

## Instability of scale-free networks under node-breaking avalanches

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**Abstract.** – The instability introduced in a large scale-free network by the triggering of node-breaking avalanches is analyzed using the fiber-bundle model as conceptual framework. We found, by measuring the size of the giant component, the avalanche size distribution and other quantities, the existence of an abrupt transition. This test of strength for complex networks like Internet is more stringent than others recently considered like the random removal of nodes, analyzed within the framework of percolation theory. Finally, we discuss the possible implications of our results and their relevance in forecasting cascading failures in scale-free networks.

The recent months have witnessed a great effort devoted to the unraveling of the properties of complex networks [1, 2]. These properties include the rules followed in the process of formation of the net, the resulting connectivity distribution of the networks and their robustness under unfavorable circumstances like the presence of acting damaging agents, etc. [3–5]. This general interest is mainly due to the fact that the subject of complex networks has a considerable impact on many branches of science and technology and also in sociology [6–11]. An important observation has been to recognize that some significant networks have a scale-free connectivity distribution, which means that the number of links emerging from one node statistically follows a power law distribution  $P_k \sim k^{-\gamma}$ . In particular, it has also been noted that the Internet belongs to this class of networks with several studies reporting an exponent  $\gamma = 2.2 \pm 0.1$  [10, 12].

In this letter, we explore the robustness of large scale-free networks in a scenario in which the failure of a node may trigger the subsequent failure of its neighbors. Two nodes are considered as nearest neighbors if they are connected by a direct link. It is thus clear that the possibility of having 2nd-step failures can in turn induce 3rd-step failures, etc. and thus a breaking avalanche can be generated. This idea is not just an entertainment of theoreticians;

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as is now well known, on 10 August 1996, a fault in two power lines in Oregon led, through an avalanche of cascades, to a large-scale blackout in the US and Canada.

To implement our model, we will use the framework of the so-called fiber-bundle models (FBM) [13–15]. In FBM a set of  $N \gg 1$  fibers (elements) is located on the sites of a —usually regular— lattice, and one assigns to each element a random strength threshold sampled from a given probability distribution (the Weibull distribution is frequently used for this purpose). Then, the set is loaded by uniformly elevating the weight acting on each element up to a certain value,  $\sigma$ . All the elements whose thresholds are lower than  $\sigma$  fail in the first instance. The individual load carried by each of the broken elements is equally transferred to their surviving nearest neighbors and therefore the rupture of an element may induce secondary failures which in turn may trigger tertiary failures and so on. These systems are usually conservative which means that when equilibrium is finally attained, *i.e.*, when there occur no more casualties, the total weight borne by the  $N_s$  surviving elements is equal to  $N_s\sigma$ . The translation of the FBM terminology to that of complex networks studied here is simple. A fiber may be viewed as a node of the net, the directions of the load transfers are now the links of the net connecting the nodes to each other and the load can represent, for instance, the intensity of electric current flowing into the nodes of the network, or the viral pressure in Internet, etc., *i.e.* any magnitude or agent able to surpass a security threshold of an individual node, break it, and then provoke an increase of potential damage in the neighborhood of that node.

Recently, several scenarios of instability have been considered for complex networks. In particular for scale-free networks the random removal of nodes in a fixed proportion and its impact on the global connectivity and functionality of the network have been explored [5] with tranquilizing conclusions: networks with  $\gamma < 3$  are completely stable. In this scenario, akin to percolation, the removal of a node never leads to cascades as it does in our model. This implies that the test of strength analyzed here is more stringent and thus offers more guarantees for the security of networks with power law connectivity distributions.

