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INTERNATIONAL ATOMIC ENERGY AGENCY

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INSTABILITIES DUE TO TEMPERATURE GRADIENTS IN COMPLEX MAGNETIC FIELD CONFIGURATIONS

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ABSTRACT

We derive an integral equation governing an instability due to ion temperature gradients. In the presence of magnetic shear, localized non-convective normal modes of instability are shown to exist if the relative temperature gradient is larger than that of density, unless the shear is exceedingly strong, i.e., the field shears through a large angle in the distance in which the temperature drops. Quasi-modes which are less localized in the direction of the gradient can be constructed from these normal modes and a large thermal diffusion may be expected. Conversely the mass diffusion is shown to be rather slow so that it is reasonable to assume that an effective "divertor" should keep the actual heat loss quite small.

INSTABILITIES DUE TO TEMPERATURE GRADIENTS IN COMPLEX MAGNETIC FIELD CONFIGURATIONS

I. INTRODUCTION

The effects of drift instabilities due to temperature gradients transverse to the magnetic field lines in complex magnetic field configurations, such as in the presence of shear, have recently become a subject of interest.[1] The reason is that relevant modes are not stabilized by ion Landau damping. In the present work we show that the introduction of magnetic shear in the equilibrium actually gives rise to unstable normal modes which are non-convective,[2] and may be strongly localized in space. These modes are expected to arise in the neighbourhood of the plasma container wall where the temperature gradient is higher than the density gradient, and may seriously contribute to diffusion of and thermal leakage from the contained plasma.

II. MODEL

We consider a one-dimensional configuration having low β with density and temperature depending on x. The main magnetic field is assumed to be along the z direction. The magnetic shear is represented by a small component along the y direction, so that $\underset{\sim}{B} = B_0 \left(\underset{\sim}{e}_z + x/L_s \, \underset{\sim}{e}_y \right)$. No electric field exists in the equilibrium so that the equilibrium distribution function can be taken as of the form[3]

$$f_0 = n \left(\frac{m}{2\pi T} \right)^{3/2} e^{-\frac{mv^2}{2T}}$$

Here n and T are assumed functions of $x + v_y/\Omega$, Ω being the gyro-frequency. Then we consider electrostatic perturbations so that $\underset{\sim}{E} = -\nabla\phi$ with normal mode solutions of the form $\phi = \tilde{\phi}(x) \exp(i\omega t + iky + ik_z z)$

We use the Vlasov equation, integrating the perturbed linearized form of it along particle orbits. In particular we shall be interested in the case $\omega \approx k_\parallel v_{thi}$; $k_\perp v_i \approx 1$.

Therefore,

$$\frac{df}{dt} = -\frac{e}{m}\left\{ 2\underset{\sim}{E}\cdot\underline{v}\,\frac{\partial f}{\partial v^2} + E_y \frac{\partial}{\partial v_y} f_o \right\} \qquad \text{and}$$

$$f = -\frac{e}{T}\left\{ \phi - i\left(\omega + \frac{K_y T}{\Omega m}\frac{\partial}{\partial x}\right)\right\} f_o \int_{-\infty}^{t} \phi(\underset{\sim}{x}')\, dt'$$

x' and t' referring to equilibrium particle trajectories. The x and y components of these are in fact

$$x' = x + \frac{V_\perp}{\Omega}\left\{ \sin(\Omega t + \psi) - \sin\psi \right\}$$

$$y' = y + \frac{V_\perp}{\Omega}\left\{ \cos(\Omega t + \psi) - \cos\psi \right\} + \frac{V_\parallel t'}{L_s}\left(x - \frac{V_\perp}{\Omega}\sin\psi \right)$$

Introducing the space Fourier transform $\phi(x) = \frac{1}{\sqrt{2\pi}}\int e^{-iK'x}\phi(K')\,dK'$

we have, referring to the ion perturbed density

$$\tilde{n}_i(K) = -\frac{ne}{T}\phi(K) + \frac{e}{T}\left(\omega + \frac{K_y T}{\Omega m}\frac{\partial}{\partial x}\right) n \left(\frac{m}{2\pi T}\right)^{3/2}$$

$$\cdot \frac{1}{2\pi}\int d^3\underset{\sim}{v}\int dx \int dK' \int_{-\infty}^{0} dt'\, e^{i\omega t'} \cdot e^{i(K-K')x}\, e^{-iK'\frac{V_\perp}{\Omega}\sin(\Omega t+\psi)} \tag{1}$$

$$e^{-iK'\frac{V_\perp}{\Omega}\sin\psi]} \cdot e^{iK_y V_\parallel \frac{t}{L_s}\left(x - \frac{V_\perp}{\Omega}\sin\psi\right) + iK_y\left\{\frac{V_\perp}{\Omega}\cos(\Omega t+\psi) - \frac{V_\perp}{\Omega}\cos\psi\right\}}$$

