Instabilities of Relativistic Stars

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Abstract Stable relativistic stars in uniform rotation form a two-parameter family, parametrized by mass and angular velocity. Limits on each of these quantities are associated with relativistic instabilities. A radial instability to gravitational collapse or explosion sets upper and lower limits on their mass, and an instability driven by gravitational waves may set an upper limit on their spin. Our summary of relativistic stability theory given here is based on and includes excerpts from the book *Rotating Relativistic Stars*, by the present authors.

1 Introduction

A neutron star in equilibrium is accurately approximated by a stationary self-gravitating perfect fluid. [1] The character of its oscillations and their stability, however, depend on bulk and shear viscosity, on the superfluid nature of its interior, and – for modes near the surface – on the properties of the crust and the strength of its magnetic field.

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[1] Departures from the local isotropy of a perfect fluid are associated with the crust; with magnetic fields that are thought to be confined to flux tubes in the superfluid interior; and with a velocity field whose vorticity is similarly confined to vortex tubes. Departures from perfect fluid equilibrium due to a solid crust are expected to be smaller than one part in $\sim 10^{-3}$, corresponding to the maximum strain that an electromagnetic lattice can support. The vortex tubes are closely spaced; but the velocity field averaged over meter scales is that of a uniformly rotating configuration. Finally, the magnetic field contributes negligibly to the pressure support of the star, even in magnetars with fields of 10^{15} G.

The stability of a rotating star is governed by the sign of the energy of its perturbations; and the amplitude of an oscillation that is damped or driven by gravitational radiation is governed by the rate at which its energy and angular momentum are radiated. Noether's theorem relates the stationarity and axisymmetry of the equilibrium star to conserved currents constructed from the perturbed metric and fluid variables. Their integrals, the canonical energy and angular momentum on a hypersurface can each be written as a functional quadratic in the perturbation, and the conservation laws express their change in terms of the flux of gravitational waves radiated to null infinity.

We begin with an action for perturbations of a rotating star from which these conserved quantities are obtained [1, 2, 3, 4, 5, 6]. We next review local stability to convection and to differential rotation. A spherical star that is stable against convection is stable to all nonradial perturbations: Only the radial instability to collapse (or explosion) can remain. Instability to collapse sets upper and lower limits on the masses of stable relativistic stars, the analog for neutron stars of the Chandrasekhar limit. A turning-point criterion governs this axisymmetric instability of spherical stars against collapse and provides a sufficient condition for instability of rotating stars. Finally, we consider the additional instabilities of rotating stars. These are nonaxisymmetric instabilities that radiate gravitational waves. They may set an upper limit on the spin of old neutron stars spun up by accretion and on nascent stars that form with rapid enough rotation.

2 Action and canonical energy

The equations governing a perfect fluid are the Einstein equation and the equation of motion of the fluid,

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}, \qquad \nabla_\beta T^{\alpha\beta} = 0, \tag{1}$$

together with an equation of state. We denote by p, ε, ρ and u^α the fluid's pressure, energy density, rest-mass density and 4-velocity, respectively, and define a tensor

$$q^\alpha{}_\beta = \delta^\alpha_\beta + u^\alpha u_\beta \tag{2}$$

that is the spatial projection operator, the projection orthogonal to u^α. The stress-energy tensor then has the form

$$T^{\alpha\beta} = \varepsilon u^\alpha u^\beta + p q^{\alpha\beta}.$$

Because the spatial projection $q^\alpha{}_\gamma \nabla_\beta T^{\beta\gamma} = 0$ is the relativistic Euler equation,

$$u^\beta \nabla_\beta u^\alpha = -\frac{q^{\alpha\beta}\nabla_\beta p}{\varepsilon + p}, \tag{3}$$

we call Eqs. (1) the Einstein-Euler equations.

One can obtain an action for stellar perturbations by introducing a Lagrangian displacement ξ^α joining each unperturbed fluid trajectory (the unperturbed worldline of a fluid element) to the corresponding trajectory of the perturbed fluid, as shown in Fig. 1

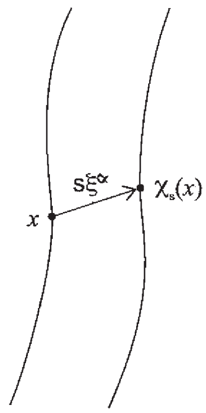

Fig. 1 For small $\mathfrak{s}$, a Lagrangian displacement ξ^α can be regarded as a vector for which $\mathfrak{s}\xi^\alpha$ joins the position x of a fluid element in an initial fluid flow to its position $\chi_{\mathfrak{s}}(x)$ in the perturbed fluid flow.

The perturbative description is made precise by introducing a family of (time dependent) solutions

$$\mathscr{Q}(\lambda) = \{g_{\alpha\beta}(\lambda), u^\alpha(\lambda), \rho(\lambda), s(\lambda)\}, \tag{4}$$

and comparing to first order in λ the perturbed variables $Q(\lambda)$ with their equilibrium values $Q(0)$. Eulerian and Lagrangian changes in the fluid variables are defined by

$$\delta\mathscr{Q} := \left.\frac{d}{d\lambda}Q(\lambda)\right|_{\lambda=0}, \qquad \Delta\mathscr{Q} = (\delta + \mathscr{L}_{\boldsymbol{\xi}})\mathscr{Q}, \tag{5}$$

with $\mathscr{L}_{\boldsymbol{\xi}}$ the Lie derivative along ξ^α.

Because oscillations of a neutron star proceed on a dynamical timescale, a timescale faster than that of heat flow, one requires that the Lagrangian change Δs in the entropy per unit rest mass vanishes. With this condition, ξ^α and $h_{\alpha\beta} := \delta g_{\alpha\beta}$ completely specify a perturbation of a perfect-fluid spacetime with an equation of state of the form $\varepsilon = \varepsilon(\rho, s), p = p(\rho, s)$. Perturbations of u^α, ρ and ε are given by

$$\Delta u^\alpha = \frac{1}{2}u^\alpha u^\beta u^\gamma \Delta g_{\beta\gamma}, \qquad \Delta\rho = -\frac{1}{2}\rho q^{\alpha\beta}\Delta g_{\alpha\beta}, \qquad \Delta\varepsilon = -\frac{1}{2}(\varepsilon + p)q^{\alpha\beta}\Delta g_{\alpha\beta}, \tag{6}$$

with $\Delta g_{\alpha\beta} = h_{\alpha\beta} + \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha$. Our restriction to adiabatic perturbations means that the Lagrangian change in the pressure is given by

$$\frac{\Delta p}{p} = \Gamma_1 \frac{\Delta \rho}{\rho} = -\frac{1}{2}\Gamma_1\, q^{\alpha\beta}\Delta g_{\alpha\beta}, \tag{7}$$

where the adiabatic index Γ_1 is defined by

$$\Gamma_1 = \frac{\partial \log p(\rho,s)}{\partial \log \rho} = \frac{\varepsilon + p}{p}\frac{\partial p(\varepsilon,s)}{\partial \varepsilon}. \tag{8}$$

The perturbed Einstein-Euler equations,

$$\delta(G^{\alpha\beta} - 8\pi T^{\alpha\beta}) = 0, \qquad \delta(\nabla_\beta T^{\alpha\beta}) = 0, \tag{9}$$

are self-adjoint in a weak and 4-dimensional sense that they are a symmetric system up to a total divergence: For any pairs $(\xi^\alpha, h_{\alpha\beta})$ and $(\widehat{\xi}^\alpha, \widehat{h}_{\alpha\beta})$, the symmetry relation has the form

$$\widehat{\xi}_\beta \delta(\nabla_\gamma T^{\beta\gamma}\sqrt{|g|}) + \frac{1}{16\pi}\widehat{h}_{\beta\gamma}\delta\left[(G^{\beta\gamma} - 8\pi T^{\beta\gamma})\sqrt{|g|}\right] = -2\mathscr{L}(\widehat{\xi},\widehat{h};\xi,h) + \nabla_\beta \Theta^\beta, \tag{10}$$

where $\mathscr{L}$ is symmetric under the interchange of (ξ,h) and $(\widehat{\xi},\hat{h})$. A symmetry relation of the form (10) implies that $\mathscr{L}^{(2)}(\xi,h) := \frac{1}{2}\mathscr{L}(\xi,h;\xi,h)$ is a Lagrangian density and

$$I^{(2)} = \int d^4x \mathscr{L}^{(2)} \tag{11}$$

is an action for the perturbed system.

The conserved canonical energy associated with the timelike Killing vector is the Hamiltonian of the perturbation, expressed in terms of configuration space variables,

$$E_c = \int_S d^3x\,(\Pi^\alpha \mathscr{L}_{\mathbf{t}}\xi_\alpha + \pi^{\alpha\beta}\mathscr{L}_{\mathbf{t}}h_{\alpha\beta} - \mathscr{L}^{(2)}), \tag{12}$$

where Π^α and $\pi^{\alpha\beta}$ are the momenta conjugate to ξ^α and $h_{\alpha\beta}$. On a spacelike hypersurface with future pointing unit normal $n^\alpha = -\alpha\nabla^\alpha t$ (where α is the lapse), the canonical momenta conjugate to ξ^α and $h_{\alpha\beta}$ are given by

$$\Pi^\alpha = \Pi^{\gamma\alpha}\nabla_\gamma t, \qquad \pi^{\alpha\beta} = \pi^{\gamma\alpha\beta}\nabla_\gamma t. \tag{13}$$

with

$$\Pi^{\alpha\beta} = \frac{1}{2}\frac{\partial \mathscr{L}(\xi,h;\xi,h)}{\partial \nabla_\alpha \xi_\beta}, \tag{14}$$

$$\pi^{\alpha\beta\gamma} = \frac{1}{2}\frac{\partial \mathscr{L}(\xi,h;\xi,h)}{\partial \nabla_\alpha h_{\beta\gamma}}. \tag{15}$$

The corresponding canonical momentum has the form

$$J_c = -\int_S d^3x(\Pi^\alpha \mathscr{L}_\phi \xi_\alpha + \pi^{\alpha\beta}\mathscr{L}_\phi h_{\alpha\beta}). \tag{16}$$

If one foliates the background spacetime by a family of spacelike but asymptotically null hypersurfaces, the difference $E_2 - E_1$ in E_c from one hypersurface to another to its future is the energy radiated in gravitational waves to future null infinity. Because this energy is positive definite, E_c can only decrease. This suggests that a condition for stability is that E_c be positive for all initial data.

This is, in fact, an appopriate stability criterion, but there is a subtlety, associated with a gauge freedom in choosing a Lagrangian displacement: There is a class of *trivial* displacements, for which the Eulerian changes in all fluid variables vanish. For a one (two) parameter equation of state, these correspond to rearranging fluid elements with the same value of ρ (and s). [2] For a trivial displacement η^α, the same physical perturbation is described by the pairs $h_{\alpha\beta}, \xi^\alpha$ and $h_{\alpha\beta}, \xi^\alpha + \eta^\alpha$, but, for nonaxisymmetric perturbations, the canonical energy is not invariant under addition of a trivial displacement, and its sign depends on this kind of gauge freedom. There is, however, a preferred class of *canonical* displacements, the displacements ξ^α that are orthogonal to all trivial displacements, with respect to the symplectic product of two perturbations,

$$W(\widehat{\xi}, \widehat{h}; \xi, h) := \int_\Sigma (\widehat{\Pi}_\alpha \xi^\alpha + \widehat{\pi}^{\alpha\beta} h_{\alpha\beta} - \Pi_\alpha \widehat{\xi}^a - \pi^{\alpha\beta}\widehat{h}_{\alpha\beta}) d^3x. \tag{17}$$

The criterion for stability can then be phrased as follows:

1. If $E_c < 0$ for some canonical data on Σ, then the configuration is unstable or marginally stable: There exist perturbations on a family of asymptotically null hypersurfaces Σ_u that do not die away in time.
2. If $E_c > 0$ for all canonical data on Σ, the magnitude of E_c is bounded in time and only finite energy can be radiated.

