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Instability of Neutron Star Matter for Pion Condensation

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The instability of normal neutron star matter is investigated from the viewpoint of collective oscillations which are coupled with condensed pions in neutron matter. It is shown that there is no inconsistency between the instability conditions obtained by two apparently different approaches, i.e., the mean field method by Sawyer et al. and the Green's function method by Migdal. The double pole condition, which determines the instability threshold in the Green's function method, is interpreted in terms of collective motions.

§ 1. Introduction

It is of great interest whether or not the pion condensate appears in superdense nuclear matter in connection with the cooling mechanism of neutron stars,[1] the understanding of transient superdense states caused by high-energy heavy-ion collisions[2] and other related problems.[3] There are two apparently different approaches to the problem of pion condensation in neutron star matter. That is, Sawyer and Scalapino[4] have worked the problem on the basis of the Hamiltonian in which the condensed π^- field has been replaced by the mean field. A series of their work has indicated the possibility that the ground state of neutron star matter would be rearranged at a slightly greater nucleon density than the normal nuclear matter density ρ_0. Then the new ground state has been prepared to be a coherent mixture of protons, neutrons and condensed negtive pions. In a preceding paper,[5] we have shown that this state can be treated with the coherent-state representation of proton particle-neutron hole. On the other hand, by using the pion Green's function, Migdal[6] has obtained the conclusion that neutral pions would be able to appear at the smaller density than ρ_0 and charged pions nearly at the same density.

It is the purpose of this paper to reproduce the results of the mean field method and the Green's function method by using the method of normal mode which was introduced by Sawada and Fukuda[7] in order to study the stability of the Hartree-Fock state within the range of the random phase approximation (RPA). They have pointed out that there exists an extremely important relation between the instability of the Hartree-Fock state and the solution of the RPA equation which describes some kind of approximate normal mode. When an infinitesimal deformation which is generated by a collective oscillation is applied to the Hartree-Fock state, there is such a case that Hartree-Fock state becomes unstable. In this case we must use a new approximate ground state in the sense of the variational principle. The instability of the Hartree-Fock state is characterized by the complex frequencies of the corresponding collective eigenmode $S^\dagger$ which satisfies the following RPA equation:

$$[S^\dagger, H] = -\omega S^\dagger, \tag{1·1}$$

where H is the total Hamiltonian of the system.

We apply the above criterion of instability to neutron star matter. Then the results which were given by Sawyer et al. and by Migdal are obtained by the method of normal mode. For simplicity, we omit the effect of (3, 3) isobar state.[6),8)]

In § 2 we apply the method of normal mode to Sawyer's model and obtain the instability condition for the appearance of condensed negative pions in neutron star matter. In § 3 we investigate the instability in Migdal's model and obtain the instability conditions for neutral and charged pions. The double pole condition for the instability threshold, which has been studied by Bertch and Johnson,[9)] is interpreted in terms of collective motions.

§ 2. Sawyer's model

In this section we apply the method of normal mode to the simple model discussed by Sawyer et al.[4)] This model is characterized by the Hamiltonian which consists of kinetic energies of neutrons and protons, the energy of negative pions, and the P-wave part of the interaction of nucleons with a single π^- mode. Using a spinor notation for nucleon operator, we have

$$H = \sum_q \varepsilon_q (n_q^\dagger n_q + p_q^\dagger p_q) + \omega_k a_k^\dagger a_k - iM_k \sum_q (p_{q-k}^\dagger \sigma_z n_q a_k^\dagger - \text{h.c.}), \tag{2·1}$$

where $n_q^\dagger$ and $p_q^\dagger$ are respectively the creation operators for neutrons and protons of momentum $\boldsymbol{q}$, $a_k^\dagger$ is the creation operator for π^- particles of momentum $\boldsymbol{k} = -k\hat{\boldsymbol{z}}$, $\varepsilon_q = q^2/2M$ is the kinetic energy of nucleons and $\omega_k = (k^2 + m_\pi^2)^{1/2}$ is the energy of pions. The last term in the Hamiltonian (2·1) is the interaction of nucleons with pions through the nonrelativistic pseudo-vector coupling, in which M_k has the form

$$M_k = \frac{fk}{m_\pi (\omega_k \Omega)^{1/2}}, \tag{2·2}$$

where f is taken to be 1.1 and Ω represents the volume.

