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ORIGINAL ARTICLE

Inquiry pedagogy to promote emerging proportional reasoning in primary students

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Abstract Proportional reasoning as the capacity to compare situations in relative (multiplicative) rather than absolute (additive) terms is an important outcome of primary school mathematics. Research suggests that students tend to see comparative situations in additive rather than multiplicative terms and this thinking can influence their capacity for proportional reasoning in later years. In this paper, excerpts from a classroom case study of a fourth-grade classroom (students aged 9) are presented as they address an inquiry problem that required proportional reasoning. As the inquiry unfolded, students' additive strategies were progressively seen to shift to proportional thinking to enable them to answer the question that guided their inquiry. In wrestling with the challenges they encountered, their emerging proportional reasoning was supported by the inquiry model used to provide a structure, a classroom culture of inquiry and argumentation, and the proportionality embedded in the problem context.

Keywords Proportional reasoning · Mathematical inquiry · Classroom argumentation

Proportional reasoning is foundational to almost all of secondary and higher mathematics (Confrey 2008). Being able to distinguish additive from multiplicative situations has been described as a major component of proportional reasoning (Confrey and Smith 1995; Van Dooren et al. 2005), but research has consistently highlighted the difficulties students experience in relation to this (Cramer et al. 1993; Dole et al. 2012a; Hilton et al. 2012; Staples and Truxaw 2012). In the early years, there is capacity to foster young children's proportional reasoning through typical multiplicative experiences such as sand and water play (if it takes three cups of sand to make one sand pie, it will take nine cups to make three sand pies). More complex situations involve students

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comparing ratios *between* situations such as in using scaled drawings or when comparing the ratio of dirt to water used by two children making different numbers of pies (Steinthorsdottir and Sriraman 2009). Such situations provide natural contexts for emphasising the critical component of proportional reasoning situations: "the multiplicative relationship that exists among the quantities that represent the situation" (Cramer et al. 1993, p. 160). Students' capacity to think proportionally is developmental, becoming more stable with maturity and experience (Clark and Kamii 1996) but, according to Hart (1981), students' intuitive strategies for solving proportion problems are typically additive. Van Dooren, de Bock and Verschaffel (2010) note that in an intermediate stage, until students' proportional reasoning is stable, they often simultaneously apply additive strategies to proportional problems and proportional strategies to additive problems.

Prerequisite knowledge for proportional reasoning has been described as being located within rational number topics such as fractions (English and Halford 1995). Lo and Watanabe (1997) discussed how developing students' understanding of ratio and proportion is difficult because students' knowledge of the concepts of multiplication, division, fractions and decimals is generally poor. Although these topics are a foundation for ratio and proportion, there is general consensus that a more targeted focus on *recognising proportional situations* is a necessary teaching point that is overlooked (Behr et al. 1992; Confrey and Smith 1995; Streefland 1985; Van Dooren et al. 2005). The essence of proportional reasoning lies in understanding the multiplicative structure of proportional situations (Behr et al. 1992; Confrey 2008). "Learning to view something 'in proportion' or 'in proportion with' precedes the acquisition of the proper concept of ratio" (Streefland 1985, p. 83). The classroom case study presented in this paper describes how inquiry pedagogy was implemented to foster students' emerging capacity to think proportionally before the application of ratio and proportion were formally addressed in the curriculum.

Learning through mathematical inquiry

Teaching with ill-structured problems

It has long been a goal of school mathematics to develop students' ability to solve problems and develop mathematical thinking so that students can conduct and apply mathematical investigations to situations beyond the school (Resnick 2010; Stacey 2006). While the goal and ideals of problem solving have been fairly clear, its implementation has often removed the heart of what problem solving was meant to achieve: mathematical thinking. "Typical mathematics lessons do not engage students in analysing, comparing, evaluating arguments, or similar thinking practices, but rather focus on routine procedures" (Staples and Truxaw 2012, p. 258). For example in their study of nine eighth-grade Australian textbooks, Stacey and Vincent (2009) argued that in most textbooks, the purpose of reasoning and explanations appeared to be to prepare students for practise exercises, rather than develop mathematics as a thinking tool: only one third of the textbooks in their study actively involved students in the reasoning process (p. 286).

Like the initial intent of mathematical problem solving, mathematical inquiry aims to re-focus learning on the mathematical thinking and reasoning that is vital for students to analyse, explain and defend their mathematical ideas as they solve problems (Cobb and McClain 2006). The inquiry approach we use in this paper is a particular type of problem solving that immerses students in addressing ill-structured problems (Makar 2012). Ill-structured problems are problems for which initial conditions and/or goals of the problem are ambiguous, or have a large number of open constraints that require negotiation (Reitman 1965). The ambiguity often requires students to mathematise a problem, a highly desired yet rarely taught skill in mathematics (Freudenthal 1981; Yoon et al. 2010). There are several other benefits for including problems with ambiguity in mathematics:

> Their inherent ambiguity allows for multiple interpretations of a question, a range of pathways, and numerous solutions with varying degrees of efficiency, applicability and elegance. This requires students to focus on decision making, analysis and justification. Rather than a 'correct' answer or strategy, there is a claim which requires evidence, explanation and defense—in short, an argument. (Fielding-Wells and Makar 2012, p. 149)

Conversely, well-structured problems tend to be abstract and decontextualised, relying more on defined rules (Jonassen 2010). Despite most problems in everyday life being ill-structured, the use of ill-structured problems is a significant departure from most approaches in school mathematics which aim for well-structured, unambiguous language (Baber 2011). Rather than promote ill-structured problems as "better" than well-structured, we argue that students need a healthy balance of both. However, school mathematics is not in balance, focusing far more on procedural fluency than mathematical reasoning and problem solving (Hollingsworth et al. 2003; Kilpatrick et al. 2001).

Inquiry approaches in mathematics and science value negotiation and reasoning, the development of collaborative norms and meaning making (Cobb and McClain 2006; Dewey 1910, 1929; Erduran and Jiménez-Aleixandre 2008). Investigation is generated through peer and class discussion and interaction with the topic, with students often refining specific questions to explore, deliberating possible approaches to investigate their questions and re-evaluating these approaches as the inquiry unfolds. This investigation deepens understanding of the complexity of the context under investigation and may generate new questions to explore (Magnusson and Palincsar 2005). The inquiry generally concludes when the results of the students' investigation sufficiently resolve or explain the problem. In mathematical inquiry, students use mathematics to argue, justify, hypothesise and to direct their inquiry (Fielding-Wells and Makar 2012). Inquiry in mathematics has been increasingly recognised for its potential to develop students' deep understanding, resilience and capacity to cope with uncertainty and setbacks, interest and engagement in mathematics, self-reliance and creativity in solving problems and ability to transfer their mathematics learning to complex and authentic problems (Fielding-Wells and Makar 2008; Goos 2004; Fielding-Wells 2013).

Challenges of inquiry-based learning

Some practices of mathematical inquiry have been sharply criticised. For example, there are dangers of assuming that creating a collaborative environment is sufficient to support student learning. Researchers have noted multiple problems when students were placed in situations with insufficient teacher guidance or when teachers did not capitalise on unexpected opportunities for learning (Rowland and Zazkis 2013; Stacey 1992). For example, Goos (2002) notes that although collaborative activity can help students to monitor and regulate each other's thinking, students can also unknowingly subvert such thinking if they have "passively accepted unhelpful suggestions, or ignored potentially useful strategies proposed by peers" (p. 300). The term inquiry is often used to describe a range of classroom environments from open-ended discovery learning to approaches that are highly structured and require little from students beyond following step-by-step instructions through an activity or pre-planned experiment. These two extremes have both been found to be quite ineffective in supporting student learning (Confrey 1991; Hattie 2009), which is why the teacher's role in questioning and supporting students is so important.