The trivial displacements are relabelings of fluid elements with the same baryon density and entropy per baryon. They are Noether-related to conservation of circulation in surfaces of constant entropy per baryon [7, 8, 9], and canonical displacements are displacements that preserve the circulation of each fluid ring – for which the Lagrangian change in the circulation vanishes.

For perturbations that are not spherical, stable perturbations have positive energy and die away in time; unstable perturbations have negative canonical energy and radiate negative energy to infinity, implying that E_c becomes increasingly negative. One would like to show that when $E_c < 0$ a perfect-fluid configuration is strictly unstable, that within the linearized theory the time-evolved data radiates infinite energy and that $|E_c|$ becomes infinite along a family Σ_u of asymptotically null hypersurfaces. There is no proof of this conjecture, but it is easy to see that if $E_c < 0$, the time derivatives $\dot{\xi}^\alpha$ and $\dot{h}_{\alpha\beta}$ must remain finitely large. Thus a configuration

[2] This is not the gauge freedom associated with infinitesimal diffeos of the metric and matter, but a redundancy in the Lagrangian- displacement description of perturbations that is already present in a Newtonian context.

with $E_c < 0$ will be strictly unstable unless it admits perturbations that are time dependent but *nonradiative*.

For spherical stars, radial perturbations have this property, but in that case, the relativistic Euler equation has the form of a Sturm-Liouville equation, and perturbations with $E_c < 0$ are in fact strictly unstable.

The symplectic form provides an alternate form of the canonical energy, used in Wald's article in this volume. Because of the quadratic structure of the second-order Lagrangian, when the field equations are satisfied, Eq. (12) is equivalent to the expression

$$\begin{aligned} E_c &= \frac{1}{2} W(\mathscr{L}_t \xi^\alpha, \mathscr{L}_t h_{\alpha\beta}, \xi^\alpha, h_{\alpha\beta}) \\ &= \frac{1}{2} \int_\Sigma (\Pi_\alpha \mathscr{L}_t \xi^\alpha + \pi^{\alpha\beta} \mathscr{L}_t h_{\alpha\beta} - \mathscr{L}_t \Pi_\alpha \xi^\alpha - \mathscr{L}_t \pi^{\alpha\beta} h_{\alpha\beta}) d^3x, \end{aligned} \tag{18}$$

From this relation and Eq. (16), one has an immediate relation between J_c and E_c for a real-frequency mode with behavior $e^{i(m\phi+\omega t)}$:

$$J_c = -\frac{m}{\omega} E_c. \tag{19}$$

3 Local stability

3.1 Convective instability

The criterion for the stability of a spherical star against convection is easy to understand. When a fluid element is displaced upward, if its density decreases more rapidly than the density of the surrounding fluid, then the element will be buoyed upward and the star will be unstable. If, on the other hand, the fluid element expands less than its surroundings it will fall back, and the star will be stable to convection.

As this argument suggests, criteria for convective stability are *local*, involving perturbations restricted to an arbitrarily small region of the star or, for axisymmetric perturbations, to an arbitrarily thin ring. For local perturbations, the Cowling approximation is valid: The change in the gravitational field can be ignored. The argument is this: A perturbation in density of order $\delta\varepsilon/\varepsilon$ that is restricted to a region of volume $V \ll R^3$ can be regarded as adding or subtracting from the source a mass δm of order $\delta\varepsilon V$. Then

$$\frac{\delta m}{M} \sim \frac{V}{R^3} \frac{\delta\varepsilon}{\varepsilon} \ll \frac{\delta\varepsilon}{\varepsilon}. \tag{20}$$

The change in the metric is then also smaller than $\delta\varepsilon/\varepsilon$ by a factor V/R^3, arbitrarily small when the support of the matter perturbation is arbitrarily small. Note that, because the metric perturbation is gauge-dependent, this statement about the smallness of the perturbed metric is also gauge-dependent. A more precise way of stating this property of a local perturbation is that a gauge can be chosen in which the metric perturbation is smaller than the density perturbation by a factor of order V/R^3.

Convective instability of spherical relativistic stars was discussed by Thorne [10] and subsequently, with greater rigor, by Kovetz [11] and Schutz [12]. An initial heuristic treatment by Bardeen [13] of convective instability of differentially rotating stars was made more precise and extended to models with heat flow and viscosity by Seguin [14].

Consider a fluid element displaced radially outward from an initial position with radial coordinate r to $r+\xi$. The fluid element expands (or, if displaced inward, contracts), with its pressure adjusting immediately – in sound travel time across the fluid element – to the pressure outside:

$$\Delta p = \boldsymbol{\xi}\cdot\nabla p = \frac{dp}{dr}\xi. \tag{21}$$

Heat diffuses more slowly, and the analysis assumes that the motion is faster than the time for heat to flow into or out of the fluid element: The perturbation is *adiabatic*:

$$\begin{aligned}\Delta\varepsilon &= \left(\frac{\partial\varepsilon}{\partial p}\right)_s \Delta p \\ &= \left(\frac{\partial\varepsilon}{\partial p}\right)_s \frac{dp}{dr}\xi = \Gamma_1\frac{\varepsilon+p}{p}\frac{dp}{dr}\xi,\end{aligned} \tag{22}$$

where we have used the adiabatic conditions (6) and (7).

The difference $\Delta_\star\varepsilon$ in the density of the surrounding star between r and $r+\xi$ is given by

$$\Delta_\star\varepsilon = \xi\frac{d\varepsilon}{dr}. \tag{23}$$

The displaced fluid element falls back if $|\Delta\varepsilon| < |\Delta_\star\varepsilon|$ – if, that is, the fluid element's density decreases more slowly than the star's density:

$$\left(\frac{\partial\varepsilon}{\partial p}\right)_s \left|\xi\frac{dp}{dr}\right| < \left|\xi\frac{d\varepsilon}{dr}\right|. \tag{24}$$

The star is then stable against convection if the inequality,

$$\left(\frac{dp}{d\varepsilon}\right)_\star := \frac{dp/dr}{d\varepsilon/dr} < \left(\frac{\partial p}{\partial\varepsilon}\right)_s \tag{25}$$

is satisfied, unstable if the inequality is in the opposite direction.

In particular, in a homentropic star with no composition gradient, the adiabatic value of $dp/d\varepsilon$ coincides with its value in the equilibrium star,

$$\left(\frac{\partial\varepsilon}{\partial p}\right)_s = \left(\frac{dp}{d\varepsilon}\right)_\star, \tag{26}$$

implying that the star is marginally unstable.

For spherical stars Detweiler and Ipser [15] (generalizing a Newtonian result due to Lebovitz [16]), argue that, apart from local instability to convection, one need only consider radial perturbations: *If a nonrotating star is stable to radial oscillations and stable against convection, the star is stable*. The Detweiler-Ipser argument, however, relies on completeness of normal modes and the assumption that all modes are continuously joined to modes of a nearly Newtonian star, for which the Lebovitz result should imply that all modes are stable. Although the result is almost certainly true, the assumptions are not: There are outgoing modes – the w-modes – analogous to the outgoing modes of black holoes, that have no Newtonian counterparts.

Research Problem. Prove that perturbations of spherical stars are stable if they are stable against convection and against radial perturbations.
This can be done by showing that, with reasonable assumptions about the EOS, the canonical energy of a nonradial perturbation is negative only if the Schwarzschild criterion is violated. The result may follow from an integral inequality (associated with Eq. (42) of [15]), that is central to the Detweiler-Ipser argument.

Within minutes after their birth, neutron stars cool to a temperature below the Fermi energy per nucleon, below 10^{12} K. Their neutrons are then degenerate, with a nearly isentropic equation of state: Convectively stable, but with convective modes having frequencies below 100 Hz, much lower than the kHz frequencies of the f- and p-modes.

3.2 Convective instability due to differential rotation: the Solberg [17] criterion

Differentially rotating stars have one additional kind of convective (local) instability. If the angular momentum per unit rest mass, $j = hu_\alpha \phi^\alpha$, decreases outward from the axis of symmetry, the star is unstable to perturbations that change the differential rotation law.

The criterion is easy to understand in a Newtonian context. Consider a ring of fluid in the star's equatorial plane that is displaced outward from r to $r+\xi$, conserving angular momentum and mass. Again the displaced ring immediately adjusts its pressure to that of the surrounding star. If the ring's centripetal acceleration is larger that the net restoring force from gravity and the surrounding pressure gradient, it will continue to move outward. Now in the unperturbed star, the centripetal acceleration is equal to the restoring force. As in the discussion of convective instability, the displaced fluid element encounters the pressure gradient and gravitational field of the uperturbed star at its new position, and the restoring force is the restoring force on a fluid element at $r+\xi$ in the unperturbed star. Thus, if the displaced fluid ring has the same value of v^2/r as the surrounding fluid it will be in equilibrium, and the star will be marginally stable. If a displaced fluid ring has larger v^2/r than its surrounding fluid the star will be unstable.

The difference in acceleration for the background star is $\Delta_\star(v^2/r) = \xi^r \dfrac{d}{dr}(v^2/r)$, and stability then requires

$$\xi^r \frac{d}{dr}\left(\frac{v^2}{r}\right) - \Delta \frac{v^2}{r} > 0, \tag{27}$$

for $\xi^r > 0$.

Because $\Delta j = 0$ and $v(j,r) = j(r)/r$, we have

$$\Delta \frac{v^2}{r} = \Delta \frac{j^2}{r^3} = j^2 \xi^r \frac{d}{dr}\frac{1}{r^3}, \tag{28}$$

while

$$\Delta_\star \frac{v^2}{r} = \xi^r \frac{d}{dr}\frac{j^2}{r^3}, \tag{29}$$

implying

$$\Delta_\star \frac{v^2}{r} - \Delta \frac{v^2}{r} = \xi^r \frac{1}{r^3}\frac{dj^2}{dr}; \tag{30}$$

and the star is stable only if $\dfrac{dj}{dr} > 0$ in the equatorial plane (for $j > 0$), or, equivalently, only if $\partial_\varpi(\varpi^2\Omega) > 0$.

Bardeen [13] gives a heuristic argument for a restricted version of this criterion, and a subsequent comprehensive and more precise treatment, including heat flow and viscosity, is due to Seguin [14]. Abramowicz [18] gives the relativistic version of the Newtonian argument summarized above; a presentation in [19] corrects some misprints and also relates the criterion to the sign of the canonical energy.

Here, for the relativistic case, we present Bardeen's simple argument. The relativistic angular momentum per unit rest mass is $j = \dfrac{\varepsilon + p}{\rho} u_\alpha \phi^a$, with ϕ^α the rotational Killing vector. The first law of thermodynamics for relativistic stars has the form

$$\delta M = \int_\Sigma \left(\frac{T}{u^t}\Delta dS + \frac{g}{u^t}\Delta dM_0 + \Omega \Delta dJ \right), \tag{31}$$

where $dM_0 = \rho dV$ and $dJ = j dM_0$. If, in a homentropic, differentially rotating star, j has an extremum as a function of radius in the the equatorial plane, then there are perturbations that conserve baryon number and that lower the energy of the system – for which $\delta M < 0$. The argument, for a homentropic star, is this: On opposite sides of the extremum, there are two rings, 1 and 2, with the same value of j and with $\Omega_2 > \Omega_1$. A perturbation that transfers matter with baryon mass δM_0 from ring 2 to ring 1 then gives $\delta M = (\Omega_1 - \Omega_2) j \delta M_0 < 0$. That is, unless j is a monotonic function, one can always find a perturbation with negative energy.

