Set-Valued Anal (2010) 18:483–511
DOI 10.1007/s11228-010-0150-z

Inradius and Circumradius of Various Convex Cones Arising in Applications

René Henrion · Alberto Seeger

Received: 2 August 2010 / Accepted: 3 August 2010 /
Published online: 25 August 2010

Abstract This note addresses the issue of computing the inradius and the circumradius of a convex cone in a Euclidean space. It deals also with the related problem of finding the incenter and the circumcenter of the cone. We work out various examples of convex cones arising in applications.

Keywords Convex cone · Incenter · Circumcenter · Inradius · Circumradius · Ball-generated cone · Fitted cone · Cone of matrices

Mathematics Subject Classifications (2010) 46B10 · 46B20 · 52A41

1 Introduction

The recent paper [20] studies four notions of center for a closed convex cone in a reflexive Banach space: the incenter, the circumcenter, the inner center, and the outer center. These four notions are different in general, but they reduce to two if the space is Hilbert. The main emphasis of [20] is on existence, unicity, and stability properties for each type of center. Such theoretical questions have deep connections with geometric aspects of infinite dimensional Banach spaces.

Dedicated to Jean-Baptiste Hiriart-Urruty on the occasion of his 60th birthday.

R. Henrion
Weierstrass Institute for Applied Analysis and Stochastics,
Mohrenstrasse 39, 10117 Berlin, Germany
e-mail: henrion@wias-berlin.de

A. Seeger (✉)
Department of Mathematics, University of Avignon,
33 rue Louis Pasteur, 84000 Avignon, France
e-mail: alberto.seeger@univ-avignon.fr

The context of the present paper is more down-to-earth: the underlying space, say $(X, \|\cdot\|)$, is assumed to be Euclidean, i.e., Hilbert and finite dimensional:

$$2 \leq \dim X < \infty.$$

Our chief aim is computing the incenter and the circumcenter of various convex cones arising in concrete applications. The notation that we use is for the most part standard: S_X and B_X are, respectively, the unit sphere and the closed unit ball of X; the symbols ∂K and $\mathrm{int}(K)$ stand, respectively, for the boundary and the interior of K, etc. We also introduce the hyperspace $\Xi(X)$ of nontrivial closed convex cones in X, and the subsets

$$\Xi_{\mathrm{sol}}(X) = \{K \in \Xi(X) : K \text{ is solid}\}$$

$$\Xi_{\mathrm{ptd}}(X) = \{K \in \Xi(X) : K \text{ is pointed}\}.$$

That a convex cone is nontrivial simply means that it is different from the singleton $\{0\}$ and different from the whole space X. A convex cone is *solid* if its topological interior is nonempty, and it is *pointed* if it contains no line.

Definition 1.1 Let $K \in \Xi_{\mathrm{sol}}(X)$. The incenter of K, denoted by $\pi_{\mathrm{inc}}(K)$, is the unique solution to the variational problem

$$\rho(K) = \max_{x \in K \cap S_X} \mathrm{dist}[x, \partial K]\,. \tag{1}$$

The coefficient $\rho(K)$ is called the inradius of K.

The inradius is well defined even if the cone is not solid, but the solidity assumption is needed for guaranteeing uniqueness of solutions to (1). Solidity has further consequences: the incenter lies in the interior of the cone and the inradius is a positive number.

Let us open a parenthesis and give a quick look at the structure of (1) when K is a polyhedral cone in the usual Euclidean space $\mathbb{R}^n$.

Example 1.2 Suppose that K is a polyhedral cone given in its canonical form

$$K = \left\{x \in \mathbb{R}^n : f_1^T x \geq 0, \ldots, f_m^T x \geq 0\right\},$$

where $\{f_1, \ldots, f_m\}$ is a finite collection of unit vectors in $\mathbb{R}^n$ and the superscript "T" indicates transposition. Since

$$\mathrm{dist}[x, \partial K] = \min_{1 \leq i \leq m} f_i^T x$$

for all $x \in K$, we are led to solve

$$\begin{aligned} &\text{maximize} \quad \min_{1 \leq i \leq m} f_i^T x \\ &\|x\| = 1 \\ &f_i^T x \geq 0 \ \ \forall i \in \{1, \ldots, m\}. \end{aligned} \tag{2}$$

If K is solid, then the maximization problem (2) admits a unique solution. Furthermore, the inequality constraints in (2) are inactive at the solution.

We close the parenthesis on polyhedral cones and come back to the general setting. The geometric meaning of the incenter and the inradius has been discussed in detail in [20] and also in references [5, 12, 14, 15, 24, 27]. We recall that (1) is equivalent to the problem of finding the radius and center of the largest ball contained in K:

$$\begin{aligned} &\text{maximize } r\\ &\|x\| = 1\\ &r \in [0, 1]\\ &x + rB_X \subset K. \end{aligned} \tag{3}$$

By an obvious reason, one asks the center of the ball to be a unit vector. The coefficient $\rho(K)$ is the radius of such largest ball and $\pi_{\mathrm{inc}}(K)$ is its center (Fig. 1).

The concept of circumcenter is somewhat dual to that of incenter. One considers instead the problem of finding the radius of the smallest ball whose generated cone contains K:

$$\begin{aligned} &\text{minimize } s\\ &\|w\| = 1\\ &s \in [0, 1]\\ &K \subset M(w, s). \end{aligned} \tag{4}$$

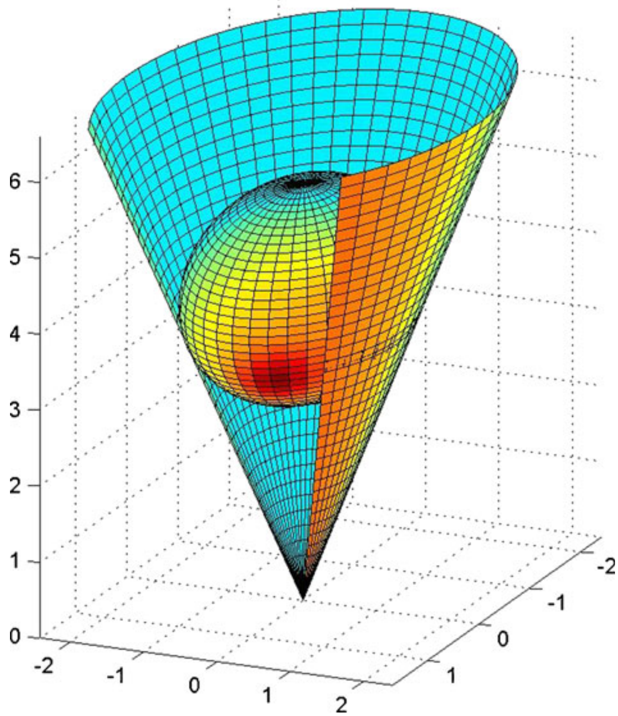


Fig. 1 The center of the largest ball is the incenter of the cone. For easy of visualization, the apex of the cone has been taken away from the origin

The notation $M(w, s)$ stands for the closed convex cone generated by the ball $w + sB_X$, that is,

$$M(w, s) = \mathrm{cl}\left[\mathbb{R}_+(w + sB_X)\right]. \tag{5}$$

The closure operation in (5) is superfluous for $s \in [0, 1[$.

Definition 1.3 Let $K \in \Xi_{\mathrm{ptd}}(X)$. The circumradius of K, denoted by $\mu(K)$, is the radius of the smallest ball whose center is a unit vector and whose generated cone contains K. The circumcenter of K, denoted by $\pi_{\mathrm{circ}}(K)$, is the center of such smallest ball.

The number $\mu(K)$ is thus the optimal value of the minimization problem (4). Such number is well defined even if the cone is not pointed, but pointedness is essential for guaranteeing the uniqueness of the circumcenter. It is clear that

$$0 \le \rho(K) \le \mu(K) \le 1$$

for all $K \in \Xi(X)$. Inspired by the definition of the condition number of a nonsingular matrix, we refer to

$$\mathrm{cond}(K) = \frac{\mu(K)}{\rho(K)}$$

as the *condition number* of a solid cone K. It can be proven in a formal way that $\mathrm{cond}(K) \approx 1$ if and only if K is "near" a ball-generated cone.

The next theorem, borrowed from [20], displays a very interesting connection between the optimal balls

$$\mathbb{B}_{\mathrm{circ}}(K) = \pi_{\mathrm{circ}}(K) + \mu(K)\, B_X \tag{6}$$

$$\mathbb{B}_{\mathrm{inc}}(K) = \pi_{\mathrm{inc}}(K) + \rho(K)\, B_X\,. \tag{7}$$

Recall that the norm $\|\cdot\|$ derives from an inner product. This is a crucial assumption indeed.

Theorem 1.4 *For all $K \in \Xi(X)$, one has*

$$\mu(K) = \sqrt{1 - [\rho\,(K^+)]^2} \quad \textit{and} \quad \rho(K) = \sqrt{1 - [\mu\,(K^+)]^2}$$

with K^+ standing for the (positive) dual cone of K. Furthermore,

$$\pi_{\mathrm{inc}}(K) = \pi_{\mathrm{circ}}(K^+) \quad \textit{if } K \textit{ is solid}$$

$$\pi_{\mathrm{circ}}(K) = \pi_{\mathrm{inc}}(K^+) \quad \textit{if } K \textit{ is pointed.}$$

It is worth mentioning that the circumcenter of a pointed cone lies in the cone itself, but not necessarily in its interior (even if the cone is solid as well). Hence, the concept of circumcenter is different from that of incenter. This remark will be illustrated with the help of Example 3.7. Anyhow, a direct consequence of Theorem 1.4 yields:

Corollary 1.5 *Let $K \in \Xi(X)$. Then*

$$\pi_{\text{inc}}(K) \in S_X \cap K^+ \cap \text{int}(K) \quad \textit{if } K \textit{ is solid}$$
$$\pi_{\text{circ}}(K) \in S_X \cap K \cap \text{int}(K^+) \quad \textit{if } K \textit{ is pointed.}$$

Inradiuses and circumradiuses are left invariant by orthogonal transformations. Incenters and circumcenters are transformed as described in the next proposition. The proof is omitted because it is easy.

Proposition 1.6 *If $U : X \to X$ is an orthogonal linear transformation, then*

$$\rho(U(K)) = \rho(K) \quad \textit{and} \quad \mu(U(K)) = \mu(K)$$

for all $K \in \Xi(X)$. Furthermore,

$$\pi_{\text{inc}}(U(K)) = U\pi_{\text{inc}}(K) \quad \textit{if } K \textit{ is solid}$$
$$\pi_{\text{circ}}(U(K)) = U\pi_{\text{circ}}(K) \quad \textit{if } K \textit{ is pointed.}$$

This is all what we need to know at a theoretical level. Once we have understood the geometric meaning of the balls (6–7) and their duality relations, we can proceed to practical computations.

2 Cones in $\mathbb{R}^n$

A wide battery of examples is always useful to build up a theory. The most intuitive results on incenters and circumcenters are obtained by restricting the attention to convex cones in the usual Euclidean space $\mathbb{R}^n$ with $n \geq 2$. We shall work out first some easy examples and then gradually increase the degree of difficulty.

