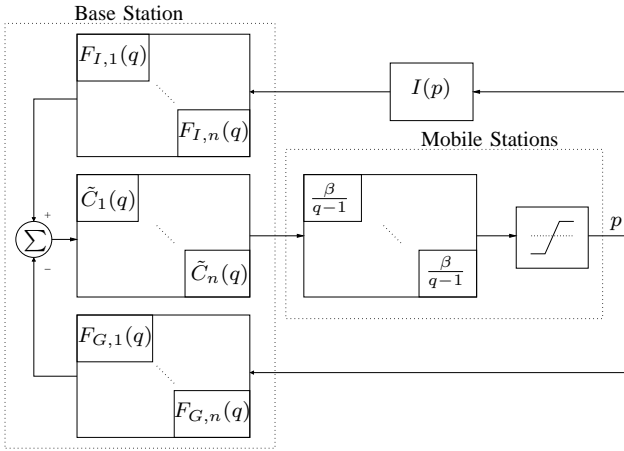


Fig. 6. System model with saturated powers.

### B. Global Analysis of Saturated Model

Define the saturation  $\text{sat}_{[B]} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  whose  $i^{\text{th}}$  component is

$$[\text{sat}_B(p)]_i = \begin{cases} p_{\max,i} & \text{if } p_i > p_{\max,i} \\ p_i & \text{if } -p_{\min,i} \leq p_i \leq p_{\max,i} \\ p_{\min,i} & \text{if } p_i < p_{\min,i}, \end{cases}$$

where the set  $B = \{p \in \mathbb{R}^n : p_{\min,i} \leq p_i \leq p_{\max,i}\}$ . Now define

$$I_{\text{sat}_{[B]}}(p) \triangleq I(\text{sat}_{[B]}(p)).$$

*Proposition 7:* Assume that  $I(p) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is two-sided scalable and that  $B$  is compact. Then  $I_{\text{sat}_{[B]}}(p)$  is Lipschitz for any  $p \in \mathbb{R}^n$  with

$$L[I_{\text{sat}_{[B]}}; \mathbb{R}^n] = L[I; B] < 1.$$

*Proof:* The saturation cuts all powers to values within the set  $B$ . This reduces the feasible set of the maximization problem giving the Lipschitz constant,

$$\begin{aligned} L[I_{\text{sat}_{[B]}}; \mathbb{R}^n] &= \sup_{z_1, z_2 \in B, z_1 \neq z_2} \frac{\|I(z_1) - I(z_2)\|_X}{\|z_1 - z_2\|_X} \\ &= L[I; B] < 1, \end{aligned}$$

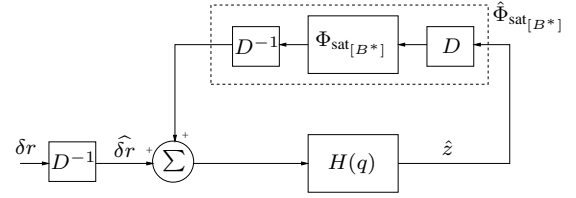


Fig. 7. The saturated system on input-output form with scale factors.

where the last inequality is given by Proposition 1. Now consider the deviations around the equilibrium point,  $z$ , and the set  $B^*$  as defined in (13). Define

$$\Phi_{\text{sat}_{[B^*]}}(z) \triangleq \Phi(\text{sat}_{[B^*]}(z)) = I(p^* + \text{sat}_{[B^*]}(z)) - I(p^*)$$

*Proposition 8:* Any  $\Phi_{\text{sat}_{[B^*]}} : X \rightarrow X$  defined by two-sided scalable interference functions is Lipschitz with

$$L[\Phi_{\text{sat}_{[B^*]}}; l_\infty] = L[\Phi_{\text{sat}_{[B^*]}}; l_{2,\infty}] = L[\Phi; B^*] < 1.$$

*Proof:* The proof follows from Proposition 3 and the observation that the optimization giving the Lipschitz constant is always performed over sets  $B_{l_\infty}^*$  and  $B_{l_{2,\infty}}^*$ .

*Theorem 2:* If  $\|H\|_1 < \frac{1}{L[\Phi; B^*]}$ , then there exists a unique power distribution  $z \in l_\infty$  for any disturbance  $\delta r$ . If it in addition holds that

$$\|H\|_{l_{2,\infty} \rightarrow l_{2,\infty}} < \frac{1}{L[\Phi; B^*]},$$

then  $z[k] \rightarrow 0$  as  $k \rightarrow \infty$ . A bound on the internal states  $z$  is given by

$$\|z\|_\infty \leq \frac{\|H\|_1}{1 - L[\Phi; B^*]\|H\|_1} \|\delta r\|_\infty$$

and

$$\|z\|_{2,\infty} \leq \frac{\|H\|_{l_{2,\infty} \rightarrow l_{2,\infty}}}{1 - L[\Phi; B^*]\|H\|_{l_{2,\infty} \rightarrow l_{2,\infty}}} \|\delta r\|_{2,\infty},$$

respectively.

*Proof:* A proof can be found in the appendix.

Note that by Proposition 8 the Lipschitz constant is the same as for the original model without saturation and by Proposition 3 we have that if  $\nabla\Phi$  is Lipschitz, then the Lipschitz constant is equal to  $\max_{z \in B^*} |\nabla\Phi(z)|_1$ . A difference for the saturated model is that we no longer need to study an interior ball to ensure that the transmitted powers are within  $B^*$ . However, by doing so, we can again guarantee that the saturation is inactive and Theorem 2 reduces to Theorem 1.

We will next introduce scalings to the model following the same steps and notation as in Section VI. The scaled model with saturation is given in Figure 7. Define

$$\hat{\Phi}_{\text{sat}_{[B^*]}} \triangleq D^{-1}\Phi_{\text{sat}_{[B^*]}}(D\hat{z}).$$

*Proposition 9:* Assume that the Jacobian of  $\Phi$  is Lipschitz on  $B^*$ . Then the scaled saturated nonlinearity  $\hat{\Phi}_{\text{sat}_{[B^*]}} : X \rightarrow X$  is Lipschitz with

$$\begin{aligned} L[\hat{\Phi}_{\text{sat}_{[B^*]}}; l_\infty] &= L[\hat{\Phi}_{\text{sat}_{[B^*]}}; l_{2,\infty}] = L[\hat{\Phi}; D^{-1}B^*] \\ &= \max_{z \in B^*} |D^{-1}\nabla\Phi(z)D|_1 = K_D. \end{aligned}$$

*Proof:* A proof is given in the appendix. ■

Next follows our main result on the saturated model.

*Corollary 2:* If  $\|H\|_1 < \frac{1}{K_D}$  then there exists a unique power distribution  $z \in l_\infty$  for any disturbance  $\delta r$ . If it in addition holds that  $\|H\|_{l_{2,\infty} \rightarrow l_{2,\infty}} < \frac{1}{K_D}$ , then  $z[k] \rightarrow 0$  as  $k \rightarrow \infty$ .

*Proof:* As in the proof of Corollary 1 we study stability in the scaled signal space. We then have  $\hat{z} = D^{-1}z$  and  $K_D$  is an upper bound of the Lipschitz constant of the scaled interference nonlinearity. The statement then follows from Theorem 2. ■

In Corollary 1 the set for guaranteed stability of the powers was a strict subset of the set over which the Lipschitz constant was taken. In Corollary 2 the sets are equal and there is no such conservatism. Another strength of the results for the saturated model is that the disturbances are allowed to be arbitrarily large in any direction. This comes at the cost of possibly large internal power levels.

### VIII. OPTIMIZATION OF LIPSCHITZ CONSTANT

In order to obtain the best possible scalings for Corollary 1 and 2 we optimize over  $D \in \mathcal{D} = \{D = \text{diag}(d_1, \dots, d_n) : d_k > 0\}$ . We hence have the following optimization problem.

$$K_D^* = \inf_{D \in \mathcal{D}} \max_{z \in B^*} |D^{-1} \nabla \Phi(z) D|_1, \quad (16)$$

where  $K_D^*$  is the minimal Lipschitz constant in the scaled signal space. A negative conclusion is that the optimization problem in (16) is neither convex nor quasiconvex.

