Progress of Theoretical Physics, Vol. 121, No. 5, May 2009

Instabilities of Kerr-AdS$_5 \times S^5$ Spacetime

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(Received December 15, 2008)

We study gravitational perturbations of the Kerr-AdS$_5 \times S^5$ spacetime with equal angular momenta. In this spacetime, we found the two types of classical instability, superradiant and Gregory-Laflamme. The superradiant instability is caused by the wave amplification via superradiance and by wave reflection due to the potential barrier of the AdS spacetime. The Gregory-Laflamme instability appears in Kaluza-Klein modes of the internal space S^5 and breaks the symmetry $SO(6)$. By taking into account these instabilities, the phase structure of Kerr-AdS$_5 \times S^5$ spacetime is revealed. The implication for the AdS/CFT correspondence is also discussed.

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§1. Introduction and summary

Recently, AdS black holes in S^5 compactified type IIB supergravity have attracted much interest because they describe strongly coupled $\mathcal{N} = 4$ thermal super Yang-Mills theory via AdS/CFT correspondence.[1)–4)] In particular, phase transitions of dual gauge theory are identified with instabilities of AdS black holes, and understanding the stability of AdS black holes is important to reveal the strongly coupled gauge theory.

The stability of Schwarzschild-AdS black holes has been shown in Refs. 5)–9). However, in the case of Kerr-AdS black holes, we can expect an instability called superradiant instability. The perturbation of Kerr-AdS black holes can be amplified by superradiance at the horizon. On the other hand, at infinity, the amplified perturbation will be reflected by the potential barrier of the AdS spacetime. This will be amplified at the horizon again. By repeating this mechanism, the initial perturbation can grow exponentially and Kerr-AdS black holes become unstable. The superradiant instability is physically reasonable, but, practically, it is difficult to find the instability by gravitational perturbation because of the difficulty in separating perturbation equations. Nevertheless, in some special cases, there are several works on the stability of Kerr-AdS black holes. In the case of four-dimensional Kerr-AdS spacetime, the superradiant instability has been found.[10),11)] In $D = 7, 9, 11, \cdots$, the same instability of Kerr-AdS black holes with equal angular momenta has been shown to exist.[12)] In the case of $(D \geq 7)$-dimensional Kerr-AdS black hole with one rotating axis, it has been shown that the superradiant instability appears in the tensor-type perturbation.[13),14),**)] However, there is no stability analysis of five-dimensional Kerr-AdS

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**) In Refs. 12) and 13), the metric perturbation is decomposed by the tensor harmonics on $(D-3)$- and $(D-4)$-dimensional base spaces, respectively. For $D = 5$, there is no tensor mode in black holes (except for a massless Kerr-AdS black holes[15]) or a scalar field perturbation[16]). To obtain relevant results for the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence, we need to study the instability of five-dimensional Kerr-AdS black holes. It is difficult to study the stability of the general Kerr-AdS_5 spacetime. However, for the equal angular momenta case, the spacetime symmetry of Kerr-AdS_5 black hole is enhanced and the separation of gravitational perturbation equations can be possible.[17)–20)] One of our purposes is to find the superradiant instability of five-dimensional Kerr-AdS black holes with equal angular momenta.

The superradiant instability is caused by a property of rotating AdS black holes, and information on the internal space S^5 is not so important for superradiant instability. However, if the internal space S^5 is taken into account, we can find another type of instability, called Gregory-Laflamme instability. Originally, the Gregory-Laflamme instability has been found in the black brane solution,[21)–24)] but in the Schwarzschild-$\mathrm{AdS}_5 \times S^5$ spacetime, the situation can be similar to the black brane system. If the horizon radius is much smaller than the radius of S^5, the internal space may be considered as $\boldsymbol{R}^5$. Then, we can consider Sch-$\mathrm{AdS}_5 \times S^5$ spacetime as a black brane, and the Gregory-Laflamme instability may appear in Kaluza-Klein modes. The Gregory-Laflamme instability of Schwarzschild-$\mathrm{AdS}_5 \times S^5$ spacetime has already been found in Ref. 25). In this paper, extending their work, we study the Gregory-Laflamme instability of Kerr-$\mathrm{AdS}_5 \times S^5$ spacetime.

We will take into account Gregory-Laflamme and superradiant instabilities and reveal the phase structure of Kerr-$\mathrm{AdS}_5 \times S^5$ spacetime. There are two types of instability in the Kerr-$\mathrm{AdS}_5 \times S^5$ spacetime. Thus, we can expect that this spacetime has a rich phase structure and it will be useful to find evidence of the AdS/CFT correspondence.

The organization and summary of this paper are as follows. In §2, we introduce Kerr-$\mathrm{AdS}_5 \times S^5$ spacetimes with equal angular momenta. In particular, the spacetime symmetry is studied. We shall see that the symmetry is $R_t \times SU(2) \times U(1) \times SO(6)$ in the case of equal angular momenta. In §3, we study the gravitational perturbation of Kerr-AdS_5 spacetime neglecting the Kaluza-Klein modes of the internal space S^5. We can obtain the master equations that are relevant to the superradiant instability. These equations are solved numerically and we find the onset of superradiant instabilities given by $\Omega_H L = 1$, where Ω_H is angular velocity of horizon and L is curvature scale of AdS spacetime. In §4, we study Gregory-Laflamme instability of Kerr-$\mathrm{AdS}_5 \times S^5$ spacetime. We consider the gravitational perturbation including Kaluza-Klein modes in order to see the Gregory-Laflamme instability and obtain the ordinary differential equations in which three variables are coupled. These equations are solved numerically and we find the Gregory-Laflamme instability. In §5, we reveal the phase structure of Kerr-$\mathrm{AdS}_5 \times S^5$ spacetime taking into account superradiant and Gregory-Laflamme instabilities. The result is in Fig. 1; Ω_H and T are angular velocity and temperature of Kerr-AdS_5 black holes, respectively. In Fig. 1, Ω_H and T are normalized using the curvature scale of the AdS spacetime, L. The

these base spaces and, thus, these formalisms are not applicable to the five-dimensional Kerr-AdS black hole.

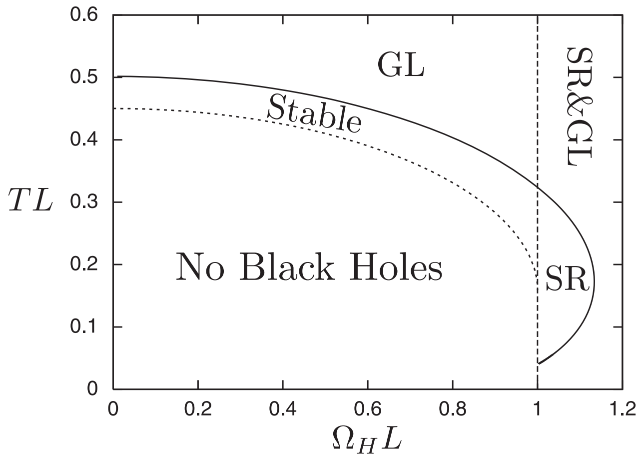

Fig. 1. This is the phase diagram of a small Kerr-AdS$_5 \times S^5$ black hole, respectively. The values Ω_H and T are angular velocity and temperature of Kerr-AdS$_5$ black holes. These are normalized using the curvature scale of the AdS spacetime, L. In the "Stable" region, Kerr-AdS black holes are stable. In the "SR" and "GL" regions, black holes are unstable against superradiant and Gregory-Laflamme instabilities, respectively. In the "SR&GL" region, black holes are unstable against both of instabilities. In the "No Black Holes" region, there is no black hole solution.

solid and dashed lines are onset of the Gregory-Laflamme and superradiant instabilities, respectively. These lines cross each other and we can see five phases in this diagram. In the "Stable" region, Kerr-AdS black holes are stable. In the "SR" and "GL" regions, black holes are unstable against superradiant and Gregory-Laflamme instabilities, respectively. In the "SR&GL" region, black holes are unstable against both of instabilities. In the "No Black Holes" region, there is no black hole solution. The final section is devoted to the conclusion.

§2. Kerr-AdS$_5$ black hole in type IIB supergravity

2.1. *Kerr-AdS$_5 \times S^5$ spacetime with equal angular momenta*

In this section, we introduce Kerr-AdS$_5$ spacetime as a solution of type IIB supergravity. The equations of motion of type IIB supergravity are given by

$$R_{MN} = \frac{1}{48} F_{MP_2P_3P_4P_5} F_N{}^{P_2P_3P_4P_5} - \frac{1}{480} g_{MN} F_{P_1P_2P_3P_4P_5} F^{P_1P_2P_3P_4P_5} \ , \qquad (2\cdot1)$$

$$\nabla_{P_1} F^{P_1P_2P_3P_4P_5} = 0 \ , \qquad (2\cdot2)$$

where $M, N, \cdots = 0, 1, \cdots, 9$. The form $F_{M_1M_2M_3M_4M_5}$ is RR 5-form satisfying $d\boldsymbol{F} = 0$ and $*\boldsymbol{F} = \boldsymbol{F}$. We concentrate on the metric and RR 5-form field in type IIB supergravity, while other components, such as dilaton, NSNS 3-form, RR 1-form, and 3-form, have been set to zero. We will consider Kerr-AdS$_5 \times S^5$ spacetime, which is a solution of (2·1) and (2·2). The Kerr-AdS$_5$ spacetime can generally have two independent angular momenta, but for simplicity, we will consider the case of two equal angular momenta. Then, the spacetime symmetry is enhanced and stability analysis will be possible. The metric of Kerr-AdS$_5 \times S^5$ spacetime with equal angular momenta is given by*)

$$ds^2 = -\left(1+\frac{r^2}{L^2}\right)dt^2 + \frac{dr^2}{G(r)} + \frac{r^2}{4}\{(\sigma^1)^2+(\sigma^2)^2+(\sigma^3)^2\} + \frac{2\mu}{r^2}\left(dt+\frac{a}{2}\sigma^3\right)^2 + L^2 d\Omega_5^2\ , \quad (2\cdot3)$$

where $G(r)$ is defined as

$$G(r) = 1+\frac{r^2}{L^2} - \frac{2\mu(1-a^2/L^2)}{r^2} + \frac{2\mu a^2}{r^4}\ . \quad (2\cdot4)$$

Then, RR 5-form is

$$\boldsymbol{F} = 2^{3/2}L^{-1}(\boldsymbol{\epsilon}_{\mathrm{AdS}_5} + \boldsymbol{\epsilon}_{S^5})\ , \quad (2\cdot5)$$

where $\boldsymbol{\epsilon}_{S^5}$ is the volume form of $L^2 d\Omega_5^2$ and $\boldsymbol{\epsilon}_{\mathrm{AdS}_5}$ is the volume form of the AdS_5 part of (2·3). Because of the relation, $*\boldsymbol{\epsilon}_{S^5} = \boldsymbol{\epsilon}_{\mathrm{AdS}_5}$, the form $\boldsymbol{F}$ satisfies the self-dual condition. In (2·3), we have defined the invariant forms σ^a $(a=1,2,3)$ of $SU(2)$ as

$$\begin{aligned}\sigma^1 &= -\sin\psi d\theta + \cos\psi\sin\theta d\phi\ ,\\ \sigma^2 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi\ ,\\ \sigma^3 &= d\psi + \cos\theta d\phi\ ,\end{aligned} \quad (2\cdot6)$$

where $0 \le \theta < \pi$, $0 \le \phi < 2\pi$, $0 \le \psi < 4\pi$. It is easy to check the relation $d\sigma^a = 1/2\epsilon^{abc}\sigma^b \wedge \sigma^c$. The dual vectors of σ^a are given by

$$\begin{aligned}e_1 &= -\sin\psi\partial_\theta + \frac{\cos\psi}{\sin\theta}\partial_\phi - \cot\theta\cos\psi\partial_\psi\ ,\\ e_2 &= \cos\psi\partial_\theta + \frac{\sin\psi}{\sin\theta}\partial_\phi - \cot\theta\sin\psi\partial_\psi\ ,\\ e_3 &= \partial_\psi\ ,\end{aligned} \quad (2\cdot7)$$

and, by the definition, they satisfy $\sigma^a_i e^i_b = \delta^a_b$.