This is a simplest example of the turning-point criterion governing axisymmetric stability: A point of marginal stability along a sequence of circular orbits of a particle is a point at whichg j is an extremum. The turning-point condition can be rephrased in terms of the particle's energy. For a particle of fixed rest mass, the difference in energy of adjacent orbits is related to the difference in its angular momentum by

$$\delta E = \Omega \delta J.$$

Then a point of marginal stability along a sequence of circular orbits of a particle of fixed baryon mass is a point at which its energy is an extremum.

4 Axisymmetric instability and turning points

For spherical stars in Newtonian gravity, instability sets in when the matter becomes relativistic, when the adiabatic index Γ_1 (more precisely, its pressure-weighted average $\bar{\Gamma}_1$) reaches the value 4/3 characteristic of zero rest-mass particles. This can be seen from the Newtonian limit of the canonical energy,

$$E_c = \int \left\{ \frac{1}{2}\rho\dot{\xi}^2 + \frac{2}{r}p'\xi^2 + \frac{\Gamma_1 p}{2r^4}\left[(r^2\xi)'\right]^2 \right\} dV. \tag{32}$$

Choosing as initial data $\xi = r$, $\dot{\xi} = 0$, gives

$$E_c = \int \left(2rp' + \frac{9}{2}\Gamma_1 p r^2\right) dV = \frac{9}{2}\int \left(\Gamma_1 - \frac{4}{3}\right) p\, dV, \tag{33}$$

implying instability for $\bar{\Gamma}_1 < 4/3$. This shows only that $\Gamma_1 < 4/3$ is a sufficient condition for instability, but spherical Newtonian polytropes with $\Gamma_1 > 4/3$ are stable.

By, in effect, deriving the relativistic canonical energy,

$$E_c = \int_0^R \frac{1}{2} e^{\lambda+\nu} \left\{ \left[\frac{4}{r}p' - \frac{p'^2}{\varepsilon + p} + 8\pi p(\varepsilon + p)\right]\xi^2 \right.$$
$$\left. + \frac{e^{3\lambda-\nu}}{r^4}\Gamma_1 p\left[(e^{-\nu}r^2\xi)'\right]^2 \right\} r^2 dr, \tag{34}$$

Chandrasekhar [20, 21] showed that the stronger gravity of general relativity implies an earlier onset of instability: Even models with the stiffest equation of state must be unstable to collapse for some value of compactness $M/R < 9/8$, the value for the most compact uniform density model. The more stringent relativistic constraint on Γ_1 for a star to be stable against radial perturbations has the form

$$\Gamma_1 < \frac{4}{3} + K\frac{M}{R}, \tag{35}$$

where K is positive and of order unity [20]. Because a gas of photons has $\Gamma_1 = 4/3$ and massive stars are radiation dominated, the instability can be important for stars with $M/R \ll 1$.

Turning point instability

The best-known instability result in general relativity is the statement that instability to collapse sets in at a point of maximum mass, along a sequence of spherical barotropic models. The configuration with maximum mass is called a turning point along the sequence, and it is also the configuration with maximum baryon mass. A similar result holds for uniformly rotating stars [22]: Instability to collapse is implied by a point of maximum mass and maximum baryon mass, along a sequence of uniformly rotating barotropic models with fixed angular momentum. As in the spherical case, stars with higher central density than that of the maximum-mass configuration are unstable. For rotating stars, however, the turning point is a sufficient but not a necessary condition for instability: The onset of instability is at a configuration with slightly lower central density (for fixed angular momentum) than that of the maximum-mass star. A formal symmetry in the way baryon mass and angular momentum occur in the first law implies that the line of turning points is also the line of extrema of angular momentum along sequences of fixed baryon mass.

For dynamical oscillations of neutron stars the adiabatic index does not coincide with the polytropic index, $\Gamma_1 \neq \Gamma := \dfrac{d\log p(r)/dr}{d\log \rho/dr}$, and the turning point criterion implies *secular* instability — an instability whose growth time is long compared to the typical dynamical time of stellar oscillations. For spherical stars, the turning-point instability proceeds on a time scale slow enough to accommodate the nuclear reactions and energy transfer that accompany the change to a nearby equilibrium. For rotating stars, the time scale must also be long enough to accommodate a transfer of angular momentum from one fluid ring to another. That is, the growth rate of the instability is limited by the time required for viscosity to redistribute the star's angular momentum. For neutron stars, this is expected to be short, probably comparable to the spin-up time following a glitch, and certainly short compared to the lifetime of a pulsar or an accreting neutron star. For this reason, it is the secular instability, that sets the upper and lower limits on the mass of spherical and uniformly rotating neutron stars.

One can easily understand why the instability sets in at an extremum of the mass by looking at a radial mode of oscillation of a nonrotating star with an equation of state $p = p(\rho), \varepsilon = \varepsilon(\rho)$. Along the sequence of spherical equilibria, a radial mode changes from stability to instability when its frequency σ changes from real to imaginary, with $\sigma = 0$ at the point of marginal stability. Now a zero-frequency mode is just a time-independent solution to the linearized Einstein-Euler equations - a perturbation from one equilibrium configuration to a nearby equilibrium with the same baryon number. From the first law of thermodynamics (31), a perturbation that keeps the star in equilibrium satisfies

$$\delta M = \frac{g}{u^t}\delta M_0, \tag{36}$$

with g the specific Gibbs free energy. The relation implies that, for a zero frequency perturbation involving no change in baryon number, the change δM in mass must vanish. This is the requirement that the mass is an extremum along the sequence of equilibria. Models on the *high-density* side of the maximum-mass instability point are unstable: Because the turning point is a star with maximum baryon number as well as maximum mass, there are models on opposite sides of the turning point with the same baryon number. Because g/u^t is a decreasing function of central density, the model on the high-density side of the turning point has greater mass than the corresponding model with smaller central density.

At the minimum mass, it is the *low-density* side that is unstable: Because the mass is a minimum, the model on the low-density side of the turning point has greater mass than the corresponding model with the same baryon number on the high-density side.

The precise statement of the turning-point criterion is the following result:

Theorem 9.1 (Friedman, Ipser, Sorkin [22]): Consider a continuous sequence of uniformly rotating stellar models based on an equation of state of the form $p = p(\varepsilon)$. Let λ be the sequence parameter and denote the derivative $d/d\lambda$ along the sequence by $(\dot{\ })$.
(i) Suppose that the total angular momentum is constant along the sequence and that there is a point λ_0 where $\dot{M} = 0$ and where $\mathscr{E} > 0$, $(\dot{\mathscr{E}}\dot{M})^{\cdot} \neq 0$. Then the part of the sequence for which $\dot{\mathscr{E}}\dot{M} > 0$ is unstable for λ near λ_0.
(ii) Suppose that the total baryon mass M_0 is constant along the sequence and that there is a point λ_0 where $\dot{M} = 0$ and where $\Omega > 0$, $(\dot{\Omega}\dot{M})^{\cdot} \neq 0$. Then the part of the sequence for which $\dot{\Omega}\dot{M} > 0$ is unstable for λ near λ_0.

In form (ii) of the theorem, the first law implies that the turning point is an extremum of angular momentum J along a sequence of constant rest mass. Ref. [22] points out the symmetry between M_0 and J that implies this maximum-J form of the theorem, and Cook, Shapiro and Teukolsky [23] first use the theorem in this form. For rotating stars, the turning point criterion is a sufficient condition for secular instability to collapse. In general, however, collapse can be expected to involve differential rotation, and the turning point identifies only nearby uniformly rotating configurations with lower energy. Rotating stars are therefore likely to be secularly unstable to collapse at densities slightly lower than the turning point density. The onset of secular instability to collapse is at or before the onset of dynamical instability along a sequence of uniformly rotating stars of fixed angular momentum, and recent work by Katami, Rezzolla and Yoshida [24] appears to show that rapidly rotating stars can also be dynamically unstable to collapse just prior to the turning point.

As illustrated in Fig. 2), they find a dynamical instability line that coincides with the turning-point line for spherical stars and that, for rapid uniform rotation, has a central density about 5% below that of the turning point. This result somewhat overstates the difference between the two lines, because it ignores the difference $\Gamma - \Gamma_1$ between the indices governing dynamical oscillations and the equilibrium equation of state. The actual dynamical instability line begins at a spherical star with higher density than the marginally unstable turning point star and probably crosses the turning-point line to lower density at some angular velocity less than Ω_K.

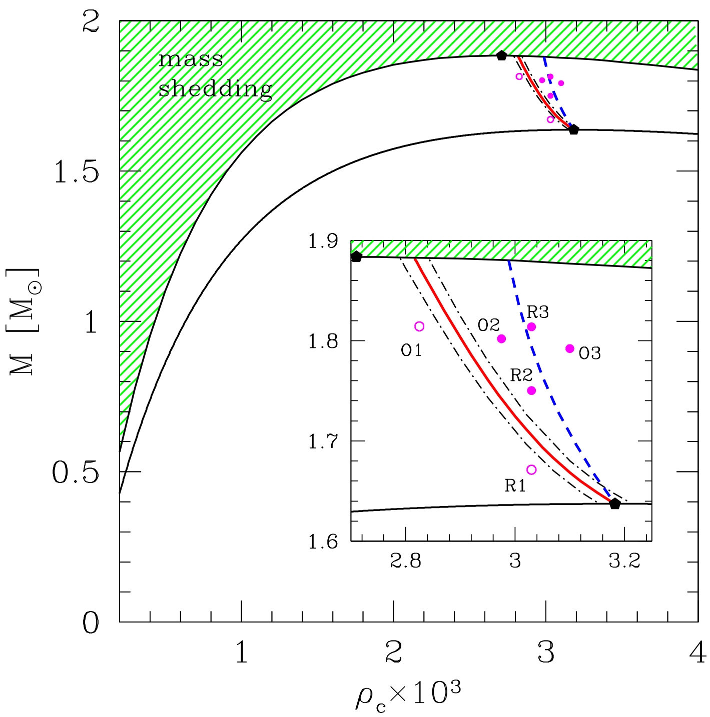

Fig. 2 Stability lines in a (ρ_c, M) diagram. The two solid black lines mark sequences with either zero (lower line) or mass-shedding angular momentum (upper line), with the filled symbols marking the corresponding maximum masses. The solid grey line is the neutral-stability line, "thickened" by the error bar (dot-dashed lines). The grey dashed line is instead the turning-point criterion for secular stability. Marked with empty or filled circles are representative models with constant angular velocity O1, O2, O3, or constant initial central rest-mass density R1, R2, R3. (Figure from Takami et al. [24]. Reproduced by permission of John Wiley and Sons.)

The greater significance of the Takami et al. result, however, is that stars along the line determined by using the equilibrium equation of state are guaranteed to be *secularly* unstable, because the diagnosed instability guarantees that the configurations have lower energy than equilibria with the same baryon mass and angular momentum. This means that the line of secular instability runs through rapidly rotating configurations with central densities more than 5% below those along the line of turning points.

5 Nonaxisymmetric instabilities

Rapidly rotating stars and drops of water are unstable to a bar mode that leads to fission in the water droplets; and a similar nonaxisymmetric instability is likely to be the reason most stars in the universe are in close binary systems. Galactic disks are unstable to nonaxisymmetric perturbations that lead to bars and to spiral structure. And a related instability of a variety of nonaxisymmetric modes, driven by gravitational waves, the Chandrasekhar-Friedman-Schutz (CFS) instability [25, 8]), may limit the rotation of neutron stars. The existence of the CFS-instability in rotating stars was first found by Chandrasekhar [25] in the case of the $l = 2$ mode in uniformly rotating, uniform density Maclaurin spheroids. Subsequently, Friedman and Schutz [26, 8] showed that this instability also appears in compressible stars and that all rotating, self-gravitating perfect fluid configurations are generically unstable to the emission of gravitational waves. We have seen that, along a sequence of stellar models, a mode changes from stable to unstable when its frequency vanishes. The generic-instability result means that zero-frequency nonaxisymmetric modes of rotating perfect-fluid stellar models are marginally stable.

