Insights into a Year 4 Student's Spatial Reasoning and Conceptual Knowledge of Rectangular Prisms

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While spatial reasoning skills have been found to predict mathematical achievement, little is known about how primary students' conceptual understanding of three-dimensional objects develops. This paper reports insights into a Year 4 student's spatial reasoning when constructing and describing the properties of rectangular prisms, using Froebel's Gifts, in an interview. Categories of the van Hiele framework were used to analyse student responses in conjunction with a specially designed analysis tool. The findings highlight the benefit of using a one-to-one interview to shed light on one student's spatial reasoning and conceptual knowledge of rectangular prisms.

Knowledge of geometrical shapes and solids is regarded as a core element of geometry and mathematics education in primary schools. However, Clements and Sarama (2011) suggest that geometry and spatial reasoning are areas of mathematics that are often ignored, or receive little attention in the early years' classrooms. Within Australian contexts there is little research relating to the use of geometric construction materials such as Froebel's Gifts (German designed geometric building blocks). By gaining insights into students' conceptual knowledge of three-dimensional objects, including rectangular prisms, we can extend our understanding of how they develop spatial reasoning and visualisation skills. Informed by literature relating to primary students' spatial reasoning and knowledge of three-dimensional objects, the rationale of this Australian pilot study was to provide new insights into students' spatial reasoning and their knowledge of prisms, using a German task-based interview.

Geometric reasoning is introduced into the Australian Mathematics Curriculum (ACM) at Year 3 when students develop the skills to make models of three-dimensional objects and can describe key features (ACARA, 2017). This implies that students need to generate the properties of three-dimensional objects via constructions. In doing so, they must consider the relationship between the different objects such as a cube is a special rectangular prism (Sinclair & Bruce, 2015; Lehrer & Curtis, 2000).

In a recent review of mathematics education in the early years MacDonald, Goff, Docket, and Perry (2016) concluded that, "there continues to be a dearth of Australasian research in the area of geometry (and space) in the early years" (p. 176). Sinclair and Bruce (2015) stressed the importance of extending geometry teaching at the primary school level shifting from an emphasis on how to name and sort shapes by properties, to learning experiences that develop children's spatial reasoning through active meaning making. Dindyal (2015) suggested that when it comes to constructing "children should be given experience with: free construction materials (clay, plasticine, ropes, boxes), geometric construction materials (lego, pattern blocks, meccano, tangrams), constructing with paper (paper folding, paper cut-

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outs), constructing on paper (drawings of shapes, patterns)" (p. 523). Also, student learning becomes richer when learning occurs through intentional guided play, and when students are encouraged to participate in geometric reasoning.

Lehrer and Curtis (2000) outlined a lesson with Grade 3 students when they experimented with finding the five "perfect" solids (Platonic solids). The students were not told the "rule" or properties of perfect solids but created solids with polydrons and conjectured about the rules depending on whether the teacher confirmed their newly created solid was "perfect" or not. The students used the language of edges, faces, vertices, congruence and angles to compare the solids and determine the rule. Similarly, Ambrose and Kenehan (2009) reported on a teaching experiment in which 8- and 9-year olds built and described polyhedra over several lessons. The results indicated that the students improved in their geometric reasoning and "began to identify, enumerate and notice relationships between component parts of polyhedral" (p. 158).

There are also connections between spatial reasoning and spatial visualisation that contribute to students' ability to decode three dimensional designs and effectively construct models of these. Previous studies (e.g., Battista & Clements, 1996; Reinhold, 2007) highlighted the important role of spatial visualisation, including the "ability to comprehend [and apply] imaginary movements in three-dimensional space or manipulate objects in imagination" (Pittalis, Mousoulides, & Christou, 2007, p. 1073). For example, Battista and Clements (1996) reported on students who failed in coordinating two different orthogonal views for the construction of a rectangular prism (e.g., a 4 x 5 x 3 prism presented as a diagram), when counting the number of single cubes. A lack of coordination was evident whenever the students were unable "to recognize how they [the orthogonal views of a prism] should be placed in proper position relative to each other" (p. 267). This prevented some students from forming "one integrated mental image of the objects" (p. 272). Their initial conception of rectangular prisms was characterised as having an uncoordinated set of faces. Students who were more successful at enumerating the number of cubes to make the figure were more likely to reconstruct the image as layers, and multiply one layer by the number of layers, or add iterations of layers (sometimes by skip counting). Those who struggled to structure the image in layers appeared to use structuring that was "local rather than global" (p. 275), that is, the students considered small sets of cubes in a side, row or column, making the structuring and counting of cubes very complex. This *local* structuring is similar to Reinhold's (2007) finding that students focus on isolated features and visualised "in bits" rather than visualise the entire structure.