The horizon radius $r=r_+$ can be determined as $G(r_+)=0$. The angular velocity of Kerr-AdS back hole is given by

$$\Omega_H = \frac{2\mu a}{r_+^4 + 2\mu a^2}\ . \quad (2\cdot8)$$

For the existence of horizon, the angular velocity has the upper boundary,

$$\Omega_H \le \left(\frac{1}{2r_+^2}+\frac{1}{L^2}\right)^{1/2} \equiv \Omega_H^{\max}\ . \quad (2\cdot9)$$

In terms of r_+ and Ω_H, the two parameters (a,μ) in the metric (2·3) can be rewritten as

$$a = \frac{r_+^2\Omega_H}{1+r_+^2/L^2}\ , \quad \mu = \frac{1}{2}\frac{r_+^2(1+r_+^2/L^2)^2}{1-(\Omega_H^2L^2-1)r_+^2/L^2}\ . \quad (2\cdot10)$$

We will mainly use parameters (r_+, Ω_H).

*) To obtain this metric from Kerr-AdS_5 spacetime given in Refs. 26)–29), we need some coordinate transformation and redefinition of a parameter. These are summarized in Appendix A.

2.2. *Spacetime symmetry*

Now, we study the symmetry of (2·3). It will be important for separating variables of the gravitational perturbation equations in §§3 and 4. Clearly, the metric (2·3) has the time translation symmetry R_t, and the $SO(6)$ symmetry comes from the S^5 part of (2·3). Additionally, the spacetime has the $SU(2)$ symmetry characterized by the Killing vectors ξ_α , $(\alpha = x, y, z)$:

$$\begin{aligned}
\xi_x &= \cos\phi\partial_\theta + \frac{\sin\phi}{\sin\theta}\partial_\psi - \cot\theta\sin\phi\partial_\phi \ , \\
\xi_y &= -\sin\phi\partial_\theta + \frac{\cos\phi}{\sin\theta}\partial_\psi - \cot\theta\cos\phi\partial_\phi \ , \\
\xi_z &= \partial_\phi \ .
\end{aligned} \tag{2·11}$$

The symmetry can be explicitly shown using the relation $\mathcal{L}_{\xi_\alpha}\sigma^a = 0$, where $\mathcal{L}_{\xi_\alpha}$ is a Lie derivative along the curve generated by the vector field ξ_α.

From the metric (2·3), we can also read off the additional $U(1)$ symmetry, which retains the part of the metric, $(\sigma^1)^2 + (\sigma^2)^2$ and is generated by e_3. The $U(1)$ generator e_3 satisfies $\mathcal{L}_{e_3}\sigma^1 = -\sigma^2$ and $\mathcal{L}_{e_3}\sigma^2 = \sigma^1$ and, thus, $\mathcal{L}_{e_3}[(\sigma^1)^2 + (\sigma^2)^2] = 0$. Therefore, the symmetry of Kerr-AdS$_5 \times S^5$ spacetime with equal angular momenta becomes $R_t \times SU(2) \times U(1) \times SO(6)$.

For later calculations, it is convenient to define the new invariant forms

$$\sigma^\pm = \frac{1}{2}(\sigma^1 \mp i\sigma^2) \ . \tag{2·12}$$

Then, the dual vectors for $\sigma^\pm$ are

$$\boldsymbol{e}_\pm = \boldsymbol{e}_1 \pm i\boldsymbol{e}_2 \ . \tag{2·13}$$

By making use of these forms, the metric (2·3) can be rewritten as

$$\begin{aligned}
ds^2 = &-\left(1 + \frac{r^2}{L^2}\right)dt^2 + \frac{dr^2}{G(r)} + \frac{r^2}{4}\{4\sigma^+\sigma^- + (\sigma^3)^2\} \\
&+ \frac{2\mu}{r^2}\left(dt + \frac{a}{2}\sigma^3\right)^2 + L^2 d\Omega_5^2 \ .
\end{aligned} \tag{2·14}$$

We will use this expression in the following sections.

§3. Superradiant instability of Kerr-AdS black holes

In the following sections, we will study the stability of Kerr-AdS$_5 \times S^5$ spacetime with equal angular momenta (2·3). In this spacetime, we can expect two types of instability. One is the superradiant instability that is caused by the wave amplification via superradiance and by wave reflection due to the potential barrier of the AdS spacetime. This instability should be observed, even if Kaluza-Klein modes are neglected. The other instability is the Gregory-Laflamme instability, which is the instability of the internal space S^5, that is, the Gregory-Laflamme instability is the instability of Kaluza-Klein modes. First, we shall see the superradiant instability of Kerr-AdS$_5$ spacetime in this section.

3.1. Perturbation equations and separability

To see the superradiant instability, we can neglect Kaluza-Klein modes of S^5. In addition, we will consider only metric fluctuation on the AdS_5 part of the spacetime, that is,

$$g'_{MN}dx^M dx^N = g_{MN}dx^M dx^N + h_{\mu\nu}(x^\mu)dx^\mu dx^\nu \ ,$$
$$\boldsymbol{F}' = 2^{3/2}L^{-1}(\boldsymbol{\epsilon}'_{\mathrm{AdS}_5} + \boldsymbol{\epsilon}_{S^5}), \tag{3·1}$$

where $\mu, \nu, \cdots$ are indexes on AdS_5 and $\boldsymbol{\epsilon}'_{\mathrm{AdS}_5}$ is volume form of $g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$. For the perturbations, (2·2) is trivially satisfied and (2·1) becomes

$$\delta G_{\mu\nu} - \frac{6}{L^2}h_{\mu\nu} = 0 \ , \tag{3·2}$$

where $\delta G_{\mu\nu}$ is perturbation of the Einstein tensor of five-dimensional metric $g_{\mu\nu}$, which is defined as

$$\delta G_{\mu\nu} = \frac{1}{2}[\nabla^\rho\nabla_\mu h_{\nu\rho} + \nabla^\rho\nabla_\nu h_{\mu\rho} - \nabla^2 h_{\mu\nu} - \nabla_\mu\nabla_\nu h$$
$$- g_{\mu\nu}(\nabla^\rho\nabla^\sigma h_{\rho\sigma} - \nabla^2 h - R^{\rho\sigma}h_{\rho\sigma}) - Rh_{\mu\nu}] \ , \tag{3·3}$$

where ∇_μ denotes the covariant derivative with respect to $g_{\mu\nu}$ and $h = g^{\mu\nu}h_{\mu\nu}$. Tensors $R_{\rho\sigma}$ and R are Ricci tensor and Ricci scalar of $g_{\mu\nu}$, respectively. We take the AdS_5 part of (2·14) as a background metric $g_{\mu\nu}$. Equation (3·2) is nothing but the perturbation of five-dimensional Einstein equations with the negative cosmological constant.

The perturbation equation (3·2) is a partial differential equation of $h_{\mu\nu}(t, r, \theta, \phi, \psi)$. However, in previous works,[17),18),20),30)] it was shown that the perturbation equations can be reduced to ordinary differential equations by focusing on the symmetry of the background spacetime, $R_t \times SU(2) \times U(1)$. Here, we will briefly review these works.

Let us define the two types of angular momentum operator,

$$L_\alpha = i\xi_\alpha \ , \quad W_a = ie_a \ , \tag{3·4}$$

where $\alpha, \beta, \cdots = x, y, z$ and $a, b, \cdots = 1, 2, 3$. They satisfy commutation relations

$$[L_\alpha, L_\beta] = i\epsilon_{\alpha\beta\gamma}L_\gamma \ , \quad [W_a, W_b] = -i\epsilon_{abc}W_c \ , \quad [L_\alpha, W_a] = 0 \ , \tag{3·5}$$

where $\epsilon_{\alpha\beta\gamma}$ and ϵ_{abc} are antisymmetric tensors that satisfy $\epsilon_{123} = \epsilon_{xyz} = 1$. The Casimir operators constructed using L_α and W_a are identical and we define $L^2 \equiv L_\alpha^2 = W_a^2$. The symmetry group, $SU(2) \times U(1)$ is generated using L_α and W_3. Here, we should note the following facts:

$$\mathcal{L}_{W_3}\sigma^\pm = \pm\sigma^\pm \ , \quad \mathcal{L}_{W_3}\sigma^3 = 0 \ . \tag{3·6}$$

It means that $\sigma^\pm$ and σ^3 have $U(1)$ charges ± 1 and 0. Since operators L^2, L_z, and W_3 commute each other, these are simultaneously diagonalizable. The eigenfunctions are called Wigner functions $D^J_{KM}(\theta, \phi, \psi)$ defined as

$$L^2 D^J_{KM} = J(J+1)D^J_{KM} \ , \quad L_z D^J_{KM} = MD^J_{KM} \ , \quad W_3 D^J_{KM} = KD^J_{KM} \ , \tag{3·7}$$

where the indexes J, K and M are defined as $J = 0, 1/2, 1, \cdots$ and $K, M = -J, -J+1, \cdots J$. The following relations are useful for later calculations:

$$W_+ D^J_{KM} = i\epsilon_K D^J_{K-1,M} \ , \quad W_- D^J_{KM} = -i\epsilon_{K+1} D^J_{K+1,M} \ , \quad W_3 D^J_{KM} = K D^J_{KM} \ , \tag{3·8}$$

where we have defined $W_\pm = W_1 \pm iW_2$ and $\epsilon_K = \sqrt{(J+K)(J-K+1)}$. From this relation, we obtain the differential rule of the Wigner function as

$$\partial_+ D^J_{KM} = \epsilon_K D^J_{K-1,M} \ , \quad \partial_- D^J_{KM} = -\epsilon_{K+1} D^J_{K+1,M} \ , \quad \partial_3 D^J_{KM} = -iK D^J_{KM} \ , \tag{3·9}$$

where we have defined $\partial_\pm \equiv e^i_\pm \partial_i$ and $\partial_3 \equiv e^i_3 \partial_i$.