In general it holds that

$$\inf_{D \in \mathcal{D}} \max_{z \in B^*} |D^{-1} \nabla \Phi(z) D|_1 \geq \max_{z \in B^*} \inf_{D \in \mathcal{D}} |D^{-1} \nabla \Phi(z) D|_1, \quad (17)$$

so a lower bound of  $K_D^*$  in (16) can easily be achieved, since it is a well known fact that (see e.g. [5])

$$\max_{z \in B^*} \inf_{D \in \mathcal{D}} |D^{-1} \nabla \Phi(z) D|_1 = \max_{z \in B^*} \rho(\nabla \Phi(z)). \quad (18)$$

However, in general the bounds in (17) are not equal.

In this section we discuss how the optimization can be performed in the case when the interference function is defined by DPC and opportunistic users as in Subsection V-A. For convenience of notation, let  $\bar{a}_{ii} = (1 - \delta_i) \bar{g}_{ii} e^{p_i^*}$  and  $\bar{a}_{ij} = \bar{g}_{ij} e^{p_j^*}$  in the expression for  $\nabla \Phi$  in (12) and let  $d$  be the vector defined by the scalings  $d = [d_1, \dots, d_n]^T$ . Defining

$$f_i(d, z) \triangleq \frac{1}{d_i} \frac{1}{\bar{\sigma}_i^2 + \sum_{j=1}^n \bar{a}_{ij} e^{z_j}} \sum_{j=1}^n d_j \bar{a}_{ij} e^{z_j}, \quad (19)$$

we have

$$|D^{-1} \nabla \Phi(z) D|_1 = \max_i f_i(d, z),$$

and hence the inner optimization in (16) is equivalently written

$$\max_{z \in B^*} \max_i f_i(d, z) = \max_i \max_{z \in B^*} f_i(d, z).$$

We propose the following heuristic algorithm to solve the optimization problem in (16).

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### Algorithm 1 Optimization of Lipschitz Constant

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- 1) Let  $d^0 = d_0$ .
  - 2) Solve  $z^* = \arg \max_i \max_{z \in B^*} f_i(d, z)$ .
  - 3) Compute step direction  $\bar{s}^k(d, z^*)$ .
  - 4) Compute  $\alpha^k$ .
  - 5) Update scalings through  $d^{k+1} = d^k - \alpha^k \bar{s}^k$ .
  - 6) Return to 2).
- 

1) : A good starting point of the algorithm is to take  $d_0$  as the eigenvector corresponding to  $\rho(\nabla \Phi(\tilde{z}))$ , where  $\tilde{z}$  is maximizing the lower bound in (18). In the case when the bounds in (17) are equal, this will be the optimal scaling vector. In general suboptimal solutions must be used, since it is a hard optimization problem to find the highest lower bound.

2) : The optimization in Step 2 can be solved using different methods. For fixed  $d$ , the problem is linear fractional in each  $f_i$ , and hence the maximizing  $z$  can be obtained by solving  $n$  equivalent linear problems [2].

In the following lemma it is also shown that for a given  $d$ , the optimal solution is on the boundary of the feasible set,  $B^*$ . For convenience of notation, denote the componentwise defined bounds of  $B^*$  by  $z_{\min,i}$  and  $z_{\max,i}$ .

*Lemma 1:* Consider given scalings  $d$ . Then a maximum solution to  $\max_{z \in B^*} f_i(d, z)$  is either  $z_i = z_{\min,i}$  or  $z_i = z_{\max,i}$ ,  $\forall i$ .

*Proof:* A proof can be found in the appendix. ■

*Remark 1:*  $f_i(d, z)$  can be interpreted as the Lipschitz constant of user  $i$ , which is a measure on the stability margin of the user. Lemma 1 proves that the worst case is achieved when all users transmit with maximum or minimum powers. In Proposition 3 it was already shown that for the case  $D = I$ , all users transmit with  $z_{\max,i}$ .

According to Lemma 1 the inner optimization problem can be solved as a combinatoric problem, which is efficient for small dimensional systems, or by using some active set method for larger problems [10].

3) : As step direction of  $d$ ,  $\bar{s}^k(d, z^*)$ , we used a weighted linear combination of the gradients of the different  $f_i$ ,

$$\bar{s}^k(d, z^*) = \sum_{i=1}^n \nabla_d f_i(d, z^*) \frac{f_i(d, z^*) - \bar{f}}{\sum_{i=1}^n |f_i(d, z^*) - \bar{f}|},$$

where  $\bar{f} = \frac{1}{n} \sum_{i=1}^n f_i(d, z^*)$ . The intuition is that we balance the functions  $f_i$  with the scalings  $d_i$  to decrease elements above the mean and increase elements below the mean. This is motivated by the fact that in the lower bound (18) all  $f_i$  are equal and furthermore, a property of non-negative matrices is that the biggest eigenvalue is bounded between the lowest and highest row sum, in this case  $\min_i f_i$  and  $\max_i f_i$ , see e.g. p. 63 in [9]. We also weight the gradients according to the distance to the mean value.

4) : As step length we used Polyak's stepsize rule

$$\alpha^k = \kappa \frac{\max_i f_i(d, z^*) - f^*}{\|\bar{s}^k\|^2},$$



where  $\kappa$  is a constant and  $f^*$  is an estimate of the optimal value. We used the highest lower bound in (18) as estimate of the optimal value.

This is not a subgradient method, but it seems to have better convergence properties in our experiments. Convergence of the algorithm can not be guaranteed. However, each iteration is feasible in the sense that it is a valid Lipschitz constant, and we can compare with the highest achieved lower bound to get an idea of optimality.

## IX. EXAMPLE

Consider a network with three users, where two of them heavily interfere the third one. Let the interconnections be given by the following gain matrix

$$G = \begin{bmatrix} 1 & 0.0003 & 0.0015 \\ 0.0758 & 1 & 0.0758 \\ 0.0030 & 0.0003 & 1 \end{bmatrix},$$

and let them use the DPC algorithm with the slight modification of having the system  $C(q) = \frac{\beta}{q-1}$  instead of  $C(q) = \frac{1}{q-1}$ . We will study the effect of delay and delay compensation on stability for the system with and without saturation on the interfering powers. Assume that  $\bar{\gamma}_i^T = 9.0$ ,  $\bar{\delta}_i = 1$  and  $\bar{\sigma}_i^2 = 0.05, \forall i$ , and that no filters are used. Given that the equilibrium point is within the unsaturated region, it is equal for the saturated and unsaturated system. The equilibrium point can be computed by

$$\bar{p}^* = (I - \bar{\Gamma}\bar{F})^{-1}\bar{\Gamma}\bar{\sigma}^2,$$

where  $\bar{\Gamma} = \text{diag}(\bar{\gamma}_1^T/(\bar{\delta}_1\bar{g}_{11}), \dots, \bar{\gamma}_n^T/(\bar{\delta}_n\bar{g}_{nn}))$  and  $\bar{\sigma}^2 = [\bar{\sigma}_1^2, \dots, \bar{\sigma}_n^2]^T$ , which gives  $\bar{p}^* = [0.4606, 1.0853, 0.4668]^T$ . We will first study the original model without saturation.

### A. System without saturation

First consider the case without delay. To analyze stability of the system we compute  $\|H\|_1$ . We have

$$H(q) = \frac{\beta}{q-1+\beta},$$

and

$$\|H\|_1 = \sum_{k=0}^{\infty} |h[k]| = \sum_{k=0}^{\infty} \beta(1-\beta)^k = \frac{\beta}{1-(1-\beta)} = 1.$$

Since by two-sided scalability the Lipschitz constant is less than one, stability follows for  $\beta \in [0, 1]$  by Theorem 1.

We will now study what happens when a delay is added to the system. We then get

$$H(q) = \frac{\beta}{q^2 - q + \beta}.$$

The norm is now dependent on the choice of  $\beta$ . To guarantee stability by Theorem 1 or Corollary 1, we need  $\|H\|_1 K < 1$  and  $\|H\|_1 K_D < 1$  respectively, where  $K = L[\Phi; B^*(\gamma)]$  and  $K_D$  is from (16). Stability of the system is dependent on both the value of  $\beta$ , and the value of  $\gamma$  or  $\hat{\gamma}$ , the bound of the deviations from the equilibrium point. This is illustrated in Figure 8. The product  $\|H\|_1 K_D$  is plotted and compared to

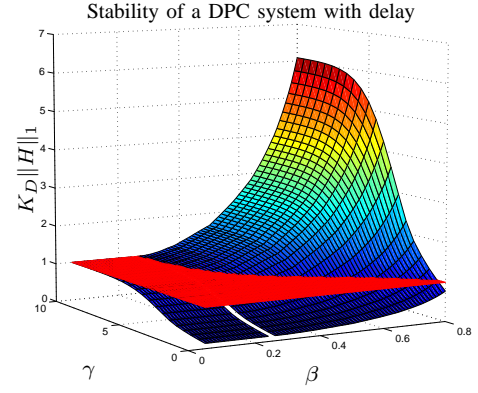


Fig. 8. Stability of DPC with delay. Stability for some disturbance is guaranteed for all values below the plane.