Inquiry teaching and learning is not a simple enterprise, requiring skills unfamiliar in conventional classrooms: ability to embrace uncertainty, capacity and skills to support student decision-making, recognition of unexpected opportunities for learning, and tolerance of periods of noise and disorganisation (Makar 2012; National Research Council 2000). For learners, the early stages of inquiry require considerable cognitive effort in formulating (or mathematising) the question for investigation, designing a plan of attack, and analysing the results in relation to their capacity to address the questions under inquiry (Diezmann et al. 2001). Inquiry requires teachers to design tasks that engage students in meaningful mathematics and take advantage of opportunities for learning in unexpected outcomes (Ainley et al. 2006; Makar 2007; National Research Council 2000; Rowland and Zazkis 2013). Inquiry challenges teachers' perceptions and expectations of mathematics learning as a hierarchical, sequential and structured process, where mathematics skills and knowledge are practiced *in preparation for* problem solving.

Argumentation culture

Because of the focus on purposeful negotiation and reasoning, the development of argumentation practices is highly beneficial when working with ill-structured problems. However, support by the teacher is crucial because of the importance of critiquing, generating and supporting alternative solutions (Allmond and Makar 2010; Cho and Jonassen 2002). For example, students often respond experientially to questions rather than seeing a need for data or research-based evidence (von Aufschnaiter et al. 2008; Zembal-Saul et al. 2013) and have difficulty in envisioning what evidence may assist with the addressing of claims (Fielding-Wells 2010, 2013; Fielding-Wells, J., *Developing argumentation in mathematics: the role of evidence and context.* Unpublished doctoral dissertation, The University of Queensland, School of Education (unpublished)). Teacher questioning and guidance are crucial in moving students beyond these impulsive responses towards more evidence-based reasoning (Ritchhart and Perkins 2005).

Argumentation offers potential for teachers to support students as they focus on the role and nature of evidence in mathematical inquiry. Berland and Reiser (2009) provide an argumentation framework that proposes levels of argumentative discourse: sense making, explanation and persuasion. At the level of sense-making, the students' goal is to understand; however, the understanding is internalised and often comes from an unchallenged perspective. As the students seek to explain their understandings, their reasoning becomes 'visible' and this affords the opportunity to challenge, deepen or strengthen understandings as necessary. At the level of persuasion, students seek to put forth their understandings in such a way that they will stand up to the scrutiny of others. To achieve this, they present a specific claim, evidence which coordinates with the claim and which is acceptable within the relevant community or discipline, and demonstrate the reasoning which enables them to link the two. Any challenge to the argument originates at the discipline level and thus there is potential for it to deepen understanding by motivating students to seek robust conclusions, supported by quality evidence, rather than those based on intuitive and impulsive ideas.

Linking inquiry with learning in mathematics

The inquiry practices used in this study embedded learning *as part of* problem-solving practices and specifically involved a balance of student control and guidance from a highly skilled teacher. Cobb and McClain (2006) argue that to be effective, guided inquiry-based learning environments need to focus on four critical aspects:

- instructional tasks that involve an investigative spirit of authentic practice
- organisation of classroom activities that emphasise whole class discussion of the problem context, work in small groups towards their solution, and sharing progress and findings back to the whole class
- use of tools and representations that focus students on thinking and reasoning of key mathematical ideas
- classroom discourse that is designed to develop critical norms of productive learning and to ensure that whole class discussions are used to advance the teachers' instructional agenda.

An inquiry framework (Table 1) was used with specific links to research on learning (see Allmond et al. 2010 for practical applications of this framework). Similar to the characteristics advocated by Cobb and McClain (2006), the mathematical inquiry used in this study was quite intentional and guided and therefore very different than "open" inquiry and discovery learning (which are often depicted as turning the reins over to students) or highly structured "cook book" style investigations that provide little opportunity for students to wrestle with and reason about problems. Because the design of the learning environment requires strong attention to the instructional agenda, the learning outcomes can be anticipated and partially guided by the teacher. Concepts were often "layered" with initial informal experiences used to generate the need for improved reasoning and more formal content, sometimes in later units (Yoon et al. 2010).

Method

Study design

The research question that is the focus of this paper is: *What is the capacity for inquiry pedagogy to foster proportional reasoning of primary students*? The paper reports on a case

Table 1 Links between inquiry and learning in literature

Inquiry phase: general aim of phase	Literature base
Discover "Hook" to engage students; immersion in the context and purpose of the inquiry; develop or understand the inquiry question and link it to students' current and/or personal knowledge; foreground mathematical "big ideas"	Cognitive engagement foundational to learning (Blumenfeld et al. 2006; Krajcek and Blumenfeld 2006) Connectedness to background knowledge supports student learning and reasoning of new concepts (NCTM 2000; Engle et al. 2012; Richland et al. 2012) Value of understanding the context of the problem (Jonassen 2010; Magnusson and Palincsar 2005) Problem purpose as a critical focus for learning mathematics (Ainley et al. 2006; Confrey 1998; Sullivan et al. 2013)
Devise Mathematise the inquiry question, develop a plan for addressing the question, envision evidence and potential conclusion; trial plans; focus on developing initial mathematics needed to address the inquiry problem	Challenge of envisioning evidence (Fielding-Wells 2010, 2013; Fielding-Wells (unpublished)) Difficulty in linking question-evidence-conclusion (Hancock et al. 1992; Makar 2004) Experience in decision-making and testing ideas supports students' learning (Blumenfeld et al. 2006; Harel and Koichu 2010) Mathematising problems is a critical skill in learning mathematics (Freudenthal 1981; Yoon et al. 2010)
Develop Carry out and revise plan; analyse data towards a conclusion; deepen and formalise mathematical ideas as they are applied and challenged; evaluation of evidence and reasoning needed to support emerging conclusion	Challenges encountered in applying knowledge to new situations (Harel and Koichu 2010; Schwartz et al. 2012; Yoon et al. 2010) Knowledge-building focus on idea improvement (Scardamalia and Bereiter 2006) Use of disequilibrium to deepen or create a need for mathematics (Harel and Koichu 2010; Makar and Fielding-Wells (unpublished)) Utility of mathematics for solving problems (Ainley et al. 2006)
Defend Report on findings, consider alternative approaches, generate new questions, reflect on learning; transfer understandings to new context	Communicating findings and anticipating critique (Andriessen, 2006; Berland and Reiser 2009; Engle et al. 2012) Making reasoning explicit (Osborne 2010; Ritchhart and Perkins 2005)
Diverge Connect to other areas of learning, follow up on new questions that emerged, transfer learning to new problems	Transfer is best achieved when students balance concrete experiences and abstractions (Day and Goldstone 2012). Connectedness of learning to other areas of mathematics, curriculum and wider world (Engle et al. 2012; NCTM 2000) Repeated opportunities to reason support learning (Harel and Koichu 2010; Ritchhart and Perkins 2005)

study within two larger studies that utilised a design research framework (Cobb et al. 2003). The aim of the larger longitudinal studies was to gain insight into teaching and learning in inquiry-based classrooms (Makar 2012; Dole et al. 2012b) and the development of student argumentation practices (Fielding-Wells 2013; Fielding-Wells (unpublished)). The first author led the argumentation study as part of her dissertation, while the other authors led the inquiry study. While the larger study was focused on pedagogical innovations, the students were learning curricular content within these interventions. A case study approach, therefore, provided an opportunity to look closely at student learning of the mathematical content. This paper focuses on that learning as it relates to proportional reasoning; the statistical aspects of this unit are reported elsewhere (Makar, K., and Fielding-Wells, J., Concept-driven disequilibrium: supporting children's early notions of inference and sampling variability. *Educational Studies in Mathematics* (unpublished); Makar et al. 2011).