Now, we consider the mode expansion of $h_{\mu\nu}$. The metric perturbations can be classified into three parts, h_{AB}, h_{Ai}, h_{ij}, where $A, B = t, r$ and $i, j = \theta, \phi, \psi$. They behave as scalar, vector, and tensor for coordinate transformation of θ, ϕ, ψ, respectively. The scalar h_{AB} can be expanded using Wigner functions immediately as

$$h_{AB} = \sum_K h^K_{AB}(x^A) D_K(x^i) \ . \tag{3·10}$$

Here, we have omitted the indexes J, M because the differential rule of Wigner function (3·9) cannot shift J, M and therefore the modes with different eigenvalues J, M are trivially decoupled in the perturbation equations.

To expand the vector part h_{Ai}, we need a device. First, we change the basis $\{\partial_i\}$ to $\{e^a\}$, that is, $h_{Ai} = h_{Aa}\sigma^a_i$ where $a = \pm, 3$. Then, because h_{Aa} is scalar, we can expand it using the Wigner function as

$$\begin{aligned} h_{Ai}(x^\mu) &= h_{A+}(x^\mu)\sigma^+_i + h_{A-}(x^\mu)\sigma^-_i + h_{A3}(x^\mu)\sigma^3_i \\ &= \sum_K \left[h^K_{A+}(x^A)\sigma^+_i D_{K-1} + h^K_{A-}(x^A)\sigma^-_i D_{K+1} + h^K_{A3}(x^A)\sigma^3_i D_K \right] \ . \end{aligned} \tag{3·11}$$

In the expansion of h_{A+}, h_{A-}, and h_{A3}, we have shifted the index K of Wigner functions, for example, h_{A+} has been expanded as $\sum_K h^K_{A+} D_{K-1}$. The reason is as follows. The invariant forms $\sigma^\pm$ and σ^3 have the $U(1)$ charges ± 1 and 0, respectively (see Eq. (3·6)), while the Wigner function D_K has the $U(1)$ charge K (see Eq. (3·7)). Therefore, by shifting the index K, we can assign the same $U(1)$ charge K to $\sigma^+_i D_{K-1}$, $\sigma^-_i D_{K+1}$, and $\sigma^3_i D_K$ in Eq. (3·11).

The expansion of tensor part h_{ij} can be carried out in a similar manner as

$$\begin{aligned} h_{ij}(x^\mu) = \sum_K \Big[& h^K_{++}\sigma^+_i\sigma^+_j D_{K-2} + 2h^K_{+-}\sigma^+_i\sigma^-_j D_K + 2h^K_{+3}\sigma^+_i\sigma^3_j D_{K-1} \\ & + h^K_{--}\sigma^-_i\sigma^-_j D_{K+2} + 2h^K_{-3}\sigma^-_i\sigma^3_j D_{K+1} + h^K_{33}\sigma^3_i\sigma^3_j D_K \Big] \ . \end{aligned} \tag{3·12}$$

To assign the same $U(1)$ charge K to each term, we have shifted the index K of Wigner functions.

Substituting Eqs. (3·10)–(3·12) into the perturbation equations (3·2), we obtain the equations for each mode labelled by J, M, K. Because of $SU(2)\times U(1)$ symmetry, different eigenmodes cannot appear in the same equation.

It is interesting that we can find master variables from the above information. First, we should note that coefficients of the expansion have different indexes K and, therefore, coefficients of components h^K_{AB}, h^K_{Aa} and h^K_{ab} are restricted as follows:

h_{++}	h_{A+}, h_{+3}	$h_{AB}, h_{A3}, h_{+-}, h_{33}$	h_{A-}, h_{-3}	h_{--}
$\lvert K-2\rvert \le J$	$\lvert K-1\rvert \le J$	$\lvert K\rvert \le J$	$\lvert K+1\rvert \le J$	$\lvert K+2\rvert \le J$

From this table, we can see that, in $K = \pm(J+2)$ mode, there is only one variable $h_{\pm\pm}$, respectively. Therefore, the $(J, M, K = \pm(J+2))$ modes always reduce to a single master equation. We will study the stability of these modes. In fact, $(J = 0, M = 0, K = 0, \pm 1)$ modes also reduce to a single master equation. The stabilities of $(J = 0, M = 0, K = 0, \pm 1)$ modes are described in Appendix B and we will see that these modes are irrelevant to see the onset of the superradiant instability.

3.2. *Master equations*

We will derive the master equation for $(J, M, K = \pm(J+2))$ modes. Because of the relation $h_{++} = h^*_{--}$, we will consider $(J, M, K = J+2)$ modes only. Then, we can set $h_{\mu\nu}$ as

$$h_{\mu\nu}(x^\mu)dx^\mu dx^\nu = h_{++}(r)e^{-i\omega t}D_J(x^i)\sigma^+\sigma^+ \ , \tag{3·13}$$

where $D_J \equiv D^J_{K=J,M}$. This h_{++} field is gauge-invariant. We substitute Eq. (3·13) into Eq. (3·2) and use the differential rule of Wigner functions (3·9). Then, $++$ component of (3·2) is given by

$$\begin{aligned}\frac{1}{2r^{10}G(r)}\Big[& -r^{10}G(r)^2 h''_{++} - r^5 G(r)(6\mu r^2\lambda a^2 - 10\mu a^2 + 6\mu r^2 - \lambda r^6 - r^4)h'_{++} \\ & - \big\{ -4\lambda^2 r^{12} + (4\lambda(3+3J+J^2)+\omega^2)r^{10} - 4(J+1)(J+2)r^8 \\ & - 2\mu(-4 + 16\lambda a^2 + 4J\lambda a^2 - 12J - 4J^2 + 4a(J+2)\omega - a^2\omega^2)r^6 \\ & + 8\mu(2\mu + 2\mu\lambda^2 a^4 + 4\mu a^2\lambda + Ja^2 + 4a^2)r^4 \\ & - 48a^2\mu^2(1+\lambda a^2)r^2 + 32\mu^2 a^4\big\}h_{++}\Big] e^{-i\omega t}D_J(\theta,\phi,\psi) = 0 \ .\end{aligned} \tag{3·14}$$

This equation can be rewritten as

$$-\frac{d^2\Phi}{dr_*^2} + V(r)\Phi = [\omega - 2(J+2)\Omega(r)]^2\Phi \ , \tag{3·15}$$

where we have introduced the new variable,

$$\Phi = \frac{(r^4 + 2\mu a^2)^{1/4}}{r^{3/2}}h_{++} \ , \tag{3·16}$$

and the tortoise coordinate,

$$dr_* = \frac{(r^4 + 2\mu a^2)^{1/2}}{r^2 G(r)}dr \ . \tag{3·17}$$

The function $\Omega(r)$ and potential $V(r)$ are given by

$$\Omega(r) = \frac{2\mu a}{r^4 + 2\mu a^2}, \tag{3·18}$$

and

$$\begin{aligned} V(r) = \frac{G(r)}{4r^2(r^4+2\mu a^2)^3}\big[&15r^{14}/L^2 + (4J+7)(4J+5)r^{12} + 6\mu(3+11a^2/L^2)r^{10} \\ &+ 2\mu a^2(16J^2+32J+5)r^8 - 4\mu^2 a^2(10-17a^2/L^2)r^6 \\ &- 4\mu^2 a^4(16J+35)r^4 + 8\mu^3 a^4(1-a^2/L^2)r^2 - 40\mu^3 a^6\big] . \end{aligned} \tag{3·19}$$

We can obtain the asymptotic form of $\Omega(r)$ and $V(r)$ as

$$\Omega(r) \to \Omega_H \quad (r \to r_+) , \quad \Omega(r) \to 0 \quad (r \to \infty) , \tag{3·20}$$

and

$$V(r) \to 0 \quad (r \to r_+) , \quad V(r) \to \frac{15r^2}{4L^4} \quad (r \to \infty) , \tag{3·21}$$

where Ω_H is the angular velocity of the horizon, which is defined in Eq. (2·8). Therefore, the asymptotic form of the solution of master equation (3·15) becomes

$$\Phi \to e^{\pm i\{\omega - 2(J+2)\Omega_H\}r_*} \quad (r \to r_+) , \quad \Phi \to r^{-1/2\pm 2} \quad (r \to \infty) . \tag{3·22}$$

We will solve (3·15) numerically and show the superradiant instability.

3.3. *Stability analysis*

3.3.1. A method of studying the stability

We will find the instability by the shooting method. Then, since the master equation (3·15) is not a self-adjoint form, we should put $\omega = \omega_R + i\omega_I$ ($\omega_R, \omega_I \in \boldsymbol{R}$) and there are two shooting parameters, ω_R and ω_I. However, if the purpose is to find the onset of instability, the number of shooting parameters can be reduced to one.[12),20),31)]

We have separated the time dependence as $h_{\mu\nu} \propto e^{-i\omega t}$ in (3·13). Therefore, the unstable mode satisfies $\mathrm{Im}\,\omega > 0$. Thus, the boundary condition for regularity at the horizon becomes

$$\Phi \to e^{-i\{\omega - 2(J+2)\Omega_H\}r_*}. \quad (r \to r_+) \tag{3·23}$$

Then, the general form of wave function at infinity becomes

$$\Phi \to Z_1 r^{-5/2} + Z_2 r^{3/2}, \quad (r \to \infty) \tag{3·24}$$

where Z_1 and Z_2 are constants. For regularity at infinity, the condition $Z_2 = 0$ must be satisfied. Therefore, the boundary conditions that the unstable mode satisfies are

$$\Phi \to e^{-i\{\omega - 2(J+2)\Omega_H\}r_*} \quad (r \to r_+) , \quad \Phi \to Z_1 r^{-5/2} \quad (r \to \infty) . \tag{3·25}$$

To study the stability, we start from the assumption that, for sufficiently small angular velocity Ω_H, Kerr-AdS$_5$ black holes are stable. This is a natural assumption because higher dimensional Schwarzschild-AdS$_5$ black hole is stable.[6)] From this assumption, for small Ω_H, wave functions with the boundary condition (3·25) must be $\mathrm{Im}\,\omega < 0$, that is, quasinormal modes. If there is the instability, for large Ω_H, some mode becomes $\mathrm{Im}\,\omega > 0$ as Ω_H becomes large. It indicates that one of the quasinormal modes must cross the real axis in the complex ω plane for some Ω_H. Therefore, if the black hole is unstable for large Ω_H, there is some critical value $\Omega_H = \Omega_H^{\mathrm{crit}}$ such that there exists a mode with $\mathrm{Im}\,\omega = 0$. We will look for such Ω_H^{crit}. For the purpose of searching Ω_H^{crit}, we can assume that $\mathrm{Im}\,\omega = 0$. In this case of $\omega \in \boldsymbol{R}$, Wronskian of Φ_K is conserved, that is,

$$\mathrm{Im}\left[\Phi^* \frac{d}{dr_*}\Phi\right]_{r=r_1}^{r=r_2} = 0 \ , \tag{3·26}$$

for any r_1 and r_2. We take $r_1 = r_+$ and $r_2 = \infty$. Then, from Eq. (3·26), we can get the relation,

$$2(J+2)\Omega_H - \omega = -4L^{-2}\,\mathrm{Im}(Z_1 Z_2^*) \ . \tag{3·27}$$

where we have used the asymptotic form of (3·23) and (3·24). To avoid divergence at infinity, $Z_2 = 0$ must be satisfied. Then, we can get

$$\omega = 2(J+2)\Omega_H \ . \tag{3·28}$$

Therefore, the equation that we should solve is

$$-\frac{d^2\Phi}{dr_*^2} + \hat{V}(r)\Phi = 0 \ , \tag{3·29}$$

where

$$\hat{V}(r) \equiv V(r) - 4(J+2)^2(\Omega_H - \Omega(r))^2 \ . \tag{3·30}$$

The boundary condition can be obtained by substituting $\Phi = \Phi(r_+) + \Phi'(r_+)(r - r_+)$ into (3·30) and it is given by

$$\frac{\Phi'(r_+)}{\Phi(r_+)} = \frac{r_+^4 + 2\mu a^2}{r_+^4}\frac{V'(r_+)}{G'(r_+)^2} \ . \tag{3·31}$$

For fixed r_+, there is only one shooting parameter, Ω_H, in (3·29).