To study the instability of scale-free networks under node-breaking avalanches, we first construct a network using the Barabási-Albert (BA) algorithm [4]. This is a stochastic growth model in which, at each time step, a new node is added and attached preferentially to the already existing ones. At the initial state, we start from a small number  $m_0$  of disconnected nodes and the network grows by adding one new node at each time step. This new node is connected preferentially to  $m$  old nodes with a probability that depends on the node connectivities  $k_i$  through the relation  $\Pi(k_i) = k_i / \sum_j k_j$ . By iterating this scheme a sufficient number of times, a network consisting of  $N$  nodes with connectivity distribution  $P_k = 2m^2k^{-3}$  and average connectivity  $\langle k \rangle = 2m$  develops [4]. It is worth noting that the preferential attachment rule introduced in the BA model accounts for the rich-gets-richer feature observed in many real complex networks. Besides, the BA model has recently been improved by adding several new ingredients in order to account for connectivity distributions with exponents  $2 < \gamma < 3$ . The main feature of the scale-free networks is that each node has a statistically significant probability of having a very large number of connections compared to the average connectivity of the network  $\langle k \rangle$ . This is not the case for other complex networks [1, 2] where the connectivity distribution is peaked at  $\langle k \rangle$  and decays exponentially fast for  $k \gg \langle k \rangle$  and  $k \ll \langle k \rangle$ . Thus, we expect that working with  $\gamma = 3$  does not alter the results one would obtain for the general case  $2 < \gamma \leq 3$ , a fact confirmed by preliminary numerical simulations in more general SF networks [16].

Let us now assume that the scale-free network previously generated is exposed to an external pressure or force  $F$  and that each node of the network represents an individual element able to support a finite amount of “load”  $\sigma$ . As noted before this could be seen as

a system where the individual elements are continuously subjected to external agents able to affect their functionality if they overcome a given security threshold. We will also assume that in the initial state this external force is equally distributed among all the nodes in the network so that each element bears a load  $\sigma = F/N$ . Furthermore, we assign to each node a statistically distributed security threshold  $\sigma_{i_{th}}$  ( $1 \leq i \leq N$ ) taken from a probability distribution. If the load acting on a node surpasses its threshold, the node fails and its load is equally transferred to the non-failed nodes directly linked to it. This may provoke other nodes to collapse and the cascades of failure events last until all the sound elements in the network bear a load lower than their threshold values. In order to assign the threshold values, we will use the Weibull distribution  $P(\sigma_{i_{th}}) = 1 - e^{-(\sigma_{i_{th}})^\rho}$ , where  $\rho$  is the so-called Weibull index which controls the degree of threshold disorder in the system (the bigger the value of  $\rho$ , the narrower the range of threshold values). This allows us to compare the stability of systems having different levels of heterogeneity in their security threshold distribution.

The threshold rule introduced above has been used for many years to study a wide class of non-equilibrium phenomena [14, 17, 18]. However, they have been extensively studied mainly for *regular* lattices. In the present model, the cascading process not only depends on the thresholds of the elements but also on the distribution of the neighbors, *i.e.*, a casualty at a node is determined by both the threshold of that node and the number (and state) of nodes directly linked to it.

We have performed large-scale numerical simulations of the cascading process produced by repeatedly applying the rules stated above. In the initial state all the elements that form the network are subjected to a small individual load  $\sigma$ . If that load is bigger than one (or several) threshold(s), a cascading process could start that lasts until the system arrives at a new equilibrium state where all the nodes support a load lower than their security thresholds. Then, the complete process is repeated again by imposing in the initial state (with all the elements in the non-failed state) a bigger load and applying the same failure rules. After each cascading event, the damage to the system increases, which affects both the properties of the network and its functionality. Each simulation is performed many times to average over the security threshold distribution and in the end a kind of phase diagram for each magnitude characterizing the final damaged state of the system is obtained.

The results obtained indicate that the system has an abrupt transition in its connectivity. Because of the similitude between this behavior and the one found in fracture systems on regular networks, we would call it a critical point. Figure 1 shows the size of the giant component in a BA network composed by  $N = 10^5$  nodes as a function of the control parameter  $\sigma$  for a Weibull distribution with two different values of  $\rho$  and for a uniform distribution of thresholds. The size of the giant component, which plays the role of an order parameter, is defined as the total number of intact nodes remaining in the largest component of the network after the cascading events divided by the system size  $N$ . We performed a depth-first search for the largest component [19] and averaged the results after many realizations to get a stable mean value of this component's size. As can be seen from the figure, for small values of the weight imposed over the system, the network remains almost intact and the giant component size is still large enough to ensure the network's functionality. However, as the load is increased, the cascading failure begins to reach more and more nodes up to the critical point where the size of the giant component suddenly falls close to zero provoking the rupture of the system in many small parts losing its properties as a functional network.