Context of the case study

The classroom case study reported here was taught by two experienced part-time inquiry teachers sharing one fourth-grade class in a state school located in a middle-class suburb in Queensland. The students (mostly 9 years old) in the class of 28 students, included nine needing additional support (through ascertained learning difficulties, English as a second language or diagnosed autistic spectrum disorder). The students had already experienced inquiry-based learning in two other short units earlier in the year. Some students had also been involved in the inquiry project the previous year and therefore had quite well-developed inquiry practices as learners. In their experiences with inquiry, students were accustomed to working collaboratively to solve ill-structured problems. Students were also accustomed to taking an ambiguous inquiry question and working it into one that could be investigated (e.g. Allmond and Makar 2010). The students had been introduced to the relationship between question, evidence, conclusion and purpose (Fig. 1).

In this unit, students wrestled with the question, “Is Barbie a Monster?”. The question came from a school-developed unit taught across the four year 4 classes in the school and was designed to integrate students' study of human proportion and design in art, and descriptive genres in English. The tongue-in-cheek question was meant to provoke students' engagement in considering whether the Barbie doll, an iconic toy for young girls, was constructed in a way that portrayed human form—or if she was not human-like (a monster) (see also Norton 2005 for a different approach with this driving question). It was intended to integrate their learning in mathematics, art and English by investigating facial and body proportions, and the creation of garments to be modelled on Barbie related to a field trip to a local museum showing the fashion designer Valentino's body of work. The girls and boys found the unit equally interesting and there was no noticeable difference in their engagement. The unit spanned several lessons (about 15 h) and was taught by the first author; the third author was also present some days to gather data for the statistics aspect of the unit and to collect data for the inquiry study. Table 2 is an overview of the phases including the proportional reasoning content involved at each stage (see Table 1 for a general description of these phases).

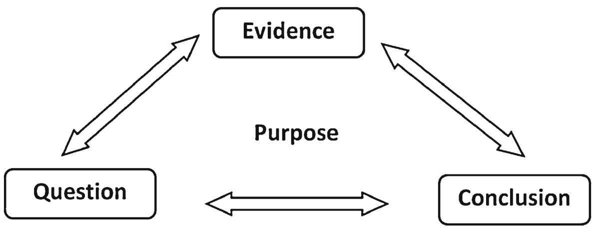

Fig. 1 Representation used in class as a frequent reminder to consider links between question, evidence, conclusion and purpose (Fielding-Wells 2010)

Table 2 Overall description of phases of inquiry unit

Inquiry phase	Description of activities	Proportional reasoning
Discover (2 h)	Engagement of students with context Meaning-making of the initial inquiry question Discussion of artistic and forensic needs for human proportion	Awareness of proportionality of face from art class Raise the issue of comparing Barbie with humans
Devise (2 h)	Initial planning ideas, sharing challenges Renegotiation of inquiry question with mathematical focus (*Does Barbie have human proportions*?) Exploration of how to compare Barbie with humans Consideration of evidence needed and how they could obtain it Writing possible "conclusions" Development of plans to address the question	Problems with additive approaches for comparing (e.g. absolute vs proportional comparisons) Link to facial proportions Transferring familiar fractions to body beyond face Clarifying language and measurement practices (e.g. "between" vs "halfway") Discussion of ratios beyond halves, quarters Direct teaching to envision ratios (e.g. 1:1, 1:2, 1:1.6) using unifix cubes with 10 cubes as one unit
Develop (5 h)	Data collection (as homework) and collation Each student assigned one ratio to analyse from collated data Exploration of ways to statistically analyse the data Analysis of their ratio: whether Barbie was "in the range" of humans Discussion of evidence in conclusion to convince peers	Calculation of ratios from measurement data (rounding to tenths) Review of envisioning ratios with unifix cubes and meaning in context
Defend (6 h)	Feedback from peers; improved forms of evidence Preparation and formal reporting on analysis of their ratio with explicit evidence; clarifying questions and critique by audience Students discussed final answer to inquiry using all ratios	Checking for reasonableness of ratios (especially outliers)
Diverge	Further work to explore the context in The Arts, English and Studies of Society and the Environment (SOSE)	Design of clothing using body proportions; Description genre

Culture of the classroom

Engagement in argumentation is contrary to the teaching and learning experiences of many students and could be considered confronting (Muller Mirza et al. 2009). Thus, a classroom culture must be carefully established in which a learning community is nurtured and supported. The class described in this paper had been together with two teachers for approximately 9 months at the time this unit was implemented. Both teachers were experienced in the use of classroom inquiry as a pedagogical tool[1]; inquiry and epistemic argumentation also matched their philosophical approaches to

[1] Both teachers had been part of the inquiry study for 5 years.

teaching, with both working to establish a learning community. Thus, the classroom was characterised by a culture that worked well with argumentation and inquiry.

Students were expected to repeatedly share their progress, seek input from their peers and re-focus towards increasingly generative solution strategies (Scardamalia and Bereiter 2006). These opportunities added accountability throughout the unit and reassured students that they would not be left on their own when challenges arose. The obstacles they encountered became points that could contribute to class discussion rather than seen as lack of progress in front of peers. At the same time, productive and insightful ideas gave students acceptance and belonging when their ideas contributed to the group. Accountability also discouraged passive learning as the collaborative nature of inquiry added to community values to contribute and try your best.

The classroom culture was created initially by encouraging, modelling and setting expectations of respect for student contributions. Students were expected and encouraged to think for themselves and to challenge others ideas and evidence respectfully in order to establish alternate viewpoints and to counter 'groupthink'. A culture in which all ideas were valued was established and students were encouraged to build on the ideas of others. As students progressed, to a large extent they lost 'ownership' of their own ideas to the larger group. Thus, when they presented ideas which were challenged or modified by the group, there was little or no offence taken. Students were heard to ask permission of their peers to argue with them, including respectfully arguing with the teacher. On one occasion, a student was even heard to say halfway through an explanation, "wait, I need to argue with myself".

Proportional reasoning in the year 4 curriculum

The state curriculum for this year level included concepts in proportional reasoning such as (Queensland Studies Authority 2007, 2008):

- Common fractions (half, quarter)
- Place value (tenths)
- Relationship between common and decimal fractions.

In later years (through year 7), the curriculum aimed to extend students' proportional reasoning to include the following content:

- Ratio expresses multiplicative relationships between like quantities (e.g. one part concentrate juice to four parts water)
- Direct proportion is the equivalence of two ratios (e.g. 1:2 and 2:4 are equivalent).

Typical learning sequences in mathematics build student understanding from simple to complex ideas. This unit took an integrated approach by "layering" content, with concepts taught in later years appearing informally at the same time that concepts in the curriculum for the current year were expected to be applied with more confidence and formality (Coffield 2000). For example, students at this grade level were expected to manipulate and work between halves and quarters with a fair amount of fluency, but only work informally with applied ratios beyond halves and quarters. This aligned with the Queensland curriculum's "Ways of Working", which aimed to more explicitly value the development of students' mathematical practices that cut across particular content. For example, this level aimed for students to be able to: "plan activities and investigations to explore concepts, pathways and strategies and solve mathematical questions, issues and problems".[2] In this way, the inquiry approach used in the unit was supported by the state curriculum.

Data collection and analysis

The focus of data collection was to seek insight and evidence to address the research question: *What is the capacity for inquiry pedagogy to foster proportional reasoning of primary students*? In particular, we were interested in understanding how the context and mathematical practices within the inquiry unit supported students' proportional reasoning. The case study focused on three key sources of data: videotaped lessons, student inquiry journals and (incomplete[3]) research notes.

Eleven lessons were video-taped, including capture of whole class discussions and a roving focus on small groups during collaborative work. Some lessons were not recorded because they either arose unexpectedly or they were not deemed to be potentially rich sources of data, such as students working independently. On most days, only one camera was positioned in the back of the class. However, when a second researcher was present, a second camera was often used to either capture students during whole class discussions or to rove between small groups.