2.1 Orthogonal Cones

The incenter of the Pareto cone (or nonnegative orthant) is a unit vector whose components are all equal, namely,

$$\widehat{\mathbf{1}}_n = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)^T.$$

A straightforward generalization of this principle to the case of an arbitrary orthogonal cone reads as follows:

Proposition 2.1 *Suppose that K is an orthogonal cone in $\mathbb{R}^n$, that is, K is generated by an orthonormal basis $\{u_1, \ldots, u_n\}$ of $\mathbb{R}^n$. Then*

$$\pi_{\text{inc}}(K) = \pi_{\text{circ}}(K) = \frac{u_1 + \ldots + u_n}{\sqrt{n}}.$$

Furthermore, $\rho(K) = \sqrt{1/n}$ and $\mu(K) = \sqrt{1 - (1/n)}$.

Proof Since K is the image of $\mathbb{R}^n_+$ under the orthogonal matrix $U = [u_1, \dots, u_n]$, one gets

$$\pi_{\mathrm{inc}}(K) = U\pi_{\mathrm{inc}}(\mathbb{R}^n_+) = U\widehat{\mathbf{1}}_n.$$

One also gets $\rho(K) = \rho(\mathbb{R}^n_+) = \sqrt{1/n}$. The remaining part of the proposition follows from Theorem 1.4 and the fact that K is self-dual. □

Recall that a *simplicial cone* in $\mathbb{R}^n$ is a polyhedral cone generated by a basis of $\mathbb{R}^n$. As shown in [3, Proposition 3.2], a simplicial cone is orthogonal if and only if it is self-dual. The next corollary is then an immediate consequence of Proposition 2.1.

Corollary 2.2 *Suppose that K is a self-dual simplicial cone in $\mathbb{R}^n$. Then $\rho(K) = \sqrt{1/n}$ and $\mu(K) = \sqrt{1-(1/n)}$.*

A word of caution is however in order: the class of self-dual polyhedral cones is wider than the class of orthogonal cones. That K is the image of a nonnegative orthant under an orthogonal transformation is crucial in the proof of Proposition 2.1.

Example 2.3 It is possible to construct a self-dual polyhedral cone in $\mathbb{R}^3$ whose inradius is different from $\sqrt{1/3}$. To see this, consider the polyhedral cone K generated by the vectors

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}.$$

This cone is self-dual (cf. [2]), but not simplicial. The representation of K as intersection of hyperplanes is

$$\begin{cases} x_1 + x_2 + x_3 \geq 0 \\ \qquad x_2 + x_3 \geq 0 \\ -x_1 + x_3 \geq 0 \\ -x_2 + x_3 \geq 0 \\ x_1 - x_2 + x_3 \geq 0. \end{cases}$$

Hence,

$$\mathrm{dist}[x, \partial K] = \min\left\{\frac{x_1+x_2+x_3}{\sqrt{3}}, \frac{x_2+x_3}{\sqrt{2}}, \frac{-x_1+x_3}{\sqrt{2}}, \frac{-x_2+x_3}{\sqrt{2}}, \frac{x_1-x_2+x_3}{\sqrt{3}}\right\}$$

for all $x \in K$. Since $u = (1/10, 0, \sqrt{99}/10)^T$ is a unit vector in K, it follows that

$$\rho(K) \geq \mathrm{dist}[u, \partial K] = \frac{1}{\sqrt{3}}\left(\frac{1+\sqrt{99}}{10}\right) > \sqrt{1/3}.$$

2.2 Revolution Cones and Elliptic Cones

Another instance where the incenter can be easily computed is that of a revolution cone. By this expression we mean a set of the form

$$\Gamma(y, \theta) = \left\{x \in \mathbb{R}^n : y^T x \geq \|x\| \cos\theta\right\},$$

where y is a unit vector that determines the revolution axis and θ is a parameter called the half-aperture angle. Revolution cones are used in various fields of applied mathematics, including mathematical programming [9] and coding theory [39]; see also [4, 17, 36].

Proposition 2.4 *Let y be a unit vector in $\mathbb{R}^n$ and $\theta \in]0, \pi/2[$. Then*

$$\pi_{\text{inc}}(\Gamma(y,\theta)) = \pi_{\text{circ}}(\Gamma(y,\theta)) = y.$$

Furthermore, $\rho(\Gamma(y,\theta)) = \mu(\Gamma(y,\theta)) = \sin\theta$.

Proof In a Euclidean space setting, the class of revolution cones coincides with the class of ball-generated cones. To be more precise, one has

$$\Gamma(y,\theta) = M(y, \sin\theta). \tag{8}$$

The above formula can be found in Goffin [17] and in other places. For a ball-generated cone, it is clear that the optimal balls (6–7) coincide. In fact,

$$\mathbb{B}_{\text{circ}}(M(y,s)) = \mathbb{B}_{\text{inc}}(M(y,s)) = y + sB_{\mathbb{R}^n} \tag{9}$$

for all $s \in]0, 1[$. The combination of (8) and (9) completes the proof. □

Example 2.5 The n - dimensional Lorentz cone

$$\Lambda_n = \left\{x \in \mathbb{R}^n : \left[x_1^2 + \cdots + x_{n-1}^2\right]^{1/2} \leq x_n\right\}$$

is a particular instance of a revolution cone. One gets

$$\pi_{\text{inc}}(\Lambda_n) = \pi_{\text{circ}}(\Lambda_n) = e_n := (0, \ldots, 0, 1)^T$$

and $\rho(\Lambda_n) = \mu(\Lambda_n) = \sqrt{2}/2$.

The analysis of elliptic cones is a bit more involved. A standard elliptic cone in $\mathbb{R}^n$ is a set of the form

$$\mathcal{E}(A) = \left\{(z,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : \sqrt{z^T A z} \leq t\right\} \tag{10}$$

with A standing for a symmetric positive definite matrix of order $n-1$. Three dimensional elliptic cones have applications in mechanics [13, 47], electromagnetic scattering [46], and many other areas. General background on higher dimensional elliptic cones can be found in [24–26, 28]. Stern and Wolkowicz [42, 43] work with a wider class of elliptic cones, namely, those that are represented as the image of the n - dimensional Lorentz cone under a nonsingular linear transformation. Note that (10) can be rewritten as

$$E(A) = M(\Lambda_n) \tag{11}$$

with M being the nonsingular linear transformation given by

$$M = \begin{bmatrix} A^{-1/2} & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}^T & 1 \end{bmatrix}.$$

Proposition 2.6 *Let A be a symmetric positive definite matrix of order $n-1$. Then*

$$\pi_{\text{inc}}(\mathcal{E}(A)) = \pi_{\text{circ}}(\mathcal{E}(A)) = e_n.$$

Furthermore,

$$\rho(\mathcal{E}(A)) = \sqrt{\frac{1}{1+\lambda_{\max}(A)}} \quad \textit{and} \quad \mu(\mathcal{E}(A)) = \sqrt{\frac{1}{1+\lambda_{\min}(A)}}$$

with $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denoting, respectively, the smallest and largest eigenvalue of A.

Proof We do not rely on the transformation mechanism (11) because M is not orthogonal. The formula for the inradius of $\mathcal{E}(A)$ has been established by Iusem and Seeger [24, Proposition 6.4]. Such formula and the expression for the incenter of $\mathcal{E}(A)$ are obtained by solving explicitly the minimization problem (3). The remaining part of the proposition follows from Theorem 1.4 and the fact that

$$\left[\mathcal{E}(A)\right]^{+} = \mathcal{E}\left(A^{-1}\right). \tag{12}$$

The relation (12) is known (cf. [24]), but it is not clear to us who proved it for the first time. □

The condition number of the elliptic cone $\mathcal{E}(A)$ is near 1 if and only if the condition number

$$\text{cond}(A) = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}$$

of the matrix A is near 1. This observation is consistent with the fact $\mathcal{E}(A)$ reduces to a ball-generated cone if all the eigenvalues of A are equal.

2.3 Epigraphical Cones

A large variety of convex cones in $\mathbb{R}^n$ are expressible as the epigraph

$$\text{epi}\, f = \left\{(z,t) \in \mathbb{R}^{n-1} \times \mathbb{R} : f(z) \le t\right\}$$

of a function $f : \mathbb{R}^{n-1} \to \mathbb{R} \cup \{\infty\}$ satisfying the following properties:

$$\begin{cases} f \text{ is sublinear,} \\ f \text{ is nonnegative,} \\ f \text{ is lower-semicontinuous,} \\ f \text{ vanishes at the origin.} \end{cases} \tag{13}$$

As usual, sublinearity is understood as the combination of subadditivity and positive homogeneity. In view of (13), the set epi f is a nontrivial closed convex cone in $\mathbb{R}^n$. One refers to epi f as the epigraphical cone associated to f.

Lemma 2.7 *Let $f : \mathbb{R}^{n-1} \to \mathbb{R} \cup \{\infty\}$ be as in (13). Then*

(a) $e_n \in \text{int}\left(\text{epi}\, f\right)$ *if and only if f is finite everywhere.*
(b) epi f *is pointed if and only if $f^{-1}(0) = \{z \in \mathbb{R}^{n-1} : f(z) = 0\}$ is pointed.*

Proof Let f be finite everywhere. Then $(z,t) \mapsto f(z) - t$ is continuous. Since $f(\mathbf{0}_{n-1}) - 1 < 0$, it follows that $f(z) - t \leq 0$ for all (z,t) in a neighborhood of e_n. This proves that e_n belongs to the interior of epi f. Conversely, suppose that $e_n \in \operatorname{int}(\operatorname{epi} f)$. Hence, there exists $\varepsilon > 0$ such that

$$e_n + \varepsilon\, \mathbb{B}_n \subset \operatorname{epi} f,$$

where $\mathbb{B}_n$ stands for the closed unit ball of $\mathbb{R}^n$. In particular, $f(\varepsilon h) \leq 1$ for all $h \in \mathbb{B}_{n-1}$. This proves that f is finite on the ball $\varepsilon\, \mathbb{B}_{n-1}$. By positive homogeneity, f is finite everywhere. Part (b) is a consequence of the equality

$$\operatorname{lin}(\operatorname{epi} f) = \left[\operatorname{lin}\left(f^{-1}(0)\right)\right] \times \{0\},$$

where $\operatorname{lin}(Q) = Q \cap -Q$ indicates the lineality space of a convex cone Q. □

The next proposition shows that, up to orthogonal transformation, each element of $\Xi_{\mathrm{sol}}(\mathbb{R}^n)$ is an epigraphical cone associated to a finite-valued function. In the sequel, we use the symbol $\mathcal{O}_n$ to indicate the group of orthogonal matrices of order n.