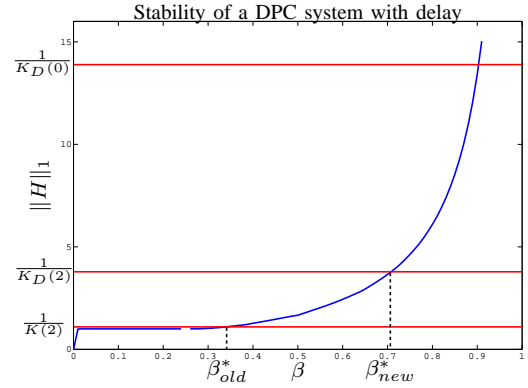


Fig. 9. Stability of DPC with delay. Here  $K_D(2)$  denotes the scaled Lipschitz constant computed for  $\gamma = 2$ , and  $K(2)$  the Lipschitz constant as computed for Theorem 1 for  $\gamma = 2$ .  $\beta_{new}^*$  and  $\beta_{old}^*$  are the maximum values of  $\beta$  given by Corollary 1 and Theorem 1 for which stability can be guaranteed for some disturbance.

the plane  $\|H\|_1 K_D = 1$ , where  $K_D$  was obtained by use of the optimization algorithm proposed in section VIII. Stability for some disturbance can be guaranteed by Corollary 1 for all  $\beta$  and  $\gamma$  where the surface of  $\|H\|_1 K_D$  is below the plane.

Consider the case  $\gamma = 2$ . This corresponds to the bounds on the powers in linear scale

$$\bar{p}_{\min} = [0.062, 0.147, 0.063]^T, \bar{p}_{\max} = [3.403, 8.020, 3.449]^T,$$

defining  $B^*$  as in (13). When the Lipschitz constant is computed as in Proposition 1 and 3, we get  $L[\Phi; B^*(\gamma)] = 0.9121$ , which implies that Theorem 1 guarantees stability for  $\beta \in [0, 0.34]$ . However, when using Corollary 1 and using optimization we obtain  $K_D = 0.2643$ . Now stability can be guaranteed for  $\beta \in [0, 0.707]$ . This is illustrated in Figure 9, where the value of  $\|H\|_1$  is plotted as a function of  $\beta \in (0, 1)$ . Stability as a function of  $\beta$  can be determined from the graph by comparing the value of  $\|H\|_1$  with the value of  $1/K$  and  $1/K_D$  for different values of  $\gamma$ . Note that as  $\gamma \rightarrow \infty$ ,  $L[\Phi; B^*(\gamma)] \rightarrow 1$ .

Now consider fixing  $\beta = 0.7$ . In this case Theorem 1 can not be applied. However, Corollary 1 guarantees stability for

powers in  $C_{l_\infty}(\hat{\gamma})$  given that the scaled disturbances are in  $C_{\delta, l_\infty}(\hat{\gamma})$ . For this example the optimal scalings are given by  $d = [0.3048, 1.1428, 0.3571]^T$ , which gives  $\hat{\gamma} = 1.7500$  and  $C(\hat{\gamma})$  is defined by the componentwise bounds

$$p_{\max} = [-0.2419, 2.0819, -0.1369]^T$$

$$p_{\min} = [-1.3086, -1.9181, -1.3867]^T.$$

We note that this set is smaller than the original set  $B^*$  used for computing the scalings. The bounds on the disturbances are given by

$$\left\| \begin{bmatrix} \delta r_1 \\ \delta r_2 \\ \delta r_3 \end{bmatrix} \right\| \leq \frac{\hat{\gamma}(1 - \|H\|_1 K_D)}{\|H\|_1} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0.0049 \\ 0.0184 \\ 0.0057 \end{bmatrix}.$$

Note that the disturbances from the first and third user are required to be much lower than the disturbances from the second user. This is because of the high cross couplings on the second row of the gain matrix which make the second user sensitive for interference by the other users.

Now consider delay and delay compensation. We then get

$$H(q) = \frac{\beta}{q(q - (1 - \beta))}.$$

By making the same calculations as in the first case, it is easy to show that  $\|H\|_1 = 1, \forall \beta \in [0, 1]$ , and stability can be guaranteed.

### B. Saturated model

Now consider the saturated model as introduced in Figure 5. Let the saturation be defined on the set  $B^*$  as in (13). By Theorem 2 and Corollary 2 stability for the saturated system is given by the same conditions on the system gain and the Lipschitz constant as in Theorem 1 and Corollary 1. Hence the same conditions on stability for the different delay cases as in the previous example are valid also for this saturated system.

However, for the saturated model we no longer have bounds on the powers and the disturbances. Furthermore the Lipschitz constant is computed on the same set as the powers.

## X. CONCLUSIONS

In this paper we model a wide class of power control algorithms including the distributed power control (DPC) and opportunistic algorithm with higher orders, giving the possibility to model delays in both the powers and interference. The paper extends the frameworks of Yates and Sung and Leung to include higher order control laws. This is highly motivated by implementations in real networks.

We believe that the structure of the power control laws are best analyzed using logarithmic variables and analyzing the system around the equilibrium point. The cumbersome definition of two-sided scalability proposed by Sung and Leung in linear scale is equivalent to a standard contraction condition on the Lipschitz constant of the interference function using the infinity norm with logarithmic variables.

We analyze one model without saturation and one model with saturated interfering powers. Existence of a unique fixed

point and sufficient conditions for convergence are given using input output analysis. To reduce conservativeness of the results scalings are introduced to exploit more structure of the interference feedback. Finding the optimal scalings is a non-convex optimization problem. However, we show that the optimization problem has a lot of structure and propose an heuristic solution algorithm.

From an example we conclude that delays have an important impact on stability, which motivates the use of higher order models. Scalings significantly reduce conservativeness compared to previous results and delay compensation can be used to stabilize a system with known delay.

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## APPENDIX: PROOFS

### A. Property of two-sided scalability

*Definition 1:* (Two-sided scalability) For all  $\alpha > 1$ ,  $(1/\alpha)\bar{p}' \leq \bar{p} \leq \alpha\bar{p}'$  implies

$$(1/\alpha)\bar{I}(\bar{p}') < \bar{I}(\bar{p}) < \alpha\bar{I}(\bar{p}').$$

*Proposition 10:* Let  $d(\cdot, \cdot)$  be defined as in (4). Then the definition of two-sided scalability is equivalent to

$$d(\bar{I}(\bar{p}), \bar{I}(\bar{p}')) < d(\bar{p}, \bar{p}').$$

*Proof:* The condition  $d(\bar{I}(\bar{p}), \bar{I}(\bar{p}')) < d(\bar{p}, \bar{p}')$  is equivalent to

$$\max_i \left\{ \max \left\{ \frac{\bar{I}_i(\bar{p})}{\bar{I}_i(\bar{p}'), \bar{I}_i(\bar{p}') \right\} \right\} < \max_i \left\{ \max \left\{ \frac{\bar{p}_i}{\bar{p}'_i}, \frac{\bar{p}'_i}{\bar{p}_i} \right\} \right\}.$$

Let

$$i^* = \arg \max_i \left\{ \max \left\{ \frac{\bar{p}_i}{\bar{p}'_i}, \frac{\bar{p}'_i}{\bar{p}_i} \right\} \right\}.$$

Let  $\bar{p} \neq \bar{p}'$  and assume w.l.o.g.  $\bar{p}'_{i^*} > \bar{p}_{i^*}$ . Now define  $\alpha = \frac{\bar{p}'_{i^*}}{\bar{p}_{i^*}} > 1$ .