Student artefacts, particularly their inquiry notebooks, were scanned to help reconstruct the unit, examine student work during the unit generally, link with specific video episodes that referred to the students' work and to use particular examples as discussion points during analysis. Lesson notes were used to elaborate on the description of the unit and rationale for particular lessons.

Video logs were created by one or two researchers of all videos. Initially, two videos were randomly selected to be logged by both researchers in order to engage them in discussion about the focus and structure of observations made when creating video logs. The full video was later transcribed once it was determined which lessons (or lesson segments) would be included in the analysis. The transcripts were descriptively annotated for global content analysis in order to initially remain open to unanticipated insights (Corbin and Strauss 2008). Next, pivotal episodes that focused on proportional reasoning were isolated. These episodes were subjected to a finer level of analysis and reflective memos to link key elements from the literature on proportional reasoning (e.g. additive thinking, rational number) and collaborative inquiry (inquiry question, planning, mathematising, disequilibrium, negotiation). A targeted set of high-quality episodes were collaboratively selected by all three authors to describe the overall flow of the lessons with an emphasis on points where proportional reasoning was illustrated, challenged or developed from the inquiry context. These episodes were then used in the initial structuring and writing of the case, which was later refined by editing out redundancies and extraneous material to lessen the volume of episodes and improve the overall focus presented in the case.

[2] http://www.qsa.qld.edu.au/downloads/p_10/qcar_el_maths_wow.pdf

[3] Shortly after this unit, the classroom was severely damaged. As a result, some of the research data (including some artefacts, the research notes, and handwritten lesson plans and/or revisions) were destroyed. The details of the research reported in this paper were reconstructed from video, student artefacts and (incomplete) records of lesson plans.

Results

Excerpts from the four main phases of the inquiry unit (Table 2) are reported here as vignettes that were illustrative of either (a) students' emerging proportional reasoning or (b) salient features of inquiry pedagogy that were seen to contribute to the development of students' proportional reasoning. Each vignette presents teacher talk (JW) and student talk (using pseudonyms) with a subheading to orient the reader to the focus of the selected aspect of the inquiry.

Discover: using proportions to understand the inquiry question

Comparing face proportions

In setting up the focus of the inquiry, the teacher drew students' attention to a previous activity they had been doing in art—drawing human faces.

> JW: Before the lunch break we had a look at facial proportion [in art class]. I explained it to you and you had a look at each other and you realised that that proportion actually holds true for all humans. So what are the proportions we talked about?
>
> Delmar: The nostrils, the eyes, the middle of the mouth, the chin and the crown of the head.
>
> JW: OK. So those were the markers. What were the actual proportions that we talked about? …
>
> Cho: Your nostrils are halfway between your chin and your eyes.
>
> JW: Excellent, so your nostrils, the bottom of your nose, are halfway between your chin and your eyes. [Indicating with hands the equivalence of the distance]. What else do we know?
>
> Oliver: A quarter way from your chin to your nostrils is your lips.
>
> JW: OK. So the bottom of your nostrils to your chin, one quarter of the way down is what part of your mouth?
>
> Oliver: Between your lips.
>
> JW: The part between your lips, where your lips meet. Excellent, so we had a look at those proportions. And then we wondered whether there would be proportions for other parts of your body didn't we? But we weren't really sure. … Look at Leanne [tiny student] and look at Mrs Wells [182 cm tall]—are we likely to have the same proportions?
>
> Sts: [Mostly no, some yes, a few were hesitant and uncertain]
>
> JW: Well, we will leave that there for a minute. [5 Oct; 03:00]

Students at this age had not met with proportion formally in the curriculum beyond simple fractions, so their use of proportion here was based primarily on their experience in art class. By linking the idea of proportion to their explorations in art, it established for students several points of reference. First, they linked the idea of proportion to familiar fractions (half, quarter) that they had worked with in mathematics, art and in everyday life. This provided important access to students with diverse levels of mathematical achievement. Second, the link to activities in art created a visual representation of these proportions that encouraged students to “see” the proportion and make sense of it. Both points of reference (familiar fractions, familiar context) began within students' comfort level but could also be extended when the proportions became more challenging mathematically.

In the discussion above, students predominantly focused on the fact that the eyes are in the middle of the face, relying on a fraction with which they were confident. Although the teacher frequently emphasised that these facial proportions were the same for all humans, she provoked some uncertainty about whether proportions beyond the face could remain the same for people of vastly different sizes. This created a space to consider one of the key differences between additive and proportional thinking: The ability to scale up. In inquiry, the use of disequilibrium is common to build a classroom culture that values the questioning and extension of ideas and to plant seeds that would later help students seek more sophisticated mathematical ideas (Makar and Fielding-Wells (unpublished)).

Connecting to the mathematics in the inquiry question

The inquiry question that was initially developed, “Is Barbie a monster?”, was accompanied by a discussion about body image and the problem with the message sent when dolls are such an unrealistic representation of a female's body. Further discussion and guidance from the teacher directed student focus away from thinking of monsters in terms of deformities (eyes where lips should be, legs and arms in the wrong place, only one eye) that would yield little to investigate. The class shifted the question to, “Is Barbie a human?”, but were unsatisfied with this question as it provoked statements that Barbie was not human because “she is made of plastic”, “she won't scream when I rip an arm or leg off”, and “she doesn't talk”. This was not a question that could be answered through mathematical investigation, nor did it align with the purpose to compare Barbie as a representation of females. Further probing by the teacher was needed to move students toward a more purposeful and measurable question.

JW: Do we need to narrow down this question a bit? …

Dominica: I think we can all see that she doesn't walk, but we are talking about the way she looks.

JW: OK, do we need to refine this a bit [pointing to the question]? I think we can all see that Barbie is not a living, breathing member of the human species. What do you think we mean by this question? …

Shana: You mean if she was a human would she be like a human?

JW: In what way?

Lee: Like her face is the same as that [pointing to facial proportions diagram from the morning art lesson].

JW: So if she was real, would her face meet these proportions? [pointing to same diagram]

Oliver: If the Barbie girl was real life size, [inaudible] … [5 Oct, 04:00]

There were two key issues the teacher wanted students to consider. First, she wanted students to mathematise the question. This required moving them from thinking about qualities of being human (comparing materials or functions to animals or inanimate objects) towards a way of quantifying the comparisons between Barbie and humans. Second, she specifically wanted them to consider Barbie in terms of proportions. Using both Lee's link to the facial proportions in art, and extreme examples (comparing Barbie with animals), she guided them towards these two goals.

JW: So let's think about what I mean by this question. Am I asking you if Barbie is a dog or a cat?

Sts: No

JW: What am I asking?

Connor: Does Barbie have the same proportions as a human?

JW: [writing] Does Barbie have, what was it? Human proportions?

Connor: Yes

JW: Does Barbie have human proportions? Is that a better question?

Sts: Yes

JW: Make more sense?

Sts: Yes

JW: Don't have to worry about whether she is a dog or a cat?

Sts: No [giggling] [5 Oct, 5:30]

Having "human proportions" was still an abstract concept. There were two ideas from an earlier discussion that assisted students in transferring their knowledge in art to this more complex problem. First, students needed to connect their visual interpretations to mathematical proportion. Second, they needed to draw on an understanding of simple proportions (1/2, 1/4) to move into more complex ratios. Because students at this age were more experienced with additive comparisons, these two considerations required significant support.

Shana: Her neck is not normal, it is too long.

JW: Why do you say that?

Shana: It looks too long.

Dominica: No. It looks about normal.

JW: So Dominica you think it's normal?

Oliver: We could test it.

JW: How?

Oliver: … Someone could bring in Barbie dolls and we could get into our groups and we could look at it and we could estimate if she was a human height whether she would be normal.