Whereas axisymmetric instability to collapse sets in at points that are nearly independent of the magnitude of viscosity or the strength of gravitational waves, the opposite is true for the nonaxisymmetric case. Gravitational radiation drives a nonaxisymmetric instability that, if no other dissipation is present, makes *every* rotating star unstable. Viscosity can drive a nonaxisymmetric instability in rapidly rotating stars for which gravitational radiation is negligible. For slowly rotating stars, however (and nearly all neutron stars rotate slowly compared to the Kepler limit), viscosity simply damps out the gravitational-wave driven instability. That is, for slow rotation, we will see that the timescale of the CFS instability is longer than the timescale for viscous damping. On the other hand, for rapidly rotating neutron stars, the instability's timescale may be short enough that it limits the rotation of young neutron stars and of old neutron stars spun up by accretion.

This review begins with a discussion of the CFS instability for perfect-fluid models and then outlines the work that has been done to decide whether the instability is present in young neutron stars and in old neutron stars spun up by accretion. For very rapid rotation and for slower but highly differential rotation, nonaxisymmetric modes can be *dynamically unstable*, with growth times comparable to the period of a star's fundamental modes, and the review ends with a brief discussion of these related dynamical instabilities.

To understand the way the CFS instability arises, consider first a stable spherical star. All its modes have positive energy, and the sign of a mode's angular momentum J_c about an axis depends on whether the mode moves clockwise or counterclockwise around the star. That is, a mode with angular and time dependence of the form $\cos(m\phi + \omega_0 t)e^{-\alpha_0 t}$, has positive angular momentum J_c about the z-axis if and only if the mode moves in a positive direction: The pattern speed, $-\dfrac{\omega_0}{m}$, is positive. Because the wave moves in a positive direction relative to an observer at infinity, the star radiates positive angular momentum to infinity, and the mode is damped. Similarly, a mode with negative angular momentum has negative pattern speed, $-\dfrac{\omega_0}{m} < 0$, and radiates negative angular momentum to infinity; and the mode is again damped.

Now consider a slowly rotating star with a backward-moving mode, a mode that moves in a direction opposite to the star's rotation. Because a short-wavelength fluid mode (a mode with a Newtonian counterpart, not a *w*-mode) is essentially a wave in the fluid, the wave moves with nearly the same speed relative to a rotating observer that it had in the spherical star. That means that an observer at infinity sees the mode dragged forward by the fluid. The real part ω_r of the frequency seen in a rotating frame is the frequency associated with the ϕ coordinate $\phi_r = \phi - \Omega t$ of a rotating observer. Then

$$m\phi + \omega t = m\phi_r + (\omega + m\Omega)t = m\phi_r + \omega_r t,$$

implying that the frequency seen by the rotating observer is

$$\omega_r = \omega + m\Omega. \tag{37}$$

For a slowly rotating star, $\omega_r \approx \omega_0$. When the star rotates with an angular velocity greater than $|\omega_r/m|$, the backward-going mode is dragged *forward* relative to an observer at infinity, and ω_r and ω have opposite signs:

$$\omega_r \omega < 0. \tag{38}$$

Because the pattern speed is now positive, the mode radiates positive angular momentum to infinity. But the canonical angular momentum is still negative, because the mode is moving backward relative to the fluid: The angular momentum of the perturbed star is smaller than the angular momentum of the star without the backward-going mode. As the star radiates positive angular momentum to infinity, J_c becomes increasingly negative, implying that the amplitude of the mode grows in time: *Gravitational radiation now drives the mode instead of damping it.*

For large m or small ω_0, the pattern speed will be positive when $\Omega \approx |\omega_0/m|$. This relation suggests two classes of modes that are unstable for arbitrarily slow rotation: Backward-moving modes with large values of m and modes with any m whose frequency is zero in a spherical star. Both classes of perturbations exist. The usual p-modes and g-modes have finite frequencies for a spherical star and are unstable for $\Omega \gtrsim |\omega_0|/m$; and r-modes, which have zero frequency for a non-rotating barotropic star, are unstable for all values of m and Ω (that is, those r-modes are unstable that are backward-moving in the rotating frame of a slowly rotating star).

We have so far not mentioned the canonical energy, but our key criterion for the onset of instability is a negative E_c. If we ignore the imaginary part of the frequency, the change in the sign of E_c follows immediately from the relation (19), $J_c = -\frac{m}{\omega}E_c$. To take the imaginary part $\mathrm{Im}\sigma = \alpha \neq 0$ of the frequency into account, we need to use the fact that energy is lost at a rate $\dot{E}_c \propto \dddot{Q}^2 \propto \sigma^6$ for quadrupole radiation, with $\dot{E}_c$ proportional to higher powers of σ for radiation into higher multipoles. Because E_c is quadratic in the perturbation, it is proportional to $e^{-2\alpha t}$, implying $\alpha \propto \sigma^6$.

Thus $\alpha/\sigma \to 0$ as $\sigma \to 0$, implying that for a normal mode E_c changes sign when ω changes sign.

Although the argument we have given so far is heuristic, there is a precise form of the statement that a stable, backward-moving mode becomes unstable when it is dragged forward relative to an inertial observer (see Friedman & Schultz [26] and Friedman & Stergioulas [19]).

Theorem. Consider an outgoing mode $(h_{\alpha\beta}(\lambda), \xi^\alpha(\lambda))$, that varies smoothly along a family of uniformly rotating perfect-fluid equilibria, labeled by λ. Assume that it has t and ϕ dependence of the form $e^{i(m\phi+\sigma t)}$, that $\omega = \mathrm{Re}\{\sigma\}$ satisfies $\omega/m+\Omega > 0$ for all λ, and that the sign of ω/m is positive for $\lambda < \lambda_0$ and negative for $\lambda > \lambda_0$. Then in a neighborhood of λ_0, $\alpha := \mathrm{Im}\{\sigma\} \le 0$; and if the mode has at least one nonzero asymptotic multipole moment with $l \ge 2$ at future null infinity, the mode is unstable ($\alpha < 0$) for $\lambda > \lambda_0$.

A corresponding result that does not rely on existence or completeness of normal modes is the statement that one can always choose canonical initial data to make $E_c < 0$ [8, 19].

The growth time τ_{GR} of the instability of a perfect fluid star is governed by the rate $\left.\dfrac{dE}{dt}\right|_{\mathrm{GR}}$ at which energy is radiated in gravitational waves:

$$\frac{1}{\tau_{\mathrm{GR}}} = -\frac{1}{2E_c}\left.\frac{dE}{dt}\right|_{\mathrm{GR}}, \tag{39}$$

where (Thorne 1980)

$$\frac{dE}{dt} = -\sum_{l\ge m} \omega_i^{2l+2} N_l \left(|\delta D_{lm}|^2 + |\delta J_{lm}|^2\right). \tag{40}$$

Here D_{lm} and J_{lm} are the asymptotically defined mass and current multipole moments of the perturbation and $N_l = \dfrac{4\pi(l+1)(l+2)}{l(l-1)[(2l+1)!!]^2}$ is, for low l, a constant of order unity. In the Newtonian limit,

$$\delta D_{lm} = \int \delta\rho\, r^l Y_{lm} d^3x. \tag{41}$$

For a star to be unstable, the growth time τ_{GR} must be shorter than the viscous damping time $\tau_{\mathrm{viscosity}}$ of the mode, and the implications of this are discussed below. In particular because the growth time is longer for larger l, only low multipoles can be unstable in neutron stars.

Modes with polar and axial parity

The spherical symmetry of a nonrotating star and its spacetime implies that perturbations can be labeled by fixed values l,m labeling an angular harmonic: The quantities $h_{\alpha\beta}, \xi^\alpha, \delta\rho, \delta\varepsilon, \delta p, \delta s$ that describe a perturbation are all proportional to scalar, vector and tensor spherical harmonics constructed from Y_{lm}, and perturbations with different l,m values decouple. Similarly, because spherical stars are invariant under parity (a map of each point P of spacetime to the diametrically opposite point on the symmetry sphere through P), perturbations with different parity decouple, the parity of a perturbation is conserved, and normal modes have definite parity. Perturbations associated with an l,m angular harmonic are said to have *polar* parity if they have the same parity as the function Y_{lm}, $(-1)^l$. Perturbations having parity $(-1)^{l+1}$, opposite to that of Y_{lm} have axial parity. In the Newtonian literature, modes of a rotating star that are continuously related to polar modes of a spherical star are commonly called *spheroidal*; while modes whose spherical limit is axial are called *toroidal*.

Every rotational scalar — ε, p, ρ, and the components in the t-r subspace of the perturbed metric $h_{\alpha\beta}$ and the perturbed fluid velocity δu^α — can be expressed as a superposition of scalar spherical harmonics $Y_{\ell m}$. As a result, modes of spherical stars that involve changes in any scalar are polar. On the other hand, the angular components of velocity perturbations can have either polar parity, with

$$\delta v = f(r)\nabla Y_{lm} \tag{42}$$

or axial parity, with Newtonian form

$$\delta v = f(r)\mathbf{r}\times\nabla Y_{lm}, \tag{43}$$

and the relativistic form $\delta u^\alpha \propto \varepsilon^{\alpha\beta\gamma\delta}\nabla_\beta t\nabla_\gamma r\nabla_\delta Y_{lm}$.

There are two families of polar modes of perfect-fluid Newtonian stars, p-modes (pressure modes) and g-modes (gravity modes). For short wavelengths, the p-modes are sound waves, with pressure providing the restoring force and frequencies

$$\sigma = c_s k, \tag{44}$$

where k is the wavenumber and c_s is the speed of sound. The short-wavelength g-modes are modes whose restoring force is buoyancy, and their frequencies are proportional to the Brunt-Väisälä frequency, related to the difference between $dp/d\varepsilon$ in the star and $c_s^2 = \partial p(\varepsilon,s)/\partial\varepsilon$. The fundamental modes of oscillation of a star (f-modes), with no radial nodes, can be regarded as a bridge between g-modes and p-modes.

Because axial perturbations of a spherical star involve no change in density or pressure, there is no restoring force in the linearized Euler equation, and the linear perturbation is a time-independent velocity field – a zero-frequency mode. [3] In a rotating star, the axial modes acquire a nonzero frequency proportional to the star's angular velocity Ω, a frequency whose Newtonian limit has the simple form

$$\sigma = -\frac{(l-1)(l+2)}{l(l+1)}m\Omega, \tag{45}$$

[3] Axial perturbations of the spacetime of a spherical star include both axial perturbations of the fluid and gravitational waves with axial parity. The axial-parity waves do not couple to the fluid perturbation, which is stationary in the sense that $\partial_t \delta u_\alpha = 0$.

where the harmonic time and angular dependence of the mode is $e^{i(m\phi-\sigma t)}$. These modes are called r-modes, their name derived from the Rossby waves of oceans and planetary atmospheres. The term r-mode can be usefully regarded as a mnemonic for a *rotationally restored* mode. Eq. (37) implies that the r-mode associated with every nonaxisymmetric multipole obeys the instability condition for every value of Ω: It is forward moving in an inertial frame and backwards moving relative to a rotating observer:

$$\sigma_r = \frac{2m}{l(l+1)}\Omega, \tag{46}$$

with sign opposite to that of σ. Because the rate at which energy is radiated is greatest for the $l=m=2$ r-mode, that is the mode whose instability grows most quickly and which determines whether an axial-parity instability can outpace viscous damping.

The instability of low-multipole r-modes for arbitrarily slow rotation is strikingly different from the behavior of the low-multipole f- and p-modes, which are unstable only for large values of Ω. The reason is that the frequencies of f- and p-modes are high, and, from Eq. (38), a correspondingly high angular velocity is needed before a mode that moves backward relative to the star is dragged forward relative to an inertial observer at infinity. Of the polar modes, f-modes with $l=m$ have the fastest growth rates; their instability points for uniformly rotating relativistic stars, found by Stergioulas & Friedman [27], are shown in Figure 3. (Work on these stability points of relativistic stars is also reported in [28, 29, 30, 31].)