$$\alpha = \frac{\bar{p}'_{i^*}}{\bar{p}_{i^*}} \Rightarrow \alpha\bar{p}_{i^*} = \bar{p}'_{i^*} \Rightarrow \alpha\bar{p}_{i^*} \geq \bar{p}'_{i^*}$$

$$\frac{1}{\alpha}\bar{p}_{i^*} = \frac{1}{\frac{\bar{p}'_{i^*}}{\bar{p}_{i^*}}}\bar{p}_{i^*} = \frac{(\bar{p}_{i^*})^2}{\bar{p}'_{i^*}} < \frac{(\bar{p}'_{i^*})^2}{\bar{p}'_{i^*}} = \bar{p}'_{i^*} \Rightarrow \frac{1}{\alpha}\bar{p}_{i^*} \leq \bar{p}'_{i^*}.$$

This holds since  $0 < \bar{p}_{i^*} < \bar{p}'_{i^*}$ . We hence have established that with this choice of  $\alpha$  we have  $\frac{1}{\alpha}\bar{p}'_{i^*} \leq \bar{p}_{i^*} \leq \alpha\bar{p}'_{i^*}$ . Similarly one can show that this relation holds for all elements of  $\bar{p}$  and  $\bar{p}'$ . We now have

$$\max_i \left\{ \max \left\{ \frac{\bar{I}_i(\bar{p})}{\bar{I}_i(\bar{p}'), \bar{I}_i(\bar{p}') \right\} \right\} < \alpha,$$

which implies that for all  $i$  we have the following relations

$$\frac{\bar{I}_i(\bar{p})}{\bar{I}_i(\bar{p}')} < \alpha, \quad \frac{\bar{I}_i(\bar{p}')}{\bar{I}_i(\bar{p})} < \alpha,$$

which is equivalent to

$$\frac{1}{\alpha}\bar{I}(\bar{p}') < \bar{I}(\bar{p}) < \alpha\bar{I}(\bar{p}').$$

### B. Proof of Proposition 1

*Proof:* Consider  $\bar{p}_1 \neq \bar{p}_2 \in B$ . Each component of  $\bar{I}$  being two-sided scalable implies by definition that

$$\max_i \left\{ \max \left\{ \frac{\bar{I}(\bar{p}_{1,i})}{\bar{I}(\bar{p}_{2,i}), \bar{I}(\bar{p}_{2,i})} \right\} \right\} < \max_i \left\{ \max \left\{ \frac{\bar{p}_{1,i}}{\bar{p}_{2,i}}, \frac{\bar{p}_{2,i}}{\bar{p}_{1,i}} \right\} \right\}.$$

Using logarithmic variables this is equivalent to

$$\begin{aligned} & \max_i \left\{ \max \left\{ I(p_{1,i}) - I(p_{2,i}), I(p_{2,i}) - I(p_{1,i}) \right\} \right\} \\ & < \max_i \left\{ \max \left\{ p_{1,i} - p_{2,i}, p_{2,i} - p_{1,i} \right\} \right\} \\ & \Leftrightarrow |I(p_1) - I(p_2)|_\infty < |p_1 - p_2|_\infty \\ & \Leftrightarrow \frac{|I(p_1) - I(p_2)|_\infty}{|p_1 - p_2|_\infty} < 1. \end{aligned}$$

■

### C. Proof of Proposition 2

*Proof:* Let us first consider the interference nonlinearity as a multivariable function  $\Phi : R_\infty^n \rightarrow R_\infty^n$ . It follows that

$$\begin{aligned} |\Phi(x) - \Phi(y)|_\infty &= \left| \int_0^1 \nabla \Phi(y + \theta(x - y))(x - y) d\theta \right|_\infty \\ &\leq \int_0^1 |\nabla \Phi(y + \theta(x - y))|_1 |x - y|_\infty d\theta \\ &\leq \sup_{z \in R^n} |\nabla \Phi(z)|_1 |x - y|_\infty. \end{aligned}$$

This gives the Lipschitz bound

$$\begin{aligned} L[\Phi; R_\infty^n] &\leq K \triangleq \sup_z |\nabla \Phi(z)|_1 \\ &= \sup_z \left[ \left| \begin{bmatrix} \pm \left[ \frac{(1-\delta_1)\bar{g}_{11}e^{p_1^*+z_1}}{\sigma_1^2 + F^1 e^{p^*+z}} \right] & \cdots & \frac{\bar{g}_{1n}e^{p_n^*+z_n}}{\sigma_1^2 + F^1 e^{p^*+z}} \right] \\ \vdots \\ \pm \left[ \frac{\bar{g}_{n1}e^{p_1^*+z_1}}{\sigma_n^2 + F^n e^{p^*+z}} \right] & \cdots & \frac{(1-\delta_n)\bar{g}_{nn}e^{p_n^*+z_n}}{\sigma_n^2 + F^n e^{p^*+z}} \right] \end{bmatrix} \right|_1 \\ &= \sup_z \max_i \frac{\bar{F}^i e^{p^*+z}}{\sigma_i^2 + \bar{F}^i e^{p^*+z}} = 1. \end{aligned}$$

We then have

$$\begin{aligned} \|\Phi(z_1) - \Phi(z_2)\|_{2,\infty} &= \sqrt{\sum_{k=0}^{\infty} |\Phi(z_1[k]) - \Phi(z_2[k])|_\infty^2} \\ &\leq K \sqrt{\sum_{k=0}^{\infty} |z_1[k] - z_2[k]|_\infty^2} \\ &= K \|z_1 - z_2\|_{2,\infty}, \end{aligned}$$

which shows that  $L[\Phi; l_{2,\infty}] \leq K = 1$ . We will next see that the bound can be achieved asymptotically by considering the  $l_{2,\infty}$ -signal,

$$z = \begin{cases} \mathbf{1}, & k = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Let  $e^{p^*+\alpha 1}$  be defined as  $e^{p^*} = [e^{p_1^*}, \dots, e^{p_n^*}]^T$ . Consider for simplicity the DPC case. We have

$$\begin{aligned} \frac{1}{\alpha} \|\Phi(\alpha z) - \Phi(0)\|_{2,\infty} &= \frac{1}{\alpha} \left( \sum_{k=0}^{\infty} |\Phi(\alpha z[k])|^2_{\infty} \right)^{1/2} \\ &= \frac{1}{\alpha} \left\| \begin{bmatrix} \ln \left( \frac{\bar{\sigma}_1^2 + F^1 e^{p^*+\alpha}}{\bar{\sigma}_1^2 + F^1 e^{p^*}} \right) \\ \vdots \\ \ln \left( \frac{\bar{\sigma}_n^2 + F^n e^{p^*+\alpha}}{\bar{\sigma}_n^2 + F^n e^{p^*}} \right) \end{bmatrix} \right\|_{\infty} \\ &= \left\| \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right\| + \left\| \begin{bmatrix} \frac{1}{\alpha} \ln \left( \frac{\bar{\sigma}_1^2 e^{-\alpha} + F^1 e^{p^*}}{\bar{\sigma}_1^2 + F^1 e^{p^*}} \right) \\ \vdots \\ \frac{1}{\alpha} \ln \left( \frac{\bar{\sigma}_n^2 e^{-\alpha} + F^n e^{p^*}}{\bar{\sigma}_n^2 + F^n e^{p^*}} \right) \end{bmatrix} \right\|_{\infty} \rightarrow 1 \end{aligned}$$

as  $\alpha \rightarrow \infty$ . A case with opportunistic or mixed users is analogous. The denominator and numerator will switch place, giving a difference in sign within the norm, but not changing the result.