By making use of the variational state which consists of a coherent mixture of neutrons, protons and condensed negative pions, Sawyer and Scalapino have found that the expectation value of the Hamiltonian (2·1) is less than the energy of normal neutron matter, if the nucleon density is greater than the critical density $\rho_{c,\text{m.f.}}$;

$$\rho_{c,\mathrm{m.f.}}=\frac{m_\pi^2\omega_k(\omega_k+\varepsilon_k)^2}{4f^2k^2}\,, \qquad (2\cdot3)$$

where the subscript m.f. signifies the result of mean field approximation for π^- fields. In Appendix A we show that, if the instability condition is satisfied, these procedures are really reasonable according to the method of normal mode.

The Hartree-Fock state in our case is the normal state of neutron matter in which all the single-particle states are filled by neutrons up to the Fermi momentum q_F set by the total nucleon density $(3\pi^2\rho)^{1/3}$:

$$|\Phi_0\rangle=\prod_{q\leq q_F} n_q^\dagger|0\rangle\,, \qquad (2\cdot4)$$

where $|0\rangle$ is the true vacuum. In order to find an approximate eigenmode associated with π^-, we define an operator as

$$S_k^\dagger=A_k a_k^\dagger+\sum_q \xi_k(q)\,n_q^\dagger\sigma_z p_{q-k}\,, \qquad (2\cdot5)$$

where the second term on the right-hand side is the particle-hole operators coupled with the π^- field. Then the coefficients A and ξ are to be determined by the RPA equation (1·1). The commutator of $S_k^\dagger$ and H is obtained as follows:

$$[S_k^\dagger, H]=-\{\omega_k A_k+2iM_k\sum_q \theta(q_F-q)\xi_k(q)\}\,a_k^\dagger$$
$$+\sum_q\{(\varepsilon_{q-k}-\varepsilon_q)\xi_k(q)-iM_kA_k\}\,n_q^\dagger\sigma_z p_{q-k}\,, \qquad (2\cdot6)$$

where θ is the ordinary step function. Identifying Eq. (2·6) with $-\omega S_k^\dagger$, we obtain a set of equations:

$$(\omega-\omega_k)A_k-2iM_k\sum_q\theta(q_F-q)\xi_k(q)=0\,, \qquad (2\cdot7\mathrm{a})$$

$$(\omega+\varepsilon_{k-q}-\varepsilon_q)\xi_k(q)-iM_kA_k=0\,. \qquad (2\cdot7\mathrm{b})$$

From Eqs. (2·7a) and (2·7b), the eigenvalue equation for ω is given by

$$1=-\frac{Mq_Ff^2k^2}{2\pi^2m_\pi^2}\,\frac{\phi(k,\omega)}{\omega_k(\omega-\omega_k)}\,, \qquad (2\cdot8)$$

where

$$\phi(k,\omega)=\frac{\omega+\varepsilon_k}{2\varepsilon_k}-\frac{q_F}{2k}\left\{\frac{(\omega+\varepsilon_k)^2}{k^2v_F^2}-1\right\}\ln\left|\frac{\omega+\varepsilon_k+kv_F}{\omega+\varepsilon_k-kv_F}\right| \qquad (2\cdot9)$$

and v_F is the Fermi velocity.

For such a density ρ that the state defined by Eq. (2·4) is still stable, the right-hand side of Eq. (2·8) is shown in Fig. 1 as a function of ω, for $\omega_k>kv_F-\varepsilon_k$. In Fig. 1 the intersecting points $A,\cdots,B,C$ and D determine the frequencies of approximate eigenmode $S_k^\dagger$. The points between A and B correspond to the continuum states of proton particle-neutron hole, and become a branch cut as $\Omega\to\infty$. The point C represents the collective motion of particle-hole pairs. The point D

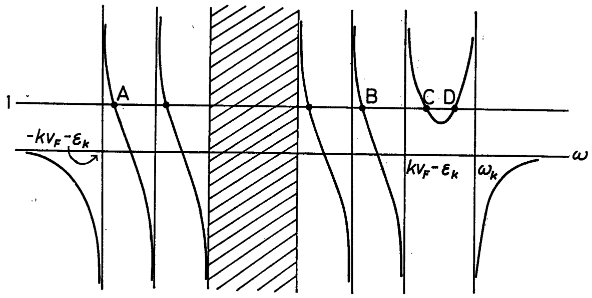

Fig. 1. The right-hand side of the eigenvalue equation (2·8) as a function of ω, for $\omega_k > kv_F - \varepsilon_k$. Eigenfrequencies are represented by the points A, B,…. Oblique lines represent the region of continuous spectra.

corresponds to the π^- state.