3.3.2. Limit of small Kerr-AdS black holes

Before the numerical calculation, it is important to solve the master equation (3·29) analytically in some limit.[32)] It may be useful to check the numerical calculation. We consider Kerr-AdS black holes in the limit of $r_+ \to 0$. Then, the master equation (3·15) can be solved accurately. The solution that approaches zero at infinity is

$$\Phi = \frac{(r/L)^{7/2+2J}}{(1+r^2/L^2)^{J+3}} F\left((J+2)\Omega_H L + J + 3, -(J+2)\Omega_H L + J + 3; 3; \frac{1}{1+r^2/L^2}\right) \ , \tag{3·32}$$

where $F(\alpha, \beta; \gamma; z)$ is Gauss hypergeometric function. Then, the asymptotic form of $r \to 0$ becomes

$$\Phi = \frac{2(2J+2)!}{\Gamma[(J+2)\Omega_H L + J + 3]\Gamma[-(J+2)\Omega_H L + J + 3]} \left(\frac{r}{L}\right)^{-2J-5/2} - \frac{4(-1)^{2J+3}}{(2J+3)!\Gamma[(J+2)\Omega_H L - J]\Gamma[-(J+2)\Omega_H L - J]} \left(\frac{r}{L}\right)^{2J+7/2} \ln\left(\frac{r}{L}\right). \quad (3\cdot33)$$

For the regularity at horizon, the first term of (3·33) must vanish. Thus, we can get $\Omega_H L = (J+3+p)/(J+2)$ where $p = 0, 1, 2, \cdots$. This calculation is to see the onset of the instability and the lowest value of Ω_H is important. The lowest value of Ω_H is given by

$$\Omega_H L = \frac{J+3}{J+2} . \quad (3\cdot34)$$

The numerical result must approach this value in the limit of $r_+ \to 0$.

3.3.3. Onset of superradiant instability

Now, we shall solve (3·29) numerically. Using the Lunge-Kutta algorithm, we integrate Eq. (3·29) from the horizon to infinity with various Ω_H. The boundary conditions at the horizon are given by (3·31). Then, the general form of the wave function at infinity is given by (3·24). We can see that, at some value of Ω_H, Z_2 flips the sign. It means that $Z_2 = 0$ mode exists. We will search such Ω_H numerically and plot the result in Ω_H-r_+ diagram. The result is shown in Fig. 2. The curves represent the borderline of stability and instability of each mode, that is, each mode is stable below the curve, while they are unstable above the curve. From this figure, we can read off that, in the limit of $r_+ \to 0$, these curves for each mode approach $\Omega_H = (J+3)/(J+2)$. This result is consistent for analytical calculation in §3.3.2. We can also see that, for higher J mode, the instability occurs at a lower angular velocity. These curves seem to approach $\Omega_H L = 1$ for large J. These properties are the same for $D = 7, 9, 11, \cdots$ cases.[12)]

It is surprising that these results have already been seen in dual gauge theory.[33),34)] In Ref. 34), the effective mass term for scalar fields of dual gauge theory has been obtained as

$$m_{\rm eff}^2 = (2J+1)^2 L^{-2} - 4\Omega_H^2 K^2 . \quad (3\cdot35)$$

Because of $|K| \leq J$, if $\Omega_H L < 1$ is satisfied, $m_{\rm eff}^2$ is positive for any J and K. However, if $\Omega_H L > 1$, $m_{\rm eff}^2$ can be negative for large J and K modes. Thus, we see that, for $\Omega_H L > 1$, the dual gauge theory is unstable and higher J mode becomes tachyonic first as Ω_H increases. These results are the same for superradiant instability of Kerr-AdS$_5$ black holes.

§4. Gregory-Laflamme instability of Kerr-AdS$_5 \times S^5$ spacetimes

4.1. *Perturbation equation*

In the previous section, we have seen the superradiant instability of Kerr-AdS$_5 \times S^5$ spacetime. The superradiant instability breaks the symmetry of Kerr-AdS$_5$. In

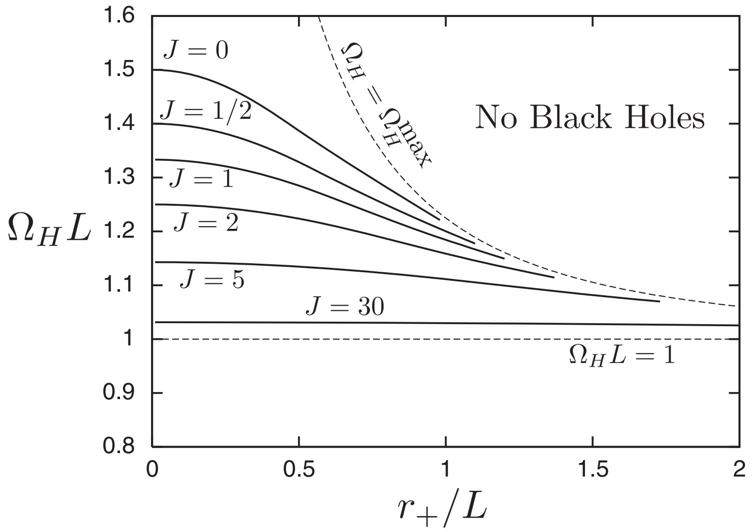

Fig. 2. The onset of superradiant instability is depicted in Ω_H-r_+ diagram. We plot onset lines for $J = 0, 1/2, 1, 2, 5, 30$ modes using solid lines. At the above dashed line, the Kerr-AdS$_5$ black holes become extreme and, in the upper-right region, there is no black hole solution. The dashed line below is $\Omega_H L = 1$. We can see that onset line of higher J modes appears at lower Ω_H and these lines approach $\Omega_H L = 1$ for $J \to \infty$.

this section, we will consider the Gregory-Laflamme instability of Kerr-AdS$_5 \times S^5$ spacetimes. This instability breaks the symmetry of S^5. Thus, we must see the Kaluza-Klein modes of the perturbations that have been neglected in the previous section.

We will consider only the metric fluctuations on the AdS$_5$ part of the spacetime, that is,

$$\begin{aligned} g'_{MN}dx^M dx^N &= g_{MN}dx^M dx^N + h_{\mu\nu}(x^\mu)Y_\ell(\Omega_5)dx^\mu dx^\nu \ , \\ \boldsymbol{F}' &= 2^{3/2}L^{-1}(\boldsymbol{\epsilon}'_{\mathrm{AdS}_5} + \boldsymbol{\epsilon}_{S^5}) \ , \end{aligned} \tag{4·1}$$

where $Y_\ell(\Omega_5)$ is spherical harmonics on S^5 that satisfy

$$\nabla^2_{S^5} Y_\ell = -\ell(\ell+4)Y_\ell \ , \tag{4·2}$$

here, $\nabla^2_{S^5}$ is the Laplacian of S^5 and $\ell = 0, 1, 2, \cdots$. The $\boldsymbol{\epsilon}'_{\mathrm{AdS}_5}$ in (4·1) is the volume form of $g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}(x^\mu)Y_\ell(\Omega_5)$. Since, in the case of Schwarzschild-AdS$_5 \times S^5$, the Gregory-Laflamme instability has been found in these fluctuations,[25] the instability of Kerr-AdS$_5 \times S^5$ must also appear in these fluctuations. Here, we should note that, in (4·1), $h_{\mu\nu}$ depends on the coordinates on S^5. It is essential to see the Gregory-Laflamme instability. Then, from (2·1), we can obtain the perturbation equations as

$$\delta G_{\mu\nu} = \frac{6}{L^2}h_{\mu\nu} - \frac{\varepsilon}{L^2}\left(h_{\mu\nu} - \frac{1}{2}h\right) \ , \tag{4·3}$$

where $\varepsilon = \ell(\ell+4)/2$ and $\delta G_{\mu\nu}$ is defined in (3·3). From (2·2), we can get

$$h(x^\mu)\partial_a Y_\ell(\Omega_5) = 0 \ , \tag{4·4}$$

where a is index on S^5. In the case $\ell = 0$, (4·4) is trivially satisfied and (4·3) reduces to (3·2). For $\ell \geq 1$, (4·4) implies $h(x^\mu) = 0$. Then, as a constraint equation of (4·3), we can get the transverse condition of $h_{\mu\nu}$ as shown in Appendix C. Thus, even for Kaluza-Klein modes, we can use transverse traceless conditions,

$$\nabla^\nu h_{\mu\nu} = g^{\mu\nu} h_{\mu\nu} = 0 \ . \tag{4·5}$$

To separate variables of Eqs. (4·3) and (4·5), we will use the formalism in §3.1 again. We can expand $h_{\mu\nu}$ using the Wigner function D^J_{KM} and obtain ordinary differential equations labeled by (J, K, M). We will not study the stability of all the modes, but we will consider only the $J = M = K = 0$ mode. The Gregory-Laflamme instability of Schwarzschild-AdS$_5 \times S^5$ spacetime was found in the s-wave of AdS$_5$[25] and, thus, we can expect that, in the Kerr-AdS$_5 \times S^5$ spacetime, the instability appears in the most symmetric mode, $J = M = K = 0$. The metric perturbation for this mode is given by

$$\begin{aligned} h_{\mu\nu}(x^\mu)dx^\mu dx^\nu = h_{tt}(r)dt^2 + 2h_{tr}(r)dtdr + h_{rr}(r)dr^2 + 2h_{t3}(r)dt\sigma^3 \\ + 2h_{r3}(r)dr\sigma^3 + 2h_{+-}(r)\sigma^+\sigma^- + h_{33}(r)\sigma^3\sigma^3 \ , \end{aligned} \tag{4·6}$$

where we assume that the metric perturbation $h_{\mu\nu}$ does not depend on t, in order to observe the onset of Gregory-Laflamme instability. The stability of $\ell = 0$ mode of this perturbation is shown in Appendix B.1 and, thus, we will consider $\ell = 1, 2, 3, \cdots$. We substitute (4·6) into (4·3). Then, from tr and $r3$ components of the perturbation equation, we can obtain

$$h_{tr} = h_{r3} = 0 \ . \tag{4·7}$$

Now, we introduce dimensionless variables, $\alpha, \beta, \delta, \eta$, and ζ, as

$$\begin{aligned} h_{tt} &= -\left(1 + \frac{r^2}{L^2} - \frac{2\mu}{r^2}\right)\alpha \ , \quad h_{rr} = \frac{\beta}{G(r)} \ , \quad h_{+-} = \frac{r^2}{4}\delta, \\ h_{33} &= \frac{r^2}{4}\left(1 + \frac{2\mu a^2}{r^4}\right)\eta \ , \quad h_{t3} = \frac{2\mu a}{r^2}\zeta \ . \end{aligned} \tag{4·8}$$