More specifically, we can write the quantity on which $\partial/\partial x$ operates as

$$J = n\left(\frac{m}{2\pi T}\right)\int_0^{2\pi} d\psi \int_0^{+\infty} v_\perp\, dv_\perp\, e^{-m\frac{v_\perp^2}{2T}} \int_{-\infty}^{+\infty} dv_{\parallel}\, e^{-\frac{m v_\parallel^2}{2T}}$$

$$\int \phi(k')\, dk' \int dx\, e^{i\left(x - \frac{v_\perp}{\Omega}\sin\psi\right)\left(k - k' + i k_y \frac{v_\parallel t}{L_s}\right)}$$

$$e^{ik\frac{v_\perp}{\Omega}\sin\psi} \quad e^{-ik'\frac{v_\perp}{\Omega}\sin(\Omega t + \psi)} \quad e^{ik_y \frac{v_\perp}{\Omega}\cos(\Omega t + \psi)}$$

If we carry out integration over $v_{\parallel}$, $x - \frac{v_\perp}{\Omega}\sin\Psi$, and $v_\perp$ neglecting terms of order ω/Ω in comparison with 1, we have

$$J = n\,\frac{\sqrt{2\pi}}{k_y}\, L_s \sqrt{\frac{m}{T}} \int_{-\infty}^{+\infty} dk'\, \phi(k') \int_{-\infty}^{0} dt'\, e^{i\omega t'}\, e^{-\frac{1}{2}(k-k')^2 \frac{m}{T}\frac{L_s^2}{k_y^2 t^2}}$$

$$e^{-\frac{T}{2m\Omega^2}\left(k^2 + k'^2 + 2k_y^2\right)} \quad I_0\left(\sqrt{k^2 + k_y^2}\,\sqrt{k'^2 + k_y^2}\,\frac{T}{m\Omega^2}\right) \tag{2}$$

Since we are interested in perturbations having phase velocity along the magnetic field less than the electron thermal velocity and on which the electron Landau damping is not important (see Section III) we take

$\tilde{n}_e = \dfrac{e\Psi(K)}{T_e}$. Then introducing the dimensionless units $t = t'\omega$,

$\bar{k} = \sqrt{\dfrac{T}{m}}\,\dfrac{1}{\Omega}\,k = \dfrac{1}{\sqrt{2}}\,a_i k$, a_i being the ion Larmor radius, and

$$\bar{\omega} = \frac{\omega}{k_y v_d} = \omega\,\frac{\Omega m}{T}\left|\frac{T}{T'}\right|\frac{1}{k_y}$$

v_d representing the ion diamagnetic velocity, we are led to consider the integral equation,

$$\phi(\bar{\kappa})\left(\frac{T}{T_e}+1\right) = i\,\frac{L_s}{r_T}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} d\bar{\kappa}'\,\phi(\bar{\kappa}')\int_{-\infty\cdot\bar{\omega}}^{0} dt\,\frac{e^{it}}{|t|}$$

$$e^{-1/2(\bar{\kappa}-\bar{\kappa}')^2\frac{L_s^2}{r_T^2}\frac{\bar{\omega}^2}{t^2}}\left[\bar{\omega}-\left\{\frac{r_T}{r_n}-\frac{1}{2}-\frac{1}{2}it-\right.\right.$$

$$\left.\left.-\frac{\kappa^2+\kappa'^2}{2}+\kappa\kappa'\frac{I_1(\kappa\kappa')}{I_0(\kappa\kappa')}\right\}\right] I_0(\kappa\kappa')\,e^{-\frac{\kappa'^2+\kappa^2}{2}}$$