Dominica: I agree with Oliver, and we could study them and see if the eyes are the same height and double check, and double check the neck as well. [5 Oct, 07:00]

Although students were making progress on moving towards mathematising the problem, operationalising their ideas continued to be a challenge. Extremes are often valuable ways to help students focus on elements that are useful (invariant) and those which are superfluous (Ainley et al. 2001). One student provided an extreme to consider.

Shana: I have an idea. Could we do Bratz[4] dolls? Wouldn't they be different and their heads are enormous!

JW: Class, let me ask you a different question, does a Bratz doll have human proportions?

Sts: No! [chorus]

JW: Why not?

Delmar: Their head is like whooo [indicating almost a full arm span]. It looks weird.

Lee: They have really, really big eyes. Really big eyes.

JW: How do you know they are bigger than a human's eyes?

Lee: [long pause] They are bigger!

JW: How do you know? [Reminding students to consider evidence]

Lee: Owh!!! [slumps frustrated]

Frances: I have a Bratz doll, and their eyes are like as big as their hands!

Shana: And Bratz have really big heads, so big they look weird.

JW: OK, so by looking at them, they are so far out of proportion, you can tell straight away. They look bizarre.

Sts: Yes.

JW: So does Barbie look bizarre?

Sts: No.

JW: So, could Barbie be in proportion?

Sts: Yes. [5 Oct, 12:00]

[4] Bratz dolls are similar to Barbie dolls in size but, “are portrayed as teenagers distinguished by large heads and skinny bodies, almond-shaped eyes adorned with eyeshadow, and lush, glossy lips” (Wikipedia)

Table 3 Nature of students' responses to varied questions

Question	Response sample
Is Barbie a monster?	She would only have one eye
Is Barbie a human?	She doesn't scream She is made of plastic
Does Barbie have human proportions?	Her neck is not normal, it is too long

In the example of Bratz dolls, there is little need to use mathematical evidence to determine whether they appear out of proportion. Visual inspection is sufficient to be convincing. This example, however, helped students to begin to make comparisons (Frances: "… their eyes are like as big as their hands") that elicited foundational proportional examples to build on.

Table 3 illustrates the changing nature of the students' responses above, suggesting that they were responding to the evolution of the question and attempting to provide evidence that corresponded to the question being asked. When the question finally adopted a mathematical focus, the students' responses took on a more mathematical aspect. This suggests that the process of negotiating the question and refining it may have assisted in highlighting the mathematical component to students and could be a valuable process for that reason (Allmond and Makar 2010).

Creating a problematic: getting beyond additive comparisons

For students who were accustomed to thinking additively in making comparisons, constructing a way to operationalise comparisons proportionally was a big leap. Students needed to be aware of two issues in order to negotiate this gap. First, they needed to appreciate shortcomings of applying their additive strategies to this problem. Second, they needed to figure out how to take a loosely related set of familiar concepts related to ratio (fractions, proportion in art) and extend it to a more complex situation. Recognising the terrain that needed to be travelled, both in terms of application context (moving from intra-facial comparisons to comparing Barbie to humans) and more challenging mathematical content (ratios other than familiar fractions), the teacher repeatedly focused on the link between the mathematics and the context to bring both issues to the forefront.

JW: What do we need to do next?

Oliver: Work out the proportions.

Shana: We don't know how to!

JW: … So we need to work out what the human proportions are. … Shana and Dominica have already had a bit of a disagreement because one says the neck is too long and the other says it's not. How can they resolve that?

Connor: By seeing if its neck is longer.

JW: Barbie's neck is 2 cm long. Is that in proportion to a human?

Delmar: No way Hosea!!

JW: I didn't ask if that was equal to a human, I said in proportion.

Sts: [Several no's]

Kody: … I think what we are trying to say is; 'is it normal as long as it was the size of a human?'. Would it be as long as our neck would be?

JW: Yes. So how would I know? I can measure Barbie's neck.

Kody: And then measure our neck then measure our bodies. As our body gets smaller our neck would get smaller.

JW: So you would compare Barbie to humans. What a good idea. So we don't just consider what length Barbie's neck is, we consider what length human necks are. [11 Oct, 06:30]

In this excerpt, Connor's initial words suggested directly comparing the measurements (additively). Comparing the neck *length* allowed the discussion to challenge direct comparisons (issue #1). Shana directly raised the issue of not knowing how to find proportions that were needed (issue #2). Although the two issues were still unresolved, students were sensitised to these issues when they moved forward into planning. The teacher was confident that in trying to plan, the students would run into these issues again and be better positioned to negotiate their resolution. Traditionally, teaching works to address issues before moving on, even before students recognise a problem exists. Inquiry often works differently. By raising these issues (but not resolving them), it allowed students time—now aware of these issues as problematic—to again wrestle with them in relation to the context.

Devise: making sense of proportions in planning

Focusing on the problems to address supported a culture of negotiation of ideas. If students are presented with an image of mathematics as unproblematic, then they would believe that any problems they encountered would simply reflect their lack of proficiency.

Developing a plan

Students moved to collaborative groups to begin to devise plans to answer the question. Group discussions initially focused on peripheral issues such as disagreements over whether Barbie had pimples or if they would need to destroy the doll (e.g. pull off her head) to find her proportions. However, the students quickly recognised that they needed to find a way to compare Barbie with a human.

The progress of one group is reported here with students debating approaches to clarify further the nature of the inquiry question. The students acknowledged that they would somehow need to compare the "proportions" on Barbie with humans, but struggled with how to operationalise that idea. With little experience calculating proportions, they initially relied on their awareness of facial proportions and direct measurement. These ideas were critical building blocks for moving forward and recognising a need for richer mathematical concepts. Although they use the term "proportion" from the inquiry question, their conception of what proportionality meant in practice was likely limited.

Gemma: We still haven't got the answer.

Connor: And we need to measure proportion. …And compare them to Barbie and see if they're close to human.

Gemma: And Mrs Wells taught us about that art thingy? We could measure the head proportions easily. We need to see if the eyes are halfway.

[Students get silly, enacting where their eyes could be on the head if they weren't halfway, and then remembered that the camera was 'watching'.] [11 Oct, 31:00]

Students had come to appreciate the importance of negotiating ideas through previous experiences in inquiry. Their work with previous inquiries had assisted them to focus on the role of evidence in an inquiry to persuade an audience and document details of how they came to their conclusion. As a regular part of the classroom, students knew that they would need to report back to the class on their planning progress and began recording their ideas.

Gemma: And then we need to work out our evidence.

Connor: …and compare with humans.

Gemma: … [Writes] Number 1. Measure human proportions. Number 2. Measure Barbie's proportions … measure head size and work out if the eyes are halfway. And do same with Barbie and see if the proportions are close.

Connor: … [But] there's lots of human proportions!

Gemma: We are looking more at the face; human proportions of the face. [To the teacher as she circulates to their group] What proportions are we measuring? Just the head? Or the whole body?

JW: Who needs to decide that?

Gemma: Our group.

Luna: Because we're pretty much looking at the face.

JW: If you just do that, will it be enough to answer the question?

Gemma/Luna: No.

[Teacher leaves] [11 Oct, 37:00]

In considering their evidence, students tried to apply their previous understandings of facial proportions to comparing Barbie and humans. The realisation that they needed to include other proportions in the body required them to find ways to extend their use of proportions into a new context. Students were enculturated into inquiry norms in which they knew the teacher would not "tell them" what to do, but that they would be supported to explore ways to extend their understandings over time. Once students confirmed that their eyes were halfway down their face for everyone in the group, new ideas started to emerge. Students began measuring random parts of one of the students (without recording), estimating that her thigh was about halfway from her shoulders to the ground. They were not satisfied, however, with their estimate using only one person.

Gemma: So basically your thigh is halfway between … [writes] *and the thigh is middle of your …*

Luna: Yeah, but we need to do more people.