Figure 1 also shows that the network with the highest degree of homogeneity in the threshold distribution is more resilient to breakdown. This behavior is the opposite to what is seen in regular networks. Although the critical load at which the network loses its functionality shifts to the right as the level of homogeneity in the thresholds is increased, the precursory

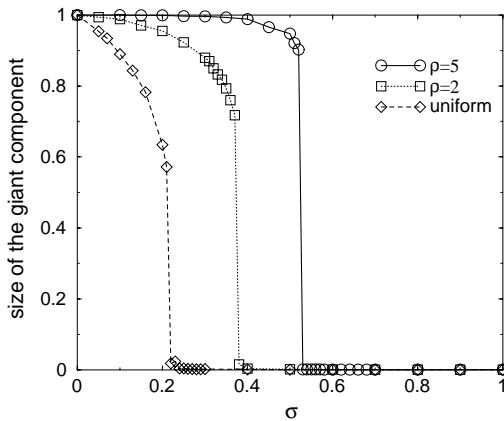


Fig. 1

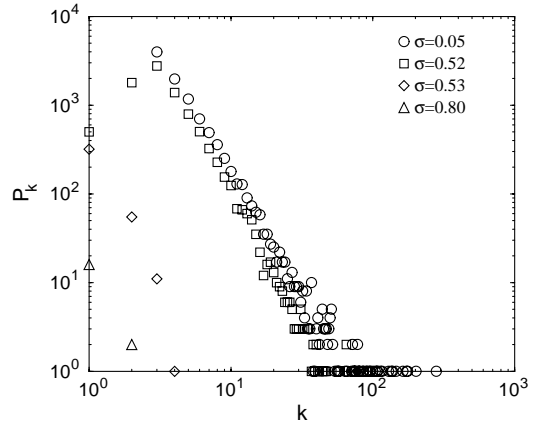


Fig. 2

Fig. 1 – Size of the giant component for a BA network consisting of  $10^5$  nodes as a function of the load imposed over the system (in dimensionless units). The values of  $\rho = 5$  and  $\rho = 2$  correspond to two different levels of heterogeneity for the security thresholds of the nodes. As the threshold distribution is more homogeneous, the critical point shifts rightward; however, the precursory activity is less intensive and the collapse of the network more catastrophic.

Fig. 2 – Connectivity distribution  $P_k$  of the network after the cascading process for several values of the load imposed on the system. The model’s parameters are:  $N = 10^5$  nodes and  $\rho = 5$ . The critical load is  $\sigma_c = 0.52$ . The connectivity distribution remains almost unaltered until the system reaches its critical point where it is split into many parts (see the distribution for  $\sigma = 0.53$ , just above  $\sigma_c$ ).

activity is less intensive and so the final breakdown of the network arises more abruptly and catastrophically, without previous warning. This result agrees with what is seen in fracture processes. Obviously, from a practical point of view, this is as unwanted as having a low critical value that makes the network very unstable. An intermediate value could satisfy both criteria; one is to have a robust network and the other to guarantee that the failure of the network is preceded by an important precursory activity which helps to foresee the cascades and the imminent collapse.

Another way to shed light on the cascading process is to inspect the change in the topology of the network as the control parameter varies. The transition from a functional network to the fragmented one is illustrated by the simulation results shown in fig. 2, where we plot the probability that a node has connectivity  $k$  when the system has reached the final equilibrium state. For very low damage pressures (for instance,  $\sigma = 0.05$  in the figure) the topology of the network remains unchanged. Right after the critical point ( $\sigma = 0.52$  in fig. 2) the system loses its properties as a scale-free network and the whole system becomes disconnected, the largest connectivity of the nodes being of about 4 links.