Hereafter, we put $L = 1$ to simplify the expressions. Then, the traceless condition $h = 0$ can be written as

$$\begin{aligned} (r^4 + r^2 - 2\mu)(r^4 + 2\mu a^2)\alpha + r^6 G(r)\beta + r^6 G(r)\delta \\ + (r^4 + r^2 - 2\mu)(r^4 + 2\mu a^2)\eta + 16\mu^2 a^2 \zeta = 0 \ , \end{aligned} \tag{4·9}$$

and the r-component of transverse condition $\nabla^\nu h_{\mu\nu} = 0$ is

$$\begin{aligned} &- (r^4 + r^2 - 2\mu)\{r^8 + 2\mu(1 + 2a^2)r^4 - 4\mu^2 a^2(1 + a^2)\}\alpha \\ &+ r^9 G(r)^2 \beta' + r^4 G(r)\{4r^6 + 3r^4 - 4\mu(1 - a^2)r^2 + 2\mu a^2\}\beta + r^8 G(r)^2 \delta \\ &- (r^4 + 2\mu a^2)\{r^8 + 2r^6 - (2\mu a^2 + 4\mu - 1)r^4 - 4\mu(a^2 + 1)r^2 + 2\mu(2\mu + 2\mu a^2 + a^2)\}\eta \\ &- 16\mu^2 a^2\{3r^4 + 2r^2 - 2\mu(1 - a^2)\}\zeta = 0 \ , \end{aligned} \tag{4·10}$$

where $' \equiv d/dr$. We can see that, using (4·9) and (4·10), α and ζ can be eliminated.*) Then, rr,$+-$ and 33 components of (4·3) are given by

$$\begin{aligned}&-r^6G(r)\beta'' - r\{13r^6+9r^4-10\mu(1-a^2)r^2+2\mu a^2\}\beta'\\ &-2\{(16-\varepsilon)r^6+6r^4-4\mu a^2\}\beta+4(r^4-2\mu a^2)\delta+4(r^4+2\mu a^2)\eta=0\ ,\end{aligned} \tag{4·11}$$

$$\begin{aligned}&4r^5G(r)\beta'+8(2r^6+r^4-2\mu a^2)\beta-r^6G(r)\delta''\\ &-r\{5r^6+3r^4-2\mu(1-a^2)r^2-2\mu a^2\}\delta'+2(\varepsilon r^6+8\mu a^2)\delta-8(r^4+2\mu a^2)\eta=0\ ,\end{aligned} \tag{4·12}$$

$$\begin{aligned}&2r^5G(r)(r^4+2\mu a^2)\beta''+2(r^4+2\mu a^2)\{11r^6+7r^4-6\mu(1-a^2)r^2-2\mu a^2\}\beta'\\ &+16r^3(3r^2+1)(r^4+2\mu a^2)\beta-2(r^8-4\mu r^6+4\mu a^2r^4+4\mu^2a^4)\delta'-8r^3(r^4+2\mu a^2)\delta\\ &-r^5G(r)(r^4+2\mu a^2)\eta''-r^4G(r)(5r^4-6\mu a^2)\eta'+2\varepsilon r^5(r^4+2\mu a^2)\eta=0\ .\end{aligned} \tag{4·13}$$

We can check that the other components of (4·3) are derived from (4·11), (4·12), and (4·13). There are three degrees of freedom in $J=M=K=0$ mode.**)

4.2. *Onset of the Gregory-Laflamme instability*

We will solve Eqs.(4·11), (4·12), and (4·13) numerically and see the onset of the Gregory-Laflamme instability. First, we derive the boundary conditions at the horizon. We substitute the asymptotic forms at the horizon of the variables,

$$\beta=b_0+b_1(r-r_+)\ ,\quad \delta=d_0+d_1(r-r_+)\ ,\quad \eta=e_0+e_1(r-r_+)\ , \tag{4·14}$$

into (4·11), (4·12) and (4·13). For these asymptotic forms at the horizon (4·14), the perturbation (4·8) does not exceed the background metric and we adopt these asymptotic forms. Then, we obtain

$$\begin{aligned}b_1&=\frac{(\varepsilon r_+^6-16r_+^6+r_+^4\varepsilon-24r_+^4-8r_+^2-4\mu)b_0+4(r_+^4+r_+^2-2\mu)d_0}{4r_+(r_+^6+2r_+^4+r_+^2-\mu)}\ ,\\ d_1&=\frac{8(r_+^6+2r_+^4+r_+^2+\mu)b_0+(\varepsilon r_+^6+\varepsilon r_+^4-4r_+^4+16\mu-4r_+^2)d_0}{2r_+(r_+^6+2r_+^4+r_+^2-\mu)}\ ,\\ e_0&=-2b_0-d_0\ .\end{aligned} \tag{4·15}$$

Free parameters b_0, d_0, e_1 remain. However, we can set $e_1=1$ by the rescale of β,δ,η. Hence, the parameters that we should set at horizon are b_0, d_0. On the other hand, at $r\to\infty$, the growing mode of β,δ,η becomes

$$\delta\simeq C_1 r^\ell\ ,\quad \eta\simeq C_2 r^\ell\ ,\quad \beta\simeq\frac{C_1+C_2}{\ell+3}\,r^{\ell-2}+C_3r^{\ell-4}\ . \tag{4·16}$$

*) We can eliminate other variables, such as (α,δ), (δ,η). However, if we eliminate these variables, the singular point will appear at $r_+<r<\infty$ in the resultant equations.[25] The variables (α,ζ) are the best variables for elimination as far as we can see.

**) In the case of $\ell=0$, the gauge freedom is restored and the degree of freedom becomes one as explained in Appendix B.1.

Thus, for large r, we can get the coefficients of the growing modes approximately as

$$C_1 = \delta/r^\ell \ , \quad C_2 = \eta/r^\ell \ , \quad C_3 = \left(\beta - \frac{C_1 + C_2}{\ell + 3}\, r^{\ell-2}\right)/r^{\ell-4} \ . \tag{4·17}$$

These C_1, C_2, C_3 must be zero at infinity.*)

Now, we can start the numerical integration. We solve (4·11), (4·12), (4·13) from $r_1 = r_+ + 1.0 \times 10^{-5}L$ to $r_2 = 1.0 \times 10^4 L$ using the Runge-Kutta algorithm. Input parameters are r_+, Ω_H, b_0, d_0 and we can get $C_i = C_i(r_+, \Omega_H, b_0, d_0)$ $(i = 1, 2, 3)$ at $r = r_2$. For fixed r_+, we look for b_0, d_0, Ω_H, which satisfy $C_i = 0$ by the Newton-Raphson method. We repeat this procedure with various r_+. The result is given in Fig. 3. These lines have a maximum value and approach $\Omega_H L = 1$ for $r_+ \to \infty$. We can see that the higher ℓ mode becomes unstable for smaller r_+ in the region of $\Omega_H L < 1$, and the $\ell = 1$ mode is relevant for the onset of the Gregory-Laflamme instability.

In the case of $\Omega_H L < 1$, the condition for the Gregory-Laflamme instability is roughly given by $r_+/L < \mathcal{O}(1)$. This result is the same as the black string solution.[21) However, in the case of $\Omega_H L > 1$, the Kerr-AdS$_5 \times S^5$ spacetime is unstable for not only $r_+/L < \mathcal{O}(1)$, but also $r_+/L >$ (some large numerical value). This is an interesting property that appears in the ultraspinning Kerr-AdS$_5 \times S^5$ spacetime.

It is remarkable that the onset lines of Gregory-Laflamme and superradiant instabilities intersect each other. Thus, both instabilities can appear in Kerr-AdS$_5 \times S^5$ spacetimes. In the limit of $\Omega_H \to 0$, we can read off the onset of the instability as $r_+/L = 0.4402(\ell = 1), 0.3238(\ell = 2), 0.2570(\ell = 3)$. It is consistent with the instability of Schwarzschild-AdS$_5 \times S^5$ spacetimes.[25)]

§5. Phase structure

In this section, taking into account superradiant and Gregory-Laflamme instabilities, the phase structure of the Kerr-AdS$_5 \times S^5$ spacetime is revealed. Here, we need some comments on the stability analyses performed in §§3 and 4. We studied some specific modes and found instabilities. However, we did not study all the modes of perturbations and, strictly speaking, onsets of superradiant and Gregory-Laflamme instabilities can be changed by the analysis of all modes. To ensure that the onset of instabilities that we derived gives a true onset of instability, we studied the stability of $(J, M, K = J+2)$, $\ell \neq 0$ modes in Appendix B.3. As a result, we found the instability whose onset is given by $\Omega_H L \simeq (J+3+\ell/2)/(J+2)$. This result suggests that the mass term of graviton lifts up the onset of instability and these are not relevant to see the onset of superradiant instability. The result of Appendix B.3 also suggests that the Gregory-Laflamme instability is not found in higher modes in the AdS$_5$ part of spacetime. Because of the above reason, we regard the result derived in §§3 and 4 as a true onset of instabilities and reveal the phase structure of Kerr-AdS$_5 \times S^5$ spacetimes.

Until the previous section, we have been using parameters (r_+, Ω_H). However,

*) For $\ell = 1, 2, 3, 4$, β is not singular even if $C_3 \neq 0$, but for the regularity of ζ, we need $C_3 = 0$.