It follows that  $L[\Phi; l_{2,\infty}] = 1$  as well as  $\|\Phi\|_{l_{2,\infty} \rightarrow l_{2,\infty}} = 1$  for both the DPC, opportunistic and mixed case. The case with the  $l_{\infty}$ -space follows using essentially the same arguments. ■

#### D. Proof of Proposition 3

*Proof:* First consider the interference nonlinearity as a function  $\Phi : R_{\infty}^n \rightarrow R_{\infty}^n$ . We have by definition that  $\Phi(z) = I(p^* + z) - I(p^*)$ , and from the proof of Proposition 1 we get

$$\begin{aligned} &\Leftrightarrow \frac{|I(p_1) - I(p_2)|_{\infty}}{|p_1 - p_2|_{\infty}} < 1 \\ &\Leftrightarrow \frac{|I(p_1) - I(p^*) - (I(p_2) - I(p^*))|_{\infty}}{|p_1 - p^* - (p_2 - p^*)|_{\infty}} < 1 \\ &\Leftrightarrow \frac{|\Phi(z_1) - \Phi(z_2)|_{\infty}}{|z_1 - z_2|_{\infty}} < 1. \end{aligned}$$

Hence we have that for powers,  $p \in B$ ,  $L[I; B] < 1$ , or equivalently for  $z \in B^*$ ,  $L[\Phi; B^*] < 1$ . Now consider  $\Phi$  as an operator  $l_{\infty}^n \rightarrow l_{\infty}^n$ .

$$\begin{aligned} L[\Phi; B_{l_{\infty}}^*] &= \sup_{z_1, z_2 \in B_{l_{\infty}}^*, z_1 \neq z_2} \frac{\|\Phi(z_1) - \Phi(z_2)\|_{\infty}}{\|z_1 - z_2\|_{\infty}} \\ &= \sup_{z_1, z_2 \in B_{l_{\infty}}^*, z_1 \neq z_2} \frac{\sup_k |\Phi(z_1[k]) - \Phi(z_2[k])|_{\infty}}{\sup_k |z_1[k] - z_2[k]|_{\infty}} \\ &\leq \sup_{z_1, z_2 \in B_{l_{\infty}}^*, z_1 \neq z_2} \frac{L[\Phi; B^*] \sup_k |z_1[k] - z_2[k]|_{\infty}}{\sup_k |z_1[k] - z_2[k]|_{\infty}} \\ &= L[\Phi; B^*] \end{aligned}$$

We now show that the inequality in fact is an equality. Since  $B^* \subset R_{\infty}^n$  is compact, there exist  $z_1^*, z_2^*$  s.t.  $|\Phi(z_1^*) - \Phi(z_2^*)|_{\infty} = L[\Phi; B^*] |z_1^* - z_2^*|_{\infty}$ . Equality is then reached above for the signals

$$z_1 = \begin{cases} z_1^*, & k = 0 \\ 0, & \text{otherwise} \end{cases}, \quad z_2 = \begin{cases} z_2^*, & k = 0 \\ 0, & \text{otherwise} \end{cases}.$$

The case where  $\Phi : l_{2,\infty} \rightarrow l_{2,\infty}$  is analogous. Hence  $L[\Phi; B_{l_{2,\infty}}^*] = L[\Phi; B_{l_{2,\infty}}^*] < 1$ .

Now assume  $\Phi$  is continuously differentiable and that the Jacobian is Lipschitz. From the first part of the proof of Proposition 2 we have that  $L[\Phi; B^*] \leq \max_{z \in B^*} |\nabla \Phi(z)|_1$ . To see that equality can be achieved we assume

$$\max_{z \in B^*} |\nabla \Phi(z)|_1 = f_{i^*}(z^*),$$

where  $i^*$  is the maximizing index and  $z^*$  is the corresponding maximizing solution. We know the existence of such since  $|\nabla \Phi(z)|_1$  is a continuous function and  $B^*$  is compact. Further, let  $\delta z$  be a unit length vector in  $R_{\infty}^n$  such that  $|\nabla \Phi(z^*) \delta z|_{\infty} = |\nabla \Phi(z^*)|_1$ . Let  $\tilde{z}$  be an interior point of  $B^*$  such that  $|\tilde{z} - z^*|_{\infty} \leq \eta$ . Now let  $y = \tilde{z}$  and  $x = \tilde{z} - \epsilon \delta z$ . We get

$$\begin{aligned} \epsilon^{-1} |\Phi(x) - \Phi(y)|_{\infty} &= \left| \int_0^1 \nabla \Phi(\tilde{z} - \epsilon \theta \delta z) \delta z d\theta \right|_{\infty} \\ &\geq |\nabla \Phi(z^*)|_1 - \left| \int_0^1 (\nabla \Phi(z^*) - \nabla \Phi(\tilde{z} - \epsilon \theta \delta z)) \delta z d\theta \right|_{\infty} \\ &\geq |\nabla \Phi(z^*)|_1 - (\epsilon + \eta) L[\nabla \Phi; B^*] \end{aligned}$$

where  $L[\nabla \Phi; B^*]$  denotes the Lipschitz bound of the Jacobian  $\nabla \Phi : B^* \rightarrow R_1^{n \times n}$  and  $R_1^{n \times n}$  is the vector space of real valued  $n \times n$  matrices equipped with the matrix  $|\cdot|_1$ -norm. Hence

$$\epsilon^{-1} |\Phi(x) - \Phi(y)|_{\infty} \geq |\nabla \Phi(z^*)|_1 - (\epsilon + \eta) L[\nabla \Phi; B^*],$$

and since  $\epsilon$  and  $\eta$  are arbitrary it follows that  $L[\Phi; B^*] \geq |\nabla \Phi(z^*)|_1$ . We conclude that  $L[\Phi; B^*] = |\nabla \Phi(z^*)|_1$ .

Now consider the DPC and opportunistic algorithms. Then the Jacobian is given in (12) and the absolute value of the row sums are on the form

$$\frac{\bar{F}^i e^{p^*+z}}{\bar{\sigma}_i^2 + \bar{F}^i e^{p^*+z}}.$$

The maximum is achieved when  $\bar{F}^i e^{p^*+z}$  is maximized. Since all coefficients are positive, the maximum solution is given by  $z_i = z_{\max, i}, \forall i$ . The Lipschitz constant is then given by the maximizing index  $i^*$ , since the order of the maximization over the index  $i$  and the set  $B^*$  can be interchanged. ■

#### E. Proof of Theorem 1

*Proof:* Define the saturation  $\text{sat}_{[-\gamma, \gamma]} : R^n \rightarrow R^n$  whose  $i^{\text{th}}$  component is

$$[\text{sat}_{[-\gamma, \gamma]}(z)]_i = \begin{cases} \gamma & \text{if } z_i > \gamma \\ z_i & \text{if } -\gamma \leq z_i \leq \gamma \\ -\gamma & \text{if } z_i < -\gamma \end{cases}$$

and let

$$\Phi_{\gamma}(z) = \Phi(\text{sat}_{[-\gamma, \gamma]}(z)).$$

Define  $F(z) = H(\delta r + \Phi_{\gamma}(z))$ , then

$$\begin{aligned} \|F(z_1) - F(z_2)\|_{\infty} &= \|H(\Phi_{\gamma}(z_1) - \Phi_{\gamma}(z_2))\|_{\infty} \\ &\leq \underbrace{\|H\|_1 L[\Phi; B^*(\gamma)]}_{< 1} \|z_1 - z_2\|_{\infty}, \end{aligned}$$

where we used Proposition 3. Hence  $F$  is a contraction on  $l_\infty$  and according to the Banach fixed point theorem there exists a unique solution  $z^*$  to the fixed point equation  $z^* = F(z^*)$ . Assume now that the bound in (15) holds. Then the fixed point  $z^*$  satisfies

$$\begin{aligned} \|z^*\|_\infty &= \|F(z^*)\|_\infty \\ &\leq \|H\|_1 \left( \|\delta r\|_\infty + L[\Phi; B^*(\gamma)] \|z^*\|_\infty \right) \\ &\leq \|H\|_1 \frac{\gamma(1 - L[\Phi; B^*(\gamma)] \|H\|_1)}{\|H\|_1} \\ &\quad + \|H\|_1 L[\Phi; B^*(\gamma)] \|z^*\|_\infty \end{aligned}$$

which is equivalent to  $\|z^*\|_\infty \leq \gamma$ . This implies that there also exists a unique power distribution with  $\|z\|_\infty \leq \gamma$  to the real system because the saturation in the definition of  $\Phi_\gamma$  is inactive.