[Measures the shoulder to toe and thigh to toe distances for the four members of the group]

Gemma: 116 and 58. Is it in the middle? … Basically we're all the same.

[Teacher claps to bring the class back together] [11 Oct, 46:00]

Review of plans

The class returned to the large group to share their progress and report on ideas so far as well as problems they were wrestling with. As each group presented, the teacher wrote key issues that groups encountered on the board. One group tried to work out how to compare Barbie and humans using direct measurement rather than proportions, but realised there was a problem with this.

Salome: The second question we had was, 'How do we make Barbies and humans the same size so we could compare the proportions?' We had an idea. Someone could lie on the ground and trace themselves on a big piece of paper. Then we'd put Barbie's proportions.

JW: … So if Barbie's height was 30 cm, would you put 30 cm on where the person is traced around?

Andrea: No [uncertain].

JW: So your problem at the moment is [writes on board]: *How do we make Barbie and human the same height so we can compare*? That's your biggest problem at the moment. Anyone have questions for that group? [11 Oct, 50:00]

All of the groups continued to struggle both with envisaging proportion and working out how to make a comparison; some focused on general measurement, some visual imagery, some absolute difference. One group suggested taking a photograph of Barbie to enlarge on the computer until it was the size of a human (allowing them to apply their previous knowledge of direct measurements). The obstacles that groups encountered were an impetus for discussion and promoted a desire to seek richer mathematical understanding that would resolve these dilemmas.

Although students were using the word "proportion" (as part of the inquiry question), clearly it did not yet hold a firm mathematical meaning for many. It appeared that their understanding was general in relation to relative sizes of parts of the body, but they still could not envision how to put these ideas into practice mathematically. The group who previously were beginning to connect other proportions to the body, tentatively presented their ideas to the group.

Gemma: We had the problem with the Barbie and human comparison and um, our solution to that, our solution to that, was to instead of actually um, measuring the height and stuff, we could, um. We could work out like, we worked out halfway between the crown and the chin is the eyes, and then we also worked out halfway between your shoulders and your toes is actually the middle of your thigh. So we could have measured if Barbie's thigh is in the middle of her body and we could measure if her eyes are in the middle of her head.

JW: OK. … So you're looking at the actual proportions rather than the measurements themselves. Of course you still have to measure to do it. Good, thank you. Keep going.

Gemma: [Pause, with students quietly discussing their response with their group.] We couldn't exactly explain how we were gonna, um, do it like, do the actual thing like, measurements and stuff. … We wrote [in our plan], “Compare Barbie and human proportions and see if they're close. Measure head size and work it out if the eyes are half way from your crown to your chin, and then do the same with the body”.

JW: OK. So, will that tell you everything you need to know to be able to answer that question? So if her eyes are halfway down her head and her thighs are halfway down her body, will that give you enough information to answer the question?

Group: (shake heads) No. [14 Oct, 03:00]

The teacher took the opportunity to build on this group's idea of locating proportions in the human body and to assist the other groups in adopting proportions.

JW: Can I get everyone to stand up. I want you to put your hands on your thighs. Freeze. Look around the classroom. [It is] somewhere between your ankles and your waist. Is that specific enough?

Sts: No. [14 Oct, 35:00]

Students had the idea that if the thigh was halfway from shoulder to toe on a human, it should be halfway on Barbie. However, they needed more precise methods to operationalise this concept. Even if the thigh itself wasn't going to be a useful “point” to measure on the body, the idea of comparing other body parts proportionally was a significant shift in thinking—from direct comparisons through measurement to recognising what proportions might be able to achieve for them. The teacher would come back to this point later in the inquiry.

After all groups had reported on their plans, the teacher summarised the issue associated with the inquiry: “Virtually every group had trouble working out a way to make a direct comparison between a human and Barbie. … Is that right?” Students agreed that this was the case. She referred back to the discussion on Barbie's neck:

JW: If we are going to say Barbie's neck is too long, do you need to be more specific than that?

Sts: Yes

JW: What might be an example of what we could say?

Seth: Barbie's neck is one centimetre shorter than a human's.

JW: Are all humans going to have exactly the same length of neck?

Sts: No

JW: OK, so we're not talking about length, we're talking about proportion.

Cho: Something such as like halfway between … We need to say that between the crown and the chin is the eyes. On Barbie, her eyes are between the crown and the chin.

JW: Everybody's eyes are *between* the crown and the chin.

Delmar: Between, but she didn't tell us where in between. …

Sts: Halfway!

JW: Are we talking about a distance? Did we say 5 cm or 10 cm? … No, I'm looking at body proportion, not body measurements. [14 Oct, 46:00]

The emphasis on proportional reasoning and not absolute measures was a key point of emphasis the teacher was trying to reinforce.

Develop: making knowledge connections with ratios

Over the following 2 weeks, students collected a set of body measurements of a parent (to represent an adult, like Barbie) and were shown how to use their calculator to calculate approximately 30 ratios (rounded to one decimal place) from these measurements; for example, height/length of foot. The teacher did not focus on the calculations any more than necessary as her purpose was rather on developing informal understanding and appreciation of the concept of proportion. To this end, the teacher conducted a directed lesson to briefly introduce students to relevant aspects of proportion. In this lesson, the teacher used unifix cubes to model multiple ratios, representing the ratios in the *1*:*x* or *x*:*1* format, and linking these to human body proportions. As an example, the teacher modelled her own elbow/fingertip to elbow/wrist ratio [48:28 cm] as 1:1.7, using 10 and 17 unifix cubes (plastic connectable cubes, approximately 2 cm in dimension) respectively, to show one whole compared to one and seven tenths [or 10 tenths and 17 tenths]. Students having difficulty with this activity were assisted to model the representations with unifix cubes for themselves.

A further week break from the inquiry was required due to the teacher's attendance at a school camp. Upon her return, the teacher refocused the inquiry and then selected some of the data the students had collected, namely the ratio of length of face/length of hand.

JW: Alright, so we worked out some proportions between some body parts on an adult. We each chose an adult and we did that. So that brings us to where we are now. … Let's look at your first measurement. You've got length of face and length of hand. What did you have Frances? When you divided length of hand into length of face, you got?

Frances: I got one point zero.[5]

JW: One point zero? So, one?

Frances: Yes.

JW: So what does that tell you about the length of that person's face and the length of that person's hand?

Frances: They are the same.

JW: They are the same, because it is a one to one ratio. The one tells you they are the same. What would two tell you?

Gemma: It says that there are two parts to one.

JW: OK. So if I said to you, “My face was 40 cm and my hand was 20 cm”, that would give me a ratio of two to one. I would either have a very long face, or a very short hand. So, if you get a two, it tells you that the thing that you measured first is twice as long as the thing that you measured second. Right? So that is just to help you understand the thing that is in front of you. [2 Nov, 12:00]

Students noted that they did not all get the same ratio (1.0) and determined they would need to find a range of values to describe human proportion of face/hand. To determine this range, the teacher asked if anyone had values higher or lower than 1.0. The highest value was 1.2. For the lowest, Darell and Oliver both had 0.9. When Lee said he got 0.1, there was immediate reaction from the majority of the class:

Sts: Whoa!!!!

Konrad: How can you get that?

JW: … What were your measures Lee?

Oliver: [quietly to Zachary] Put your hand in front of your face. That's like impossible!

Zachary: [Back to Oliver] But then your face would be about this big (holds finger and thumb approximately 2 centimetres apart)

JW: Just out of curiosity, when Lee said he got 0.1, you said ‘How could you get that?’ … Why did you think that Konrad?

Konrad: Because it is like a really, really, really low number! [2 Nov, 17:00]

The teacher asked if this ratio could be possible, to which some students stated that it could be depending upon the doll or person they were measuring as they didn't have enough data to make a firm conclusion.