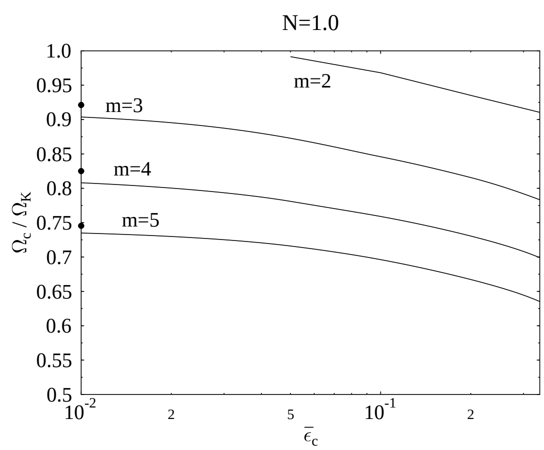

Fig. 3 Critical angular velocity Ω/Ω_K vs. the dimensionless central energy density $\bar{\varepsilon}_c$ for the $m=2$, 3, 4 and 5 neutral modes of $N=1.0$ polytropes. The filled circles on the vertical axis are the Newtonian values of the neutral points for each mode. (Reproduced from [27].)

The figure shows that, for uniform rotation, the $l=m=2$ f-mode is unstable only for stars with relatively high central density or high mass. For tabulated EOSs, this practically applies to all neutron stars with masses greater than 1.3 $M_\odot$ and

$T/|W| > 0.06$ [32]. Because neutron stars, rotate differentially at birth, and the $l = 2$ mode, as well as higher modes, could be initially unstable for a larger range of parameters.

Implications of the instability

The nonaxisymmetric instability may limit the rotation of nascent neutron stars and of old neutron stars spun up by accretion; and the gravitational waves emitted by unstable modes may be observable by gravitational wave detectors. Whether a limit on spin is in fact enforced depends on whether the instability of perfect-fluid models implies an instability of neutron stars; and the observability of gravitational waves also requires a minimum amplitude and persistence of an unstable mode. We briefly review observational support for an instability-enforced upper limit on spin and then turn to the open theoretical issues.

Evidence for an upper limit on neutron-star spin smaller than the Keplerian frequency Ω_K comes from nearly 30 years of observations of neutron stars with millisecond periods, seen as pulsars and as X-ray binaries. The observations reveal rotational frequencies ranging upward to 716 Hz and densely populating a range of frequencies below that. Selection biases against detection of the fastest millisecond radio pulsars have made conclusions about an upper limit on spin uncertain, but Chakrabarty argues that the class of sources whose pulses are seen in nuclear bursts (nuclear powered accreting millisecond X-ray pulsars) constite a sample without significant bias [33]. Their distribution of spins, together with the spins of other millisecond pulsars, is shown in Fig. 4

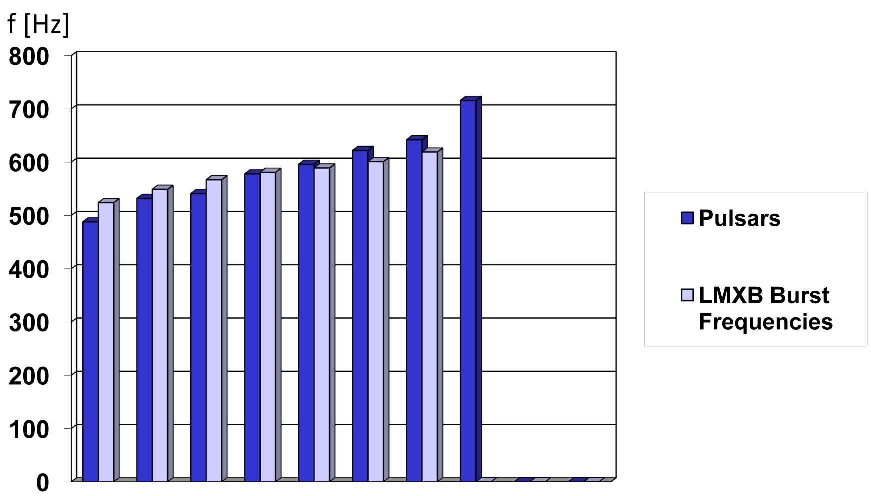

Fig. 4 Highest observed neutron-star spin frequencies.

A magnetic field of order 10^8 G can limit the spin of an accreting millisecond pulsar. Because matter within the magnetosphere corotates with the star, only matter that accretes from outside the magnetosphere can spin up the star, leading to an equilibrium period given approximately by Ghosh & Lamb [34] [4]

$$P_{\rm eq} \sim 2\times 10^{-3}{\rm s}\left(\frac{B}{10^8{\rm G}}\right)^{6/7}\left(\frac{\dot M}{10^{-10}M_\odot{\rm yr}^{-1}}\right)^{-3/7}. \tag{47}$$

Because this period depends on the magnetic field, a sharp cutoff in the frequency of accreting stars is not an obvious prediction of magnetically limited spins. For a magnetically set maximum rotation rate of order 700-800 Hz the range of magnetic fields would need to have a corresponding minimum cutoff value of about 10^8 G; and the highest observed spin rates should correspond to the lowest magnetic fields. The required cutoff and a fairly narrow range of observed frequencies has made gravitational-wave limited spin a competitive possibility for accreting neutron stars. Arguments for and against this based on available observations are given by White and Zhang [36] and by Patruno et al. [37], respectively.

Under what circumstances the CFS instability could limit the spin of recycled pulsars has now been studied in a large number of papers. References to this work can be found in the treatment in FS, on which the present review is based and in comprehensive earlier discussions by Stergioulas [38], by Andersson and Kokkotas [39], and by Kokkotas and Ruoff [40], while briefer reviews of more recent work are given in [41, 42]. References in the present review are generally limited to initial work and to a late paper that contains intervening references.

Whether the instability survives the complex physics of a real neutron star has been the focus of most recent work, but it remains an open question. Studies have focused on:

- Dissipation from bulk and shear viscosity and mutual friction in a superfluid interior;
- magnetic field wind-up;
- nonlinear evolution and the saturation amplitude; and
- the possiblity that a continuous spectrum replaces r-modes in relativistic stars.

We discuss these in turn and then summarize their implications for nascent, rapidly rotating stars and for old stars spun up by accretions.

Viscosity

When viscosity is included, the growth-time or damping time τ of an oscillation has the form

$$\frac{1}{\tau} = \frac{1}{\tau_{GR}} + \frac{1}{\tau_b} + \frac{1}{\tau_s}, \tag{48}$$

with τ_b and τ_s the damping times due to bulk and shear viscosity. Bulk viscosity is large at high temperatures, shear viscosity at low temperatures. This leaves a

[4] Shapiro and Teukolsky [35] give a clear, simplified version, and Eq. (47) is their Eq. (15.2.22), with $M = 1.4M_\odot$, $R = 10$ km, and a ratio ω_s of the angular velocity to Ω_K at the inner edge of the disk set to 1.

window of opportunity in which a star with large enough angular velocity can be unstable. The window for the $l = m = 2$ r-mode is shown in Fig. 5, for a representative computation of viscosity. The highest solid curves on left and right mark the critical angular velocity Ω_c above which the $l = m = 2$ r-mode is unstable. The curves on the left, show the effect of shear viscosity at low temperature, allowing instability when $\Omega < \Omega_K$ only for $T > 10^6$K; the curve on the right shows the corresponding effect of bulk viscosity, cutting off the instability at temperatures above about 4×10^{10}K.

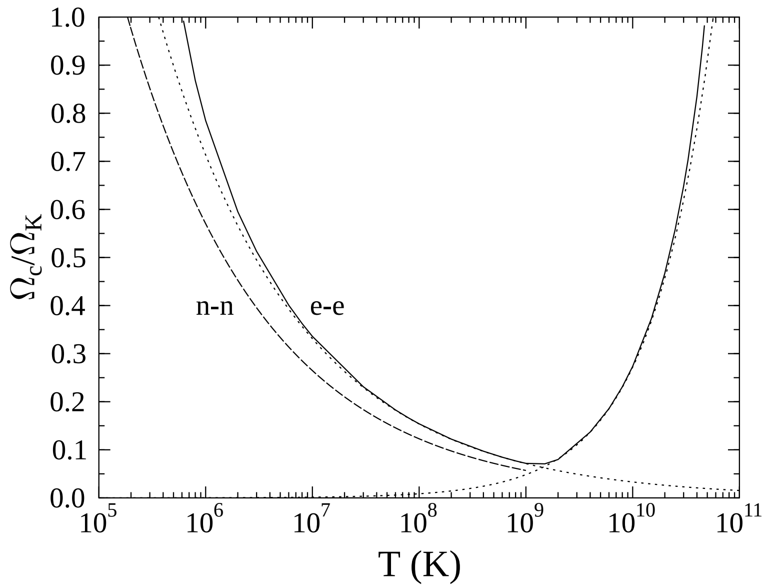

Fig. 5 Critical angular velocity for the onset of the r-mode instability as a function of temperature (for a 1.5 $M_\odot$ neutron star model). The solid line corresponds to the $O(\Omega^2)$ result using electron-electron shear viscosity, and modified URCA bulk viscosity. The dashed line corresponds to the case of neutron-neutron shear viscosity. Dotted lines are $O(\Omega)$ approximations. (Reproduced from [43].)

There is substantial uncertainty in the positions of both of these curves.

Bulk viscosity arises from nuclear reactions driven by the changing density of an oscillating fluid element, with neutrons decaying, $n \to p + e + \bar{\nu}_e$, as the fluid element expands and protons capturing electrons, $p + e \to n + \nu_e$, as it contracts. The neutrinos leave the star, draining energy from the mode. The rates of these *URCA* reactions increase rapidly with temperature and are fast enough to be important above about 10^9K, with an expected damping time τ_b given by

$$\frac{1}{\tau_\text{b}} = \frac{1}{2E_c} \int \zeta (\delta\theta)^2 d^3x, \tag{49}$$

where $\theta = \nabla_\alpha u^\alpha$ is the divergence of the fluid velocity and the coefficient of bulk viscosity ζ is given by [44]

$$\zeta = 6\times 10^{25} \rho_{15}^2 T_9^6 \left(\frac{\omega_r}{1\text{Hz}}\right)^{-2} \ \text{g cm}^{-1}\ \text{s}^{-1}, \tag{50}$$

where $T_9 = T/(10^9\text{K})$. With these values, bulk viscosity suppresses the instability in all modes above a few times 10^{10}K (see also Ipser & Lindblom [45, 46] and Yoshida & Eriguchi [47]). [5]

These equations and Fig. 5 assume that only *modified URCA* reactions can occur, that the URCA reactions require a collision to conserve four-momentum, and this will be true when the proton fraction is less than about 1/9. Should the equation of state be unexpectedly soft (and if the mass is large enough), direct URCA reactions would be allowed, suppressing the instability for uniformly rotating stars at roughly 10^9K [51]. A soft equation of state would also more likely lead to stars with hyperons in their core with an additional set of nuclear reactions that dissipate energy and increase the bulk viscosity [52, 53, 54, 55, 56] or quarks [57, 58, 59, 60, 61]. However, the observation of a 2 $M_\odot$ pulsar makes the existence of hyperons or quarks in the core of 1.4 $M_\odot$ neutron stars less probable.