Proposition 2.8 *Let $K \in \Xi_{\mathrm{sol}}(\mathbb{R}^n)$. Then $K = U(\operatorname{epi} f)$ for some $U \in \mathcal{O}_n$ and some nonnegative sublinear function $f : \mathbb{R}^{n-1} \to \mathbb{R}$.*

Proof By Corollary 1.5, the vector $c = \pi_{\mathrm{inc}}(K)$ belongs to K^+. Pick a matrix $U \in \mathcal{O}_n$ such that $U^T c = e_n$. In such a case, the closed convex cone $Q = U^T(K)$ is contained in the half-space $\mathbb{R}^{n-1} \times \mathbb{R}_+$. Since $c \in \operatorname{int}(K)$, it follows that $e_n \in \operatorname{int}(Q)$. We claim that Q is an epigraphical cone. For proving this, one just needs to examine the function

$$z \in \mathbb{R}^{n-1} \mapsto f(z) = \inf\{t : (z,t) \in Q\}.$$

One can easily show that f is nonnegative and sublinear. The condition $e_n \in \operatorname{int}(Q)$ implies that f is finite everywhere (and therefore it is a continuous function). It remains to check that $Q = \operatorname{epi} f$, but this is a mere routine. □

As explained in the next lemma, computing the dual cone of epi f amounts to evaluating a certain function $f^\circ : \mathbb{R}^{n-1} \to \mathbb{R} \cup \{\infty\}$ defined by

$$f^\circ(w) = \inf\left\{s \geq 0 : w^T z \leq s f(z) \ \text{ for all } z \in \operatorname{dom} f\right\}$$

with $\operatorname{dom} f = \{z \in \mathbb{R}^{n-1} : f(z) < \infty\}$ denoting the effective domain of f. In the convex analysis literature (cf. [37]), one refers to f° as the *polar function* of f. One can check that

$$f^\circ(w) = \inf\left\{s \geq 0 : w \in s\,\Omega_f\right\}, \tag{14}$$

where

$$\Omega_f \;=\; \left\{w \in \mathbb{R}^{n-1} : w^T z \leq f(z) \text{ for all } z \in \mathbb{R}^{n-1}\right\} \;=\; \{f \leq 1\}^\circ$$

is a closed convex set containing the origin. In the above line we employ the notation

$$\{f \leq 1\} = \left\{z \in \mathbb{R}^{n-1} : f(z) \leq 1\right\}$$
$$C^\circ = \left\{w \in \mathbb{R}^{n-1} : w^T z \leq 1 \text{ for all } z \in C\right\}.$$

By the way, the right-hand side of (14) corresponds to the usual definition of the *gauge function* of Ω_f. It is not difficult to show that f° satisfies all the properties listed in (13).

Remark 2.9 If f is a norm on $\mathbb{R}^{n-1}$, then

$$w \in \mathbb{R}^{n-1} \mapsto f^\circ(w) = \max_{z \neq 0} \frac{w^T z}{f(z)}$$

is nothing but the polar (or dual) norm of f. Of course, the polar of f° is f itself.

Lemma 2.10 *Let* $f : \mathbb{R}^{n-1} \to \mathbb{R} \cup \{\infty\}$ *be as in* (13). *Then the dual cone of* epi f *is again an epigraphical cone, namely*

$$(\mathrm{epi}\, f)^+ = \left\{(w, s) \in \mathbb{R}^{n-1} \times \mathbb{R} : f^\circ(-w) \leq s\right\}. \tag{15}$$

If, in addition, f is an even function, then so is f° and

$$(\mathrm{epi}\, f)^+ = \mathrm{epi}\, f^\circ. \tag{16}$$

Proof Formula (15) can be obtained by working out [37, Theorem 14.4], but we prefer to give a short and self-contained proof. Let $(w, s) \in (\mathrm{epi}\, f)^+$, that is,

$$w^T z + st \geq 0 \quad \text{for all } (z, t) \in \mathrm{epi}\, f.$$

This condition breaks down into two pieces:

$$\begin{aligned} s &\geq 0 \\ w^T z + sf(z) &\geq 0 \quad \text{ for all } z \in \mathrm{dom}\, f. \end{aligned}$$

The combination of these two pieces is equivalent to saying that $f^\circ(-w) \leq s$. This takes care of the equality (15). Since f° satisfies (13), so does the function

$$w \in \mathbb{R}^{n-1} \mapsto f^\ominus(w) = f^\circ(-w).$$

Hence, the right-hand side of (15) is an epigraphical cone. Finally, suppose that f is even, i.e., $f(-z) = f(z)$ for all $z \in \mathbb{R}^{n-1}$. In such a case, $-\mathrm{dom}\, f = \mathrm{dom}\, f$ and

$$\begin{aligned} f^\circ(-w) &= \inf\left\{s \geq 0 : (-w)^T z \leq sf(z) \text{ for all } z \in \mathrm{dom}\, f\right\} \\ &= \inf\left\{s \geq 0 : w^T(-z) \leq sf(-z) \text{ for all } z \in -\mathrm{dom}\, f\right\} \\ &= f^\circ(w) \end{aligned}$$

for all $w \in \mathbb{R}^{n-1}$. This and (15) lead to (16). □

The elliptic cone (10) is perhaps the most prominent example of an epigraphical cone. The duality formula (12) can be recovered as a particular case of (16). Another interesting example of epigraphical cone is

$$\Phi_{p,n} = \left\{(z, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : \|z\|_p \leq t\right\} \tag{17}$$

with $p \in [1, \infty]$ and $\| \cdot \|_p$ standing for the ℓ^p - norm in $\mathbb{R}^{n-1}$. One refers to (17) as the n - dimensional ℓ^p - cone. As a second application of (16), one obtains the well known relation (cf. [9, 18])

$$\Phi_{p,n}^{+} = \Phi_{q,n}$$

with $p, q \in [1, \infty]$ standing for a pair of conjugate numbers. This means that $p^{-1} + q^{-1} = 1$ with the usual convention $1/\infty = 0$ being in force.

The next theorem provides a formula for computing the inradius of an epigraphical cone. Everything boils down to being able to evaluate an expression of the form

$$\ell(f) = \sup_{\|u\| \leq 1} \left\{ f(u) + \sqrt{1 - \|u\|^2} \right\}. \tag{18}$$

The above maximization problem is quite interesting by itself, so we shall come back to the analysis of (18) in a moment.

Theorem 2.11 *Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be a nonnegative sublinear function. Suppose, in addition, that f is even. Then*

$$\pi_{\text{inc}}(\text{epi}\, f) = e_n \quad \textit{and} \quad \rho(\text{epi}\, f) = 1/\ell(f)\,. \tag{19}$$

Proof The function f satisfies not only (13), but more than that. The fact that f is finite everywhere implies that epi f is a solid cone. The unicity of its incenter is then guaranteed. Having said this, we now work out the maximization problem (3), which takes here the form

$$\text{maximize } r \tag{20}$$

$$\|z\|^2 + t^2 = 1 \tag{21}$$

$$r \in [0, 1] \tag{22}$$

$$(z, t) + r\mathbb{B}_n \subset \text{epi}\, f. \tag{23}$$

That f is an even function imposes a certain symmetry on the feasible set

$$M = \left\{ (z, t, r) \in \mathbb{R}^{n+1} : (z, t, r) \text{ satisfies } (21\text{–}23) \right\}.$$

Indeed, one can show that

$$(z, t, r) \in M \quad \Longrightarrow \quad (-z, t, r) \in M. \tag{24}$$

Let $(\bar{z}, \bar{t}, \bar{r})$ be the unique solution to the maximization problem (20–23). In view of the symmetry property (24), it follows that $\bar{z} = \mathbf{0}_{n-1}$, in which case $\bar{t} = 1$. This proves the first equality in (19). The inradius of epi f is then the optimal value of the unidimensional maximization problem

$$\text{maximize } r \tag{25}$$

$$r \in [0, 1] \tag{26}$$

$$e_n + r\mathbb{B}_n \subset \text{epi}\, f. \tag{27}$$

The constraint (27) says that $f(ru) \leq 1 + r\tau$ for all $(u, \tau) \in \mathbb{B}_n$. By a convexity argument, it is enough to let (u, τ) range over the boundary of $\mathbb{B}_n$. We also know that f is positively homogeneous. Hence, (27) can be written in the form

$$rf(u) \leq 1 + r\tau \quad \text{whenever } \|u\|^2 + \tau^2 = 1.$$

This is yet equivalent to saying that

$$r\left[f(u) + \sqrt{1 - \|u\|^2} \right] \leq 1 \quad \text{for all } u \in \mathbb{B}_{n-1}.$$

Hence, the maximum in (25–27) is attained at $r = 1/\ell(f)$, showing the second equality in (19). □

Which is the geometric meaning of $\ell(f)$ and how to compute this expression in practice? As we saw already, under the assumptions of Theorem 2.11, one has

$$\ell(f) = \frac{1}{\operatorname{dist}\left[e_n, \partial(\operatorname{epi} f)\right]}\,.$$

Evaluating the distance from e_n to the boundary of epi f is not always an easy task, so we propose below an alternative characterization of $\ell(f)$.

Proposition 2.12 *Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be a sublinear function. Then*

$$\ell(f) = \sqrt{1 + \|\Omega_f\|^2} \tag{28}$$

with $\|\Omega_f\| = \max_{w \in \Omega_f} \|w\|$.

Proof The starting point of the proof is the observation that

$$\ell(f) = \sup_{u \in \mathbb{R}^{n-1}} \{f(u) - \kappa(u)\}\,,$$

where $\kappa : \mathbb{R}^{n-1} \to \mathbb{R} \cup \{\infty\}$ is the convex lower-semicontinuous function given by

$$\kappa(u) = \begin{cases} -\sqrt{1 - \|u\|^2} & \text{if } u \in \mathbb{B}_{n-1} \\ \infty & \text{otherwise.} \end{cases}$$

By applying the Toland-Singer duality theorem (cf. [11, 21, 45]), one gets

$$\ell(f) = \sup_{w \in \mathbb{R}^{n-1}} \left\{\kappa^*(w) - f^*(w)\right\}, \tag{29}$$

where the superscript $*$ refers to the operation of Legendre-Fenchel conjugation. The conjugate of κ is given by $\kappa^*(w) = \sqrt{1 + \|w\|^2}$, whereas the conjugate of the sublinear function f is the indicator function of the set Ω_f, that is,

$$f^*(w) = \begin{cases} 0 & \text{if } w \in \Omega_f \\ \infty & \text{otherwise.} \end{cases}$$

By the way, Ω_f is bounded because f is finite everywhere. By plugging this information in (29), one gets the equality

$$\ell(f) = \max_{w\in\Omega_f} \sqrt{1+\|w\|^2},$$

which leads in turn to (28). □

The next theorem provides a formula for computing the circumradius of an epigraphical cone.