As the density is increased, the loop between C and D is lifted up and therefore the points C and D come closer to each other. At the critical density ρ_c they coincide with each other, and it becomes a double root of Eq. (2·8). The complex eigenfrequencies appear at the densities above ρ_c. Then, according to the criterion, the Hartree-Fock state (2·4) is no longer stable with respect to this kind of collective oscillation. This statement followed from the method of normal mode corresponds to the double pole condition in the Green's function method.

For $\omega + \varepsilon_k \gg kv_F$, we obtain

$$\phi(k,\omega) \simeq \frac{2\pi^2 \rho}{Mq_F(\omega+\varepsilon_k)} . \tag{2·10}$$

In this case C and D coincide with each other at

$$\omega = \frac{1}{2}(\omega_k - \varepsilon_k), \tag{2·11}$$

and the corresponding critical density is

$$\rho_c = \frac{m_\pi^2 \omega_k (\omega_k + \varepsilon_k)^2}{4f^2k^2}, \tag{2·12}$$

which agrees with Eq. (2·3) obtained by the mean field method. If the instability is arisen, the variational procedure to obtain the new ground state is given in Appendix A in connection with the π^- condensed state used by Sawyer et al.[4]

§ 3. Migdal's model

In the same way as discussed in § 2, we apply the instability criterion based on the method of normal mode to the pion-nucleon system with the Hamiltonian given by

$$H = H_0 + H_{\pi N} + H_{NN} + H' . \tag{3·1}$$

H_0 consists of the kinetic energy of nucleons and the energy of pions:

$$H_0 = \sum_q \varepsilon_q C_q^\dagger C_q + \sum_k \omega_k \boldsymbol{\varphi}_k^\dagger \cdot \boldsymbol{\varphi}_k , \tag{3·2}$$

where $C_q^\dagger$ is the 4-component creation operator for nucleons defined by

$$C_q^\dagger = (p_{q\uparrow}^\dagger p_{q\downarrow}^\dagger n_{q\uparrow}^\dagger n_{q\downarrow}^\dagger), \tag{3·3}$$

and $\varphi_k^\dagger$ the pion field operator. Then the creation operators for π^+, π^- and π^0 are respectively represented by

$$\phi_k^{(+)\dagger} = (\varphi_{k,x}^\dagger + i\varphi_{k,y}^\dagger)/\sqrt{2}, \quad \phi_k^{(-)\dagger} = (\varphi_{k,x}^\dagger - i\varphi_{k,y}^\dagger)/\sqrt{2}, \quad \phi_k^{(0)\dagger} = \varphi_{k,z}^\dagger . \tag{3·4}$$

$H_{\pi N}$ is the P-wave pion-nucleon interaction:

$$H_{\pi N} = (i/\sqrt{2}) \sum_k M_k \sum_q C_{q-k}^\dagger (\hat{\boldsymbol{k}}\cdot\boldsymbol{\rho}^1)(\boldsymbol{\rho}^2\cdot\boldsymbol{\varphi}_k^\dagger) C_q + \text{h.c.} \tag{3·5}$$

H_{NN} is the effective nucleon-nucleon interaction which excludes one-pion exchange terms,[10] and therefore H_{NN} is mainly the short range interaction:

$$H_{NN} = \frac{1}{2}\sum_\alpha \sum_k \frac{v^\alpha(k)}{\Omega} \sum_q \sum_{q'} C_q^\dagger \rho^\alpha (C_{q'-k}^\dagger \rho^\alpha C_{q'}) C_{q-k} . \tag{3·6}$$

In Eqs. (3·5) and (3·6), ρ^α are the matrices defined by

$$\boldsymbol{\rho}^1 = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}; \quad \rho_x^2 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \rho_y^2 = \begin{pmatrix} 0 & -i\mathbf{1} \\ i\mathbf{1} & 0 \end{pmatrix},$$

$$\rho_z^2 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}; \quad \rho^3 = \boldsymbol{\rho}^1\boldsymbol{\rho}^2; \quad \rho^4 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \tag{3·7}$$

where $\mathbf{1}$ is the 2×2 unit matrix. H' contains the other interactions, e.g., S-wave pion-nucleon interaction and pion-pion interaction, which are neglected in this paper.