(3)

where $\kappa = \sqrt{b+\bar{\kappa}^2}$

$\kappa' = \sqrt{b+\bar{\kappa}'^2}$, $b = \frac{1}{2}(k_y a_i)^2$, $r_n = -\left(\frac{d\ln n}{dx}\right)^{-1}$

and $r_T = -\left(\frac{d\ln T}{dx}\right)^{-1}$

We have derived this equation in view of studying the localized modes due to temperature gradient, which are found in Section III in the fluid approximation for $T'/T \gg n'/n$ (strong temperature gradient) and $L_s \gg r_T$ (relatively weak shear). In particular we are interested in the critical value of r_T/L_s at which stability occurs. We know[4] that the instability disappears when $n'/n > T'/T$. On the other hand for $T'/T > n'/n$ we shall see in Section III that stability cannot be achieved as long as $L_s \gg r_T$. To determine the exact critical value for L_s/r_T will require a numerical solution of Eq. (3) which will be undertaken in a later work. We note here only that for $L_s/r_T \ll 1$ no unstable eigenvalue may be found as the real part of the right-hand side becomes very small for all unstable $\bar{\omega}$ and cannot balance the left-hand side.

III. THE FLUID APPROXIMATION

The most immediate limit in which Eq. (3) can be solved corresponds to wavelengths longer than the ion Larmor radius so that $\bar{k} < 1$ and $b < 1$, and to neglecting the effects of ion Landau damping so that

$$\Phi(\bar{k}') \approx \Phi(\bar{k}) + (\bar{k}' - \bar{k}) \frac{d\Phi}{d\bar{k}} + \frac{1}{2} (\bar{k} - \bar{k}')^2 \frac{d^2\Phi}{d\bar{k}^2}$$

$$k'^2 \approx \bar{k}^2 + b + 2\bar{k}(\bar{k}' - \bar{k}) + (\bar{k}' - \bar{k})^2 \tag{4}$$

Then we have in lowest order

$$\left(1 + \frac{T_i}{T_e}\right)\Phi(\bar{k}) = -i \frac{L_s}{r_T} \frac{1}{\sqrt{\pi}} \int_{-\infty \cdot \bar{\omega}}^{0} dt \, \frac{e^{it}}{|t|} \Bigg[\left\{ \bar{\omega} - \left(\frac{r_T}{r_n} - \frac{1}{2} - \frac{i}{2} t - \bar{k}^2 - b\right) \right\}$$

$$\cdot (1 - \bar{k}^2 - b)\, \Phi(\bar{k}) \int_{-\infty}^{+\infty} d\xi \; e^{-\frac{1}{2} \xi^2 \frac{L_s^2}{r_T^2} \frac{\bar{\omega}^2}{t^2}} \; - \tag{5}$$

$$- \left\{ \left(\bar{\omega} - \left(\frac{r_T}{r_n} - \frac{1}{2} - \frac{i}{2} t\right) - 1\right) \bar{k} \frac{d\Phi}{d\bar{k}} - \left(\bar{\omega} - \left(\frac{r_T}{r_n} - \frac{1}{2} - \frac{it}{2}\right)\right) \frac{d^2\Phi}{d\bar{k}^2} \right\}$$

$$\int_{-\infty}^{+\infty} d\xi \; e^{-\frac{1}{2} \xi^2 \frac{L_s^2}{r_T^2} \frac{\bar{\omega}^2}{t^2}} \, \xi^2 \Bigg] ,$$

where $\xi = k' - k$.

Carrying out the ξ integration we have

$$\left(\frac{r_T}{L_s}\right)^2 \frac{1}{\bar{\omega}^3} \left\{ \left(1 + \frac{r_T}{r_n} - \bar{\omega}\right) \frac{d^2\phi}{d\bar{K}^2} + \left(4 - \frac{r_T}{r_n} + \bar{\omega}\right) \bar{K} \frac{d\phi}{d\bar{K}} \right\} =$$

$$= \left(\frac{T_i}{T_e} + \frac{1}{\bar{\omega}} \frac{r_T}{r_n} \right) \phi - (\bar{K}^2 + b) \frac{1}{\bar{\omega}} \left(1 + \frac{r_T}{r_n} - \bar{\omega}\right) \phi \tag{6}$$