In contrast to bulk viscosity, shear viscosity increases as the temperature drops. In terms of the shear tensor $\sigma_{\alpha\beta} = (\delta_\alpha^\gamma + u_\alpha u^\gamma)(\delta_\beta^\delta + u_\beta u^\delta)(\nabla_\gamma u_\delta + \nabla_\delta u_\gamma - \frac{2}{3} g_{\gamma\delta}\nabla_\varepsilon u^\varepsilon)$, the damping time is given by

$$\frac{1}{\tau_s} = \frac{1}{E_c}\int \eta\, \delta\sigma^{\alpha\beta}\delta\sigma_{\alpha\beta}\, d^3x, \tag{51}$$

where η is the coefficient of shear viscosity. For nascent neutron stars hotter than the superfluid transition temperature (about 10^9K), a first estimate of the neutron-neutron shear viscosity coefficient is [62]

$$\eta_n = 2\times 10^{18} \rho_{15}^{9/4} T_9^{-2} \ \text{g cm}^{-1}\ \text{s}^{-1}, \tag{52}$$

where $\rho_{15} = \rho/(10^{15}\text{g cm}^{-3})$. Below the superfluid transition temperature, electron-electron scattering determines the shear viscosity in the superfluid core, giving [63]

$$\eta_e = 2.5\times 10^{18} \left(\frac{x_p}{0.1}\rho_{15}\right)^2 T_9^{-5/3} \ \text{g cm}^{-1}\ \text{s}^{-1}. \tag{53}$$

Shear viscosity may be greatly enhanced after formation of the crust in a boundary layer (Ekman layer) between crust and core [64, 65, 66, 67, 68]. The enhancement depends on the extent to which the core participates in the oscillation, parametrized by the slippage at the boundary. The uncertainty in this slippage appears to be the greatest current uncertainty in dissipation of the mode by shear vis-

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[5] At temperatures above roughly 10^{10}K, another complication appears: neutrino absorption increases with increasing temperature [48, 49]), and the modified URCA bulk viscosity no longer rises, but is reduced by an order of magnitude between 10^{10} K and 10^{11}K, allowing the instability to operate in very hot proto-neutron stars [50].

cosity, and it significantly affects the critical angular velocity of the r-mode instability in accreting neutron stars.

For f-modes, the part of the instability window in Fig. 5 to the left of 10^9 K is thought to be removed by another dissipative mechanism that comes into play below the superfluid transition temperature. Called mutual friction, it arises from the scattering of electrons off magnetized neutron vortices. Work by Lindblom and Mendell [69] shows that mutual friction in the superfluid core completely suppresses f- and p-mode instabilities below the transition temperature. For the r-mode instability, subsequent work by the same authors [70] finds that the mutual friction is much smaller, with a damping time of order 10^4 s, too long to be important.

In a recent paper, Gaertig et al. [31] point out the possibility of an interaction between vortices and quantized flux tubes that would result in a much smaller value for the mutual friction. They argue that the resulting uncertainty is great enough that shear viscosity could be the dominant dissipative mechanism for f-modes as well as r-modes.

Magnetic field windup

At second-order in the perturbation, the nonlinear evolution of an unstable mode includes an axisymmetric part that describes a growing differential rotation. Because differential rotation will wind up magnetic field lines, the mode's energy could be transferred to the star's magnetic field [71, 72, 73, 74, 75]. Again there is large uncertainty about the strength of a toroidal magnetic field that will be generated by the differential rotation, what magnetic instabilities will arise, and what the effective dissipation will be. Apart from the studies cited here (all of which deal with r-modes) nearly all the remaining work on the evolution of unstable modes ignores magnetic fields.

Relativistic r-modes and a possible continuous spectrum

Relativistic r-modes have been computed by a number of authors [76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 40, 86, 87] Where the Newtonian approximation has purely axial $l = m$ r-modes for barotropic stars at lowest order in Ω, in the full theory all rotationally restored modes include a polar part. The change in the structure of the computed r-modes are small, but that may not be the end of the story.

For non-barotropic stars Kojima found a single second-order eigenvalue equation for the frequency, to lowest nonvanishing order in Ω. The coefficient of the highest derivative term in that equation vanishes at some value of the radial coordinate r, for typical candidate neutron-star equations of state, and that singular behavior gives a continuous spectrum. Lockitch, Andersson & Watts [82] consider the question of the continuous spectrum and the existence of r-modes in some detail. They argue that the singularity in the Kojima equation is an artifact of the slow-rotation approximation and is not present if one includes terms of order Ω^2. Their work is a strong argument for the existence of r-modes in non-barotropic models.

Showing the existence of the mode, however, does not decide the question of whether a continuous spectrum is also present or whether the existence of a continuous or nearly continuous spectrum significantly alters the evolution of an initial perturbation.

Nonlinear evolution

Linear perturbation theory is valid only for small-amplitude oscillations; as the amplitude of an unstable mode grows, couplings to other modes become increasingly important, and the mode ultimately reaches a saturation amplitude or is disrupted, losing coherence. The first nonlinear studies of the r-mode instability involved fully nonlinear 3+1 evolutions by Stergioulas & Font [88], in which the r-mode was set at a large initial amplitude or Newtonian evolutions by Lindblom, Tohline & Vallisneri [89, 90] in which the r-mode was driven to large amplitude by an artificially large gravitational-radiation reaction term. On a few tens of dynamical timescales, saturation was seen only at an amplitude of order unity. Subsequently, simulations on longer timescales showed a coupling to daughter modes [91, 92], suggesting that the actual saturation amplitude of the r-mode is smaller than the amplitude at which gravitational-radiation reaction was switched off in the short-timescale simulations.

The grid resolution of 3+1 simulations, however, is currently too low to see couplings to short-wavelength modes, and they cannot run for a time long enough to see the growth from a realistic radiation-reaction term. The alternative is to examine the nonlinear evolution in the context of higher-order perturbation theory. To do this, the Cornell group (initially with S. Morsink) [93, 94, 95] constructed a second-order perturbation theory for rotating Newtonian stars, and then used the formalism to study the nonlinear evolution of an unstable r-mode. Their series of papers leaves little doubt that nonlinear couplings sharply limit the amplitude of an unstable r-mode, with a possible range of 10^{-1}-10^{-5} (see Bondarescu, Teukolsky & Wasserman [96, 97] and references therein).

The nonlinear development of the f-mode instability has been modeled in three-dimensional, hydrodynamical simulations (in a Newtonian framework) by Ou, Tohline & Lindblom [98] and by Shibata & Karino [99], essentially confirming previous approximate results obtained by Lai & Shapiro [48]. Kastaun, Willburger & Kokkotas [100] report an initial nonlinear study of f-modes in general relativity. In the framework of a 3+1 simulation in a Cowling approximation (a fixed background metric of the unperturbed rotating star), they find limits on the amplitude of less than 0.1, set by wave-breaking and by coupling to inertial modes. This can be regarded as an upper limit on the amplitude, with second-order perturbative computations still to be done.

Instability scenarios in nascent neutron stars and in old accreting stars

Both r-modes and f-modes may be unstable in nascent neutron stars that are rapidly rotating at birth. Recent work on f-modes in relativistic models [101, 31] finds growth times substantially shorter than previously computed Newtonian values. In a particular model (where the $l=m=2$ mode becomes unstable only very near the mass-shedding limit), the $l=m=3$ and $l=m=4$ f-modes have growth times of 10^3-10^5 s for Ω near Ω_K. In a typical scenario, a star with rotation near the Kepler limit becomes unstable within a minute of formation, when the temperature has dropped below 10^{11}K. As the temperature drops further, the instability grows to saturation amplitude in days or weeks. Loss of angular momentum to gravitational waves spins down the star until the critical angular velocity is reached below which the star is stable, at or before the time at which the core becomes a superfluid. The $l=m=3$ mode (or the $l=m=2$ mode in models with different masses or equations of state than the one studied above) could be a source of observable gravitational waves for supernovae in or near the Galaxy (but with an uncertain event rate).

The time over which the instability is active depends on the saturation amplitude, the cooling rate, and the superfluid transition temperature, and all of these have large uncertainties. The time at which a superfluid transition occurs could be shorter than a year, but recent analyses of the cooling of a neutron star in Cassiopeia A [102, 103] suggest a superfluid transition time for that star of order 100 years.

The scenario for the $l = m = 2$ r-mode instability of a nascent star is similar. The r-mode instability itself was pointed out by Andersson [77], with a mode-independent proof for relativistic stars given by Friedman and Morsink [104]. First computations of the growth and evolution were reported by Lindblom, Owen & Morsink [105] and by Andersson, Kokkotas & Schutz [106], with effects of a crust discussed by Lindblom, Owen & Ushomirsky [65]. Intervening work is referred to by Bondarescu, Teukolsky & Wasserman [97]; the simulations reported by Bondarescu et al. include nonlinear couplings that saturate the amplitude and the alternative possibilities for viscosity that we have discussed above. The r-mode's saturation amplitude is likely to be lower than that of the f-modes, and it is likely to persist longer because of its low mutual friction.

As mentioned above, the r-mode instability of neutron stars spun up by accretion has been more intensively studied in connection with the observed spins of LMXBs. Papaloizou & Pringle [107] suggested the possibility of accretion spinning up a star until it becomes unstable to the emission of gravitational waves and reaches a steady state, with the angular momentum gained by accretion equal to the angular momentum lost to gravitational waves. Following the discovery of the first millisecond pulsar, Wagoner examined the mechanism in detail for CFS unstable f-modes [108]. Although mutual friction appears to rule out the steady-state picture for f-modes, it remains a possibility for r-modes [109, 110, 66, 111]. Levin [112] and (independently) Spruit [71], however, pointed out that viscous heating of the neutron star by its unstable oscillations will lower the shear viscosity and so increase the mode's growth rate, leading to a runaway instability. The resulting scenario is a cycle in which a cold, stable neutron star is spun up over a few million years until it becomes unstable; the star then heats up, the instability grows, and the star spins down until it is again stable, all within a few months; the star then cools, and the cycle repeats.

This scenario would rule out r-modes in LMXBs as a source of detectable gravitational waves because the stars would radiate for only a small fraction of the cycle. A small saturation amplitude, however, lengthens the time spent in the cycle, possibly allowing observability [113]. The steady state itself remains a possible alternative in stars whose core contains hyperons or free quarks (or if the "neutron stars" are really strange quark stars) [59, 53, 111, 114, 55, 56]. Heating the core increases the bulk viscosity, and with an exotic core, this growth in the bulk viscosity is large enough to prevent the thermal runaway and allow a steady state. In Bondarescu et al. [96] the nonlinear evolutions (restricted to 3 coupled modes) include neutrino cooling, shear viscosity, hyperon bulk viscosity and dissipation at the core-crust boundary layer, with parameters to span a range of uncertainty in these various quantities. They display the regions of parameter space associated with the alternative scenarios just outlined – steady state, cycle, and fast and slow runaways. In all cases, the r-mode amplitude remains very small ($\sim 10^{-5}$), but because of the long duration of the instability, such systems are still good candidates for gravitational wave detection by advanced LIGO class interferometers [96, 115, 42].

Dynamical nonaxisymmetric instability

Work on dynamical nonaxisymmetric instabilities is largely outside the scope of this review. They are most likely to be relevant to protoneutron stars and to the short-lived hypermassive neutron stars that form in the merger of a double neutron star system. Unless the star has unusually high differential rotation, instability requires a large value of the ratio $T/|W|$ of rotational kinetic energy to gravitational binding energy: comparable to the value $T/|W| = 0.27$ that marks the dynamical instability of the $l = m = 2$ mode of uniformly rotating uniform density Newtonian models (the Maclauring spheroids). This bar instability, if present, will emit strong gravitational waves with frequencies in the kHz regime. The development of the instability and the resulting waveform have been computed numerically in the context of both Newtonian gravity and in full general relativity (see [116, 117, 118, 119] for representative studies).