Theorem 2.13 *Let $f : \mathbb{R}^{n-1} \to \mathbb{R} \cup \{\infty\}$ be as in (13). Suppose, in addition, that f is even and vanishes only at the origin. Then*

$$\pi_{\mathrm{circ}}(\mathrm{epi}\, f) = e_n \quad \text{and} \quad \mu(\mathrm{epi}\, f) = \sqrt{1 - \left[\frac{1}{\ell(f^\circ)}\right]^2}.$$

Proof That f vanishes only at the origin has two important consequences. First of all, the cone $\mathrm{epi}\, f$ is pointed, and therefore it admits a unique circumcenter. And, secondly, the polar function f° is finite everywhere. By combining Lemma 2.10 and Theorem 1.4, one gets

$$\pi_{\mathrm{circ}}(\mathrm{epi}\, f) = \pi_{\mathrm{inc}}\left(\mathrm{epi}\, f^\circ\right) \quad \text{and} \quad \mu(\mathrm{epi}\, f) = \sqrt{1 - \left[\rho\left(\mathrm{epi}\, f^\circ\right)\right]^2}.$$

The rest of the proof consists in applying Theorem 2.11 to the function f°. □

Example 2.14 Consider again the cone $\Phi_{p,n}$ given by (17). This in an epigraphical cone with $f(z) = \|z\|_p$ and $\Omega_f = \{w \in \mathbb{R}^{n-1} : \|w\|_q \le 1\}$. A matter of computation yields

$$\|\Omega_f\| = \begin{cases} (n-1)^{\frac{1}{p}-\frac{1}{2}} & \text{if } p \in [1,2] \\ 1 & \text{if } p \in [2,\infty]. \end{cases}$$

Hence (Fig. 2),

$$\rho(\Phi_{p,n}) = \begin{cases} \left[1 + (n-1)^{(2-p)/p}\right]^{-1/2} & \text{if } p \in [1,2] \\ \sqrt{2}/2 & \text{if } p \in [2,\infty]. \end{cases}$$

$$\mu(\Phi_{p,n}) = \begin{cases} \sqrt{2}/2 & \text{if } p \in [1,2] \\ \left[1 - \dfrac{1}{1 + (n-1)^{(p-2)/p}}\right]^{1/2} & \text{if } p \in [2,\infty]. \end{cases}$$

Remark 2.15 The class of Bishop–Phelps cones include the revolution cones, the elliptic cones, and the ℓ^p-cones as well. If X is a vector space equipped with a certain norm $\|\cdot\|_X$, then

$$\Gamma_{\mathrm{BP}}(y, c) = \{x \in X : c\,\|x\|_X \le y(x)\}$$

is referred to as the Bishop–Phelps cone with parameters y and c (cf. [7, 31, 35]). Here $y : X \to \mathbb{R}$ is a linear function whose (operator) norm is equal to 1 and c is a

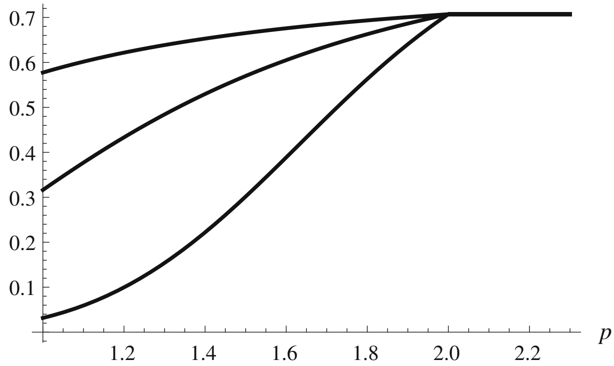


Fig. 2 Behavior of $\rho(\Phi_{p,n})$ as function of p. One considers $n = 3$ (*upper curve*), $n = 10$ (*middle curve*), and $n = 1000$ (*lower curve*)

positive real. The interest of this class of cones is that the norm $\|\cdot\|_X$ does not need to be Euclidean. By using a suitable orthogonal transformation, any Bishop–Phelps cone can be brought to an epigraphical form.

2.4 Fitted Cones

A common way of constructing a nontrivial closed convex cone in $\mathbb{R}^n$ is to pick a closed convex set $C \subset \mathbb{R}^{n-1}$ containing the origin and form

$$F(C) = \mathrm{cl}\left[\mathbb{R}_+(C \times \{1\})\right] \tag{30}$$

$$= \mathrm{cl}\left[\left\{(z,t) \in \mathbb{R}^{n-1} \times \mathbb{R} :\ t \geq 0,\ \ z \in tC\right\}\right]. \tag{31}$$

The equality (30) is the definition of $F(C)$, whereas (31) is a useful characterization. Clearly, $F(C)$ belongs to $\Xi(\mathbb{R}^n)$. One says that $F(C)$ is the *cone fitted* by C. This terminology, although not widely spread in the literature, is used by a number of authors [18, 29, 40]. Be aware, however, that not everyone asks the same properties to the ingredient set C. See Fig. 3 for a geometric representation of a fitted cone.

The whole section on epigraphical cones can be translated into the language of fitted cones. Conversely, every result concerning a fitted cone has a counterpart in the realm of epigraphical cones.

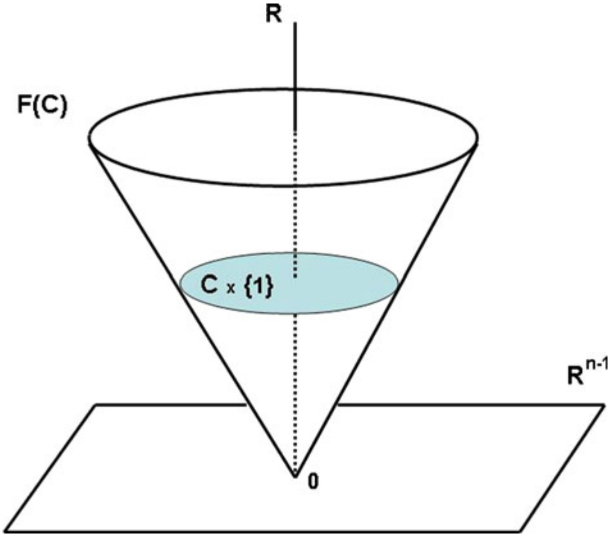
Lemma 2.16 *Fitted and epigraphical cones are the same mathematical objects. More precisely:*

(a) *If $f : \mathbb{R}^{n-1} \to \mathbb{R} \cup \{\infty\}$ is as in (13), then its sublevel set $C_f = \{f \leq 1\}$ is a closed convex set containing the origin and* $\mathrm{epi}\, f = F(C_f)$.
(b) *If $C \subset \mathbb{R}^{n-1}$ is a closed convex set containing the origin, then its gauge function*

$$z \in \mathbb{R}^{n-1} \mapsto f_C(z) = \inf\{t \geq 0 : z \in t\,C\}$$

satisfies the properties listed in (13) and $F(C) = \mathrm{epi}\, f_C$.

Fig. 3 Cone fitted by the set C



The proof of the above lemma is easy and therefore omitted. It is essentially a matter of exploiting the theory of gauge functions as developed in Rockafellar's book [37]. Without further ado we reformulate the main results of Section 2.3. The next two corollaries are obtained by applying Theorems 2.11 and 2.13, respectively, to the gauge function f_C.

Corollary 2.17 *If $C \subset \mathbb{R}^{n-1}$ is a symmetric closed convex set containing the origin in its interior, then $F(C)$ is solid and*

$$\pi_{\text{inc}}(F(C)) = e_n, \quad \rho(F(C)) = \left[1 + \|C^\circ\|^2\right]^{-1/2}.$$

Corollary 2.18 *If $C \subset \mathbb{R}^{n-1}$ is a symmetric compact convex set containing the origin, then $F(C)$ is pointed and*

$$\pi_{\text{circ}}(F(C)) = e_n, \quad \mu(F(C)) = \sqrt{1 - \frac{1}{1 + \|C\|^2}}.$$

2.5 Permutation Invariant Cones

The Pareto cone can be embedded in the larger class of permutation invariant cones. Let $\mathbb{P}_n$ denote the group of permutation matrices of order n. A set V in $\mathbb{R}^n$ is called *permutation invariant* if $P(V) = V$ for all $P \in \mathbb{P}_n$.

Proposition 2.19 *Let $K \in \Xi_{\text{sol}}(\mathbb{R}^n)$ be permutation invariant. Then*

$$\pi_{\text{inc}}(K) = \begin{cases} \widehat{\mathbf{1}}_n & \text{if } \widehat{\mathbf{1}}_n \in K \\ -\widehat{\mathbf{1}}_n & \text{if } \widehat{\mathbf{1}}_n \notin K. \end{cases}$$

Proof Since K is permutation invariant, so are the sets $\{x \in K : \|x\| = 1\}$ and ∂K. Hence, $\mathrm{dist}[\,\cdot\,, \partial K]$ is a permutation invariant function, i.e.,

$$\mathrm{dist}[Px, \partial K] = \mathrm{dist}[x, \partial K] \quad \text{for all } P \in \mathbb{P}_n.$$

With this information at hand, one readily sees that the solution set to (1) is permutation invariant. Since this solution set contains $\pi_{\rm inc}(K)$ as unique element, it follows that $P\pi_{\rm inc}(K) = \pi_{\rm inc}(K)$ for all $P \in \mathbb{P}_n$. Therefore, $\pi_{\rm inc}(K) = a\,\widehat{\mathbf{1}}_n$ for some $a \in \mathbb{R}$. Since $\pi_{\rm inc}(K)$ is a unit vector, the scalar a must be either -1 or 1. □

Corollary 2.20 *Let $K \in \Xi_{\rm ptd}(\mathbb{R}^n)$ be permutation invariant. Then*

$$\pi_{\rm circ}(K) = \begin{cases} \widehat{\mathbf{1}}_n & \text{if } \widehat{\mathbf{1}}_n \in K \\ -\widehat{\mathbf{1}}_n & \text{if } \widehat{\mathbf{1}}_n \notin K. \end{cases}$$

Proof That $K \in \Xi_{\rm ptd}(\mathbb{R}^n)$ is permutation invariant implies that $K^+ \in \Xi_{\rm sol}(\mathbb{R}^n)$ is permutation invariant. It suffices then to combine Theorem 1.4 and Proposition 2.19. □

Proposition 2.19 gives no information on the inradius of K. It settles however the question of identifying the incenter. To evaluate $\rho(K)$ is now a matter of computing

$$\rho(K) = \begin{cases} \mathrm{dist}\left[\widehat{\mathbf{1}}_n, \partial K\right] & \text{if } \widehat{\mathbf{1}}_n \in K \\ \mathrm{dist}\left[-\widehat{\mathbf{1}}_n, \partial K\right] & \text{if } \widehat{\mathbf{1}}_n \notin K. \end{cases}$$

The example below illustrates the use of this formula.