Gutiérrez, Jaime, and Fortuny (1991) described the van Hiele levels (1986) in terms of three-dimensional objects, which aimed to ascertain a students' level of geometric accuracy by measuring the degree to which students' thinking adhered to a level or levels. The following condensed version of the specific three-dimensional descriptors for the van Hiele Levels 1 to 3 (Gutiérrez et al., 1991, p. 242) informed the data analysis in our study.

> Level I (Recognition): Students consider three-dimensional objects as a whole and recognise and name solids (prisms, cones, pyramids, etc.). They distinguish a given solid from others on a visual basis using reasoning of the type "it looks like..." They do not explicitly consider the components or properties based in order to identify or name a solid.
>
> Level 2 (Analysis): Students identify the components of solids (faces, edges, etc.), and describe properties of three-dimensional objects in an informal way. They are unable to logically relate properties to each other, nor logically classify solids or families of solids. Students are able to discover properties of the solids by experimentation.

Level 3 (Informal deduction): Students are able to logically classify families of solids (e.g., classes of prisms or rounded solids, regular polyhedra). Definitions are meaningful for students, and they are able to handle equivalent definitions for the same concept.

When teaching geometry, it is important for teachers to be aware of a possible trajectory that shows the growth of students' geometric understanding (e.g., van Hiele, 1986) as well as being able to select rich tasks that can extend student learning. A study by Moss, Hawes, Naqvi, and Caswell (2015), provided the kind of professional learning that helped teachers to focus on rich tasks and to look for student understanding within their actions of constructing three-dimensional objects.

As part of ongoing research exploring Australian primary students' interactions with Froebel's Gifts, we interviewed 23 students. For the purpose of examining one case study as an example of the rich insights gained through this research, we present one Year 4 student's interview data. The following research question underpins this study:

What insights can be gained from a Year 4 student's construction of rectangular prisms, regarding his spatial reasoning and conceptual knowledge of prisms?

Method

This exploratory case study is of one student (Mark) who was part of a pilot project that investigated students' spatial reasoning and conceptual knowledge of cubes and cuboids (rectangular prisms). The project involved Year 3 and 4 students (aged 8-10) from two Australian primary schools (N=23). Students were asked to respond to a one-to-one geometric reasoning interview, designed by German authors (Reinhold & Wöller, 2016), using Froebel's Gifts. The Australian teachers reported that wooden blocks were not used as part of the school's curriculum and it was unlikely that students had experienced tasks of this nature. Mark (pseudonym) was selected because his responses and reasoning after constructing rectangular prisms were influenced by a "learnt definition" of the properties of prisms. His responses were also representative of those of other students.

The interview was initially used with German students to assess Year 3 and 4 students' geometric knowledge and spatial reasoning. As the term cuboid (used in the original interview) is not used in the Australian curriculum it was substituted for rectangular prism. The following questions were asked in the interview:

1. Close your eyes and imagine a rectangular prism. Describe what you see.
2. Please finish this sentence," A rectangular prism is…."
3. I want you to build a rectangular prism using these blocks (2cm cubes).
4. How do you know this is a rectangular prism?
5. Can you build a different rectangular prism?
6. How do you know this is a rectangular prism?
7. I want you to build a rectangular prism using these blocks (small rectangular prisms). Please explain what you are doing.
8. How do you know this is a rectangular prism?
9. Can you build a different rectangular prism?
10. How do you know this is a rectangular prism?
11. Compare this cube (one 2cm cube) with this rectangular prism (small rectangular prism). How are they the same and different?
12. Is this construction (4 cubes together in single layer) is a rectangular prism? Why? Why not?