Kody: But some parents are tall and some are smaller than others.

[5] Students often ignored the 1 in the ratio 1:x and just read the ratio as “x”.

JW: So might that make a difference if we've got short people and tall people? Might that make a difference to their proportion?

Sts: (Mostly yes, one insistent maybe)

Kody: Your hand might be bigger but not your face.

JW: So your hand would be bigger but your face wouldn't, if you were taller.

Kody: No, your face would be as well.

JW: So if your hand is bigger and your face is bigger, does the proportion change?

Connor: No, because they're both near the same.

JW: … How can we work out what might be reasonable?

Kody: up to 2.

JW: 1 to 2 ratio, your hand could be two times bigger than your face.

Sts: [no reaction]

JW: I will help you visualise this because I think this is something you need to be able to see. You haven't done enough decimal work yet to 'see' this. [2 Nov, 27:30]

The teacher, being unsure if students understood the ratios they were suggesting, particularly the decimal values, spent further time modelling some of the students' proportions using unifix cubes. The teacher first placed her own hand over her face and students agreed that the length of these two measures were similar. She then modelled this with two stacks of 10 unifix cubes and compared them. Students agreed that the stacks were the same, like her face and hand. For the ratio 1.2, she created one stack of 12 unifix cubes and compared this to a second stack of 10 cubes. The students related this to their own hand and face and agreed that these still seemed reasonable. Then to model the ratio of 2.0, she created one stack of 20 cubes and a second stack with 10 cubes. Most students quickly realised that this ratio was an unreasonable proportion. To exaggerate, the teacher asked what would happen if they got a ratio of 10.0? This use of extreme values appeared to be quite effective in terms of eliciting student discussion.

Sts: Whoa!!!

JW: How might I know that there was a mistake in your measurement? What might I think if I went ahead and said anywhere in this range is OK for Barbie's face.

Connor: Because your head would be humongous or your hand would be teeny![2 Nov, 28:30]

The discussion continued with the teacher asking students to provide the ratios that they calculated. The ratios ranged between 0.9 and 1.2 and students saw that the ratio of 0.1 was likely an error. The focus of the rest of the unit was in looking at the distributions of ratios to determine a "reasonable range" based on statistical properties in order to see if each of Barbie's ratios were "in the range" or "out of the range". Students then made a call as to when Barbie's ratio would be reasonable (see Fielding-Wells 2013).

Defend: deliberating proportional evidence in context

Each student put together a presentation to the class to explain and provide evidence for their conclusion. These presentations, while lengthy, offered insight into individual student's understandings and enabled their evidence to be challenged as necessary using the context as a reference. Two excerpts are provided below; in the first, Sadie is using proportional reasoning to justify her decision to exclude an outlying ratio.

> Connor: What did the spread of data look like?
>
> JW: [to Sadie] … show them the dot plot so they can see the spread of data. So what was your actual range of all the data you collected?
>
> Sadie: 0.1 to 8.2.
>
> JW: OK. You said you had a score of 8.2, so why did you not consider 8.2 in your range? You are saying normal is 0.9 to 1.0 so why do you think that 8 is not normal?
>
> Sadie: Because no one can have their arm span eight times their height. [17 Nov, 4:30]

In this second excerpt, Geneva recognised that her data was unlikely, but was willing to use the context to argue that it is possible, even if unlikely. It was apparent early on that she had visualised the proportion accurately: "they could be really tall but not have very long legs". However, she was unwilling to state that this was not possible and later suggested they may have an injury to their legs.

> JW: Ok, Geneva, one more question for you, …you said you did navel to foot, to height. If a person had a ratio of 1:3, what might they look like? Could you visualise that person?
>
> Geneva: Maybe they, um, maybe they were like, they could be like really tall, but not have very long legs, or something like that?
>
> JW: Ok. So let me [T stands up]. … you could use me as an example. Navel to foot [indicates on herself the distance from her navel to her foot], that would be 1, and my height would be 3 times that [indicates the distance that would be 3 times the navel to foot]. If I had a ratio of 3, navel to foot, my height would be three times that. So here's my navel. Here's my foot. [Draws a stick figure on the board]. And my overall height would be one, two, three [draws].
>
> Geneva: They'd have to have very short legs, though.
>
> Lee: Holy moley!
>
> Oliver: Whoa!
>
> Delmar: That's impossible.

Konrad: That doesn't exactly seem right.

JW: Ok, Geneva's pointed out that it could be a person with very short legs. … What would you do now? If you are thinking that this isn't possible or that you are not certain?

Geneva: Check out the person measured at 3.0 and make sure they measured it correctly.

JW: Anything else?

Geneva: Make sure there wasn't something with the person; they didn't have an injury or something. [22 Nov, 3:00]

These excerpts suggest that students' visualisations of the ratios were developing to a point where they could use the data from their distributions in conjunction with their visualising of proportion to estimate a suggested human range for each proportion.

At the completion of their presentations, the unit culminated with the students tallying the total number of proportions with which they judged Barbie to be human-like (14), and the number of proportions for which she did not display human-like attributes (12). The majority of the class concluded that Barbie did not have human proportions as the slight majority of her proportions 14:12 were outside the human range. However, one student argued that the total number was irrelevant "say you have 30 proportions and 10 of them aren't human and 20 are human proportions then she's not a human".

Discussion

Students' difficulties with proportion are well-documented. As outlined above, proportional reasoning emerges from the study of rational number, but calls have been made for more explicit focus on *recognising proportional situations* (e.g. Behr et al. 1992; Confrey and Smith 1995; Streefland 1985; Van Dooren et al. 2005). In the case study reported here, the oscillating and tenuous nature of students' proportional reasoning is evidenced as they grappled with ways to make comparisons between Barbie and humans.

The inquiry approach used in this case study embraced the four aspects put forth by Cobb and McClain (2006) as critical for effective, inquiry-based learning environments. First, the task invited an investigative spirit that mirrored the intellectual rigour (at an appropriate level) of authentic practice. Second, the classroom organisation balanced and capitalised on both large group discussion and small group work. Iterations between these two structures added variety, infused accountability and helped to maintain the momentum throughout the long solution process over several weeks. The teacher's ability to skillfully question students and gently refocus the direction of the work in both whole class and small groups invited ownership and redirected attention to what the students wanted to find out. Third, tools and representations focused students on key mathematical ideas, drawing on both concrete and abstract representations of ratio to visualise the reasonableness of their work. The heavy reliance on the context, human proportion, acted as an important tool for challenging and generating students' ideas for scaling up. Finally, the classroom discourse was purpose-built for encouraging productive practices and norms. Whole class discussions provided both accountability and support through the teacher's guidance. Her agenda to promote proportional reasoning allowed her to privilege ideas that would channel discussion towards that outcome.

In the beginning, we asked the question "*What is the capacity for inquiry pedagogy to foster proportional reasoning of primary students*?" We analyse three insights into this question with regards to: The inquiry approach itself, guided by the inquiry model (Table 1) and a highly experienced teacher; the proportionality embedded in the context of the inquiry task that drove a need for understanding ratio and generated "teachable moments" (Table 2); and the beginning of a focus on argumentation (built on in later units; Fielding-Wells (unpublished)) to collectively seek and constructively critique the solution pathways.

Inquiry model

The inquiry model provided a design structure that valued student engagement and purpose as foundational to learning (Table 1). The "Discover" phase provided a hook for students to make the problem both accessible and challenging. The boldness of the initial question, "Is Barbie a Monster?" engaged students both intellectually and affectively. While they knew the question was a bit silly, it gave them permission to rely on personal knowledge and experiences in a way that generated debate (Engle et al. 2012). Without engagement, learning opportunities are limited (Krajcek and Blumenfeld 2006), particularly for students who have been less successful in traditional mathematics. This is an important point in at a time when the role of mathematics as a gatekeeper needs to be more aggressively scrutinised (Confrey 2010; Walshaw 2007).