First, we look for an approximate collective eigenmode associated with neutral pions of momentum $\boldsymbol{k}$. We introduce an operator

$$S_k^{(0)\dagger} = X_{1,k}^{(0)} \phi_k^{(0)\dagger} + X_{2,k}^{(0)} \phi_{-k}^{(0)} + \sum_q [\boldsymbol{\eta}_{1,k}(q)\cdot C_q^\dagger \boldsymbol{\rho}^1 C_{q-k} + \boldsymbol{\eta}_{2,k}(q)\cdot C_q^\dagger \boldsymbol{\rho}^1 \rho_z^2 C_{q-k}]. \tag{3·8}$$

Then, utilizing the fact that the $v(k)$'s are mainly the short range interaction, we have

$$\begin{aligned}
[S_k^{(0)\dagger}, H] = &-[\omega_k X_{1,k}^{(0)} + i\sqrt{2}\, M_k \hat{\boldsymbol{k}}\cdot \sum_q \{\theta(q_F - q) - \theta(q_F - |\boldsymbol{q}-\boldsymbol{k}|)\} \\
&\times \{\boldsymbol{\eta}_{1,k}(q) - \boldsymbol{\eta}_{2,k}(q)\}]\phi_k^{(0)\dagger} + [\omega_k X_{2,k}^{(0)} - i\sqrt{2} M_k \hat{\boldsymbol{k}} \\
&\times \sum_q \{\theta(q_F - q) - \theta(q_F - |\boldsymbol{q}-\boldsymbol{k}|)\}\cdot\{\boldsymbol{\eta}_{1,k}(q) - \boldsymbol{\eta}_{2,k}(q)\}]\phi_{-k}^{(0)} \\
&+ \sum_q [(\varepsilon_{q-k} - \varepsilon_q)\boldsymbol{\eta}_{1,k}(q) + (w_{1,k}/2\Omega) \sum_{q'} \{\theta(q_F - q') \\
&- \theta(q_F - |\boldsymbol{q}' - \boldsymbol{k}|)\}\{\boldsymbol{\eta}_{1,k}(q') - \boldsymbol{\eta}_{2,k}(q')\}]\cdot C_q^\dagger \boldsymbol{\rho}^1 C_{q-k} \\
&+ \sum_q [(\varepsilon_{q-k} - \varepsilon_q)\boldsymbol{\eta}_{2,k}(q) + (i/\sqrt{2}) M_k \hat{\boldsymbol{k}} (X_{1,k}^{(0)} - X_{2,k}^{(0)}) \\
&+ (w_{3,k}/2\Omega) \sum_{q'} \{\theta(q_F - q') - \theta(q_F - |\boldsymbol{q}' - \boldsymbol{k}|)\} \\
&\times \{\boldsymbol{\eta}_{2,k}(q') - \boldsymbol{\eta}_{1,k}(q')\}]\cdot C_q^\dagger \boldsymbol{\rho}^1 \rho_z^2 C_{q-k} ,
\end{aligned} \tag{3·9}$$

where the w_k's are given by some linear combinations of the $v(k)$'s and the $v(0)$'s.

By equating Eq. (3·9) with $-\omega S_k^{(0)\dagger}$, the equations for obtaining the coefficients in Eq. (3·8) are as follows:

$$(\omega-\omega_k)X_{1,k}^{(0)}=i\sqrt{2}M_k\hat{\boldsymbol{k}}\cdot\sum_q\{\theta(q_F-q)-\theta(q_F-|\boldsymbol{q}-\boldsymbol{k}|)\}\times\{\eta_{1,k}(q)-\eta_{2,k}(q)\}, \tag{3·10a}$$

$$(\omega+\omega_k)X_{2,k}^{(0)}=i\sqrt{2}M_k\hat{\boldsymbol{k}}\cdot\sum_q\{\theta(q_F-q)-\theta(q_F-|\boldsymbol{q}-\boldsymbol{k}|)\}\times\{\eta_{1,k}(q)-\eta_{2,k}(q)\}, \tag{3·10b}$$

$$(\omega+\varepsilon_{q-k}-\varepsilon_q)\eta_{1,k}(q)=-(w_{1,k}/2\Omega)\sum_{q'}\{\theta(q_F-q')-\theta(q_F-|\boldsymbol{q}'-\boldsymbol{k}|)\}\times\{\eta_{1,k}(q')-\eta_{2,k}(q')\}, \tag{3·10c}$$