Froebel's Gift Sets 3 (congruent cubes) and 4 (congruent rectangular prisms) were used in the interview. Reinhold, Downton, Livy, and Wöller (2017) provide further explanations of the Gift Sets including their geometrical features and pedagogical intentions underpinning the use of these blocks in primary mathematics education.

Data collection and analysis

The Australian authors conducted the one-on-one interviews with each student. One provided the Froebel block sets and asked the interview questions, while the other filmed the student's constructions and took field notes. Each interview took approximately 30 minutes to complete. Grounded theory was used to develop guidelines for detecting student interview responses. Each author independently used open coding while viewing the videos, identifying key themes related to each student's response. In collaboration, the authors conducted a further cycle of refined coding (Table 1) to reach agreement on interpretations and categories. The first column of our data analysis tool shows three categories and corresponding questions from the interview.

Table 1.
Categories and Codes Used for Analysing Student Constructions of Rectangular Prisms.

Category (and interview question numbers)	Codes
Explanation, argument and justification.	1.1No response
	1.2 Incorrect response
	1.3 Using informal language
(Questions: 1, 2, 4, 6, 8, 10, 11, 12)	1.4 Informal language and enumerating
	1.5 Name properties and use correct mathematical terms and enumerate
	1.6 Reference to 2D and 3D relationship
	1.7 Other responses
Construction processes using cubes/ prisms	2.1 No Construction
	2.2 Single cubes/prisms randomly placed
	2.3 Single cubes/prisms randomly placed leading to rectangles
(Questions: 3 and 5 when using cubes)	2.4 Single cubes/prisms forming rows or columns
	2.5 Multiple cubes/prisms forming rows or columns.
	2.6. Orientating the prisms when building layers or walls
(Questions: 7 and 9 when using rectangular prisms)	2.7 Starting with cubes/prisms then "stretching"
	2.8 Other
Completed product	3.1 No response
	3.2 Incorrect response (incomplete rectangular prism)
	3.3 Uses one piece (small rectangular prism)
	3.4 One layer (e.g., using 2 cubes)
	3.5 Two layers or more with four rectangular and two square faces
	3.6 Six rectangular faces

The following results report Mark's responses to questions 2 to 10 and 12. Questions 3 to 6 refer to his construction of rectangular prisms using small cubes and questions 7 to 10 refer to his construction of rectangular prisms using small rectangular prisms.

Results and Discussion

The results include the interview questions and reference to the codes in Table 1. The discussion of results makes links to the theoretical framework (van Hiele's levels) presented in the review of literature.

When Mark was asked to complete Question 2, "A rectangular prism is…." he said, "It is longer and bigger than a square and has eight corners and six sides." We coded this response as 1.4 Informal language and enumerating (Table 1) because Mark did not use correct mathematical language for the properties of his rectangular prisms such as faces and vertices or the geometric names of the faces.

Constructing a Rectangular Prism with Cubes

For Question 3, Mark was given a set of blocks (cubes) and asked, "Build a rectangular prism." First, he used six blocks to make a layer 2 (length) by 3 (width) by 1 (height) (Figure 1a), then he added a second layer making a 2 by 3 by 2 (Figure 1b). He rotated his construction (Figures 1c and 1d) and his completed product includes two square faces and four rectangular faces (Figure 1e). We coded Mark's construction process as 2.4 Single cubes (one by one) forming rows, and 2.6 as he rotated the construction part way through to position the square face directly in front of him. The completed product was coded as 3.5 Two layers or more with four rectangular and two square faces.

Figure 1. Mark's sequence when correctly constructing a rectangular prism with cubes (a, b, c, d, e).