By comparing the last two terms on the right-hand side we see that we have to choose $r_T / r_n < 1$ for consistency with $b + k^2 < 1$. Now if we introduce the transformation

$$\phi(\bar{K}) = g(\bar{K}) \Phi(\bar{K}) \quad \text{with} \quad 2(1 - \bar{\omega}) g'(\bar{K}) + \bar{K}(4 + \bar{\omega}) g(K) = 0$$

we have

$$\left(\frac{r_T}{L_s}\right)^2 \frac{1}{\bar{\omega}^3} (1 - \bar{\omega}) \frac{d^2\Phi}{d\bar{K}^2} = \left\{ \left(\frac{T_i}{T_e} + \frac{r_T}{\bar{\omega} r_n} \right) + \frac{\bar{K}^2}{4} \left(\frac{r_T}{L_s}\right)^2 \frac{1}{\bar{\omega}^3} \right.$$

$$\left. \cdot \frac{(4 + \bar{\omega})^2}{(1 + \bar{\omega})} + \frac{1}{2} \left(\frac{r_T}{L_s} \right)^2 \frac{1}{\bar{\omega}^3} (4 - \bar{\omega}) - (\bar{K}^2 + b) \frac{1}{\bar{\omega}} (1 - \bar{\omega}) \right\} \Phi(\bar{K}) = 0 \tag{7}$$

We have verified a posteriori that in the limit of small r_T / L_S, in comparison with 1, the first and the last term on the right-hand side of Eq. (7) are the ones of lowest order in r_T / L_S. Then Eq. (7) can be further simplified and if we retransform it to the x space we have

$$\left\{ \frac{a_i^2}{2} \left(\frac{\partial^2}{\partial x^2} - K_y^2 \right) + \bar{\omega} \frac{T_i}{T_e} + \frac{r_T}{r_n} \right.$$

$$\left. + \frac{1}{2} \left(\frac{2}{K_y a_i} \right)^2 \left(\frac{r_T}{\bar{\omega}} \right)^2 \frac{K_y^2 x^2}{L_s^2} \right\} \tilde{\Phi}(x) = 0 \tag{8}$$

We notice that Eq. (8) can be derived more simply taking the Fourier transform of Eq. (3) and considering the limit where $k_y^2 a_i^2 \sim a_i^2 \frac{\partial^2}{\partial x^2} \ll 1$ and

$v_{th i} < \omega/k_{\parallel} < v_{th e}$ with $k_{\parallel} = k_y\, x/L_S$. This is in fact the fluid approximation so that moment equations can also be used for it. Therefore, a simple fluid derivation will be given at the end of this section.

Looking for solutions of the form $H_{\bar{n}}(\hat{k}) \exp -\hat{k}^2/2$, where H_n indicate Hermite polynomials and $\hat{k}^2 = \bar{\sigma}\,\bar{k}^2$, we obtain, in the limit indicated above,

$$\bar{\omega}\frac{T_i}{T_e} + \frac{r_T}{r_n} - b = -i\left(\frac{r_T}{L_s}\right)\left(\frac{1}{\bar{\omega}} - 1\right)(2\bar{n}+1) \tag{9}$$

and

$$\bar{\sigma} = (1-\bar{\omega})(2\bar{n}+1)\Big/\left(\bar{\omega}\frac{T_i}{T_e} + \frac{r_T}{r_n} - b\right) \tag{10}$$

recalling that $b = \frac{1}{2}k_y^2 a_i^2$.

For the solution to be bounded in space and for consistency with the assumption of long wavelengths, $\bar{k}^2 < 1$, we require $\mathrm{Re}\,\bar{\sigma} > 1$, and then $\bar{\omega}\; T_i/T_e < 1$.

Assuming $T_i/T_e \sim 1$, we obtain

$$\bar{\omega} = \frac{1}{2}\frac{T_e}{T_i}\left\{ b - \frac{r_T}{r_n} + \sqrt{\left(\frac{r_T}{r_n} - b\right)^2 - i\left(4\frac{T_i}{T_e}\frac{r_T}{L_s}\right)(2\bar{n}+1)} \right\} \tag{11}$$

Therefore

1) in the limit where $\left(4\frac{T_i}{T_e}\;\frac{r_T}{L_s}\right) < \left(\frac{r_T}{r_n} - b\right)^2$ we obtain a purely growing mode with growth rate

$$-\mathrm{Im}\,(\omega) = \frac{1}{2}\,k_y\, a_i\, V_{thi}\left(\frac{r_n}{L_s}\right)\left(\frac{1}{r_T - b\, r_n}\right)(2\bar{n}+1) \tag{12}$$

2) in the limit $\left(4 \frac{T_i}{T_e} \frac{r_T}{L_s}\right) > \left(\frac{r_T}{r_n} - b\right)^2$ we have

$$-\mathrm{Im}(\omega) = \mathrm{Re}(\omega) \approx \frac{1}{2} k_y a_i v_{thi} \left(\frac{1}{2} \frac{T_e}{T_i} \frac{1}{r_T L_s}\right)^{1/2} \tag{13}$$