$$(\omega+\varepsilon_{q-k}-\varepsilon_q)\eta_{2,k}(q)=(w_{3,k}/2\Omega)\sum_{q'}\{\theta(q_F-q')-\theta(q_F-|\boldsymbol{q}'-\boldsymbol{k}|)\}\times\{\eta_{1,k}(q')-\eta_{2,k}(q')\}-(i/\sqrt{2})M_k\hat{\boldsymbol{k}}(X_{1,k}^{(0)}-X_{2,k}^{(0)}). \tag{3·10d}$$

From Eqs. (3·10c) and (3·10d), we obtain

$$\eta_{1,k}(q)-\eta_{2,k}(q)=\frac{i}{\sqrt{2}}\frac{M_k\hat{\boldsymbol{k}}(X_{1,k}^{(0)}-X_{2,k}^{(0)})}{(\omega+\varepsilon_{q-k}-\varepsilon_q)\{1+g^{(0)}(k)\chi_0(k,\omega)\}}, \tag{3·11}$$

where

$$\chi_0(k,\omega)=\frac{1}{N(q_F)}\sum_q\frac{\theta(q_F-q)-\theta(q_F-|\boldsymbol{q}-\boldsymbol{k}|)}{\omega+\varepsilon_{q-k}-\varepsilon_q}$$
$$=\frac{1}{2}\left[1-\frac{q_F}{2k}\left\{\frac{(\omega+\varepsilon_k)^2}{k^2v_F^2}-1\right\}\ln\left|\frac{\omega+\varepsilon_k+kv_F}{\omega+\varepsilon_k-kv_F}\right|-\frac{q_F}{2k}\left\{\frac{(\omega-\varepsilon_k)^2}{k^2v_F^2}-1\right\}\ln\left|\frac{\omega-\varepsilon_k-kv_F}{\omega-\varepsilon_k+kv_F}\right|\right], \tag{3·12}$$

$$g^{(0)}(k)=\frac{N(q_F)}{2\Omega}(w_{1,k}+w_{3,k}) \tag{3·13}$$

and $N(q_F)=\Omega Mq_F/(2\pi^2)$ is the density of states per unit energy for a nucleon on the Fermi surface.

The solubility condition for the resultant equations, which are obtained by substituting Eq. (3·11) into Eqs. (3·10a) and (3·10b), leads to the eigenvalue equation for ω:

$$1=\frac{\Pi_0(k,\omega)}{\omega^2-\omega_k^2}, \tag{3·14}$$

where $\Pi_0(k,\omega)$ is the polarization operator of π^0 in neutron star matter and is given by

$$\Pi_0(k,\omega)=-\frac{2N(q_F)f^2k^2}{\Omega m_\pi^2}\frac{\chi_0(k,\omega)}{1+g^{(0)}(k)\chi_0(k,\omega)}\,. \tag{3·15}$$

As is seen from Eqs. (3·14) and (3·15) by taking $q_F\to 0$, there may exist a branch of ω according to the following equation:

$$1+g^{(0)}(k)\chi_0(k,\omega)=0\,, \tag{3·16}$$

which is spin sound branch.[6] In Appendix B, we briefly show that the spin sound branch can be obtained from Eq. (3·16), if $g^{(0)}(k)$ is a short range and repulsive interaction. However, since we are interested in the instability threshold, we are not concerned with this branch.

The right-hand side of Eq. (3·14) is illustrated in Fig. 2 as a function of ω^2, for $\omega_k>\varepsilon_k+kv_F$ and for the case that the Hartree-Fock state (2·4) is still stable. The point A represents the collective motion of particle-hole pairs. The points between B and C become a branch cut as $\Omega\to\infty$. The point D corresponds to the state of π^0.

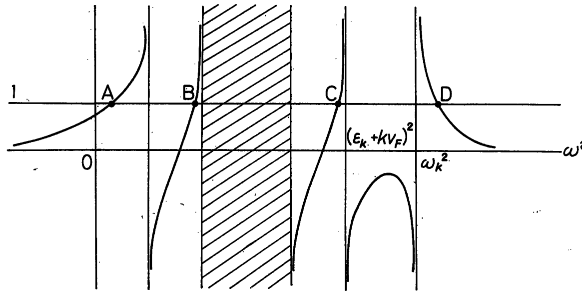

Fig. 2. The right-hand side of the eigenvalue equation (3·14) as a function of ω^2, for $\omega_k>\varepsilon_k+kv_F$.