For Question 4 when asked to explain, "How do you know this is a rectangular prism?" Mark said, "It has 6 sides, 8 corners, it is larger and more stretched out than a cube." We coded this response as 1.4 (Informal language and enumerating) rather than 1.5 (Name properties and use correct mathematical terms) because he referred to faces as sides and vertices as corners.

For Question 5 when asked to construct a different rectangular prism using these cubes he constructed a 3 by 4 by 1 prism, then added a second layer (Figure 2). The second prism was wider and longer. Rather than placing cubes one by one as he had done for Question 3, he placed three blocks down row by row when making the base and this was coded as 2.5 Multiple cubes forming rows or columns.

Figure 2: Marks two responses when making rectangular prisms with cubes.

The final product in Figure 2 (3 by 4 by 2) was coded as 3.6 Six rectangular faces. This response was the first time he had made a prism with no square faces.

When asked to justify (Question 6) why the response in Figure 2 was a rectangular prism and what was the same or different, Mark replied, "They both have 6 sides and 8 corners. They are different because one is larger than the other." Note he did not make reference to the shape of the faces. This response was also coded as 1.4.

He was then prompted to notice the faces of each construction. Mark identified the faces of the larger prism (in front, Figure 2) as rectangles and said, “The other has rectangles and squares [faces]” (see, Figure 1e).

Next, he was asked if they were both rectangular prism and stated, “No not exactly... because this is not a square [pointing to the face of the larger prism].”

Both constructions (Figures 1e and 2) were correct, although Mark was not convinced. Mark’s explanation suggests he had an incomplete or prototypical view of prisms. His perception seemed to be that all rectangular prisms have square faces at the ends and he had difficulty recognising and naming a construction with six rectangular faces as a prism.

Constructing a Rectangular Prism with Rectangular Prisms

For Questions 7 and 8 Mark was asked to construct a rectangular prism using rectangular blocks and to justify. First, he used nine blocks to construct a rectangular prism 3 by 1 by 3 (see, Figure 3 a). He then took blocks away leaving a 2 by 1 by 2 (Figure 3b) and said, “This is not a rectangular prism because this [face] is not a square,” [pointing to the end of the prism - Figure 3b]. Mark then removed two blocks and said, “This is a rectangle prism because it has six sides and eight corners, this is a square and this is a rectangle prism,” [pointing to the faces Figure 3c].

Figure 3. Mark’s sequences when making a rectangular prism with rectangular prisms (a, b, c).

When asked why Figure 3b was not a rectangular prism Mark said, “Well if you add these two up it is not a square [and pointed to the front face of Figure 2b which is not a square] but it doesn’t have to be, but I prefer it to be.” This construction was coded as 2.5 Multiple prisms forming rows or columns. The completed product was coded as 3.5 Two layers with four rectangular and two square faces. Mark’s justification as to why Figure 3c was a rectangular prism suggests his explanation is 1.4 because he is still relying on informal language to describe the properties of prisms.

The dialogue of this section of the interview reflects a shift in Mark’s thinking relating to the properties of a rectangular prism. He confirms that a prism does not require square ends but that he “prefers it to be.” As the interview progresses he starts to further articulate aspects of geometric reasoning when constructing and justifying his responses.

For Question 9, “Can you build a different rectangular prism?” Mark rotated the blocks placing them sideways rather than flat and made a 3 by 1 by 2 prism (Figure 4a).

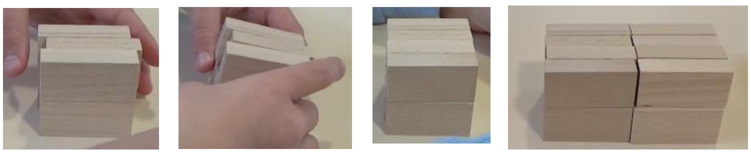

Figure 4. Mark’s second construction using rectangular prisms (a, b, c, d).

Mark tilted the blocks (Figure 4b), suggesting he was attending to the shape of the front face (rectangle) rather than the other faces. Although he had made a prism, Mark chose to continue constructing adding another column (Figure 4c), then extended the width making a 4 by 2 by 2 prism (Figure 4d). This construction was coded as 2.5 multiple prisms and 2.6

orienting the prism when building layers. The completed product was coded as 3.5 as it had two square faces and four rectangular faces.