Furthermore, in correspondence with Eq. (8) we have

$$\bar{\sigma} = (2\bar{n}+1) \Big/ \left[\frac{1}{2} \frac{T_e}{T_i} \left\{ \frac{r_T}{r_n} - b + \sqrt{\left(\frac{r_T}{r_n} - b\right)^2 - i\left(\frac{4 r_T}{L_s}\right)(2\bar{n}+1)} \right\}\right] \tag{14}$$

Now we notice that the expansion given in Eq. (4) is justified if

$\frac{1}{2}(\kappa - \kappa')^2 \frac{\bar{\sigma}}{2} = \frac{1}{2} \xi^2 \frac{\bar{\sigma}}{2} < 1$ and that, considering Eq. (5), we have $\xi^2 \approx \frac{r_T^2}{L_s^2} \frac{2}{\bar{\omega}^2}$. From Eqs. (10) and (11) we then obtain, for $T_i = T_e$

$$\frac{1}{2}\left(\frac{r_T}{L_s}\right)^{1/2} < 1$$

as the corresponding condition.

In conclusion we see that in the limit of relatively small shear and large temperature gradient no stabilizing influence is found. In this case the shear has only the effect of introducing strongly localized modes. A criterion for stability can only be obtained in the regime where the full integral equation is valid and for $L_S / r_T \approx 1$.

In order to show the derivation of Eq. (8) from moment equations we choose the case where shear is absent and $k_\parallel = k_z$. In addition we consider the limit of long wavelengths where $k_y a_i < k_\parallel v_{thi} / \omega$. Then the linearized equation of motion for ions along the lines of force is

$$i\omega (nM) \tilde{v}_z = -i k_z (\tilde{n} T + n \tilde{T}) + e \tilde{E}_z n$$

and the mass conservation equation is

$$i\omega \tilde{n} + i k_z n \tilde{v}_z = 0$$

On the other hand the equation of state for ions gives

$$i\omega n\tilde{T} + 2nTik_z v_z + \frac{\tilde{E}_y}{B_0} nT' = 0 \tag{15}$$

where we have assumed $\gamma = 3$ treating a one-dimensional case, and taken zero electric field in the equilibrium. Now, for perturbations with phase velocity less than the electron thermal velocity $\omega/k_{\parallel} \ll v_{the}$, the electrons have a Boltzmann distribution so that

$$\tilde{n}_e = \frac{n}{T} e\tilde{\phi} = \tilde{n}_i$$

where $\underset{\sim}{\tilde{E}} = -\nabla\tilde{\phi}$.

From these equations we derive the dispersion relation

$$\omega^3 - 4\omega k_{\parallel}^2 \frac{T}{M} = -k_{\parallel}^2 \frac{T}{M}\,\omega_T \tag{16}$$

with $\omega_T \approx k a_i \, v_{th} \, T'/T$.

Now we recall that in a uniform plasma we have ion waves along $\underset{\sim}{B}$ with

$$\left(\frac{\omega}{k_{\parallel}}\right)^2 = \frac{1}{M}\left(\gamma_i T + \gamma_e T\right) = 4\frac{T}{M}$$

for $\gamma_i = 3$ and γ_e (electrons) = 1. The term on the right-hand side of Eq. (16) represents an additional pumping of temperature in the ion waves due to transversal transport of heat in a non-uniform plasma (see last term in Eq. (15)).

For $\omega < \omega_T$ we have an unstable root from

$$\omega^3 \cong -k_{\parallel}^2 \frac{T}{M}\,\omega_T$$

which is the same dispersion relation as that derived in Ref. 5. [Note that this is in agreement with Eq. (8) if we set $\frac{k_y x}{L_s} = k_{\parallel}$, $\frac{\partial}{\partial x} = 0$.]

This unstable root corresponds to the possibility of always having the additional pumping due to $\tilde{E}_y/B_o \; dT/dx$ in phase with the growing temperature for the ion sound waves.

We note also that as this is primarily an ion instability we would not expect significant finite β modifications until $\beta \approx 1$, in contrast with the density gradient instability which can change character for $\beta \approx m/M$.