Uniformly rotating neutron stars have maximum values of $T/|W|$ smaller than 0.14, apparently precluding dynamical nonaxisymmetric instability. For highly differential rotation, however, Centrella et al. [120] found a one-armed ($m = 1$) instability for smaller rotation, for $T/|W| \sim 0.14$, but for a polytropic index of $N = 3$ which is not representative for neutron stars. Remarkably, Shibata, Karino & Eriguchi [121, 122] then found found an $m = 2$ instability for $T/|W|$ as low as 0.01, for models with polytropic index $N = 1$, representing a stiffness appropriate to neutron stars. These instabilities appear to be related to the existence of corotation points, where the pattern speed of the mode matches the star's angular velocity [123, 124]; Ou and Tohline [125] tie the growth of the instability to a resonant cavity associated with a minimum in the vorticity to density ratio (the so-called vortensity) . Collapsing cores in supernovae are differentially rotating, and these instabilities of protoneutron stars arise in simulations of rotating core collapse [126, 127]. Because they can radiate more energy in gravitational waves than the post-bounce burst signal itself, interest in these dynamical instabilities is strong.

Acknowledgments

J.F.'s work is supported in part by NSF Grant PHY 1001515. N.S.'s is supported in part by an Excellence Grant for Basic Research (Research Committee of the Aristotle University of Thessaloniki – 87896) and by an IKY–DAAD exchange grant (IKYDA 2012).

References

1. A.H. Taub, *General relativistic variational principal for perfect fluids*, Phys. Rev. **94**, 1468 (1954)
2. A.H. Taub, *Stability of general relativistic gaseous masses and variational principles*, Commun. Math. Phys. **15**, 235 (1969)
3. B.F. Schutz, *Perfect fluids in general relativity: velocity potentials and a variational principle*, Phys. Rev. D **2**, 2762 (1970)
4. B.F. Schutz, Jr., *Linear pulsations and stability of differentially rotating stellar models. II. General-relativistic analysis*, Astrophys. J. **24**, 343 (1972)
5. B. Carter, *Elastic perturbation theory in general relativity and variational principle for a rotating solid star*, Commun. Math. Phys. **30**, 261 (1973)
6. B.F. Schutz, R.D. Sorkin, *Variational aspects of relativistic field theories with applications to perfect fluids*, Ann. Phys. **107**, 1 (1977)
7. M.G. Calkin, *An action principle for magnetohydrodynamics*, Can. J. Phys. **41**, 2241 (1963)
8. J.L. Friedman, *Generic instability of Rotating Relativistic Stars*, Commun. Math. Phys. **62**, 247 (1978)
9. J.L. Friedman, B.F. Schutz, *Lagrangian perturbation theory of nonrelativistic fluids*, Astrophys. J. **221**, 937 (1978)
10. K.S. Thorne, *Validity in general relativity of the Schwarzschild criterion for convection*, Astrophys. J. **144**, 201 (1966)
11. A. Kovetz, *Schwarzschild's criterion for convective instability in general relativity*, Z. Astrophys. **66**, 446 (1967)
12. B.F. Schutz, Jr., *Taylor instabilities in relativistic stars*, Astrophys. J. **161**, 1173 (1970)
13. J.M. Bardeen, *A variational principle for rotating stars in general relativity*, Astrophys. J. **162**, 71 (1970)
14. F.H. Seguin, *The stability of nonuniform rotation in relativistic stars*, Astrophys. J. **197**, 745 (1975)
15. S.L. Detweiler, J.R. Ipser, *A variational principle and a stability criterion for the non-radial modes of pulsation of stellar models in general relativity*, Astrophys. J. **185**, 685 (1973)
16. N.R. Lebovitz, *On Schwarzschild's criterion for the stability of gaseous masses*, Astrophys. J. **142**, 229 (1965)
17. H. Solberg, *Le mouvement d'inertie de l'atmosphere stable et son role dans la theorie des cyclones*, in *Proces Verbaux de l' Association de Météorologie. International Union of Geodesy and Geophysics. 6th General Assembly (Edinburgh)*, vol. 2 (International Union of Geodesy and Geophysics, Edinburgh, 1936), pp. 66–82
18. M.A. Abramowicz, *Rayleigh and Solberg criteria reversal near black holes: the optical geometry explanation* (2004). ArXiv:astro-ph/0411718
19. J.L. Friedman, N. Stergioulas, *Rotating relativistic stars* (Cambridge University Press, 2013)
20. S. Chandrasekhar, *The dynamical instability of gaseous masses approaching the Schwarzschild limit in general relativity*, Astrophys. J. **140**, 417 (1964)
21. W.A. Fowler, *The stability of supermassive stars*, Astrophys. J. **144**, 180 (1966)
22. J.L. Friedman, J.R. Ipser, R.D. Sorkin, *Turning-point method for axisymmetric stability of rotating relativistic stars*, Astrophys. J. **325**, 722 (1988)
23. G.B. Cook, S.L. Shapiro, S.A. Teukolsky, *Spin-up of a rapidly rotating star by angular momentum loss - Effects of general relativity*, Astrophys. J. **398**, 203 (1992)
24. K. Takami, L. Rezzolla, S. Yoshida, *A quasi-radial stability criterion for rotating relativistic stars*, Mon. Not. R. Astron. Soc. **416**, L1 (2011)
25. S. Chandrasekhar, *Solutions of two problems in the theory of gravitational radiation*, Phys. Rev. Lett. **24**, 611 (1970)
26. J.L. Friedman, B.F. Schutz, *Secular instability of rotating Newtonian stars*, Astrophys. J. **222**, 281 (1978)
27. N. Stergioulas, J.L. Friedman, *Nonaxisymmetric neutral modes in rotating relativistic stars*, Astrophys. J. **492**, 301 (1998)

28. S. Yoshida, Y. Eriguchi, *Neutral points of oscillation modes along equilibrium sequences of rapidly rotating polytropes in general relativity: application of the Cowling approximation*, Astrophys. J. **490**, 779 (1997)
29. S. Yoshida, Y. Eriguchi, *A numerical study of normal modes of rotating neutron star models by the Cowling approximation*, Astrophys. J. **515**, 414 (1999)
30. B. Zink, O. Korobkin, E. Schnetter, N. Stergioulas, *Frequency band of the f-mode Chandrasekhar-Friedman-Schutz instability*, Phys. Rev. D **81**(8), 084055 (2010)
31. E. Gaertig, K. Glampedakis, K.D. Kokkotas, B. Zink, *The f-mode instability in relativistic neutron stars*, Phys. Rev. Lett. **107**, 101102 (2011)
32. S. Morsink, N. Stergioulas, S. Blattnig, *Quasi-normal modes of rotating relativistic stars - neutral modes for realistic equations of state*, Astrophys. J. **510**, 854 (1999)
33. D. Chakrabarty, *The spin distribution of millisecond X-ray pulsars*, in *A Decade of Accreting Millisecond X-Ray Pulsars*, *AIP Conference Proceedings*, vol. 1068, ed. by R. Wijnands, et al. (2008), pp. 67–74
34. P. Ghosh, F.K. Lamb, *Accretion by rotating magnetic neutron stars. II - Radial and vertical structure of the transition zone in disk accretion*, Astrophys. J. **232**, 259 (1979)
35. S.L. Shapiro, S.A. Teukolsky, *Black Holes, White Dwarfs and Neutron Stars* (Wiley, New York, 1983)
36. N.E. White, W. Zhang, *Millisecond X-ray pulsars in low-mass X-ray binaries*, Astrophys. J. **490**, L87 (1997)
37. A. Patruno, B. Haskell, C. D'Angelo, *Gravitational waves and the maximum spin frequency of neutron stars*, Astrophys. J. **746**, 9 (2012)
38. N. Stergioulas, *Rotating stars in relativity*, Living Rev. Relativity **6**, 3 (2003)
39. N. Andersson, K.D. Kokkotas, *The r-mode instability in rotating neutron stars*, Int. J. Mod. Phys. D **10**, 381 (2001)
40. K.D. Kokkotas, J. Ruoff, *Instabilities of relativistic stars*, in *2001: A relativistic spacetime Odyssey* (2002). Firenze 2001
41. N. Andersson, V. Ferrari, D.I. Jones, et al., *Gravitational waves from neutron stars: promises and challenges*, Gen. Rel. Grav. **43**, 409 (2011)
42. B.J. Owen, *How to adapt broad-band gravitational-wave searches for r-modes*, Phys. Rev. D **82**, 104002 (2010)
43. K.D. Kokkotas, N. Stergioulas, *Analytic desctription of the r-mode instability in uniform density stars*, Astron. Astrophys. **341**, 110 (1999)
44. C. Cutler, L. Lindblom, R.J. Splinter, *Damping times for neutron star oscillations*, Astrophys. J. **363**, 603 (1990)
45. J.R. Ipser, L. Lindblom, *On the adiabatic pulsations of accretion disks and rotating stars*, Astrophys. J. **379**, 285 (1991)
46. J.R. Ipser, L. Lindblom, *The oscillations of rapidly rotating Newtonian stellar models. II – Dissipative effects*, Astrophys. J. **373**, 213 (1991)
47. S. Yoshida, Y. Eriguchi, *Gravitational radiation driven secular instability of rotating polytropes*, Astrophys. J. **438**, 830 (1995)
48. D. Lai, S.L. Shapiro, *Gravitational radiation from rapidly rotating nascent neutron stars*, Astrophys. J. **442**, 259 (1995)
49. S. Bonazzola, J. Frieben, E. Gourgoulhon, *Spontaneous symmetry breaking of rapidly rotating stars in general relativity*, Astrophys. J. **460**, 379 (1996)
50. D. Lai, *Secular bar-mode evolution and gravitational waves from neutron stars*, in *Astrophysical sources for ground-based gravitational wave detectors*, *AIP Conference Proceedings*, vol. 575 (2001), pp. 246–257
51. J.L. Zdunik, *Damping of GRR instability by direct URCA reactions*, Astron. Astrophys. **308**, 828 (1996)
52. P.B. Jones, *Comment on "Gravitational radiation instability in hot young neutron stars"*, Phys. Rev. Lett. **86**, 1384 (2001)
53. L. Lindblom, B.J. Owen, *Effect of hyperon bulk viscosity on neutron-star r-modes*, Phys. Rev. D **65**, 063006 (2002)