The last statement in the theorem follows from the bound

$$\|z^*\|_{2,\infty} \leq \frac{\|H\|_{l_{2,\infty} \rightarrow l_{2,\infty}}}{1 - L[\Phi; B^*(\gamma)] \|H\|_{l_{2,\infty} \rightarrow l_{2,\infty}}} \|\delta r\|_{2,\infty}$$

which is derived in the same fashion as the previous bound. ■

#### F. Proof of Proposition 5

*Proof:* From the derivations when introducing the scalings it is clear that  $L[\hat{\Phi}; D^{-1}B^*] \leq K_D$ . To see that equality can be achieved we assume

$$K_D = \max_{z \in B^*} |D^{-1} \nabla \Phi(z) D|_1 = f_{i^*}(z^*),$$

where  $i^*$  is the maximizing index and  $z^*$  is the maximizing argument. Such index and argument exists, since  $|D^{-1} \nabla \Phi(z) D|_1$  is a continuous function and the optimization is over a compact set. Let  $\hat{\delta}z = D^{-1} \delta z$  be a unit length vector in  $R_\infty^n$  such that  $|D^{-1} \nabla \Phi(z^*) D \hat{\delta}z|_\infty = |D^{-1} \nabla \Phi(z^*) D|_1$ . Let  $\tilde{z}$  be an interior point of  $B^*$  such that  $|\tilde{z} - z^*|_\infty \leq \eta$ . Furthermore, let  $\hat{x} = D^{-1} \tilde{z} - \epsilon \hat{\delta}z$  and  $\hat{y} = D^{-1} \tilde{z}$ . We then get

$$\begin{aligned} \frac{1}{\epsilon} |\hat{\Phi}(\hat{x}) - \hat{\Phi}(\hat{y})|_\infty &= \frac{1}{\epsilon} \left| \int_0^1 \nabla \hat{\Phi}(\hat{y} + \theta(\hat{x} - \hat{y})) (\hat{x} - \hat{y}) d\theta \right|_\infty \\ &= \frac{1}{\epsilon} \left| \int_0^1 D^{-1} \nabla \Phi(D\hat{y} + D\theta(\hat{x} - \hat{y})) D(\hat{x} - \hat{y}) d\theta \right|_\infty \\ &= \left| \int_0^1 D^{-1} \nabla \Phi(\tilde{z} - \theta \epsilon \delta z) D \hat{\delta}z d\theta \right|_\infty \\ &\geq |D^{-1} \nabla \Phi(z^*) D|_1 \\ &\quad - \left| \int_0^1 D^{-1} (\nabla \Phi(z^*) - \nabla \Phi(\tilde{z} - \theta \epsilon \delta z)) D \hat{\delta}z d\theta \right|_\infty \\ &\geq |D^{-1} \nabla \Phi(z^*) D|_1 - (\eta + \epsilon \tilde{d}) L[D^{-1} \nabla \Phi D; B^*] \end{aligned}$$

where  $\tilde{d} = \max_i d_i$  and  $L[D^{-1} \nabla \Phi D; B^*]$  denotes the Lipschitz bound of the scaled Jacobian of  $\Phi$ ,  $D^{-1} \nabla \Phi D : B^* \rightarrow R_1^{n \times n}$  and  $R_1^{n \times n}$  is the vector space of real valued  $n \times n$  matrices equipped with the matrix  $|\cdot|_1$ -norm. By assumption  $\nabla \Phi$  is Lipschitz on  $B^*$ , and hence we have that also  $D^{-1} \nabla \Phi D$  is Lipschitz. Since  $\epsilon$  and  $\eta$  are arbitrary it follows that  $L[\hat{\Phi}; D^{-1}B^*] \geq K_D$ . ■

#### G. Proof of Theorem 2

*Proof:* Define  $F(z) = H(\delta r + \Phi_{\text{sat}_{[B^*]}}(z))$ , then

$$\begin{aligned} \|F(z_1) - F(z_2)\|_\infty &= \|H(\Phi_{\text{sat}_{[B^*]}}(z_1) - \Phi_{\text{sat}_{[B^*]}}(z_2))\|_\infty \\ &\leq \underbrace{\|H\|_1 L[\Phi; B^*]}_{<1} \|z_1 - z_2\|_\infty, \end{aligned}$$

where we used Proposition 8. Hence  $F$  is a contraction on  $l_\infty$  and according to the Banach fixed point theorem there exists a unique solution  $z^*$  to the fixed point equation  $z^* = F(z^*)$ . Assume now a fixed  $\delta r$ , then the fixed point  $z^*$  satisfies

$$\begin{aligned} \|z^*\|_\infty &= \|F(z^*)\|_\infty \\ &\leq \|H\|_1 \left( \|\delta r\|_\infty + L[\Phi; B^*] \|z^*\|_\infty \right), \end{aligned}$$

which is equivalent to

$$\|z^*\|_\infty \leq \frac{\|H\|_1}{1 - \|H\|_1 L[\Phi; B^*]} \|\delta r\|_\infty.$$

The case where  $\Phi : l_{2,\infty} \rightarrow l_{2,\infty}$  is analoguous. ■

#### H. Proof of Proposition 9

*Proof:* First consider

$$\begin{aligned} L[\hat{\Phi}_{\text{sat}_{[B^*]}}; R_\infty^n] &= \sup_{D\hat{z}_1, D\hat{z}_2 \in B^*} \frac{|D^{-1} \Phi(D\hat{z}_1) - D^{-1} \Phi(D\hat{z}_2)|_\infty}{|\hat{z}_1 - \hat{z}_2|_\infty} \\ &= \sup_{\hat{z}_1, \hat{z}_2 \in D^{-1}B^*} \frac{|D^{-1} \Phi(D\hat{z}_1) - D^{-1} \Phi(D\hat{z}_2)|_\infty}{|\hat{z}_1 - \hat{z}_2|_\infty} \\ &= L[\hat{\Phi}; D^{-1}B^*]. \end{aligned}$$

This can be expanded to  $L[\hat{\Phi}_{\text{sat}_{[B^*]}}; l_\infty^n]$  and  $L[\hat{\Phi}_{\text{sat}_{[B^*]}}; l_{2,\infty}^n]$  by the same argument as is Proposition 3. The last equalities is a restatement of Proposition 5. ■

#### I. Proof of Lemma 1

*Proof:* Consider fixed values of  $d$  and study the function  $f_i(z) := f_i(d, z)$ , as defined in (19). First we show that there is no stationary point in the interior of the feasible region. That gives us that at least one index,  $k_j$ , of  $z$  must be either  $z_{\max, k_j}$  or  $z_{\min, k_j}$ . We then fix the value for  $z_{k_j}$  to  $z_{k_j} = \{z_{\max, k_j}, z_{\min, k_j}\}$  and study the optimization problem again. Recursively we prove that for the reduced problems, with some variables fixed, stationary points only exist on a set of measure zero of the  $d$ -space. Furthermore this set is independent of the remaining variables. Hence the remaining variables can also be taken on the boundary. If the scalings  $d$  do not belong to this set of measure zero, the optimal solution at the boundary is unique.

1) *First step:* First consider

$$f_i^{(n)}(z) = \frac{\frac{d_1}{d_i} \bar{a}_{i1} e^{z_1} + \dots + \frac{d_n}{d_i} \bar{a}_{in} e^{z_n}}{\bar{a}_{i1} e^{z_1} + \dots + \bar{a}_{in} e^{z_n} + \bar{\sigma}_i^2},$$

where  $n$  is denotes that there are  $n$  free variables and  $\bar{a}_{ii} = (1 - \bar{\delta}_i) \bar{g}_{ii} e^{p_i^*}$  and  $\bar{a}_{ij} = \bar{g}_{ij} e^{p_j^*}$ . Now differentiate w.r.t.  $z_j$

and set to zero.

$$\begin{aligned} \frac{\partial f_i^{(n)}}{\partial z_j}(z) &= \frac{\frac{d_j}{d_i} \bar{a}_{ij} e^{z_j} (\bar{a}_{i1} e^{z_1} + \dots + \bar{a}_{in} e^{z_n} + \bar{\sigma}_i^2)}{(\bar{a}_{i1} e^{z_1} + \dots + \bar{a}_{in} e^{z_n} + \bar{\sigma}_i^2)^2} \\ &\quad - \frac{\bar{a}_{ij} e^{z_j} (\frac{d_1}{d_i} \bar{a}_{i1} e^{z_1} + \dots + \frac{d_n}{d_i} \bar{a}_{in} e^{z_n})}{(\bar{a}_{i1} e^{z_1} + \dots + \bar{a}_{in} e^{z_n} + \bar{\sigma}_i^2)^2} \\ &= 0 \end{aligned}$$

Note that the denominator is always positive so we can focus on the numerator. First multiply all terms with  $d_i$ , then divide by  $\bar{a}_{ij} e^{z_j}$ , which is always strictly greater than zero.