Example 2.21 Let $x^{\uparrow}$ denote the vector which is obtained by rearranging in nondecreasing order the components of $x \in \mathbb{R}^n$. Consider the closed convex cone

$$K_{p,n} = \left\{x \in \mathbb{R}^n :\, x_1^{\uparrow} + \ldots + x_p^{\uparrow} \geq 0\right\}. \tag{32}$$

Note that $K_{1,n}$ is the Pareto cone in $\mathbb{R}^n$. The case $p = n$ is of no interest because

$$\begin{aligned} K_{n,n} &= \left\{x \in \mathbb{R}^n :\, x_1^{\uparrow} + \ldots + x_n^{\uparrow} \geq 0\right\} \\ &= \left\{x \in \mathbb{R}^n :\, x_1 + \ldots + x_n \geq 0\right\} \end{aligned}$$

is simply a half-space. The intermediate case $2 \leq p \leq n-1$ appears in concrete problems of optimization [34] and principal components analysis [38]. Note that $K_{p,n}$ is pointed, solid, permutation invariant, and contains the vector $\widehat{\mathbf{1}}_n$. Hence,

$$\pi_{\rm inc}(K_{p,n}) = \pi_{\rm circ}(K_{p,n}) = \widehat{\mathbf{1}}_n.$$

For computing $\rho(K_{p,n})$ we evaluate the distance from $\widehat{\mathbf{1}}_n$ to the boundary of $K_{p,n}$. In other words, we solve

$$\begin{aligned} &\text{minimize} \quad \left[\left(u_1 - \frac{1}{\sqrt{n}}\right)^2 + \ldots + \left(u_n - \frac{1}{\sqrt{n}}\right)^2\right]^{1/2} \\ &u_1^{\uparrow} + \ldots + u_p^{\uparrow} = 0. \end{aligned} \tag{33}$$

If one permutes the components of a solution to (33), then one gets another solution to (33). So, there is no loss of generality in searching for a solution whose components are already arranged in nondecreasing order:

$$\begin{aligned}
&\text{minimize} \quad \left[\left(u_1 - \frac{1}{\sqrt{n}}\right)^2 + \ldots + \left(u_n - \frac{1}{\sqrt{n}}\right)^2\right]^{1/2} \\
&u_1 \leq \ldots \leq u_n \\
&u_1 + \ldots + u_p \ = 0.
\end{aligned} \tag{34}$$

The later problem admits as solution the vector u given by

$$u_i = \begin{cases} 0 & \text{if } \ i \in \{1, \ldots, p\} \\ n^{-1/2} & \text{if } \ i \in \{p+1, \ldots, n\}. \end{cases}$$

By plugging this vector in the cost function of (34) and simplyfing, one gets $\rho(K_{p,n}) = \sqrt{p/n}$.

Remark 2.22 Computing the circumradius of $K_{p,n}$ requires first to find the generators of $K_{p,n}$. For instance, the cone $K_{n-1,n}$ is generated by n unit vectors, namely, the permutations of the vector

$$w = \frac{1}{\sqrt{n^2 - 3n + 3}} \left(1, \ldots, 1, 2 - n\right)^T.$$

These generators serve to describe the dual cone $K^+_{n-1,n}$ as intersection of homogeneous half-spaces. One gets in this way

$$\rho\left(K^+_{n-1,n}\right) = \operatorname{dist}\left[\widehat{\mathbf{1}}_n, \partial\left(K^+_{n-1,n}\right)\right] = \left(\widehat{\mathbf{1}}_n\right)^T w = \frac{1}{\sqrt{n\left(n^2 - 3n + 3\right)}}.$$

By using Theorem 1.4, one gets finally

$$\mu(K_{n-1,n}) = \left[1 - \frac{1}{n\left(n^2 - 3n + 3\right)}\right]^{1/2}.$$

2.6 Monotonic Cones

We continue next with some convex cones arising in maximum likelihood estimation. The upward monotonic cone and the downward monotonic cone are defined respectively by

$$\begin{aligned}
K_n^{\text{up}} &= \left\{x \in \mathbb{R}^n : x_1 \leq \ldots \leq x_n\right\} \\
K_n^{\text{down}} &= \left\{x \in \mathbb{R}^n : x_1 \geq \ldots \geq x_n\right\}.
\end{aligned}$$

These cones are used, for instance, in the modeling of isotone regression problems [6, 16]. Both cones are solid. Where are their incenters? Since these cones are mutually opposite, we just need to examine the upward monotonic case.

Proposition 2.23 *One has*

$$\rho\left(K_n^{\mathrm{up}}\right) = h/\sqrt{2}$$

$$\pi_{\mathrm{inc}}\left(K_n^{\mathrm{up}}\right) = (a, a+h, a+2h, \ldots, a+(n-1)h)^T \tag{35}$$

with

$$= \sqrt{\frac{12}{n(n-1)(n+1)}} \quad \text{and} \quad a = -\sqrt{\frac{3(n-1)}{n(n+1)}}.$$

In particular, the components of $\pi_{\mathrm{inc}}(K_n^{\mathrm{up}})$ *are equidistant and symmetrically distributed around* 0.

Proof The variational problem (1) for the choice $K = K_n^{\mathrm{up}}$ takes the form

$$\begin{aligned} &\text{maximize} \quad \min\left\{\frac{x_2 - x_1}{\sqrt{2}}, \ldots, \frac{x_n - x_{n-1}}{\sqrt{2}}\right\} \\ &x_1 \leq \ldots \leq x_n \\ &x_1^2 + \ldots + x_n^2 = 1. \end{aligned} \tag{36}$$

This is a matter of placing a collection of points $x_1, \ldots, x_n$ on the interval $[-1, 1]$ in such a way that the smallest distance between two succesive points is a large as possible. The unique solution $\bar{x}$ to (36) has necessarily the form (35) i.e., the components of $\bar{x}$ are equidistant. Since $\bar{x}$ must be a unit vector, one has

$$\sum_{k=0}^{n-1}(a+kh)^2 = 1.$$

After simplification, one gets $na^2 + 2t_n ah + s_n h^2 = 1$ with

$$t_n = \sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$$

$$s_n = \sum_{k=1}^{n-1} k^2 = \frac{n(n-1)(2n-1)}{6}.$$

We are led to a maximization problem with two decision variables subject to an elliptic constraint:

$$\text{maximize}\left\{h \Big/ \sqrt{2} : a \in \mathbb{R}, h \geq 0,\ na^2 + 2t_n ah + s_n h^2 = 1\right\}. \tag{37}$$

For completing the proof one just needs to observe that

$$a = -\frac{t_n}{n}\left[s_n - \frac{t_n^2}{n}\right]^{-1/2} = -\left[\frac{3(n-1)}{n(n+1)}\right]^{1/2}$$

$$h = \left[s_n - \frac{t_n^2}{n}\right]^{-1/2} = \left[\frac{12}{n(n-1)(n+1)}\right]^{1/2}$$

is the optimal solution to (37). □

Table 1 Analysis of the upward monotonic cone

n	$\rho(K_n^{\mathrm{up}})$	$\pi_{\mathrm{inc}}(K_n^{\mathrm{up}})$
2	1	$\frac{1}{\sqrt{2}}\,(-1,1)^T$
3	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}\,(-1,0,1)^T$
4	$\frac{1}{\sqrt{10}}$	$\frac{1}{\sqrt{20}}\,(-3,-1,1,3)^T$
5	$\frac{1}{\sqrt{20}}$	$\frac{1}{\sqrt{10}}\,(-2,-1,0,1,2)^T$

Table 1 displays the incenter and inradius of K_n^{up} as function of n. Note that the components of $\pi_{\mathrm{inc}}(K_n^{\mathrm{up}})$ are equidistant and symmetrically distributed around 0.

2.7 Unimodal Cones

In unimodal regression theory [8, 44], a vector $x \in \mathbb{R}^n$ is called unimodal with a peak at the q-th component (q-unimodal, for short) if

$$x_1 \leq \ldots \leq x_{q-1} \leq x_q \geq x_{q+1} \geq \ldots \geq x_n.$$

For applications of the concept of unimodality in other areas of mathematics, see the interesting survey by Stanley [41]. The set

$$U_{q,n} = \left\{x \in \mathbb{R}^n : x \text{ is } q\text{-unimodal}\right\}$$

is a polyhedral convex cone because it is expressible as intersection of $n-1$ homogeneous half-spaces.

The extreme cases $U_{1,n} = K_n^{\mathrm{up}}$ and $U_{n,n} = K_n^{\mathrm{down}}$ have been considered already in Section 2.6. Another configuration of interest is the one in which the peak x_q occurs at a central component of the vector x, i.e.,

$$q = \begin{cases} (n+1)/2 & \text{if } n \text{ is odd} \\ n/2 \text{ or } (n/2)+1 & \text{if } n \text{ is even.} \end{cases} \tag{38}$$

The next proposition concerns an odd dimension n.

Proposition 2.24 *Let n be odd. For $q = (n+1)/2$, one has*

$$\rho(U_{q,n}) = h/\sqrt{2}$$
$$\pi_{\mathrm{inc}}(U_{q,n}) = (a, a+h, \ldots, a+(q-2)h, a+(q-1)h, a+(q-2)h\ \ldots, a+h, a)^T$$

with

$$h = \sqrt{\frac{3(2q-1)}{q(q-1)\left(q^2-q+1\right)}} \quad \textit{and} \quad a = -\sqrt{\frac{3(q-1)^3}{q(2q-1)\left(q^2-q+1\right)}}.$$

In particular, the components of $\pi_{\mathrm{inc}}(U_{q,n})$ are equidistant (but they are not symmetrically distributed around 0*).*

Proof The variational problem (1) for the choice $K = U_{q,n}$ takes the form

$$\begin{aligned} &\text{maximize} \quad \min\left\{\frac{x_2 - x_1}{\sqrt{2}}, \ldots, \frac{x_q - x_{q-1}}{\sqrt{2}}, \frac{x_q - x_{q+1}}{\sqrt{2}}, \ldots, \frac{x_{n-1} - x_n}{\sqrt{2}}\right\} \qquad (39) \\ &x \in U_{q,n} \\ &\|x\| = 1. \end{aligned}$$

Since $U_{q,n}$ is solid, this maximization problem admits a unique solution, say $\bar{x}$. In order to solve (39) explicitly, we exploit the fact that q indexes the central coordinate of $\bar{x}$. By a symmetry argument, $\bar{x}$ must satisfy

$$\bar{x}_{k+1} = \begin{cases} \bar{x}_k + h & \text{if } k \in \{1, \ldots, q-1\} \\ \bar{x}_k - h & \text{if } k \in \{q, \ldots, n-1\} \end{cases}$$

with $h \geq 0$ and $\bar{x}_1 = a$ linked by the normalization condition

$$\sum_{k=0}^{q-2} (a + kh)^2 \; + \; (a + (q-1)h)^2 \; + \sum_{k=1}^{n-q} (a + (q-1)h - kh)^2 = 1. \tag{40}$$

Since $\bar{x}_{q-j} = \bar{x}_{q+j}$ for all $j \in \{1, \ldots, q-1\}$, the equality (40) reduces to

$$2\sum_{k=0}^{q-2} (a + kh)^2 \; + \; (a + (q-1)h)^2 = 1.$$

This is equivalent to $na^2 + 2\tilde{t}_q ah + \tilde{s}_q h^2 = 1$ with

$$\begin{aligned} \tilde{t}_q &= \sum_{k=1}^{q-1} k + \sum_{k=1}^{q-2} k + = (q-1)^2 \\ \tilde{s}_q &= \sum_{k=1}^{q-1} k^2 + \sum_{k=1}^{q-2} k^2 = \frac{1}{3}(q-1)\left(2q^2 - 4q + 3\right). \end{aligned}$$

The remaining part of the proof is as in Proposition 2.23. □

If $q \in \{1, n\}$ or if q is as in (38), then the components of $\pi_{\text{inc}}(U_{q,n})$ are equidistant. Such an equidistance principle does not hold if q is not chosen as mentioned above. This fact is illustrated with Fig. 4 and, more formally, with the statement of Theorem 2.26.