As the density is increased, the point A moves toward the left in figure and intersects $\omega=0$ at the critical density $\rho_c^{(0)}$. At the densities above $\rho_c^{(0)}$ the eigenfrequencies corresponding to the point A become imaginary, so that the ground state of neutron star matter must be rearranged according to the criterion discussed before. Thus the instability condition is given by

$$\omega_k^2+\Pi_0(k,0)=0\,. \tag{3·17}$$

The critical density is determined by Eq. (3·17).

For $\omega+\varepsilon_k\gg kv_F$, we have

$$\chi_0(k,\omega)\simeq-\frac{\pi^2\rho k^2}{M^2q_F(\omega^2-\varepsilon_k^2)}\,. \tag{3·18}$$

Therefore, if we neglect the nucleon-nucleon interaction, we obtain

$$\rho_c^{(0)}=\frac{m_\pi^2\omega_k^2}{4Mf^2}\,. \tag{3·19}$$

On the manner quite parallel to the previous discussions, we investigate the instability of neutron star matter with respect to the collective oscillations caused by the condensed π^-, which is the only charged pion to be coupled with the collective motion of particle-hole in neutron matter. Let us introduce an operator

$$S_k^{(-)\dagger}=X_{1,k}^{(-)}\phi_k^{(-)\dagger}+X_{2,k}^{(-)}\phi_{-k}^{(+)}+\sum_q(\boldsymbol{\zeta}_{1,k}(q)\cdot \boldsymbol{C}_q^\dagger\rho^1\rho_x^2\boldsymbol{C}_{q-k}$$
$$+\boldsymbol{\zeta}_{2,k}(q)\cdot \boldsymbol{C}_q^\dagger\rho^1\rho_y^2\boldsymbol{C}_{q-k}]. \qquad (3\cdot20)$$

Then, by performing tedious but straightforward calculations with the RPA equation, we obtain the eigenvalue equation as follows:

$$1=\frac{\Pi_-(k,\omega)}{\omega^2-\omega_k^2}, \qquad (3\cdot21)$$

where $\Pi_-(k,\omega)$ is the polarization operator of π^- in neutron star matter and is given by

$$\Pi_-(k,\omega)=-\frac{2N(q_F)f^2k^2}{\Omega m_\pi^2}\frac{\chi_-(k,\omega)}{1+g^{(-)}(k)\chi_-(k,\omega)} \qquad (3\cdot22)$$

with

$$\chi_-(k,\omega)=\frac{2}{N(q_F)}\sum_q\frac{\theta(q_F-q)}{\omega+\varepsilon_{q-k}-\varepsilon_q-\Delta}$$
$$=\frac{\omega+\varepsilon_k-\Delta}{2\varepsilon_k}-\frac{q_F}{2k}\left\{\frac{(\omega+\varepsilon_k-\Delta)^2}{k^2v_F^2}-1\right\}\ln\left|\frac{\omega+\varepsilon_k+kv_F-\Delta}{\omega+\varepsilon_k-kv_F-\Delta}\right| \qquad (3\cdot23)$$

and

$$g^{(-)}(k)=\frac{N(q_F)}{2\Omega}w_{3;k}. \qquad (3\cdot24)$$

The quantity Δ is a function of the $v(0)$'s and represents the energy shift which originates from asymmetry of isospin in the Hartree-Fock state (2·4).

The right-hand side of the eigenvalue equation (3·21) is shown in Fig. 3 as a function of ω, for $\omega_k>kv_F-\varepsilon_k+\Delta$. The points between A and B become a branch cut as $\Omega\to\infty$. The point C represents the collective motion of proton particle-neutron hole pairs. The point D corresponds to the state of π^-. The point E corresponds to the state of π^+, which has been excluded from the Hamiltonian used by Sawyer et al.[4] (See also Fig. 1.)