For Question 10 he was asked, "How do you know if this is a rectangular prism?" Mark said, "It is longer and bigger than a square," highlighting his incomplete understanding of the properties of a prism. He named the properties of the prism using informal language and was coded 1.4.

Finally, Mark was asked to clarify if the construction was a rectangular prism. Mark again demonstrated uncertainty as illustrated in the following dialogue:

> This could be a square but it is not a cube. Yes, we could call it a rectangular prism [then he changed his mind] … it was not because the sides are the same [pointing to the four sides] … this side is not larger than the other [focusing on the property of length]. It is a shape but it might be an irregular shape and I cannot name it.

While Mark demonstrated some understanding of prisms (e.g., he constructed [and possibly visualised them] in layers; understood the attributes of a square though not its inclusiveness with rectangles; and that prisms have 8 corners and 6 sides [faces]), he incorrectly believed that all rectangular prisms have two square faces and that those square faces should be orientated at the front and back of the prism. Mark's spatial reasoning suggests he is operating at van Hiele Level 2 because he described the properties of a prism using informal language and was unable to logically relate the properties of his constructions to one another.

In summary, this study provided an in-depth analysis of Mark's spatial reasoning and lack of conceptual knowledge when constructing prisms. Our findings contribute to the literature by highlighting the importance of students constructing prisms and deriving the properties of prisms through experimentation.

Conclusion and Implications

A critical analysis of a Year 4 student's responses to a one-on-one interview provided insights into Mark's conceptual knowledge of the properties of a rectangular prism. Having to justify and explain his construction assisted with a shift in his understanding of the properties of a rectangular prism, rather than relying on learnt knowledge that a rectangular prism has 6 faces and 8 corners. We also acknowledge only one student was reported here, however the results and discussion revealed the importance of questioning and probing student thinking to elucidate their knowledge about rectangular prisms.

Importantly, Mark's ability to draw on spatial and mathematical language to identify and describe the features of and relationship between two- and three-dimensional shapes (square, rectangle, vertices) limited his capacity to engage in more sophisticated, reasoned responses to our questions. Mark focused on isolated bits of information rather than being able to visualise the entire structure of the constructions, suggesting an inability to simultaneously integrate multiple aspects of visual or cognitive information, which concurs with earlier findings (Battista & Clements, 1996; Reinhold, 2007).

Our findings suggest that construction of three-dimensional objects supports students' development of geometric reasoning. As highlighted within the ACM (ACARA, 2017) students need to generate the properties of three-dimensional objects via construction. In doing so they are able to logically classify families of solids and work toward achieving Level 3 of the van Hiele framework. As previous research indicated students' geometric reasoning improves when they have opportunities to construct solids with materials; use spatial language to describe the properties of prisms; and begin to notice relationships between the properties of prisms (Ambrose & Kenehan, 2009; Lehrer & Curtis 2000).

Implications for teachers include engaging students in investigations that involve generating the properties of three-dimensional objects via construction with a range of materials. Second, that teachers engage students in exploring non-prototypical forms and to look for logical connections and relationships between objects. Third, conducting an interview could also provide teachers with an opportunity to notice how students explain and justify their constructions, and identify their stages of geometric thought. As Moss et al. (2015) found, professional learning helped teachers to notice student understanding within students' actions of constructing three-dimensional objects.

Further exploration of our research within an Australian context would extend the work of earlier studies (Reinhold & Wöller, 2016; Sinclair & Bruce, 2015). Ongoing in-depth analysis of individual students' geometrical concepts of rectangular prisms will also aim to strengthen these findings and connections with the van Hiele levels. In addition, the data analysis tool used to analyse and code student interview responses adds to the literature and could be useful for teachers when assessing students' understanding of three-dimensional objects. In subsequent papers, we will report further on the use of these categories and codes after analysing the other 22 (and future) student data.

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