We then get the following equivalent equation.

$$\begin{aligned} d_j \bar{a}_{i1} e^{z_1} + \dots + d_j \bar{a}_{in} e^{z_n} + d_j \bar{\sigma}_i^2 \\ - d_1 \bar{a}_{i1} e^{z_1} - \dots - d_n \bar{a}_{in} e^{z_n} = 0, \end{aligned}$$

which is equivalent to

$$\bar{a}_{i1} (d_j - d_1) e^{z_1} + \dots + \bar{a}_{in} (d_j - d_n) e^{z_n} = -\bar{\sigma}_i^2 d_j.$$

Differentiating  $f_i$  w.r.t. all  $z_j$ ,  $j = 1, \dots, n$ , we get the following system of  $n$  equations, corresponding to  $\nabla f_i(z) = 0$ ,

$$\begin{aligned} (d_1 - d_1)x_1 + \dots + (d_1 - d_n)x_n &= -\bar{\sigma}_i^2 d_1 \\ &\vdots \\ (d_n - d_1)x_1 + \dots + (d_n - d_n)x_n &= -\bar{\sigma}_i^2 d_n, \end{aligned}$$

where  $x_j = \bar{a}_{ij} e^{z_j}$ . Assuming the dimension  $n > 1$  we get the following relation

$$\left( \sum_{i=0}^n x_i \right) d - (d^T x) \mathbf{1} = -\bar{\sigma}_i^2 d,$$

which has no solution for positive  $x$  and  $d$ . Hence there is no extreme point in the interior, and the maximum must be at the boundary. This implies that at least one element of the  $z$ -vector must be either  $z_{\max, k_j}$  or  $z_{\min, k_j}$ .

2) *Second step:* Let  $k_1$  be an index of a variable at the boundary. Define  $K_1 = \bar{a}_{ik_1} e^{z_{k_1}}$ , where  $z_{k_1} \in \{z_{\max, k_1}, z_{\min, k_1}\}$ . Note that  $K_1 > 0$ . With the index  $k_1$  taken away, order the remaining variables from 1 to  $n-1$ . We then study the stationary point of the function

$$f_i^{(n-1)}(z) = \frac{\frac{d_1}{d_i} \bar{a}_{i1} e^{z_1} + \dots + \frac{d_{n-1}}{d_i} \bar{a}_{i, n-1} e^{z_{n-1}} + \frac{d_{k_1}}{d_i} K_1}{\bar{a}_{i1} e^{z_1} + \dots + \bar{a}_{i, n-1} e^{z_{n-1}} + \bar{\sigma}_i^2 + K_1}.$$

Following the same steps as before, we differentiate w.r.t.  $z_j$ ,  $j \in [1, n-1]$  and set to zero

$$\frac{\partial f_i^{(n-1)}}{\partial z_j}(z) =$$

$$\begin{aligned} \frac{\frac{d_j}{d_i} \bar{a}_{ij} e^{z_j} (\bar{a}_{i1} e^{z_1} + \dots + \bar{a}_{i, n-1} e^{z_{n-1}} + \bar{\sigma}_i^2 + K_1)}{(\bar{a}_{i1} e^{z_1} + \dots + \bar{a}_{i, n-1} e^{z_{n-1}} + \bar{\sigma}_i^2 + K_1)^2} \\ - \frac{\bar{a}_{ij} e^{z_j} (\frac{d_1}{d_i} \bar{a}_{i1} e^{z_1} + \dots + \frac{d_{n-1}}{d_i} \bar{a}_{i, n-1} e^{z_{n-1}} + \frac{d_{k_1}}{d_i} K_1)}{(\bar{a}_{i1} e^{z_1} + \dots + \bar{a}_{i, n-1} e^{z_{n-1}} + \bar{\sigma}_i^2 + K_1)^2} = 0. \end{aligned}$$

As before, the denominator is always positive so we can focus on the numerator. Multiply all terms with  $d_i$  and divide by  $\bar{a}_{ij} e^{z_j} > 0$ . We then get the equation

$$\begin{aligned} d_j \bar{a}_{i,1} e^{z_1} + \dots + d_j \bar{a}_{i, n-1} e^{z_{n-1}} + d_j \bar{\sigma}_i^2 + d_j K_1 \\ - d_1 \bar{a}_{i,1} e^{z_1} - \dots - d_{n-1} \bar{a}_{i, n-1} e^{z_{n-1}} - d_{k_1} K_1 = 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \bar{a}_{i,1} (d_j - d_1) e^{z_1} + \dots + \bar{a}_{i, n-1} (d_j - d_{n-1}) e^{z_{n-1}} = \\ - (\bar{\sigma}_i^2 + K_1) d_j + d_{k_1} K_1. \end{aligned}$$

Differentiating  $f_i^{(n-1)}$  w.r.t. all  $z_j$ ,  $j = 1, \dots, n-1$ , we get a system of  $n-1$  equations, now corresponding to  $\nabla f_i^{(n-1)}(z) = 0$ . Again, let  $\bar{a}_{ij} e^{z_j} = x_j$ .

$$\begin{aligned} (d_1 - d_1)x_1 + \dots + (d_1 - d_{n-1})x_{n-1} \\ = -(\bar{\sigma}_i^2 + K_1)d_1 + d_{k_1}K_1 \\ \vdots \\ (d_{n-1} - d_1)x_1 + \dots + (d_{n-1} - d_{n-1})x_{n-1} \\ = -(\bar{\sigma}_i^2 + K_1)d_{n-1} + d_{k_1}K_1 \end{aligned}$$

Reorganizing the equations gives the following relation that must be fulfilled for existence of an interior stationary point

$$\begin{aligned} \left( \sum_{i=1}^{n-1} x_i \right) \begin{bmatrix} d_1 \\ \vdots \\ d_{n-1} \end{bmatrix} - (d^T x) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \\ -(\bar{\sigma}_i^2 + K_1) \begin{bmatrix} d_1 \\ \vdots \\ d_{n-1} \end{bmatrix} + d_{k_1} K_1 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \end{aligned}$$

Consider first the case when at least one  $d_i \neq d_j$ ,  $(i, j) \in \{1, \dots, n-1\}$ . Then the coefficients of the vectors must be the same, giving the conditions

$$\sum_{j=1}^{n-1} x_j = \sum_{j=1}^{n-1} \bar{a}_{ij} e^{z_j} = -(\bar{\sigma}_i^2 + K_1) = -(\bar{\sigma}_i^2 + \bar{a}_{i, k_1} e^{z_{k_1}}),$$

and

$$\sum_{j=1}^{n-1} d_j x_j = \sum_{j=1}^{n-1} d_j \bar{a}_{ij} e^{z_j} = -d_{k_1} K_1 = -d_{k_1} \bar{a}_{i, k_1} e^{z_{k_1}},$$

which cannot be fulfilled for positive  $x, d$ . Now consider the case  $d_i = \alpha, i = 1, \dots, n-1$ . We then get

$$\alpha \sum_{i=1}^{n-1} x_i - \alpha \sum_{i=1}^{n-1} x_i = -\alpha (\bar{\sigma}_i^2 + K_1) + d_{k_1} K_1,$$

which reduces to

$$\alpha = \frac{d_{k_1} K_1}{\bar{\sigma}_i^2 + K_1}.$$

Hence there is a stationary point on the set of measure zero of the  $d$ -space when  $d_i = \alpha, \forall i \neq k_1$ . However, the condition on the scalings for the stationary point is independent of the values of the remaining powers, which implies that they can be taken on the boundary.

Assuming the fixed scalings do not fulfill the condition for a stationary point, another variable can be fixed to the boundary.