Obtaining an explicit formula for $\rho(U_{q,n})$, with q arbitrary, is a cumbersome task. This can be done, however, with a bit of patience and effort. We need to state first a lemma.

Lemma 2.25 *Suppose that K is a solid polyhedral cone as in Example* 1.2. *Then $\bar{x} = \pi_{\text{inc}}(K)$ if and only if*

$$\bar{x} \in K, \quad \|\bar{x}\| = 1, \quad \bar{x} \in \text{cone}\{f_i : i \in I(\bar{x})\}, \tag{41}$$

where "cone" *stands for convex conic hull, and $I(\bar{x})$ denotes the set of indices $j \in \{1, \ldots, m\}$ such that $f_j^T \bar{x} = \min_{1 \leq i \leq m} f_i^T \bar{x}$.*

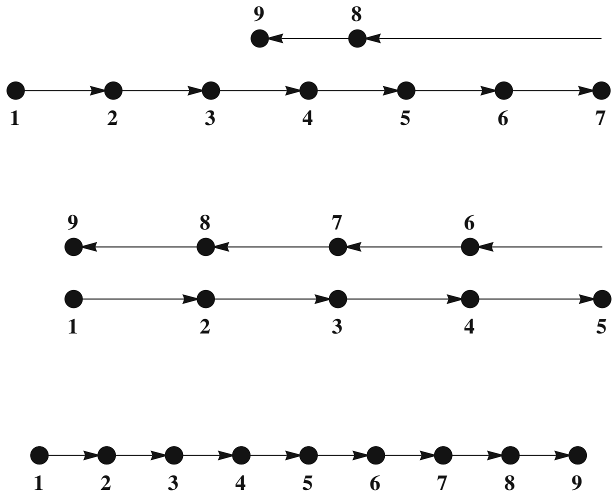


Fig. 4 Distribution of the components of $\pi_{\mathrm{inc}}(U_{q,9})$ when $q = 7$ (*upper case*), $q = 5$ (*middle case*), and $q = 9$ (*lower case*). Note that the components of $\pi_{\mathrm{inc}}(U_{7,9})$ are not equally spaced

Proof Let us take a closer look at the maximization problem (2). By positive homogeneity, the normalization condition $\|x\| = 1$ can be written in the inequality form $1 - \|x\|^2 \geq 0$. Hence, (2) is about maximizing the concave function

$$x \mapsto c(x) = \min_{1 \leq i \leq m} f_i^T x$$

on the convex set

$$F = \left\{ x \in \mathbb{R}^n : 1 - \|x\|^2 \geq 0,\ f_1^T x \geq 0, \ldots, f_m^T x \geq 0 \right\}.$$

The Slater qualification condition holds because K is solid. In such a case, a natural thing to do is to introduce the Lagrangean function

$$L(x, y_0, y_1, \ldots, y_m) = c(x) - y_0 \left(1 - \|x\|^2\right) - \sum_{i=1}^{m} y_i f_i^T x$$

and work out the standard Karush-Kuhn-Tucker optimality conditions

$$\begin{cases} 0 \in -\partial^{\mathrm{fenchel}}(-c)(\bar{x}) - 2y_0 \bar{x} + \sum_{i=1}^m y_i f_i\,, \\ \bar{x} \in F, \\ y_0 \geq 0,\ y_0 \left(1 - \|\bar{x}\|^2\right) = 0, \\ y_i \geq 0,\ y_i f_i^T \bar{x} = 0 \text{ for all } i \in \{1, \ldots, m\}. \end{cases}$$

Here, $\partial^{\mathrm{fenchel}}$ refers to the subdifferential operator in the sense of convex analysis, i.e.,

$$-\partial^{\mathrm{fenchel}}(-c)(\bar{x}) = \mathrm{co}\{ f_i : i \in I(\bar{x})\}$$

with “co” standing for the convex hull operation. Since the constraints $f_i^T x \geq 0$ are inactive at the solution $\bar{x}$, the multipliers $y_1, \ldots, y_m$ must be equal to zero. By contrast, y_0 is positive because otherwise the pointedness of K^+ would be contradicted. For completing the proof, it is enough to reformulate

$$\bar{x} \in \frac{1}{2y_0} \operatorname{co}\{f_i : i \in I(\bar{x})\}, \quad y_0 > 0$$

as a conic hull inclusion. □

Theorem 2.26 *Let n and $q \in \{1, \ldots, n\}$ be arbitrary, but not as in Proposition* 2.24. *Then $\bar{x} = \pi_{\text{inc}}(U_{q,n})$ and $\rho(U_{q,n})$ can be explicitly computed as follows:*

(a) *For $q \geq (n/2) + 1$, one has $\rho(U_{q,n}) = h/\sqrt{2}$ and*

$$\bar{x}_i = \begin{cases} (2i - q - 1)h/2 & \text{if } i \in \{1, \ldots, q\} \\ (n + q + 1 - 2i)h/2 & \text{if } i \in \{q + 1, \ldots, n\} \end{cases}$$

with $h = 2\sqrt{3}[n(n^2 - 3nq + 3q^2 - 1]^{-1/2}$.

(b) *For $q < (n/2) + 1$, one has $\rho(U_{q,n}) = d/\sqrt{2}$ and*

$$\bar{x}_i = \begin{cases} (2i - q)d/2 & \text{if } i \in \{1, \ldots, q - 1\} \\ (n + q - 2i)d/2 & \text{if } i \in \{q, \ldots, n\} \end{cases}$$

with $d = 2\sqrt{3}[n(n^2 - 3nq + 3n - 6q + 3q^2 + 2]^{-1/2}$.

Proof We prove only (a), as (b) follows in an absolutely analogous way. Note that

$$U_{q,n} = \left\{x \in \mathbb{R}^n : f_1^T x \geq 0, \ldots, f_{n-1}^T x \geq 0\right\}$$

with

$$f_i = \begin{cases} (e_{i+1} - e_i)/\sqrt{2} & \text{if } i \in \{1, \ldots, q - 1\} \\ (e_i - e_{i+1})/\sqrt{2} & \text{if } i \in \{q, \ldots, n - 1\} \end{cases}$$

and $\{e_1, \ldots, e_n\}$ standing for the canonical basis of $\mathbb{R}^n$. According to Lemma 2.25, we are done if we can prove the three conditions listed in (41). The first condition can be easily checked from the very definition of $\bar{x}$. By the way, it took us a long time to figure out which was the general structure of the candidate vector $\bar{x}$. The announced expression of h corresponds to the positive root of the quadratic equation

$$(h/2)^2 \left[\sum_{i=1}^{q} (2i - q - 1)^2 + \sum_{i=q+1}^{n} (n + q + 1 - 2i)^2\right] = 1.$$

Such choice of h guarantees that $\|\bar{x}\| = 1$. Concerning the last condition in (41), one has

$$\sqrt{2}\, f_i^T \bar{x} = \begin{cases} h & \text{if } i \neq q \\ (q - (n/2))\, h & \text{if } i = q, \end{cases}$$

and, therefore,

$$I(\bar{x}) = \begin{cases} \{1, \ldots, n - 1\} \backslash \{q\} & \text{if } q > (n/2) + 1 \\ \{1, \ldots, n - 1\} & \text{if } q = (n/2) + 1. \end{cases} \tag{42}$$

Let us consider the first case in (42). The second case can be treated in a similar way. We must show that

$$\bar{x} = \sum_{i=1}^{q-1} y_i(e_{i+1} - e_i) + \sum_{i=q+1}^{n-1} y_i(e_i - e_{i+1}) \tag{43}$$

for suitable coefficients $y_1, \ldots, y_{q-1}, y_{q-1}, \ldots, y_{n-1} \geq 0$. By writing (43) componentwisely, one gets

$$\begin{aligned}
\bar{x}_1 &= -y_1 \\
\bar{x}_i &= y_{i-1} - y_i \quad \text{for } i = 2, \ldots, q-1 \\
\bar{x}_q &= y_{q-1} \\
\bar{x}_{q+1} &= y_{q+1} \\
\bar{x}_i &= y_i - y_{i-1} \quad \text{for } i = q+2, \ldots, n-1 \\
\bar{x}_n &= -y_{n-1}.
\end{aligned}$$

The above linear system admits

$$y_i = \begin{cases} i(q-i)h/2 & \text{for } i = 1, \ldots, q-1 \\ (n-i)(i-q)h/2 & \text{for } i = q+1, \ldots, n-1 \end{cases} \tag{44}$$

as solution. By using the very definition of $\bar{x}$ one can check that (44) is a collection of nonnegative coefficients. □

Theorem 2.26 shows the components of $\pi_{\text{inc}}(U_{q,n})$ are symmetrically distributed around 0, provided that n and q are not as in Proposition 2.24. We end this section with an easy corollary.

Corollary 2.27 *One has:*

(a) $\rho(U_{q,n}) = \rho(U_{n-q+1,n})$ *for all* $q \in \{1, \ldots, n\}$.
(b) *The function* $q \in \{1, \ldots, n\} \mapsto \rho(U_{q,n})$ *is unimodal, achieving its peak at* $q = (n+1)/2$ *if* n *is odd, and at* $q \in \{n/2, (n/2)+1\}$ *if* n *is even.*
(c) *In particular, among all unimodal cones in* $\mathbb{R}^n$*, the monotonic cones* K_n^{up} *and* K_n^{down} *are those that have the smallest inradius.*

3 Cones of Symmetric Matrices

The linear space $\mathcal{S}_n$ of symmetric matrices of order n is equipped with the trace inner product $\langle A, B \rangle = \text{tr}(AB)$. In such a Euclidean space there are plenty of interesting and widely used convex cones. Some of them are quite simple and some others have an amazingly complex structure.

3.1 Spectral Cones

A convex cone $\mathcal{K}$ in $\mathcal{S}_n$ is called *spectral* (or weakly unitarily invariant) if

$$A \in \mathcal{K} \quad \Longrightarrow \quad U^T A U \in \mathcal{K} \;\; \text{for all } U \in \mathcal{O}_n.$$

In fact, the concept of spectrality applies to arbitrary sets in $\mathcal{S}_n$ and not just to convex cones. The next lemma is taken from [26]. The notation $\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))^T$ stands for the vector of eigenvalues of $A \in \mathcal{S}_n$ arranged in nondecreasing order, and $\mathrm{Diag}(x)$ stands for the diagonal matrix whose entries on the diagonal are the components of the vector $x \in \mathbb{R}^n$.