IV QUASI-MODES

To show the physical relevance of the modes found above, we can construct a perturbation which is localized along the magnetic field[6,7] out of a superposition of modes centered around successive values x_o of x and propagating at each point perpendicularly to the magnetic field so that $\underline{k} \cdot \underline{B}(x_o) = 0$. We suppose that all points x_o lie within an interval where T, and hence ω, have a relatively small variation. Then we can obtain perturbations of the form

$$\tilde{\Phi}(x, y, z, t) = w(x) \tilde{\psi}(\xi, t)\, e^{ik_y(y - \frac{x}{L_s}\xi)}\, e^{i\omega(x)t}$$

where $w(x)$ is a weighting function expressing an x-dependence milder than that of the elementary modes, and $\xi = z + xy/L_s$ is a co-ordinate following a magnetic field line. The function $\xi(\xi, t)$ is given by the Fourier transform of the elementary mode $\tilde{\varphi}(x)$ so that, for $t = 0$, $\xi = k_x L_s/k$, and k_x is the Fourier variable.[7]

In particular, if we consider, for simplicity, the $\bar{n} = 0$ mode with the space dependence $\exp(-\frac{1}{2}\sigma x^2)$, we obtain, for $\omega(x_o) \approx \omega(x) + (x_o - x)\omega' + \frac{1}{2}(x_o - x)^2 \omega''$

$$\tilde{\psi}(\xi, t) = \exp\left\{-\frac{1}{2}\left(\frac{k\xi}{L_s} + \frac{d\omega}{dx}t\right)^2 \Big/ \left(\sigma - i\frac{d^2\omega}{dx^2}t\right)\right\}$$

These types of perturbation behave as modes for all times such that $k\xi/L_s > d\omega/dx\,t$ and $\sigma > i\, d^2\omega/dx^2 t$, giving rise to convective cells elongated over the magnetic field and spread in the x-direction over a region larger than the width of localization of the single modes. The width Δ of localization along ξ can be computed as $\Delta^{-2} \approx 1/2\,(k/L_s)^2\, \mathrm{Re}\,(1/\sigma)$ so that in the limit where the growth rate is given by Eq. (11) we have

$$\Delta \approx L_s^{3/4} \, r_T^{1/4} \, \frac{2}{K a_i}$$

We notice that since we have made, from the start, the "local" approximation treating $n(x)$ and dn/dx as constants, we cannot compute correctly $d\omega/dx$ and $d^2\omega/dx^2$ from our dispersion relations.

V. QUASI-LINEAR ESTIMATES

Since the temperature gradient instability is hard to stabilize by shear* it is very important to be able to anticipate the non-linear consequences of such linear instability. In order to do this we shall apply the conventional method of the quasi-linear theory in non-uniform plasma, developed previously for the investigation of the density gradient universal instability.[8] We shall restrict ourselves to consideration of cases with not very strong shear and wavelength greater than ion Larmor radius. In this case the relevant modes are not strongly localized in x and we can neglect terms of order of $k_i^2 a_i^2$ or higher. As usual, we shall represent the distribution function in the form of a sum of slowly and rapidly varying parts $f = \tilde{f} + f_\sim$, where $f_\sim$ satisfies the linear equation while, in the equation for the slowly varying function $\tilde{f}$, we shall take into account the averaged quadratic terms, describing the influence of rapidly varying processes. Since we consider the case of long waves we can use the drift approximation of the kinetic equation:

$$\frac{\partial f}{\partial t} + V_z \frac{\partial f}{\partial z} + c\frac{E_y}{H_0}\frac{\partial f}{\partial x} - c\frac{E_x}{H_0}\frac{\partial f}{\partial y} + \frac{eE_z}{M}\frac{\partial f}{\partial V_z} = 0 \tag{17}$$

which for the slowly varying part $\tilde{f}$ will have a form

$$\frac{\partial \tilde{f}}{\partial t} + \left\langle \left(c\frac{E_y}{H_0}\frac{\partial}{\partial x} - c\frac{E_x}{H_0}\frac{\partial}{\partial y} + \frac{eE_z}{M}\frac{\partial}{\partial V_z}\right) f_\sim \right\rangle = 0 \tag{18}$$

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* We expect that other topological methods of stabilization, (e.g., "rippling" of the field) are also quite ineffective.