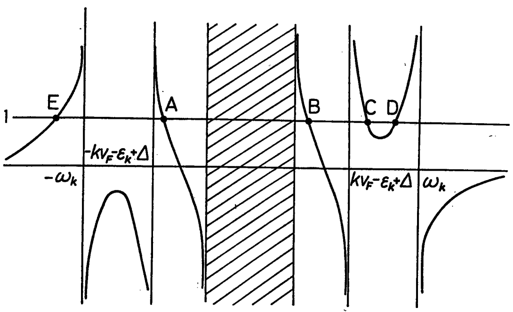

Fig. 3. The right-hand side of the eigenvalue equation (3·21) as a function of ω, for $\omega_k>kv_F-\varepsilon_k+\Delta$.

The complex eigenfrequencies appear at the densities above the critical density $\rho_c^{(-)}$, at which C and D coincide with each other. Thus the instability condition is found as follows:

$$2\omega_c=\left[\frac{\partial\Pi_-(k,\omega)}{\partial\omega}\right]_{\omega=\omega_c}, \qquad (3\cdot25)$$

where ω_c is a root of Eq. (3·21).

For $\omega+\varepsilon_k\gg kv_F$, neglecting the nucleon-nucleon interaction, we obtain

$$\chi_-(k,\omega)\simeq\frac{2\pi^2\rho}{Mq_F(\omega+\varepsilon_k)}. \tag{3·26}$$

Therefore the instability condition in this case gives the critical density as follows:

$$\rho_c^{(-)}=\frac{m_\pi^2\omega_k^3}{3\sqrt{3}f^2k^2}\left(1+\frac{\varepsilon_k^2}{3\omega_k^2}\right)^{3/2}+\frac{m_\pi^2\varepsilon_k\omega_k^2}{3f^2k^2}\left(1-\frac{\varepsilon_k^2}{9\omega_k^2}\right), \tag{3·27}$$ *)

and then the corresponding frequency is

$$\omega_c=\frac{\omega_k}{\sqrt{3}}\left\{\left(1+\frac{\varepsilon_k^2}{3\omega_k^2}\right)^{1/2}-\frac{\varepsilon_k}{\sqrt{3}\omega_k}\right\}. \tag{3·28}$$

The instability conditions for the cases of neutral and charged pions, (3·17) and (3·25), are identical with the double pole conditions in the Green's function method.[6),9),11)]

§ 4. Conclusion

For the investigation of the instability of neutron star matter, the method of normal mode introduced by Sawada and Fukuda has been applied to the models discussed by Sawyer and Scalapino and by Migdal. The application to the former gives the same critical nucleon density for the instability threshold as the one obtained by means of the mean field method. On the other hand, the application to the latter reproduces the results which have been obtained by means of the Green's function method. Also the double pole condition in the Green's function method has been explained from the viewpoint based on the collective eigenmode.

Appendix A

In this Appendix, according to the method of normal mode, we give a procedure to obtain a new ground state for the system described by the Hamiltonian (2·1), if the instability is arisen by the approximate collective eigenmode (2·5). Sawada and Fukuda have shown that, if the system is unstable with respect to the collective oscillation characterized by an approximate eigenmode $S^\dagger$, the new ground state can be obtained from the variational principle by making use of trial functions

$$|\Psi(\alpha)\rangle=\exp[i\{S(\alpha)^\dagger+S(\alpha)\}]|\Phi_0\rangle, \tag{A·1}$$

where $S(\alpha)$ has the same structure as S and the α's are variational parameters.

In our case the variational trial function is

$$|\Psi\rangle=\exp[Aa_k^\dagger-A^*a_k+\sum_q(B_qp_{q-k}^\dagger\sigma_zn_q-B_q^*n_q^\dagger\sigma_zp_{q-k})]|\Phi_0\rangle. \tag{A·2}$$

*) The difference between numerical factors in Eqs. (2·12) and (3·27) in the lowest order approximation is referred to the description of pions by the Schrödinger equation or the Klein-Gordon equation.

Here it is impotant to note that we must pay attention to the charge neutrality condition of the state (A·2). We divide the operator on the right-hand side of Eq. (A·2) into

$$U_\pi = \exp(A a_k^\dagger - A^* a_k) \tag{A·3}$$

and

$$U_N = \exp(i\tilde{S}), \tag{A·4}$$

where

$$\tilde{S} = -i \sum_q \theta_q (e^{i\varphi} p_{q-k}^\dagger \sigma_z n_q - \text{h.c.}) \tag{A·5}$$

with the replacement of $B_q = \theta_q \exp(i\varphi)$. U_π is the operator which can generate the coherent state of π^-, i.e., the condensed state of π^-. Then, if we take the π^- condensed state as the trial function, the variational ground state energy is