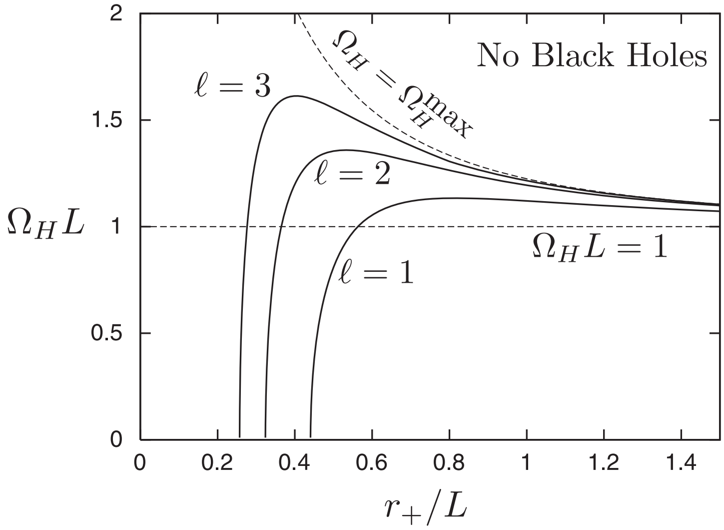

Fig. 3. The onset lines of Gregory-Laflamme instability are depicted. We depicted them for $\ell = 1, 2, 3$ using solid lines. In the regions above each line, Kerr-AdS$_5$ is unstable against the Gregory-Laflamme instabilities of each mode. These lines have a maximum value and approach $\Omega_H L = 1$ for $r_+ \to \infty$. At the above dashed line, the Kerr-AdS$_5$ black holes become extreme and, in the upper-right region, there are no black hole solutions. The dashed line below is $\Omega_H L = 1$, which is the onset of the superradiant instability.

for comparison with the gauge theory, the horizon radius r_+ is not a good parameter because r_+ is not defined in the gauge theory. Therefore, we see the phase diagram using thermodynamical parameters, the temperature T and angular velocity Ω_H. The temperature is defined as

$$T = \frac{2(1 - \Omega_H^2 L^2) r_+^2 / L^2 + 1}{2\pi r_+} \sqrt{\frac{1 + r_+^2/L^2}{(1 - \Omega_H^2 L^2) r_+^2/L^2 + 1}} \, . \tag{5·1}$$

Using this equation, we can map Fig. 3 onto the T-Ω_H diagram. The result is given in Fig. 1. The solid and dashed lines indicate the onsets of Gregory-Laflamme and superradiant instabilities, respectively. These lines cross each other and we can see five phases in this diagram. In the "Stable" region, Kerr-AdS black holes are stable. In the "SR" and "GL" regions, black holes are unstable against superradiant and Gregory-Laflamme instabilities, respectively. In the "SR&GL" region, black holes are unstable against both of them. In the "No Black Holes" region, there is no black hole solution.

To get this phase diagram, we need some following attentions. In Fig. 4, we plot the temperature as a function of r_+. In the case of $\Omega_H L < 1$, there are two r_+ giving the same temperature. We will call these phases small and large black hole phases. There is no one-to-one correspondence for r_+ and T. Since the Gregory-Laflamme instability appears in the small black hole phase, we chose the small black hole phase to depict the phase diagram. We can also see that the temperature has minimal value $T_{\min}(\Omega_H) > 0$. The "No Black Holes" phase in Fig. 1 comes from this minimum value

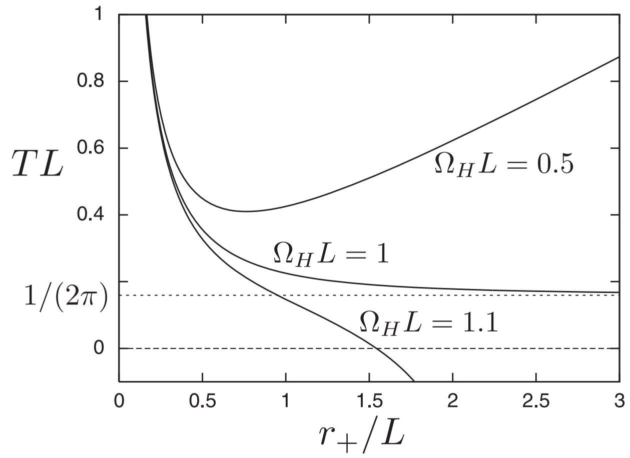

Fig. 4. We plot the temperature of Kerr-AdS$_5$ black holes as a function of r_+ for $\Omega_H L = 0.5, 1, 1.1$. For $\Omega_H L < 1$, there are two r_+ giving the same temperature, and the temperature T has a minimal value. For $\Omega_H L = 1$, the temperature monotonically decreases and approaches $TL = 1/(2\pi)$. For $\Omega_H > 1$, the temperature becomes zero at some value of r_+.

of T. In the case of $\Omega_H L = 1$, the temperature T has no minimal value, but it is bounded by $T > 1/(2\pi)$. For $\Omega_H L > 1$, T becomes zero at some value of r_+. Thus, the "No Black Holes" phase is vanishing for $\Omega_H L > 1$ in Fig. 1. The $\Omega_H = \Omega_H^{\rm max}$ line in Fig. 3, has been mapped onto a ray of $T = 0$ and $\Omega_H > 1$.

§6. Conclusions and discussion

We have studied gravitational perturbations of Kerr-AdS$_5$ $\times$ S^5 spacetimes with equal angular momenta. First, we studied the stability of Kerr-AdS$_5$ neglecting Kaluza-Klein modes and found the superradiant instability. We could see that the onset is given by $\Omega_H L = 1$. We also studied the stability including Kaluza-Klein modes of S^5 and found the Gregory-Laflamme instability. From these results, we derived the phase diagram of Kerr-AdS$_5$ $\times$ S^5 spacetime in Fig. 1 and found five phases in this diagram.

It is surprising that superradiant and Gregory-Laflamme instabilities can be understood on the basis of the dual gauge theory. In the dual gauge theory, the angular velocity of the horizon Ω_H is regarded as a chemical potential.[26),33),34)] It has been shown that, for $\Omega_H L > 1$, the gauge theory is unstable and the higher J mode becomes tachyonic first as Ω_H increases.[34)] In the gravity theory, we found the same property, that is, we can see that, from Fig. 2, higher J mode becomes unstable first as Ω_H increases. This remarkable coincidence of gravity and gauge theories provides a strong evidence for the AdS/CFT correspondence.

There are also several works on the Gregory-Laflamme instability from the gauge theory point of view.[35)–38)] In particular, in Ref. 38), $\mathcal{N} = 4$ SYM on S^3, which is dual to the Schwarzschild-AdS$_5$ $\times$ S^5 spacetime, is studied with weak 't Hooft coupling. In their work, at high temperature, they found a new saddle point in which $SO(6)$

R-symmetry is spontaneously broken and $SO(5)$ symmetry remains. The $SO(6)$ R-symmetry in the gauge theory corresponds to the symmetry of internal space S^5 in the dual gravity theory and, thus, this appearance of the new saddle point was related to the Gregory-Laflamme instability. For the Kerr-AdS$_5 \times S^5$ spacetime, our result in Fig. 1 gives a prediction for the phase structure of the gauge theory. However, unfortunately, there is no work concerning the Gregory-Laflamme instability in view of the gauge theory for the rotating black hole. It is interesting to extend the work described in Ref. 38) to Kerr-AdS$_5 \times S^5$ spacetime and compare with our result in Fig. 1.

In the dual gauge theory, we can still introduce R-symmetry chemical potentials. This theory corresponds to the R-charged black hole solution obtained in Ref. 39). For highly R-charged black holes, thermodynamical instability was found[34),39),40)] and this instability can be understood on the basis of the dual gauge theory.[34),40)] However, no dynamical instability is found through gravitational perturbations. In the gauge theory, this instability is described from the appearance of a tachyonic mode of scalar fields and, thus, this instability breaks the R-symmetry, $SO(6)$. On the other hand, in gravity theory, $SO(6)$ symmetry comes from internal space S^5 and, therefore, we can expect that this instability appears in Kaluza-Klein modes of S^5. The Kaluza-Klein modes can be regarded as a charged field in the effective theory in AdS$_5$. In the system of a charged black hole and charged field, superradiance occurs and superradiant instability is caused in the AdS spacetime. It is challenging to study the stability of R-charged black hole, while taking into account the Kaluza-Klein modes of S^5.

In this paper, because of practical reasons, we could not discuss the stability of Kerr-AdS$_5$ spacetimes with independent angular momenta. However, for Kerr-AdS$_5$ with one rotation, we may be able to find a new type of instability. In the case of asymptotically flat spacetimes, it was suggested that there is a phase transition between the five-dimensional Kerr black hole and black ring solutions.[41)] For asymptotically AdS spacetimes, a perturbative solution of black ring has been found[42)] and there may be a transition between the Kerr-AdS$_5$ black hole and the AdS black ring.

Acknowledgements

We are grateful to J. Soda and R. A. Konoplya for careful reading of this paper. We would also like to thank Veronika E. Hubeny for providing us with the numerical method to find the Gregory-Laflamme instability of Schwarzschild-AdS$_5 \times S^5$ spacetime. The work was supported by JSPS Grant-in-Aid for Scientific Research No. 19 · 3715 and a Grant-in-Aid for the Global COE Program "The Next Generation of Physics, Spun from Universality and Emergence" from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan.

Appendix A
—— *Kerr-AdS Black Hole with Independent Angular Momenta* ——

The Kerr-AdS$_5$ spacetime with independent angular momenta is given by[26)–29)]

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt + \frac{a_1\sin^2\theta_1}{\Xi_1}d\phi_1 + \frac{a_2\cos^2\theta_1}{\Xi_2}d\phi_2\right)^2 + \frac{\Delta_\theta\sin^2\theta_1}{\rho^2}\left(a_1 dt + \frac{(\bar{r}^2+a_1^2)}{\Xi_1}d\phi_1\right)^2$$
$$+\frac{\Delta_{\theta_1}\cos^2\theta_1}{\rho^2}\left(a_2 dt + \frac{(\bar{r}^2+a_2^2)}{\Xi_2}d\phi_2\right)^2 + \frac{\rho^2}{\Delta}d\bar{r}^2 + \frac{\rho^2}{\Delta_{\theta_1}}d\theta_1^2$$
$$+\frac{(1+\bar{r}^2/L^2)}{\bar{r}^2\rho^2}\left(a_1 a_2 dt + \frac{a_2(\bar{r}^2+a_1^2)\sin^2\theta_1}{\Xi_1}d\phi_1 + \frac{a_1(\bar{r}^2+a_2^2)\cos^2\theta_1}{\Xi_2}d\phi_2\right)^2 , \qquad \text{(A·1)}$$

where

$$\Delta = \frac{1}{\bar{r}^2}(\bar{r}^2+a_1^2)(\bar{r}^2+a_2^2)(1+\bar{r}^2/L^2) - 2m ,$$
$$\Delta_\theta = 1 - a_1^2 L^{-2}\cos^2\theta_1 - a_2^2 L^{-2}\sin^2\theta_1 ,$$
$$\rho^2 = \bar{r}^2 + a_1^2\cos^2\theta_1 + a_2^2\sin^2\theta_1 ,$$
$$\Xi_i = 1 - a_i^2/L^2 . \quad (i=1,2) \qquad \text{(A·2)}$$

This spacetime has symmetry $R_t \times U(1)^2$ generated by ∂_t, ∂_{ϕ_1}, and ∂_{ϕ_2}. Now, we consider Kerr-AdS spacetime with equal angular momenta, $a_1 = a_2 \equiv a$. We introduce new coordinates and parameters,

$$\theta = 2\theta_1 , \quad \phi = \phi_2 - \phi_1 , \quad \psi = \phi_1 + \phi_2 - 2at ,$$
$$r^2 = (\bar{r}^2 + a^2)/(1-a^2) , \quad \mu = m/(1-a^2)^3 . \qquad \text{(A·3)}$$

As a result, we can get the AdS part of (2·3).