Lemma 3.1 *A convex cone $\mathcal{K}$ in $\mathcal{S}_n$ is spectral if and only if there is a permutation invariant convex cone Q in $\mathbb{R}^n$ such that*

$$\mathcal{K} = \{A \in \mathcal{S}_n : \lambda(A) \in Q\}.$$

Furthermore, such Q is unique and it is given by

$$Q_{\mathcal{K}} = \left\{x \in \mathbb{R}^n : \mathrm{Diag}(x) \in \mathcal{K}\right\}.$$

A similar lemma could be stated for general convex sets (cf. [32, 33]), but we focus the attention on convex cones. A list of examples of spectral convex cones is provided in [26]. What makes a spectral convex cone $\mathcal{K}$ so attractive is that everything boils down to working with the corresponding permutation invariant convex cone $Q_{\mathcal{K}}$. For instance, one can write

$$\begin{aligned} \mathrm{int}(\mathcal{K}) &= \{A \in \mathcal{S}_n : \lambda(A) \in \mathrm{int}(Q_{\mathcal{K}})\} \\ \partial\mathcal{K} &= \{A \in \mathcal{S}_n : \lambda(A) \in \partial Q_{\mathcal{K}}\} \\ \mathcal{K}^+ &= \left\{A \in \mathcal{S}_n : \lambda(A) \in Q_{\mathcal{K}}^+\right\} \end{aligned}$$

and many other formulas of the same kind. One can also reverse the order and write instead

$$\begin{aligned} \mathrm{int}\,(Q_{\mathcal{K}}) &= \{x \in \mathbb{R}^n : \mathrm{Diag}(x) \in \mathrm{int}(\mathcal{K})\} \\ \partial Q_{\mathcal{K}} &= \{x \in \mathbb{R}^n : \mathrm{Diag}(x) \in \partial\mathcal{K}\} \\ Q_{\mathcal{K}}^+ &= \{x \in \mathbb{R}^n : \mathrm{Diag}(x) \in \mathcal{K}^+\}. \end{aligned}$$

For the sake of completeness, we recall also the Iusem-Seeger commutation principle for optimization problems with spectral data (cf. [26, Lemma 4]).

Lemma 3.2 *Let Q be a spectral set in $\mathcal{S}_n$ and $\Phi : \mathcal{S}_n \to \mathbb{R}$ be a spectral function, i.e.,*

$$\Phi(U^T A U) = \Phi(A) \quad \text{for all } A \in \mathcal{S}_n \text{ and } U \in \mathcal{O}_n.$$

Let $\bar{A}, \bar{B} \in \mathcal{S}_n$. If $\bar{B}$ is a local minimum (or a local maximum) over Q of the function $\langle \bar{A}, \cdot \rangle + \Phi(\cdot)$, then $\bar{A}$ and $\bar{B}$ commute.

With all this information at hand, we are now ready to state:

Theorem 3.3 *Let $\mathcal{K} \in \Xi(\mathcal{S}_n)$ be spectral. Then*

(a) $\rho(\mathcal{K}) = \rho(Q_{\mathcal{K}})$ *and* $\mu(\mathcal{K}) = \mu(Q_{\mathcal{K}})$.
(b) *K is solid if and only if $Q_{\mathcal{K}}$ is solid. In such a case,* $\pi_{\mathrm{inc}}(\mathcal{K}) = \mathrm{Diag}\,(\pi_{\mathrm{inc}}(Q_{\mathcal{K}}))$.
(c) *K is pointed if and only if $Q_{\mathcal{K}}$ is pointed. In such a case,* $\pi_{\mathrm{circ}}(\mathcal{K}) = \mathrm{Diag}\,(\pi_{\mathrm{circ}}(Q_{\mathcal{K}}))$.

Proof

Part (a) Spectrality is used at several stages. The starting point is the observation that

$$\rho(\mathcal{K}) = \max_{\substack{A\in\mathcal{K}\\ \|A\|=1}} \operatorname{dist}[A,\partial\mathcal{K}]$$
$$= \max_{\substack{U\in\mathcal{O}_n,\, x\in Q_{\mathcal{K}}\\ \|U\operatorname{Diag}(x)U^T\|=1}} \operatorname{dist}\left[U\operatorname{Diag}(x)U^T,\partial\mathcal{K}\right]. \tag{45}$$

Since $\|\cdot\|$ and $\operatorname{dist}[\,\cdot\,,\partial\mathcal{K}]$ are spectral functions, one can get rid of the maximization variable $U\in\mathcal{O}_n$ and write simply

$$\rho(\mathcal{K}) = \max_{\substack{x\in Q_{\mathcal{K}}\\ \|\operatorname{Diag}(x)\|=1}} \operatorname{dist}[\operatorname{Diag}(x),\partial\mathcal{K}].$$

In other words, a solution to (45) can be found in the subspace of diagonal matrices. A clever application of Lemma 3.2 shows that

$$\operatorname{dist}[\operatorname{Diag}(x),\partial\mathcal{K}] = \operatorname{dist}\left[x^{\uparrow},\partial Q_{\mathcal{K}}\right]$$

for all $x\in\mathbb{R}^n$. Since $Q_{\mathcal{K}}$ is permutation invariant, so is the function $\operatorname{dist}\left[\,\cdot\,,\partial Q_{\mathcal{K}}\right]$. So, after simplification, one ends up with

$$\rho(\mathcal{K}) = \max_{\substack{x\in Q_{\mathcal{K}}\\ \|x\|=1}} \operatorname{dist}[x,\partial Q_{\mathcal{K}}]. \tag{46}$$

This establishes the equality between the inradiuses of $\mathcal{K}$ and $Q_{\mathcal{K}}$. As a by-product, one gets

$$\mu(\mathcal{K}) = \left[1-\left[\rho\left(\mathcal{K}^+\right)\right]^2\right]^{1/2} = \left[1-\left[\rho\left(Q_{\mathcal{K}^+}\right)\right]^2\right]^{1/2} = \left[1-\left[\rho\left(Q_{\mathcal{K}}^+\right)\right]^2\right]^{1/2}$$
$$= \mu(Q_{\mathcal{K}}).$$

Part (b) As a consequence of the previous part, one obtains

$$\mathcal{K}\text{ is solid} \;\Leftrightarrow\; \rho(\mathcal{K})>0 \;\Leftrightarrow\; \rho(Q_{\mathcal{K}})>0 \;\Leftrightarrow\; Q_{\mathcal{K}}\text{ is solid.}$$

If $\mathcal{K}$ is solid, then the solution to (45) is unique. In fact, it has the form $\pi_{\text{inc}}(\mathcal{K}) = \operatorname{Diag}(\bar{x})$ with $\bar{x}\in\mathbb{R}^n$. As stated implicitly in the proof of (a), such $\bar{x}$ must be the unique solution to (46).

Part (c) Similarly,

$$\mathcal{K}\text{ is pointed} \;\Leftrightarrow\; \mu(\mathcal{K})<1 \;\Leftrightarrow\; \mu(Q_{\mathcal{K}})<1 \;\Leftrightarrow\; Q_{\mathcal{K}}\text{ is pointed.}$$

If $\mathcal{K}$ is pointed, then

$$\pi_{\text{circ}}(\mathcal{K}) = \pi_{\text{inc}}\left(\mathcal{K}^+\right) = \operatorname{Diag}\left(\pi_{\text{inc}}\left(Q_{\mathcal{K}^+}\right)\right) = \operatorname{Diag}\left(\pi_{\text{inc}}(Q_{\mathcal{K}}^+)\right)$$
$$= \operatorname{Diag}\left(\pi_{\text{circ}}(Q_{\mathcal{K}})\right).$$

This completes the proof. □

Example 3.4 The most familiar example of spectral cone is the Loewner cone of positive semidefinite matrices:

$$\mathcal{P}_n = \left\{ A \in \mathcal{S}_n : \; x^T A x \geq 0 \;\; \text{for all } x \in \mathbb{R}^n \right\}.$$

It has $\mathbb{R}^n_+$ as associated permutation invariant cone. One obtains

$$\pi_{\rm inc}(\mathcal{P}_n) = \pi_{\rm circ}(\mathcal{P}_n) = \widehat{I}_n$$

with $\widehat{I}_n = I_n/\sqrt{n}$ denoting the normalized identity matrix of order n. One gets also $\rho(\mathcal{P}_n) = \sqrt{1/n}$ and $\mu(\mathcal{P}_n) = \sqrt{1-(1/n)}$.

Example 3.5 For each $p \in \{2, \ldots, n-1\}$, consider the positively homogeneous concave function

$$A \in \mathcal{S}_n \mapsto g_p(A) = \text{sum of the } p \text{ smallest eigenvalues of } A$$

and the corresponding closed convex cone $\mathcal{K}_{p,n} = \{A \in \mathcal{S}_n : g_p(A) \geq 0\}$. This cone has been studied under different angles by many authors [1, 26, 34]. It is known that $\mathcal{K}_{p,n}$ is a spectral cone with (32) as associated permutation invariant cone. Hence,

$$\pi_{\rm inc}(\mathcal{K}_{p,n}) = \pi_{\rm circ}(\mathcal{K}_{p,n}) = \widehat{I}_n \quad \text{and} \quad \rho(\mathcal{K}_{p,n}) = \sqrt{p/n}\,.$$

3.2 Cone of Copositive Matrices and Variants

The next proposition is specially tailored for dealing with the cone of copositive matrices

$$\mathcal{C}_n = \left\{ A \in \mathcal{S}_n : x^T A x \geq 0 \;\; \text{for all } x \in \mathbb{R}^n_+ \right\},$$

but it can also be applied to more sophisticated cones like

$$\begin{aligned}
\mathcal{E}_n &= \{A \in \mathcal{S}_n : A \text{ is stochastically copositive}\}\\
\mathcal{P}_{n,r} &= \{A \in \mathcal{S}_n : \text{any principal submatrix of } A \text{ of order } r \text{ is positive semidefinite}\}\\
\mathcal{C}_{n,r} &= \{A \in \mathcal{S}_n : \text{any principal submatrix of } A \text{ of order } r \text{ is copositive}\}
\end{aligned}$$

with $r \in \{1, \ldots, n\}$. For a gentle introduction to copositivity the reader may consult the survey papers [22, 23]. See the book by Jacobson [30] for the definition and main properties of stochastic copositivity.

Proposition 3.6 *Let K be a closed convex cone in $\mathcal{S}_n$ such that*

$$\mathcal{P}_n \subset K \subset \{A \in \mathcal{S}_n : a_{i,i} \geq 0 \; \textit{for all } i \in \{1, \ldots, n\}\}.$$

Then $\pi_{\rm inc}(K) = \widehat{I}_n$ and $\rho(K) = 1/\sqrt{n}$.

Proof The proof of this result appears implicitly in [22, Section 6.6]. □

The next example shows that the circumcenter of a solid pointed convex cone may fall in the boundary of the cone. As a consequence, the concept of circumcenter is different from that of incenter.