Taking into account that $\vec{E} = -\nabla\varphi$ and using the Fourier transform

$$\phi = \sum \phi_k(x) e^{ik_z z + ik_y y + i\omega t}$$

Substituting $f_{\sim} = \sum f_k(v, x)\, e^{ik_z z + ik_y y}$ where f_k satisfies the linear equation

$$i(\omega + k_z v_z) f_{k\sim} - c\,\frac{ik_y\phi_k}{H_0}\,\frac{\partial \tilde{F}_z}{\partial x} - \frac{e i k_z \phi_k}{M}\,\frac{\partial \tilde{F}}{\partial v_z} = 0 \tag{19}$$

into Eq. (18) we shall have the final quasi-linear equation in the form

$$\frac{\partial \tilde{F}}{\partial t} = \frac{e^2}{M^2}\sum_{k_y, k_z}\left(k_z \frac{\partial}{\partial v_z} + \frac{k_y}{\Omega}\frac{\partial}{\partial x}\right) \cdot |\phi_k(x)|^2\, \mathrm{Im}\left(\frac{1}{\omega + k_z v_z}\right)\left(k_z\frac{\partial}{\partial v_z} + \frac{k_y}{\Omega}\frac{\partial}{\partial x}\right)\tilde{F} \tag{20}$$

Only the imaginary part remains, of course, after summing over $f_{\sim k}$ and $f_{\sim -k}$.

Knowing the spectrum of electric field fluctuations $|\varphi_k|^2$ one can describe the time evolution of plasma behaviour, but the problem of finding the spectrum involves mode-mode coupling phenomena, which we expect for the temperature gradient instability, would lead to strong plasma turbulence case. However, we can derive some important conclusions using only the quasi-linear apprxomation. Firstly, it is interesting to know the relationship between the turbulent diffusion and the thermal conductivity coefficients.

In order to estimate diffusion we have to take the $1 \times d\vec{v}$ moment of Eq. (20)

$$\frac{\partial n}{\partial t} \equiv \int \frac{\partial \tilde{f}}{\partial t}\, dv = \frac{e^2}{m^2} \sum \frac{k_y}{\Omega} \frac{\partial}{\partial x} |\phi_k(x)|^2\, \mathrm{Im} \int \frac{dv}{\omega + k_z v_z}$$

$$\cdot \left(k_z \frac{\partial}{\partial v_z} + \frac{k_y}{\Omega} \frac{\partial}{\partial x} \right) \tilde{f} \tag{21}$$

Now we see that the integral from the right-hand side of Eq. (4)

$$\frac{1}{M} \int \frac{dv}{\omega + k_z v_z} \left(k_z \frac{\partial}{\partial v_z} + \frac{k_y}{\Omega} \frac{\partial}{\partial x} \right) \tilde{f} \tag{22}$$

is exactly the ions' density contribution in the linear dispersion equation of temperature gradient instability. Due to quasi-neutrality this is equal to the electron contribution

$$-\frac{1}{m} \int \frac{dv}{\omega + k_z v_z} \left(k_z \frac{\partial}{\partial v_z} + \frac{k_y}{\Omega_-} \frac{\partial}{\partial x} \tilde{f}_e \right) \tag{23}$$

For $\omega \ll k_{\parallel} v_{the}$ we can describe the electron density perturbation as Boltzmannian. The integral of (23) in that approximation obviously does not have an imaginary part. Therefore, the first non-vanishing contribution to the diffusion equation comes from the next order expansion in ($\omega / (k_{\parallel} v_{t}^{e})$) in the integral of (23). For the fastest growing waves $\omega / (k_{\parallel} v_{thi}) \sim 1$ and therefore we expect $\omega / (k_{\parallel} v_{the}) \approx \sqrt{m/M}$.

Now for thermal diffusion we can take the $\int v^2 dv$ - moment of the quasi-linear Eq. (22)

$$\frac{\partial nT}{\partial t} \equiv \int M \frac{v_{\parallel}^2}{2} \tilde{f}\, dv = \frac{e^2}{M} \sum \Big\{ -k_z \left|\phi_k(x)\right|^2 Im \cdot$$

$$\cdot \int \frac{v_z\, dv_z}{\omega + k_z v_z} \left(k_z \frac{\partial}{\partial v_z} + \frac{k_y}{\Omega} \frac{\partial}{\partial x} \right) \tilde{f} + \frac{1}{2} \frac{k_y}{\Omega} \frac{\partial}{\partial x} \cdot \left|\phi_k(x)\right|^2 Im \cdot$$

$$\cdot \int \frac{v_z^2\, dv_z}{\omega + k_z v_z} \left(k_z \frac{\partial}{\partial v_z} + \frac{k_y}{\Omega} \frac{\partial}{\partial x} \right) \tilde{f} \tag{24}$$

The second integral in Eq. (24), describing the heat conductivity, does not cancel in the first approximation in $\omega/(k_{\parallel} v_{th}^{e})$. Thus, we can expect the thermal conductivity coefficient to be $(v_{th}^{e}\ k_{\parallel})/\omega \approx \sqrt{M/m}$ times greater than the diffusion coefficient.