Appendix B
—— *Stability Analysis for Other Modes* ——

In §§3 and 4, we studied specific modes and found superradiant and Gregory-Laflamme instabilities. However, Gregory-Laflamme instabilities may be changed by the analysis of all modes. In this appendix, we will study the stability of other modes that can be reduced to a single master equation. These are $(J=0, M=0, K=0,1)$ with $\ell=0$ modes and $(J, M, K=J+2)$ with any ℓ mode. As the result of stability analysis, we can see some evidence that the onset of instabilities derived in §3 and §4 gives a true onset of instability.

B.1. $(J=0, M=0, K=0)$ *with* $\ell=0$ *mode*

In the perturbation equation (3·2), $(J=0, M=0, K=0,1)$ and $(J, M, K = J+2)$ modes can be reduced to a single master equation. The stabilities of $(J, M, K = J+2)$ modes are studied in §3. Here, we shall consider $(J=0, M=0, K=0)$ mode.*)

*) This mode has already been studied using another formalism.[43)]

As we have seen in §3.1, there exist $h_{tt}, h_{tr}, h_{rr}, h_{t3}, h_{r3}, h_{+-}, h_{33}$ fields in this mode. We set $h_{\mu\nu}$ as

$$h_{\mu\nu}dx^\mu dx^\nu = e^{-i\omega t}\big[h_{tt}(r)dt^2 + 2h_{tr}(r)dtdr + h_{rr}(r)dr^2 + 2h_{t3}(r)dt\sigma^3 \\ + 2h_{r3}(r)dr\sigma^3 + 2h_{+-}(r)\sigma^+\sigma^- + h_{33}(r)\sigma^3\sigma^3\big] \ . \quad \text{(B·1)}$$

With the gauge parameters

$$\xi_A(x^\mu) = \xi_A(r)e^{-i\omega t} \ , \quad \xi_i(x^\mu) = \xi_3(r)e^{-i\omega t}\sigma^3_i \ , \quad \text{(B·2)}$$

the gauge transformations $\delta h_{\mu\nu} = \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$ for these components are given by

$$\delta h_{tt} = -2i\omega\xi_t - \frac{4\mu G(r)}{r^3}\xi_r \ , \quad \delta h_{tr} = \xi_t' - \frac{4\mu}{r^3G(r)}\xi_t - i\omega\xi_r + \frac{8\mu}{r^5G(r)}\xi_3 \ ,$$
$$\delta h_{t3} = -\frac{2G(r)\mu a}{r^3}\xi_r - i\omega\xi_3 \ , \quad \delta h_{rr} = 2\xi_r' + \frac{4\mu(r^2-2a^2)}{r^5G(r)}\xi_r \ ,$$
$$\delta h_{r3} = -\frac{4\mu a}{r^3G(r)}\xi_t + \xi_3' - \frac{2(r^4-2\mu r^2-2\mu a^2)}{r^5G(r)}\xi_3' \ ,$$
$$\delta h_{+-} = rG(r)\xi_r \ , \quad \delta h_{33} = \frac{G(r)(r^4-2\mu a^2)}{2r^3}\xi_r \ . \quad \text{(B·3)}$$

Our gauge choices are

$$h_{tt} = h_{t3} = h_{33} = 0 \ . \quad \text{(B·4)}$$

One can check that these are complete gauge fixings from (B·3). After the gauge fixing, four fields $h_{tr}, h_{rr}, h_{r3}, h_{+-}$ remain. However, all of them do not have degree of freedom. Substituting Eqs. (B·1) and (B·4) into Eq. (3·2), we can get three constraint equations, and one degree of freedom remains. Therefore, we can get a single master equation. The equation can be written in the Schrödinger form as*)

$$-\frac{d^2\Phi_0}{dr_*^2} + V_0(r)\Phi_0 = \omega^2\Phi_0 \ , \quad \text{(B·5)}$$

where

$$\Phi_0 \equiv \frac{(r^4-2\mu a^2)(r^4+2\mu a^2)^{1/4}}{r^{3/2}(3r^4+2\mu a^2)}h_{+-} \ , \quad \text{(B·6)}$$

and the tortoise coordinate r_* is defined in (3·17). The potential $V_0(r)$ is determined as

$$V_0(r) = \frac{G(r)}{4(3r^4+2\mu a^2)^2(r^4+2\mu a^2)^3r^2} \\ \times\big[135r^{22}/L^2 + 315r^{20} + 18\mu(9+43a^2/L^2)r^{18} + 2430\mu a^2r^{16} \\ + 8\mu^2a^2(174+55a^2/L^2)r^{14} + 5400\mu^2a^4r^{12} + 16\mu^3a^4(363-193a^2/L^2)r^{10} \\ + 2608\mu^3a^6r^8 + 80\mu^4a^6(76-49a^2/L^2)r^6 - 2064\mu^4a^8r^4 \\ + 32\mu^5a^8(1-a^2/L^2)r^2 - 160\mu^5a^{10}\big] \ . \quad \text{(B·7)}$$

*) The detailed calculations are very similar to those in Ref. 20) and we have omitted most of them.

We consider the stability of this $J = M = K = 0$ mode. In this mode, the master equation (B·5) is in the Schrödinger form. Therefore, the positivity of V_0 indicates the stability of this mode. The typical profile of V_0 is shown in Fig. 5. We can see the positivity of this potential from this figure. In fact, the positivity can be checked from the expression (B·7). From Eqs. (2·9) and (2·10), we obtain

$$a^2 \le \frac{r_+^4}{(1+r_+^2/L^2)^2}\left(\frac{1}{2r_+^2}+\frac{1}{L^2}\right) . \tag{B·8}$$

The right-hand side is an increasing function of r_+, which approaches L^2 in the limit of $r_+ \to \infty$. Thus, we can get the inequality $a^2 \le L^2$. Therefore, the negative terms in the large brackets of Eq. (B·7) are r^4 and r^0 terms. To see the positivity of $V_0(r)$, we focus on r^6, r^4 and r^0 terms in the large bracket of Eq. (B·7). After dividing them by $16\mu^4 a^6$, these terms become

$$f(r) = 5(76 - 49a^2/L^2)r^6 - 129a^2r^4 - 10\mu a^4 . \tag{B·9}$$

If $f(r)$ is positive, $V_0(r)$ is also positive. Now, we substitute Eq. (2·10) into Eq. (B·9). Because of $\Omega \le \Omega_H^{\max}$ and $r \ge r_+$, we can put $\Omega_H = s^2/(1+s^2)\Omega_H^{\max}$ for $s \ge 0$ and $r^2 = x^2 + r_+^2$. Then, we can obtain

$$\begin{aligned} f(r) =& \frac{L^6}{2(1+s^2)^2\alpha^2}\big[\{760\alpha^2 + 1520\alpha^2 s^2 + (760+1275\beta+270\beta^2)s^4\}x^6 \\ &+ \beta\{2280\alpha^2 + 4560\alpha^2 s^2 + 3(717+1189\beta+270\beta^2)s^4\}x^4 \\ &+ \beta^2\{2280\alpha^2 + 4560\alpha^2 s^2 + 3(674+1103\beta+270\beta^2)s^4\}x^2 \\ &+ \beta^3\gamma\{1520\alpha^3 + 6080\alpha^3 s^2 + 2\alpha(4051+7477\beta+3310\beta^2)s^4 \\ &+ 4\alpha(1011+1397\beta+270\beta^2)s^6 + (626+997\beta+250\beta^2)s^8\}\big] , \end{aligned} \tag{B·10}$$

where $\alpha = 1 + r_+^2/L^2$, $\beta = r_+^2/L^2$ and $\gamma = 1/(2\alpha + 4s^2\alpha + s^4)$. We can see $f(r) \ge 0$ explicitly. It indicates the stability of the $J = M = K = 0$ mode.

B.2. $(J = 0, M = 0, K = 1)$ *with* $\ell = 0$ *mode*

In $(J = 0, M = 0, K = 1)$ mode, there are h_{t+}, h_{r+}, h_{+3} fields. We set $h_{\mu\nu}$ as

$$h_{\mu\nu}dx^\mu dx^\nu = e^{-i\omega t}\big[2h_{t+}(r)dt\,\sigma^+ + 2h_{r+}(r)dr\,\sigma^+ + 2h_{+3}(r)\sigma^+\sigma^3\big] . \tag{B·11}$$

With the gauge parameter $\xi_i(x^\mu) = e^{-i\omega t}\xi_+(r)\sigma_i^+$, the gauge transformations for these components are given by

$$\delta h_{t+} = -i\omega\xi_+ + \frac{4i\mu a}{r^4}\xi_+ , \quad \delta h_{r+} = \xi_+' - \frac{2}{r}\xi_+ , \quad \delta h_{+3} = \frac{2i\mu a^2}{r^4}\xi_+ . \tag{B·12}$$

Our gauge choice is

$$h_{+3} = 0 . \tag{B·13}$$

This condition fixes the gauge completely. After the gauge fixing, two fields h_{t+} and h_{r+} remain. However, because one constraint exists in this mode, the physical degree

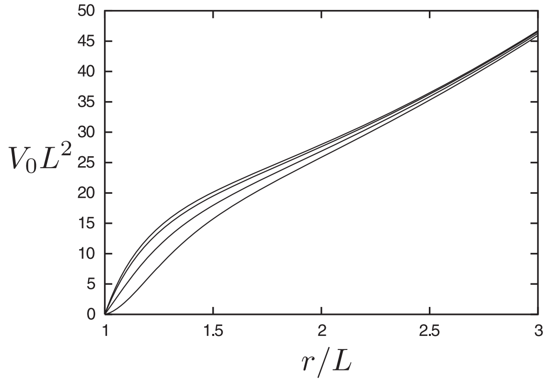

Fig. 5. Typical profiles for the potential V_0 are depicted. We put $r_+ = 1.0L$. From top to bottom, each curve represents the potential for $\Omega_H/\Omega_H^{\max} = 0.1, 0.7, 0.9, 0.99$. We see the positivities of these potentials.

of freedom becomes one. Therefore, we can get one master equation. Substituting Eq. (B·11) and Eq. (B·13) into Eq. (3·2), we can get the master equation for this mode,

$$-\frac{d^2\Phi_1}{dr_*^2} + V_1(r)\Phi_1 = [\omega - 2\Omega_1(r)]^2\Phi_1 \ , \tag{B·14}$$

where we have defined a new variable,*)

$$\Phi_1 = \frac{r^{5/2}(r^4+2\mu a^2)^{5/4}}{(r^{10}+2\mu a^2 r^6+\mu^2 a^6)^{1/2}}\left[\left(-i\omega + \frac{4i\mu a}{r^4+2\mu a^2}\right)\frac{h_{r+}}{r^2} - \left(\frac{h_{t+}}{r^2}\right)'\right] \tag{B·15}$$

and the functions Ω_1 and V_1 are given by

$$\Omega_1(r) = \frac{2\mu a}{r^4+2\mu a^2}\left(1 - \frac{a^2 r^4(5r^4+6\mu a^2)G}{4(r^{10}+2\mu a^2 r^6+\mu^2 a^6)}\right) \ , \tag{B·16}$$

and

$$\begin{aligned}
V_1(r) = {} & \frac{G(r)}{4r^2(2\mu a^2+r^4)^3(r^{10}+2\mu a^2 r^6+\mu^2 a^6)^2} \\
& \times [15r^{34}/L^2 + 35r^{32} + 18\mu(1+7a^2/L^2)r^{30} + 310\mu a^2 r^{28} \\
& + 8a^2\mu^2(20+57a^2/L^2)r^{26} + 2\mu^2 a^4(596-75a^2/L^2)r^{24} \\
& + 2\mu^2 a^4(152\mu + 456\mu a^2/L^2 - 75a^2)r^{22} + 4\mu^3 a^6(767-240a^2/L^2)r^{20} \\
& - 16\mu^3 a^6(8\mu - 63\mu a^2/L^2 + 60a^2)r^{18} + 24\mu^4 a^8(217-94a^2/L^2)r^{16} \\
& - \mu^4 a^8(-480\mu a^2/L^2 - 35a^4/L^2 + 480\mu + 2128a^2)r^{14} \\
& + 3\mu^4 a^{10}(1424\mu - 768\mu a^2/L^2 + 5a^2)r^{12} - 2\mu^5 a^{12}(827-77a^2/L^2)r^{10} \\
& + 2\mu^5 a^{12}(432\mu - 432\mu a^2/L^2 + 25a^2)r^8 - 12\mu^6 a^{14}(14-15a^2/L^2)r^6
\end{aligned}$$

*) This choice of master variable is important. If we use an other master variable, the $\omega^3, \omega^4, \cdots$ terms may appear in the resultant master equation. The good master variable (B·15) can be found by moving into the Hamiltonian formalism as explained in Ref. 20).