The Devise and Develop phases of the inquiry built on the initial momentum to deepen the mathematics in the unit more explicitly. Planning was not just to give students ownership; in order to plan, students were expected to hypothesise potential conclusions and envision evidence needed to generate and justify their findings (Fig. 1; Fielding-Wells 2010, 2013). By looking ahead, they were confronted with several challenges. First, they needed to make meaning of and mathematise the question to give the investigation an evidence-based focus. Second, attempting to envision the conclusion and evidence helped students seek ways to operationalise a solution. The gap between this goal and their ability to fulfil it created a need for the mathematical tools that would eventually support them towards their goal. The defend phase, on the horizon throughout the unit, reminded students that they would need to do the intellectual work and make the effort to have a solution to present to peers at the end (Andriessen 2006). This presentation added a level of cognitive engagement to ensure their solution was not only visible but also coherent (Berland and Reiser 2009; Osborne 2010; Ritchhart and Perkins 2005). Reflection and opportunities for transfer (enacted or hypothetical) in the unit added depth to the learning experience (Engle et al. 2012).

Classroom culture of inquiry and argumentation

The classroom culture was one of both inquiry and emerging argumentation. The focus on argumentation supported the development of the 'best' possible conclusion to a negotiated question. This required the classroom to work as a community to put forward reasoned claims which best fit the evidence at hand. When the question was being refined, the students offered intuitive suggestions, such as “she is made of plastic”. However, this was their response when they were working with the question ‘Is Barbie a human?’, a question which reasonably relies on observational evidence (which was the only evidence the students had available). Similar to the findings of von Aufschnaiter et al. (2008) and Zembal-Saul et al. (2013), their responses were intuitively evidenced by what the students had available to them. To improve their responses, it was necessary to refine the question to something that was mathematically researchable; as this was done, the students' responses followed the nature of the questions (Table 3). This recognition of the need for evidence, in conjunction with the nature of the context, enabled the teacher to increasingly shift the focus towards the mathematics required, and establish a need for mathematics they did not yet have, but for which they could now envisage a need.

The ‘public’ nature of argumentation stimulated a need for evidence that may not have otherwise been apparent. When Lee provided a ratio that was not possible, he was quickly challenged by students, urging the class to re-examine his evidence and engage more deeply with the content and context. It is this level of challenge that Berland and Reiser (2009) describe as persuasion, when those engaged in the argument attempt to convince others of the mathematical acceptability of their evidence and reasoning. At this level of persuasion, students are positioned to defend the quality of their evidence. As such, they need to anticipate potential challenges and prepare to address or circumvent them before they arise. Over a period of time, students came to the realisation that explanations were expected with answers and that challenges were a normal classroom activity. This characteristic is illustrated in the final excerpt when the teacher is challenging Geneva to help her to see the unlikelihood of her response. Geneva's responses were valid and reasoned; she stands her ground well and justifies her thinking even though the teacher and other students are trying to have her see the unlikelihood of her responses. This culture is necessary for argumentation to be effective: students need to develop the risk-taking necessary to go against the crowd, the creativity to consider alternate views (as we saw with Geneva), the articulation and self-confidence to state their point, and yet be secure enough to openly have their claims, evidence and/or reasoning challenged. These qualities of argumentation need to be developed in an environment where ownership of ideas are relinquished so students focus externally on collective understanding rather than inwardly on their own beliefs.

Proportionality of the task

In order to promote the teachers' instructional agenda to develop students' proportional reasoning, the inquiry problem being addressed required a proportional context and supported students in learning to recognise proportional situations (Behr et al. 1992; Van Dooren et al. 2005). Although they did not initially recognise the situation as a proportional, the comparison of Barbie's body to humans required scaling up Barbie's measurements, explicitly challenging their initial additive ideas (Hart 1981). The excerpts above illustrate how students grappled with proportional comparison as their comments slowly shifted to include increasingly proportional ideas. As expected, they continued to oscillate between additive and proportional reasoning (Van Dooren et al.

2005). For example, when the teacher asked if Barbie's 2 cm neck was proportional to her own, Delmar stated “No way!” (additive response), while Kody noted “As our body gets smaller our neck would get smaller” (proportional response). When Lee stated his ratio was 0.1, there was clear recognition by many students that this would be an impossible ratio.

Students had several mathematical concepts to work from within the curriculum, including simple fractions (halves, quarters), place value to tenths and relationships between fractional and decimal forms of these simple fractions (QSA 2008). The state curriculum at the time did not ask students to work with proportions in applied situations for two or more years allowing for these ideas to be developed informally without demands of fluency. The design of the unit (Table 2) provided students with opportunities to explore ways to develop more complex proportional comparisons between Barbie and humans, despite their lack of formal knowledge to do so. These informal experiences would likely support them in later years to apply their proportional reasoning more formally (Coffield 2000).

Although they had worked with simple proportions of the face in art lessons, the transfer to more complex ratios on the body was not trivial, requiring a move from simple fractions (halves, quarters) to more complex ratios (decimals to tenths) and a move from proportions on the face to other proportions within the body. One group's endeavour to test whether the thigh was halfway between their shoulders and the floor illustrated an attempt to move from simple facial proportions to body proportions. In doing so, students were challenged in the whole class discussion to seek more quantitative methods and language. This discussion provided guidance to students that furthered the teachers' agenda. First, it privileged ideas that the teacher knew would be productive in deepening their proportional reasoning and encouraged to students to build in this direction. This assisted groups who had not yet located productive avenues to proportionally compare Barbie (for example, the group who suggested tracing their outline onto large paper and then “compare it” somehow to Barbie). Second, the focus on locating specific points in the body rather than large regions such as a thigh, would be important if they were to identify and work with an accurate range of ratios. More quantitative language was developed to further support students by shifting them from using qualitative descriptions (“between”, “in the middle”) to more proportional language (“halfway”).

The need for mathematics is an important element to operationalise learning (Harel and Koichu 2010). The challenges that students encountered created teachable moments for direct instruction in ratios. Early on, Shana expressed concern that they did not know how to work with proportions. The teacher did not at this point intervene because she recognised that few students had yet come to this realisation. A few days after the above discussion about the thigh, the need to work with more precise ratios was apparent to students more broadly. The teaching of ratio comparisons such as 1:1, 1:2 and 1:1.6 within a mathematics lesson at this point had meaning for students. For one, they could see the utility (Ainley et al. 2006) of these ratios that would have been less clear had they only been taught the skill of finding these ratios abstractly. Second, the connection between these ratios and body proportions was built into the lesson, giving further guidance and building confidence in students to transfer their work with decimal ratios and facial fractions to more complex body proportions. When students realised they needed adult measurements and were directed to collect data from home, their proportional reasoning continued in their checking for measurement error in the outliers by considering the reasonableness of the proportions within the context. Exploration of extreme values emphasised the need to interpret the ratio in context, making the absurdity of some proportions more clear: as summed up by Connor, “because your head would be humongous or your hand would be teeny”.

The results of the study described here provide insights into the opportunities of an inquiry-based environment for developing students' emerging proportional reasoning. The design of the inquiry, based on a model developed by members of the research team (Allmond et al. 2010; Makar and Fielding-Wells 2011; Makar 2012) provided important ingredients supported by research on learning (Table 1). The classroom culture and focus on evidence encouraged students to move beyond surface explanations and seek more powerful mathematical tools. Finally, the context of the task generated a need for proportionality and allowed students to draw on their background knowledge and understanding of the broader context to support their learning. Contrary to “open inquiry”, which suggests a lack of structure, this study used the structures of an inquiry model, argumentation research and design of a relevant task to illustrate ways that an inquiry approach can be used to support students' learning in mathematics.

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