Example 3.7 One says that $B \in \mathcal{S}_n$ is completely positive if one can find an integer m and a matrix F of size $n \times m$ with nonnegative entries such that $B = FF^T$. As shown by Hall [19],

$$\mathcal{D}_n = \left\{B \in \mathcal{S}_n : B \text{ is completely positive}\right\}$$

is the dual cone of $\mathcal{C}_n$. By combining Proposition 3.6 and Theorem 1.4, one gets $\pi_{\mathrm{circ}}(\mathcal{D}_n) = \widehat{I}_n$. Clearly, $\pi_{\mathrm{circ}}(\mathcal{D}_n)$ is a completely positive matrix. On the other hand, since any matrix in the interior of $\mathcal{D}_n$ is positive entrywise, it follows that $\pi_{\mathrm{circ}}(\mathcal{D}_n) \in \partial\mathcal{D}_n$.

4 By Way of Conclusion

We have derived explicit formulas for the optimal balls (6–7) in a large variety of situations. Table 2 gives a long list of examples, but it is not exhaustive. The most remarkable results were obtained in the context of the Euclidean space $\mathbb{R}^n$, but some advances were made also in the context of the Euclidean space $\mathcal{S}_n$.

We would like to point out that a challenging question concerning the cone of copositive matrices has been left unsolved, namely, the problem of computing its circumcenter and its circumradius. In view of Theorem 1.4, this amounts to computing the incenter and the inradius of the cone of completely positive matrices. The cone of copositive matrices can be viewed as a particular case of

$$\mathcal{C}_{n,Q} = \left\{A \in \mathcal{S}_n : x^T A x \geq 0 \text{ for all } x \in Q\right\}, \tag{47}$$

where Q stands for an arbitrary set in $\mathbb{R}^n$. The closed convex cone (47) has been explored in detail by [10], but not from the point of view of its incenter and its circumcenter. Such a study remains to be done.

Table 2 Inradius and circumradius for various cones in $\mathbb{R}^n$ and $\mathcal{S}_n$

K	$\pi_{\mathrm{inc}}(K)$	$\rho(K)$	$\pi_{\mathrm{circ}}(K)$	$\mu(K)$
$\mathbb{R}^n_+$	$\widehat{\mathbf{1}}_n$	$\sqrt{1/n}$	$\widehat{\mathbf{1}}_n$	$\sqrt{1-(1/n)}$
$\Gamma(y,\theta)$	y	$\sin\theta$	y	$\sin\theta$
$\mathcal{E}(A)$	e_n	$\left[1+\lambda_{\max}(A)\right]^{-1/2}$	e_n	$\left[1+\lambda_{\min}(A)\right]^{-1/2}$
$\Phi_{p,n}$ $(1 \leq p \leq 2)$	e_n	$\left[1+(n-1)^{(2-p)/p}\right]^{-1/2}$	e_n	$\sqrt{2}/2$
$\Phi_{p,n}$ $(p \geq 2)$	e_n	$\sqrt{2}/2$	e_n	$\left[1-\frac{1}{1+(n-1)^{(p-2)/p}}\right]^{1/2}$
$K_{p,n}$	$\widehat{\mathbf{1}}_n$	$\sqrt{p/n}$	$\widehat{\mathbf{1}}_n$	?
K_n^{up}	Cf. Proposition 2.23	$\left[\frac{6}{n(n-1)(n+1)}\right]^{1/2}$	Not unique	1
$U_{q,n}$	Cf. Theorem 2.26	Cf. Theorem 2.26	Not unique	1
$\mathcal{P}_n$	$\widehat{I}_n$	$\sqrt{1/n}$	$\widehat{I}_n$	$\sqrt{1-(1/n)}$
$\mathcal{K}_{p,n}$	$\widehat{I}_n$	$\sqrt{p/n}$	$\widehat{I}_n$	?
$\mathcal{C}_n$	$\widehat{I}_n$	$\sqrt{1/n}$	?	?

References

1. Azé, D., Hiriart-Urruty, J.-B.: Optimal Hoffman-type estimates in eigenvalue and semidefinite inequality constraints. J. Glob. Optim. **24**, 133–147 (2002)
2. Barker, G.P., Foran, J.: Self-dual cones in Euclidean spaces. Linear Algebra Appl. **13**, 147–155 (1976)
3. Barker, G.P., Loewy, R.: The structure of cones of matrices. Linear Algebra Appl. **12**, 87–94 (1975)
4. Bauschke, H.H., Kruk, S.G.: Reflection-projection method for convex feasibility problems with an obtuse cone. J. Optim. Theory Appl. **120**, 503–531 (2004)
5. Belloni, A., Freund, R.M.: A geometric analysis of Renegar's condition number, and its interplay with conic curvature. Math. Program. **119**, 95–107 (2009)
6. Best, M.J., Chakravarti, N.: Active set algorithms for isotonic regression; a unifying framework. Math. Program. **47**, 425–439 (1990)
7. Bishop, E., Phelps, R.R.: The support functionals of a convex set. Proc. Symp. Pure Math. **7**, 27–35 (1963)
8. Boyarshinov, V., Magdon-Ismail, M.: Linear time isotonic and unimodal regression in the L^1 and L^∞ norms. J. Discret. Algorithms **4**, 676–691 (2006)
9. Boyd, S., Vandenberghe, L.: Convex Optimization. Cambridge University Press, Cambridge (2004)
10. Eichfelder, G., Jahn, J.: Set-semidefinite optimization. J. Convex Anal. **15**, 767–801 (2008)
11. Ellaia, R., Hiriart-Urruty, J.B.: The conjugate of the difference of convex functions. J. Optim. Theory Appl. **49**, 493–498 (1986)
12. Epelman, M., Freund, R.M.: A new condition measure, preconditioners, and relations between different measures of conditioning for conic linear systems. SIAM J. Optim. **12**, 627–655 (2002)
13. Feng, Z.Q., Hjiaj, M., Mróz, Z., de Saxcé, G.: Effect of frictional anisotropy on the quasistatic motion of a deformable solid sliding on a planar surface. Comput. Mech. **37**, 349–361 (2006)
14. Freund, R.M.: On the primal-dual geometry of level sets in linear and conic optimization. SIAM J. Optim. **13**, 1004–1013 (2003)
15. Freund, R.M., Vera, J.R.: Condition-based complexity of convex optimization in conic linear form via the ellipsoid algorithm. SIAM J. Optim. **10**, 155–176 (1999)
16. Gao, W., Shi, N.Z.: I-projection onto isotonic cones and its applications to maximum likelihood estimation for log-linear models. Ann. Inst. Stat. Math. **55**, 251–263 (2003)
17. Goffin, J.L.: The relaxation method for solving systems of linear inequalities. Math. Oper. Res. **5**, 388–414 (1980)
18. Güler, O.: Barrier functions in interior point methods. Math. Oper. Res. **21**, 860–885 (1996)
19. Hall, M.: Combinatorial Theory. Blaisdell Publishing Co., Boston (1967)
20. Henrion, R., Seeger, A.: On properties of different notions of centers for convex cones. Set-Value Var. Anal. **18**, 205–231 (2010)
21. Hiriart-Urruty, J.B.: A general formula on the conjugate of the difference of functions. Can. Math. Bull. **29**, 482–485 (1986)
22. Hiriart-Urruty, J.B, Seeger, A.: A variational approach to copositive matrices. SIAM Rev. **52** (2010, in press)
23. Ikramov, K.D., Savel'eva, N.V.: Conditionally definite matrices. J. Math. Sci. **98**, 1–50 (2000)
24. Iusem, A., Seeger, A.: Axiomatization of the index of pointedness for closed convex cones. Comput. Appl. Math. **24**, 245–283 (2005)
25. Iusem, A., Seeger, A.: Measuring the degree of pointedness of a closed convex cone: a metric approach. Math. Nachr. **279**, 599–618 (2006)
26. Iusem, A., Seeger, A.: Angular analysis of two classes of non-polyhedral convex cones: the point of view of optimization theory. Comput. Appl. Math. **26**, 191–214 (2007)
27. Iusem, A., Seeger, A.: Normality and modulability indices. Part I: convex cones in normed spaces. J. Math. Anal. Appl. **338**, 365–391 (2008)
28. Iusem, A., Seeger, A.: Searching for critical angles in a convex cone. Math. Program. **120**, 3–25 (2009)
29. Iusem, A., Seeger, A.: Distances between closed convex cones: old and new results. J. Convex Anal. **17** (2010, in press) (online since 2009)
30. Jacobson, D.H.: Extensions of Linear-Quadratic Control, Optimization and Matrix Theory. Academic, London (1977)

31. Jahn, J.: Bishop–Phelps cones in optimization. Int. J. Optim. Theory Methods Appl. **1**, 123–139 (2009)
32. Lewis, A.S.: Convex analysis on the Hermitian matrices. SIAM J. Optim. **6**, 164–177 (1996)
33. Martinez-Legaz, J.E.: On convex and quasiconvex spectral functions. In: Proceed. of the 2nd Catalan Days on Applied Mathematics (Odeillo, 1995), pp. 199–208, Collect. Études, Presses Univ. Perpignan, Perpignan (1995)
34. Overton, M., Womersley, R.S.: Optimality conditions and duality theory for minimizing sums of the largest eigenvalues of symmetric matrices. Math. Program. **62**, 321–357 (1993)
35. Petschke, M.: On a theorem of Arrow, Barankin, and Blackwell. SIAM J. Control Optim. **28**, 395–401 (1990)
36. Pinto da Costa, A., Seeger. A.: Numerical resolution of cone-constrained eigenvalue problems. Comput. Appl. Math. **28**, 37–61 (2009)
37. Rockafellar, R.T.: Convex Analysis. Princeton University Press, Princeton (1970)
38. Saporta, G.: Probabilités, Analyse des Donnés et Statistique. Editions Technip, Paris (1990)
39. Shannon, C.: Probability of error for optimal codes in a Gaussian channel. Bell Syst. Tech. J. **38**, 611–656 (1959)
40. Shimizu, S.: A remark on homogeneous convex domains. Nagoya Math. J. **105** , 1–7 (1987)
41. Stanley, R.P.: Log-concave and unimodal sequences in algebra, combinatorics, and geometry. Graph theory and its applications: East and West (Jinan, 1986). Ann. N.Y. Acad. Sci. **576**, 500–535 (1989)
42. Stern, R.J., Wolkowicz, H.: Exponential nonnegativity on the ice cream cone. SIAM J. Matrix Anal. Appl. **12**, 160–165 (1991)
43. Stern, R.J., Wolkowicz, H.: Invariant ellipsoidal cones. Linear Algebra Appl. **150**, 81–106 (1991)
44. Stout, Q.F.: Unimodal regression via prefix isotonic regression. Comput. Stat. Data Anal. **53**, 289–297 (2008)
45. Toland, J.F.: A duality principle for nonconvex optimisation and the calculus of variations. Arch. Ration. Mech. Anal. **71**, 41–61 (1979)
46. Wang, D.S., Medgyesi-Mitschang, L.N.: Electromagnetic scattering from finite circular and elliptic cones. IEEE Trans. Antennas Propag. **33**, 488–497 (1985)
47. Zhang, H.W., He, S.Y., Li, X.S., Wriggers, P.: A new algorithm for numerical solution of 3D elastoplastic contact problems with orthotropic friction law. Comput. Mech. **34**, 1–14 (2004)