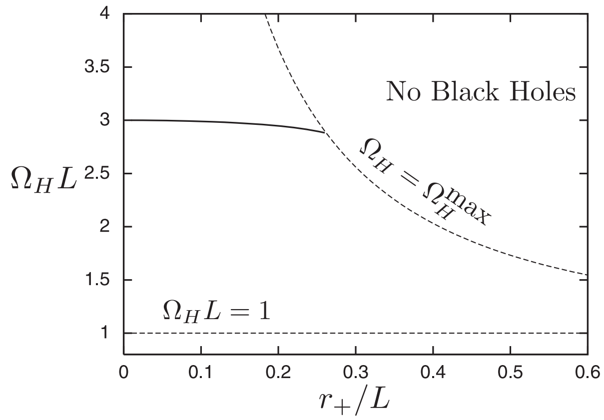

Fig. 6. The onset line of the superradiant instability for the $(J = 0, M = 0, K = 1)$ mode is depicted. The solid line is the onset of the instability. In the upper-right region, there is no black hole solution.

$$+ 68\mu^6 a^{16} r^4 - 24\mu^7 a^{16}(1 - a^2/L^2)r^2 + 56\mu^7 a^{18}] \ . \tag{B·17}$$

We have used the tortoise coordinate defined in Eq. (3·17).

We can get the asymptotic form of $\Omega_1(r)$ and $V_1(r)$ as

$$\Omega_1(r) \to 0 \quad (r \to \infty) \ , \quad \Omega_1(r) \to \Omega_H \quad (r \to r_+) \ , \tag{B·18}$$

and

$$V_1(r) \to 0 \quad (r \to r_+) \ , \quad V_1(r) \to \frac{15r^2}{4L^4} \quad (r \to \infty) \ . \tag{B·19}$$

Therefore, the asymptotic form of the solution of the master equation (B·14) becomes

$$\Phi_1 \to e^{\pm i\{\omega - 2\Omega_H\}r_*} \quad (r \to r_+) \ , \quad \Phi_1 \to r^{-1/2\pm 2} \quad (r \to \infty) \ . \tag{B·20}$$

We can study the onset of superradiant instability of this mode by the same method as that in §3.3. The result is depicted in Fig. 6. We see that the onset of the instability is $\Omega_H L \simeq 3$. On the other hand, in §3.3, it was shown that the onset is $\Omega_H L = 1$ for $(J, M, K = J + 2)$ modes. Thus, the $(J = 0, M = 0, K = 1)$ mode is irrelevant for the onset of the instability.

B.3. $(J, M, K = J + 2)$ *with* $\ell > 0$ *modes*

To see the effect of Kaluza-Klein modes for superradiant instability, we study the stability of $(J, M, K = J + 2)$ with $\ell > 0$ modes. The metric perturbations for these modes are given by

$$h_{MN}(x^M)dx^M dx^N = h_{++}(r)e^{-i\omega t} D_J(x^i) Y_\ell(\Omega_5)\sigma^+\sigma^+ \ , \tag{B·21}$$

where $D_J(x^i) \equiv D^J_{K=J,M}(x^i)$ and $Y_\ell(\Omega_5)$ is spherical harmonics on S^5 defined using (4·2). We define a new variable as

$$\Phi = \frac{(r^4 + 2\mu a^2)^{1/4}}{r^{3/2}} h_{++} \ . \tag{B·22}$$

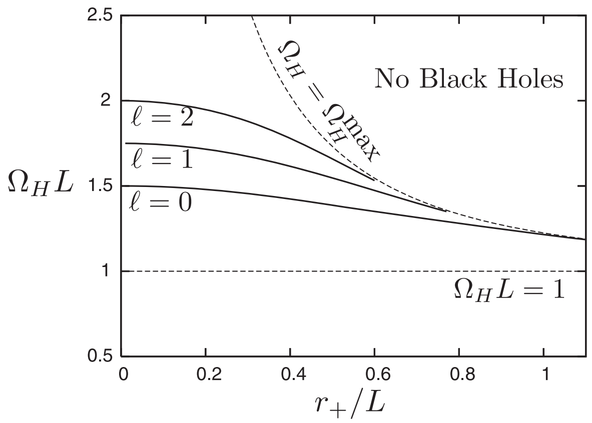

Fig. 7. The onset lines of the superradiant instability for the $(J = 0, M = 0, K = 2)$ and $\ell = 0, 1, 2$ modes are depicted. The solid line is the onset of instability of each mode. In the upper-right region, there is no black hole solution.

Then, using (4·3), we can obtain the equation for these modes as

$$-\frac{d^2\Phi}{dr_*^2} + V(r)\Phi = [\omega - 2(J+2)\Omega(r)]^2\Phi \ , \tag{B·23}$$

where the functions $\Omega(r)$ and $V(r)$ are determined as

$$\Omega(r) = \frac{2\mu a}{r^4 + 2\mu a^2}, \tag{B·24}$$

and

$$\begin{aligned} V(r) =& \frac{G(r)}{4r^2(r^4+2\mu a^2)^3}\big[(15+8\varepsilon)r^{14}/L^2 + (4J+5)(4J+7)r^{12} \\ &+ 2\mu(9+33a^2/L^2+16\varepsilon a^2/L^2)r^{10} + 2(16J^2+32J+5)\mu a^2 r^8 \\ &+ (-40+32\varepsilon a^2/L^2+68a^2/L^2)\mu^2 a^2 r^6 - 4\mu^2 a^4(16J+35)r^4 \\ &+ 8(1-a^2/L^2)\mu^3 a^4 r^2 - 40\mu^3 a^6\big] \ . \end{aligned} \tag{B·25}$$

Equation (B·23) return to (3·15) for $\ell = 0$. By a similar method to that in §3.3.2, we can see that, for small black holes, the onset of superradiant instability is given by $\Omega_H L = (J + 3 + \ell/2)/(J + 2)$. For any value of r_+, we solve (B·23) numerically by the same method as that in §3.3.3 and obtain Fig. 7. We plot the onset of instability for $(J = 0, M = 0, K = 2)$ and $\ell = 0, 1, 2$ modes. From this result, we can see that the superradiant instability of Kaluza-Klein modes appears at Ω_H higher than that in zero mode. Thus, this result suggests that Kaluza-Klein modes are not relevant to see the onset of superradiant instability.

Appendix C
—— *Transverse Traceless Condition for Kaluza-Klein Graviton* ——

In this appendix, we prove that we can impose the transverse traceless condition (4·5) in (4·3).

From the Bianchi identity, we can get

$$\begin{aligned} 0 &= \delta(g^{\rho\sigma}\nabla_\rho G_{\sigma\mu}) \\ &= -h^{\rho\sigma}\nabla_\rho G_{\sigma\mu} + g^{\rho\sigma}(-\delta\Gamma^\lambda_{\rho\sigma}G_{\lambda\mu} - \delta\Gamma^\lambda_{\rho\mu}G_{\lambda\sigma}) + g^{\rho\sigma}\nabla_\rho\delta G_{\sigma\mu} \ , \end{aligned} \tag{C·1}$$

where $\delta\Gamma^\rho_{\mu\nu}$ is perturbation of the Christoffel symbol defined as

$$\delta\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\nabla_\mu h_{\nu\sigma} + \nabla_\nu h_{\sigma\mu} - \nabla_\sigma h_{\mu\nu}) \ . \tag{C·2}$$

The background equation is given by $G_{\mu\nu} = 6L^{-2}g_{\mu\nu}$ and, thus, we can get $\nabla_\rho G_{\sigma\mu} = 0$. Therefore, the first term in the second line of (C·1) vanishes and we can get

$$g^{\rho\sigma}\nabla_\rho\delta G_{\sigma\mu} = 6L^{-2}(g^{\rho\sigma}g_{\lambda\mu}\delta\Gamma^\lambda_{\rho\sigma} + \delta\Gamma^\rho_{\rho\mu}) = 6L^{-2}\nabla^\rho h_{\rho\mu} \ , \tag{C·3}$$

where we have used the expression (C·2) at the last equality. Hence, from the divergence of the perturbation equation (4·3),

$$\nabla^\rho(h_{\rho\mu} - \frac{1}{2}g_{\rho\mu}h) = -\varepsilon^{-1}L^2\nabla^\rho(\delta G_{\rho\mu} - 6L^{-2}h_{\mu\nu}) = 0 \ . \tag{C·4}$$

It is the constraint equation of (4·3).

Now, we consider the trace of (4·3). The trace of perturbation of the Einstein tensor is given by

$$g^{\rho\sigma}\delta G_{\rho\sigma} = -\frac{3}{2}\nabla^\rho\nabla^\sigma(h_{\rho\sigma} - g_{\rho\sigma}h) + \frac{5}{2}h_{\rho\sigma}\Big(R^{\rho\sigma} - \frac{1}{5}g^{\rho\sigma}R\Big) \ . \tag{C·5}$$

Because of $R_{\mu\nu} = -4L^{-2}g_{\mu\nu}$ and $R = -20L^{-2}$, Ricci tensor and Ricci scalar terms in (C·5) cancel each other. Thus, by making use of the constraint equation (C·4), (C·5) becomes

$$g^{\rho\sigma}\delta G_{\rho\sigma} = \frac{3}{4}\nabla^2 h \ . \tag{C·6}$$

Thus, from the trace of (4·3), we can get the equation for the trace part of $h_{\mu\nu}$ as

$$\nabla^2 h = 2(\varepsilon + 2)L^2 h \ . \tag{C·7}$$

Therefore, the trace part of $h_{\mu\nu}$ is decoupled from other components of $h_{\mu\nu}$ and we can put $h = 0$ consistently. Then, the constraint equation becomes $\nabla^\rho h_{\rho\mu} = 0$.

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