54. P. Haensel, K.P. Levenfish, D.G. Yakovlev, *Bulk viscosity in superfluid neutron star cores. III. Effects of Σ^- hyperons*, Astron. Astrophys. **381**, 1080 (2002)
55. M. Nayyar, B.J. Owen, *R-modes of accreting hyperon stars as persistent sources of gravitational waves*, Phys. Rev. D **73**, 084001 (2006)
56. B. Haskell, N. Andersson, *Superfluid hyperon bulk viscosity and the r-mode instability of rotating neutron stars*, Mon. Not. R. Astron. Soc. **408**, 1897 (2010)
57. J. Madsen, *How to identify a strange star*, Phys. Rev. Lett. **81**, 3311 (1998)
58. J. Madsen, *Probing strange stars and color superconductivity by r-mode instabilities in millisecond pulsars*, Phys. Rev. Lett. **85**, 10 (2000)
59. N. Andersson, D.I. Jones, K.D. Kokkotas, *Strange stars as persistent sources of gravitational waves*, Mon. Not. R. Astron. Soc. **337**, 1224 (2002)
60. P. Jaikumar, G. Rupak, A.W. Steiner, *Viscous damping of r-mode oscillations in compact stars with quark matter*, Phys. Rev. D **78**, 123007 (2008)
61. G. Rupak, P. Jaikumar, *Constraining phases of quark matter with studies of r-mode damping in compact stars*, Phys. Rev. C **82**, 055806 (2010)
62. E. Flowers, N. Itoh, *Transport properties of dense matter*, Astrophys. J. **206**, 218 (1976)
63. P.S. Shternin, D.G. Yakovlev, *Shear viscosity in neutron star cores*, Phys. Rev. D **78**, 063006 (2008)
64. L. Bildsten, G. Ushomirsky, *Viscous boundary-layer damping of r-modes in neutron stars*, Astrophys. J. **529**, L33 (2000)
65. L. Lindblom, B.J. Owen, G. Ushomirsky, *Effect of a neutron-star crust on the r-mode instability*, Phys. Rev. D **62**, 084030 (2000)
66. N. Andersson, D.I. Jones, K.D. Kokkotas, N. Stergioulas, *R-mode runaway and rapidly rotating neutron stars*, Astrophys. J. **534**, L75 (2000)
67. K. Glampedakis, N. Andersson, *Crust-core coupling in rotating neutron stars*, Phys. Rev. D **74**, 044040 (2006)
68. K. Glampedakis, N. Andersson, *Ekman layer damping of r modes revisited*, Mon. Not. R. Astron. Soc. **371**, 1311 (2006)
69. L. Lindblom, G. Mendell, *Does gravitational radiation limit the angular velocities of superfluid neutron stars*, Astrophys. J. **444**, 804 (1995)
70. L. Lindblom, G. Mendell, *R-modes in superfluid neutron stars*, Phys. Rev. D **61**, 104003 (2000)
71. H.C. Spruit, *Gamma-ray bursts from X-ray binaries*, Astron. Astrophys. **341**, L1 (1999)
72. L. Rezzolla, F.K. Lamb, S.L. Shapiro, *R-mode oscillations in rotating magnetic neutron stars*, Astrophys. J. **531**, L139 (2000)
73. L. Rezzolla, F.K. Lamb, D. Marković, S.L. Shapiro, *Properties of r modes in rotating magnetic neutron stars. I. Kinematic Secular Effects and Magnetic Evolution Equations*, Phys. Rev. D **64**, 104013 (2001)
74. L. Rezzolla, F.K. Lamb, D. Marković, S.L. Shapiro, *Properties of r modes in rotating magnetic neutron stars. II. Evolution of the r modes and stellar magnetic field*, Phys. Rev. D **64**, 104014 (2001)
75. C. Cuofano, A. Drago, *Magnetic fields generated by r-modes in accreting millisecond pulsars*, Phys. Rev. D **82**, 084027 (2010)
76. Y. Kojima, *Quasi-toroidal oscillations in rotating relativistic stars*, Mon. Not. R. Astron. Soc. **293**, 49 (1998)
77. N. Andersson, *A new class of unstable modes of rotating relativistic stars*, Astrophys. J. **502**, 708 (1998)
78. Y. Kojima, M. Hosonuma, *The r-mode oscillations in relativistic rotating stars*, Astrophys. J. **520**, 788 (1999)
79. Y. Kojima, M. Hosonuma, *Approximate equation relevant to axial oscillations on slowly rotating relativistic stars*, Phys. Rev. D **62**, 044006 (2000)
80. K.H. Lockitch, N. Andersson, J.L. Friedman, *Rotational modes of relativistic stars: Analytic results*, Phys. Rev. D **63**, 024019 (2001)
81. K.H. Lockitch, J.L. Friedman, N. Andersson, *Rotational modes of relativistic stars: Numerical results*, Phys. Rev. D **68**, 124010 (2003)

82. K.H. Lockitch, N. Andersson, A.L. Watts, *Regularizing the r-mode problem for non-barotropic relativistic stars*, Class. Quant. Grav. **21**, 4661 (2004)
83. J. Ruoff, K.D. Kokkotas, *On the r-mode spectrum of relativistic stars in the low-frequency approximation*, Mon. Not. R. Astron. Soc. **328**, 678 (2001)
84. J. Ruoff, K.D. Kokkotas, *On the r-mode spectrum of relativistic stars: the inclusion of the radiation reaction*, Mon. Not. R. Astron. Soc. **330**, 1027 (2002)
85. J. Ruoff, A. Stavridis, K.D. Kokkotas, *Inertial modes of slowly rotating relativistic stars in the Cowling approximation*, Mon. Not. R. Astron. Soc. **339**, 1170 (2003)
86. S. Yoshida, U. Lee, *Relativistic r-modes in slowly rotating neutron stars: numerical analysis in the Cowling approximation*, Astrophys. J. **567**, 1112 (2002)
87. W. Kastaun, *Inertial modes of rigidly rotating neutron stars in Cowling approximation*, Phys. Rev. D **77**, 124019 (2008)
88. N. Stergioulas, J.A. Font, *Nonlinear r-modes in rapidly rotating relativistic stars*, Phys. Rev. Lett. **86**, 1148 (2001)
89. L. Lindblom, J.E. Tohline, M. Vallisneri, *Non-linear evolution of the r-modes in neutron stars*, Phys. Rev. Lett. **86**, 1152 (2001)
90. L. Lindblom, J.E. Tohline, M. Vallisneri, *Numerical evolutions of nonlinear r-modes in neutron stars*, Phys. Rev. D **65**, 084039 (2002)
91. P. Gressman, L.M. Lin, W.M. Suen, N. Stergioulas, J.L. Friedman, *Nonlinear r-modes in neutron stars: instability of an unstable mode*, Phys. Rev. D **66**, 041303 (2002)
92. L.M. Lin, W.M. Suen, *Non-linear r-modes in neutron stars: a hydrodynamical limitation on r-mode amplitudes*, Mon. Not. R. Astron. Soc. **370**, 1295 (2006)
93. A.K. Schenk, P. Arras, E.E. Flanagan, S.A. Teukolsky, I. Wasserman, *Nonlinear mode coupling in rotating stars and the r-mode instability in neutron stars*, Phys. Rev. D **65**, 024001 (2002)
94. P. Arras, E.E. Flanagan, S.M. Morsink, et al., *Saturation of the r-mode instability*, Astrophys. J. **591**, 1129 (2003)
95. S.M. Morsink, *Relativistic precession around rotating neutron stars: Effects due to frame-dragging and stellar oblateness*, Astrophys. J. **571**, 435 (2002)
96. R. Bondarescu, S.A. Teukolsky, I. Wasserman, *Spin evolution of accreting neutron stars: Nonlinear development of the r-mode instability*, Phys. Rev. D **76**, 064019 (2007)
97. R. Bondarescu, S.A. Teukolsky, I. Wasserman, *Spinning down newborn neutron stars: Nonlinear development of the r-mode instability*, Phys. Rev. D **79**, 104003 (2009)
98. S. Ou, J.E. Tohline, L. Lindblom, *Nonlinear development of the secular bar-mode instability in rotating neutron stars*, Astrophys. J. **617**, 490 (2004)
99. M. Shibata, S. Karino, *Numerical evolution of secular bar-mode instability induced by the gravitational radiation reaction in rapidly rotating neutron stars*, Phys. Rev. D **70**, 084022 (2004)
100. W. Kastaun, B. Willburger, K.D. Kokkotas, *Saturation amplitude of the f-mode instability*, Phys. Rev. D **82**, 104036 (2010)
101. E. Gaertig, K.D. Kokkotas, *Gravitational wave asteroseismology with fast rotating neutron stars*, Phys. Rev. D **83**, 064031 (2011)
102. D. Page, P. M., L.J. M., S.A. W., *Rapid cooling of the neutron star in Cassiopeia A triggered by neutron superfluidity in dense matter*, Phys. Rev. Lett. **106**, 081101 (2011)
103. P.S. Shternin, D.G. Yakovlev, C.O. Heinke, W.C.G. Ho, D.J. Patnaude, *Cooling neutron star in the Cassiopeia A supernova remnant: evidence for superfluidity in the core*, Mon. Not. R. Astron. Soc. **412**, L108 (2011)
104. J.L. Friedman, S.M. Morsink, *Axial instability of rotating relativistic stars*, Astrophys. J. **502**, 714 (1998)
105. L. Lindblom, B.J. Owen, S.M. Morsink, *Gravitational radiation instability in hot young neutron stars*, Phys. Rev. Lett. **80**, 4843 (1998)
106. N. Andersson, K.D. Kokkotas, B.F. Schutz, *Gravitational radiation limit on the spin of young neutron stars*, Astrophys. J. **510**, 846 (1999)
107. J. Papaloizou, J.E. Pringle, *Gravitational radiation and the stability of rotating stars*, Mon. Not. R. Astron. Soc. **184**, 501 (1978)

108. R.V. Wagoner, *Gravitational radiation from accreting neutron stars*, Astrophys. J. **278**, 345 (1984)
109. L. Bildsten, *Gravitational radiation and rotation of accreting neutron stars*, Astrophys. J. **501**, L89 (1998)
110. N. Andersson, K.D. Kokkotas, N. Stergioulas, *On the relevance of the r-mode instability for accreting neutron stars and white dwarfs*, Astrophys. J. **516**, 307 (1999)
111. R.V. Wagoner, *Conditions for steady gravitational radiation from accreting neutron stars*, Astrophys. J. **578**, L63 (2002)
112. Y. Levin, *Runaway heating by r-modes of neutron stars in low-mass X-ray binaries*, Astrophys. J. **517**, 328 (1999)
113. J. Heyl, *Low-mass X-ray binaries may be important laser interferometer gravitational-wave observatory sources after all*, Astrophys. J. **574**, L57 (2002)
114. A. Reisenegger, A. Bonacić, *Millisecond pulsars with r-modes as steady gravitational radiators*, Phys. Rev. Lett. **91**, 201103 (2003)
115. A.L. Watts, B. Krishnan, *Detecting gravitational waves from accreting neutron stars*, Adv. Space Res. **43**, 1049 (2009)
116. J.L. Houser, J.M. Centrella, S.C. Smith, *Gravitational radiation from nonaxisymmetric instability in a rotating star*, Phys. Rev. Lett. **72**, 1314 (1994)
117. J.E. Tohline, R.H. Durisen, M. McCollough, *The linear and nonlinear dynamic stability of rotating $n = 3/2$ polytropes*, Astrophys. J. **298**, 220 (1985)
118. M. Shibata, *Axisymmetric simulations of rotating stellar collapse in full general relativity - criteria for prompt collapse to black holes*, Prog. Theor. Phys. **104**, 325 (2000)
119. G.M. Manca, L. Baiotti, R. De Pietri, L. Rezzolla, *Dynamical non-axisymmetric instabilities in rotating relativistic stars*, Class. Quant. Grav. **24**, S171 (2007)
120. J.M. Centrella, K.C.B. New, L.L. Lowe, J.D. Brown, *Dynamical rotational instability at low T/W*, Astrophys. J. **550**, L193 (2001)
121. M. Shibata, S. Karino, Y. Eriguchi, *Dynamical instability of differentially rotating stars*, Mon. Not. R. Astron. Soc. **334**, L27 (2002)
122. M. Shibata, S. Karino, Y. Eriguchi, *Dynamical bar-mode instability of differentially rotating stars: effects of equations of state and velocity profiles*, Mon. Not. R. Astron. Soc. **343**, 619 (2003)
123. A.L. Watts, N. Andersson, D.I. Jones, *The nature of low $T/|W|$ dynamical instabilities in differentially rotating stars*, Astrophys. J. **618**, L37 (2005)
124. M. Saijo, S. Yoshida, *Low $T/|W|$ dynamical instability in differentially rotating stars: diagnosis with canonical angular momentum*, Mon. Not. R. Astron. Soc. **368**, 1429 (2006)
125. S. Ou, J.E. Tohline, *Unexpected dynamical instabilities in differentially rotating neutron stars*, Astrophys. J. **651**, 1068 (2006)
126. C.D. Ott, S. Ou, J.E. Tohline, A. Burrows, *One-armed spiral instability in a low-$T/|W|$ postbounce supernova core*, Astrophys. J. **625**, L119 (2005)
127. C.D. Ott, *The gravitational-wave signature of core-collapse supernovae*, Class. Quant. Grav. **26**, 063001 (2009)