3) *l:th step*: In the  $l$ :th step we have the free variables  $z_1, \dots, z_l$  and the fixed variables  $z_{k_1}, \dots, z_{k_{n-l}}$  giving

$$f_i^{(l)}(z) = \frac{\frac{d_1}{d_i} \bar{a}_{i1} e^{z_1} + \dots + \frac{d_l}{d_i} \bar{a}_{i,l} e^{z_l} + \sum_{j=1}^{n-l} \frac{d_{k_j}}{d_i} K_j}{\bar{a}_{i1} e^{z_1} + \dots + \bar{a}_{i,l} e^{z_l} + \bar{\sigma}_i^2 + \sum_{j=1}^{n-l} K_j}.$$

Differentiate w.r.t.  $z_j, j \in \{1, \dots, l\}$  and set equal to zero we get

$$\begin{aligned} \frac{\partial f_i^{(l)}}{\partial z_j}(z) &= \frac{\frac{d_j}{d_i} \bar{a}_{ij} e^{z_j} (\bar{a}_{i1} e^{z_1} + \dots + \bar{a}_{i,l} e^{z_l} + \bar{\sigma}_i^2 + \sum_{j=1}^{n-l} K_j)}{(\bar{a}_{i1} e^{z_1} + \dots + \bar{a}_{i,l} e^{z_l} + \bar{\sigma}_i^2 + \sum_{j=1}^{n-l} K_j)^2} \\ &\quad - \frac{\bar{a}_{ij} e^{z_j} (\frac{d_1}{d_i} \bar{a}_{i1} e^{z_1} + \dots + \frac{d_l}{d_i} \bar{a}_{i,l} e^{z_l} + \sum_{j=1}^{n-l} \frac{d_{k_j}}{d_i} K_j)}{(\bar{a}_{i1} e^{z_1} + \dots + \bar{a}_{i,l} e^{z_l} + \bar{\sigma}_i^2 + \sum_{j=1}^{n-l} K_j)^2} \\ &= 0. \end{aligned}$$

As before, multiply all terms with the denominator and  $d_i$  and divide by  $\bar{a}_{ij} e^{z_j}$ . We then get

$$\begin{aligned} d_j \bar{a}_{i1} e^{z_1} + \dots + d_j \bar{a}_{i,l} e^{z_l} + d_j \bar{\sigma}_i^2 + d_j \sum_{j=1}^{n-l} K_j \\ - d_1 \bar{a}_{i1} e^{z_1} - \dots - d_l \bar{a}_{i,l} e^{z_l} - \sum_{j=1}^{n-l} d_{k_j} K_j = 0. \end{aligned}$$

This is equivalent to

$$\begin{aligned} (d_j - d_1) \bar{a}_{i1} e^{z_1} + \dots + (d_j - d_l) \bar{a}_{i,l} e^{z_l} = \\ - (\bar{\sigma}_i^2 + K_1 + \dots + K_{n-l}) d_j + d_{k_1} K_1 + \dots + d_{k_{n-l}} K_{n-l}. \end{aligned}$$

Differentiating w.r.t. all  $j \in \{1, \dots, l\}$  we get the following system of  $l$  equations, corresponding to  $\nabla f_i^{(l)}(z) = 0$ . Again, let  $\bar{a}_{ij} e^{z_j} = x_j$ .

$$\begin{aligned} (d_1 - d_1) x_1 + \dots + (d_1 - d_l) x_l \\ = - (\bar{\sigma}_i^2 + \sum_{j=1}^{n-l} K_j) d_1 + \sum_{j=1}^{n-l} d_{k_j} K_j \\ \vdots \\ (d_l - d_1) x_1 + \dots + (d_l - d_l) x_l \\ = - (\bar{\sigma}_i^2 + \sum_{j=1}^{n-l} K_j) d_l + \sum_{j=1}^{n-l} d_{k_j} K_j \end{aligned}$$

Reorganizing gives

$$\begin{aligned} \left( \sum_{i=1}^l x_i \right) \begin{bmatrix} d_1 \\ \vdots \\ d_l \end{bmatrix} - (d^T x) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \\ - \left( \bar{\sigma}_i^2 + \sum_{j=1}^{n-l} K_j \right) \begin{bmatrix} d_1 \\ \vdots \\ d_l \end{bmatrix} + \left( \sum_{j=1}^{n-l} d_{k_j} K_j \right) \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}. \end{aligned}$$

Again, consider first the case where  $d_i \neq \alpha, i = 1, \dots, l$ . Then the coefficients of the vectors must be the same, giving the conditions

$$\sum_{j=1}^l x_j = \sum_{j=1}^l \bar{a}_{ij} e^{z_j} = - (\bar{\sigma}_i^2 + \sum_{j=1}^{n-l} K_j) = - \left( \bar{\sigma}_i^2 + \sum_{j=1}^{n-l} \bar{a}_{i,k_j} e^{z_{k_j}} \right),$$

and

$$\sum_{j=1}^l d_j x_j = \sum_{j=1}^l d_j \bar{a}_{ij} e^{z_j} = - \left( \sum_{j=1}^{n-l} d_{k_j} K_j \right) = - \sum_{j=1}^{n-l} d_{k_j} \bar{a}_{i,k_j} e^{z_{k_j}},$$

which cannot be fulfilled for positive  $x, d$ . Now consider  $d_i = \alpha, i = 1, \dots, l$ . We then get

$$\alpha \sum_{j=1}^l x_j - \alpha \sum_{j=1}^l x_j = - \alpha \left( \bar{\sigma}_i^2 + \sum_{j=1}^{n-l} K_j \right) + \sum_{j=1}^{n-l} d_{k_j} K_j,$$

giving

$$\alpha = \frac{\sum_{j=1}^{n-l} d_{k_j} K_j}{\bar{\sigma}_i^2 + \sum_{j=1}^{n-l} K_j}.$$

Hence again there is a set of measure zero in the  $d$ -space, given by  $d_i = \alpha, i = 1, \dots, l$ , which is a stationary point. As before the condition on the scalings for the stationary point is independent of the values of the remaining powers, which implies that they can be taken on the boundary.

4) *Final step*: If the scalings are not on the special set giving a stationary point, we proceed in the same way fixing the variables on the boundary until the number of free variables reach one.

For  $n = 1$  we have

$$f_i^{(1)}(z) = \frac{\frac{d_1}{d_i} \bar{a}_{i1} e^{z_1} + \sum_{j=1}^{n-1} \frac{d_{k_j}}{d_i} \bar{a}_{i,k_j} e^{z_{k_j}}}{\bar{a}_{i1} e^{z_1} + \bar{\sigma}_i^2 + \sum_{j=1}^{n-1} \bar{a}_{i,k_j} e^{z_{k_j}}}.$$

Differentiate w.r.t.  $z_1$  and set to zero

$$\begin{aligned} \frac{\partial f_i^{(1)}}{\partial z_1}(z) &= \frac{\frac{d_1}{d_i} \bar{a}_{i1} e^{z_1} (\bar{a}_{i1} e^{z_1} + \bar{\sigma}_i^2 + \sum_{j=1}^{n-1} \bar{a}_{i,k_j} e^{z_{k_j}})}{(\bar{a}_{i1} e^{z_1} + \bar{\sigma}_i^2 + \sum_{j=1}^{n-1} \bar{a}_{i,k_j} e^{z_{k_j}})^2} \\ &\quad - \frac{\bar{a}_{i1} e^{z_1} (\frac{d_1}{d_i} \bar{a}_{i1} e^{z_1} + \sum_{j=1}^{n-1} \frac{d_{k_j}}{d_i} \bar{a}_{i,k_j} e^{z_{k_j}})}{(\bar{a}_{i1} e^{z_1} + \bar{\sigma}_i^2 + \sum_{j=1}^{n-1} \bar{a}_{i,k_j} e^{z_{k_j}})^2} \\ &= 0. \end{aligned}$$

This is equivalent to

$$\begin{aligned} d_1 \bar{a}_{i1} e^{z_1} + d_1 \bar{\sigma}_i^2 + d_1 \sum_{j=1}^{n-1} \bar{a}_{i,k_j} e^{z_{k_j}} - d_1 \bar{a}_{i1} e^{z_1} \\ - \sum_{j=1}^{n-1} d_{k_j} \bar{a}_{i,k_j} e^{z_{k_j}} = 0, \end{aligned}$$

which gives the relation

$$d_j = \frac{d_{k_1} \bar{a}_{i,k_1} e^{z_{k_1}} + \dots + d_{k_{n-1}} \bar{a}_{i,k_{n-1}} e^{z_{k_{n-1}}}}{\bar{a}_{i,k_1} e^{z_{k_1}} + \dots + \bar{a}_{i,k_{n-1}} e^{z_{k_{n-1}}} + \bar{\sigma}_i^2}.$$

Note that it is independent of the last free variable  $z_1$  and only fulfilled on a set of measure zero of the  $d$ -space. This

implies that almost surely all  $z_i$  will be on the boundary. On the set of measure zero, all  $z_i$  will be on the boundary, except possibly one index. For this index,  $k_j$ ,  $f_i^{(1)}(z_{k_j})$  has a stationary point independent of the value of  $z_{k_j}$ , and hence we can take any value of  $z_{k_j}$  without changing the function value of  $f_i^{(1)}$ , including the boundary. ■