In order to make a very crude quantitative estimate and reproduce Kadomtsev's result of the thermal diffusion coefficient, we see from Eq. (24) that $\mathcal{D} \approx \frac{c^2 \mathcal{E}_r^2}{B^2} Im(\omega)/\omega^2$. We may guess that for limiting turbulence $\overline{v_x^2} = \overline{v_y^2} = \overline{v_d^2}$ where $v_d = a_i/r_T\ v_{thi}$ is the original velocity drift due to the temperature gradient. Moreover, from Eq. (16), Im $\omega \approx$ Re $\omega \approx k_{\parallel} v_{thi} \approx (k_y x)/L_s\ v_{thi} \approx k_y v_d$. Hence the scale of turbulence we expect will be given by $k_y \approx k_x \approx 1/x \approx r_T/L_s\ 1/a_i$. Estimating ω from this we infer

$$\mathcal{D} \approx \frac{a_i L_s}{r_T^2} \mathcal{D}_{Bohm} \tag{25}$$

As noted above we expect the mass diffusion to be smaller than given in Eq. (25) by a factor $\sqrt{m/M}$.

This leads to the speculation that this type of loss may be radically cut down by a good divertor. We assume that such a divertor would be capable of reducing the particle density at the wall to a small fraction of the density at the centre. On the other hand, even the best possible divertor may still have some small density of cold particles near the wall so that T'/T is indeed much greater than n'/n and this instability will result near the wall. What is important, however, is that the heat loss is proportional to n and hence will be quite low in this region in spite of the high thermal conductivity. Moreover, as we have shown, the actual density diffusion is low so that a density profile which is low near the wall will not be strongly affected by this instability, thereby keeping the heat loss small, i. e., proportional to the plasma density near the divertor.

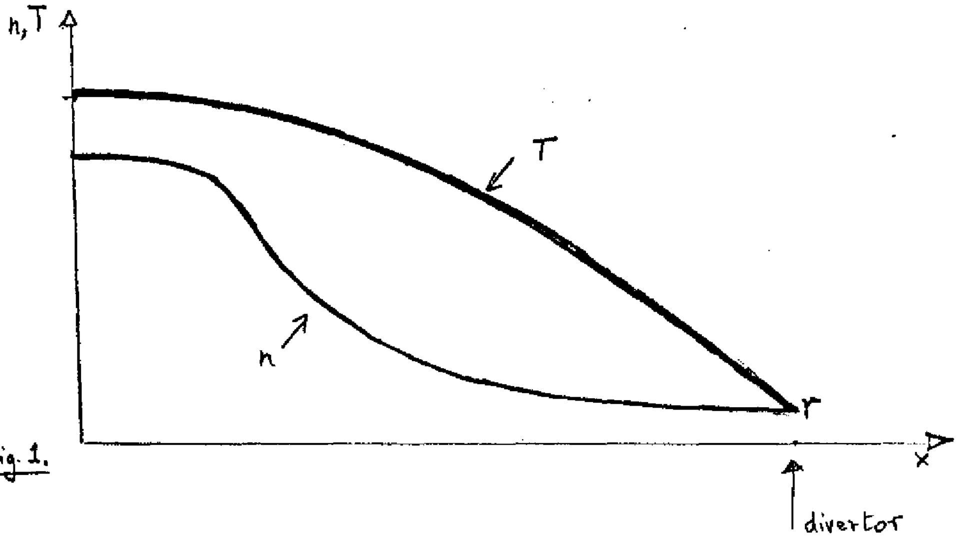

Fig. 1.

Thus a density temperature profile as indicated in Fig. 1 would result in a relatively small heat loss as n_{wall} is small, and could be stably maintained since particle diffusion is slow.

VI. CONCLUSION

We have derived an integral equation governing an instability due to ion temperature gradients. We show that, in the presence of shear, localized non-convective normal modes of instability exist if $T'/T \;\; n/n' > 1$ unless the shear is exceedingly strong, i. e., the field shears through a large angle in the distance in which the temperature drops. Quasi-modes can be constructed from these normal modes and one may expect a large thermal diffusion. On the other hand, mass diffusion is rather slow which affords a good hope that an effective divertor could keep the actual heat loss quite small.

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