$$\langle \Psi | H | \Psi \rangle = \langle \Phi_{\text{cond}} | U_N^{-1} H U_N | \Phi_{\text{cond}} \rangle , \tag{A·6}$$

where

$$|\Phi_{\text{cond}}\rangle = U_\pi |\Phi_0\rangle \tag{A·7}$$

is the state with the π^- condensate in the Fermi sea of neutrons. Although the state $|\Phi_{\text{cond}}\rangle$ is the state with electric charge, the condition of charge neutrality as a whole can be considered as a constraint condition on the variational calculations.

In the calculation of (A·6), it is seen that the π^- fields in H defined by Eq. (2·1) may be replaced by the square root of the number of condensed pions, namely, the mean field. The unitary transformation of the Hamiltonian left in Eq. (A·6) is the analogous transformation with the one introduced by Yoshida on the theory of superconductivity,[12] and also is equivalent to the canonical transformation performed by Sawyer et al.[4] Based on the coherent state representation of proton particle-neutron hole pairs, the variational calculations to be continued have been given in the preceding paper by making use of the saddle-point method.[5]

Appendix B

We shall briefly review the presence of the spin sound branch, when the condition (3·16) is satisfied. If $g^{(0)}(k)$ is the short range and repulsive interaction, it behaves like a positive constant in the limit $k \to 0$. Therefore, ω must tend to zero in the limit $k \to 0$ in order that Eq. (3·16) is satisfied. Accordingly we have

$$\chi_0(k, \omega) \simeq 1 - \frac{\omega}{2kv_F} \ln \frac{\omega + kv_F}{\omega - kv_F}, \tag{B·1}$$

and the condition (3·16) is

$$\frac{\Omega}{N(q_F)(w_{1,k} + w_{3,k})} = \frac{\omega}{2kv_F} \ln \frac{\omega + kv_F}{\omega - kv_F} - 1 . \tag{B·2}$$

When we take the limit $q_F \to 0$ in Eq. (B·2), we have a branch of ω in the region of long wave-length with the energy

$$\omega = kv_F\,, \tag{B·3}$$

which is the spin sound in the Fermi system.

References

1) J. N. Bahcall and R. A. Wolf, Phys. Rev. **140** (1965), 953.
J. Kogut and J. T. Manassah, Phys. Letters **41A** (1972), 129.
2) G. Baym, *Proceedings of the 5th International Conference on High-Energy Physics and Nuclear Structure, Uppsala, Sweden* (1973), p. 436.
3) R. G. Palmer and P. W. Anderson, Phys. Rev. **D9** (1974), 3281.
V. R. Pandharipande and R. A. Smith, Nucl. Phys. **A237** (1975), 507.
4) R. F. Sawyer, Phys. Rev. Letters **29** (1972), 382.
D. J. Scalapino, Phys. Rev. Letters **29** (1972), 386.
R. F. Sawyer and D. J. Scalapino, Phys. Rev. **D7** (1973), 953.
R. F. Sawyer and A. C. Yao, Phys. Rev. **D7** (1973), 1578.
R. F. Sawyer, Phys. Rev. Letters **31** (1973), 1556.
5) A. Suzuki, Y. Futami and Y. Takahashi, Prog. Theor. Phys. **53** (1975), 1204.
6) A. B. Migdal, Soviet Phys.-JETP **36** (1973), 1052.
A. B. Migdal, Phys. Letters **45B** (1973), 448.
A. B. Migdal, Phys. Rev. Letters **31** (1973), 257.
A. B. Migdal, Nucl. Phys. **A210** (1973), 421.
See also, G. Baym and E. Flowers, Nucl. Phys. **A222** (1974), 29.
7) K. Sawada and N. Fukuda, Prog. Theor. Phys. **25** (1961), 653.
8) S. Barshay, G. Vagradov and G. E. Brown, Phys. Letters **43B** (1973), 359.
9) G. F. Bertch and M. B. Johnson, Phys. Letters **48B** (1974), 397.
10) C. B. Dover, J. Hüfner and R. H. Lemmer, Ann. of Phys. **66** (1971), 248.
11) C. K. Au, G. Baym and E. Flowers, Phys. Letters **51B** (1974), 1.
12) K. Yoshida, Phys. Rev. **111** (1958), 1255.