We have also monitored the nodes that collapse more frequently. This can be observed in fig. 3 where we have represented the fraction of broken nodes  $n_k$  with respect to their connectivities for the same parameters of fig. 2. The existence of a critical point is again clearly demonstrated. As the load supported by the system approaches the breakdown point from below, the nodes with higher connectivities are the most affected by the cascading process although the system still conserves a few hubs that make possible the existence of a large giant component that keeps a significant fraction of nodes interconnected. Right after the critical point only nodes forming isolated clusters remain, none of them having a large connectivity.

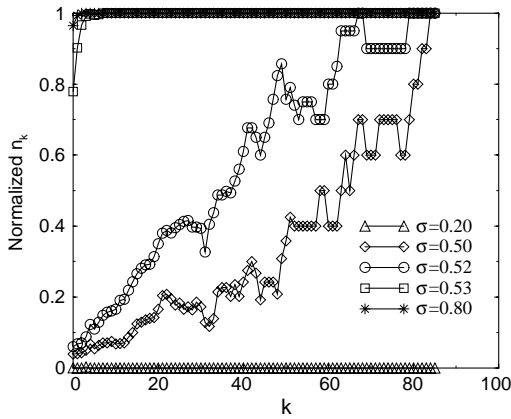


Fig. 3

Fig. 3 – Normalized fraction of broken nodes with connectivity  $k$  as a function of their connectivity. The existence of a critical point is again clearly manifested. Note that at the critical point there remain in the network a few hubs that ensure the presence of a giant component in the system. The model parameters are the same as fig. 2.

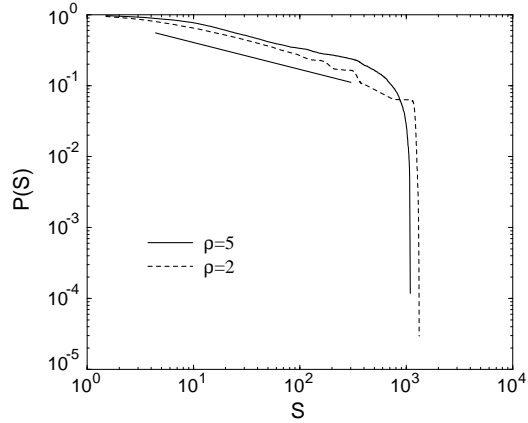


Fig. 4

Fig. 4 – Cumulative avalanche size distributions at the critical point for  $\rho = 5$  and  $\rho = 2$ . The system consists of  $N = 10^5$  elements and each curve has been obtained after averaging over  $10^4$  different realizations of the disorder. A full line with a slope of 0.12 has been drawn for comparison. Above the critical point the probability distribution function splits into two parts and is characterized by the excess of large avalanches. For very homogeneous threshold distributions the avalanche size distribution becomes strongly peaked.

While the fragility of scale-free networks with respect to the removal of highly connected nodes has recently been documented by several authors [5], it is worth noting that the above result, although pointing in the same direction, is obtained as a consequence of the model's rules instead of being imposed from the outside, *i.e.*, the system is subjected to an external force and it evolves according to a simple threshold rule in contrast to other models [5] where the removal of hubs is directed.

Finally, we have characterized the cascading process itself by measuring the avalanche distribution. The size  $s$  of an avalanche is defined as the total number of nodes that break simultaneously. The cumulative distributions  $P(s)$  of avalanche sizes for a network composed of  $10^5$  nodes have been represented in fig. 4 for two different values of the threshold disorder parameter. In both cases, the avalanche size distribution was measured at the critical points which are in these cases  $\sigma_c = 0.52$  and  $\sigma_c = 0.37$  for  $\rho = 5$  and  $\rho = 2$ , respectively. The distributions can be fitted, for low values of  $\rho$ , by a power law of the form  $P(s) \sim s^{-\tau}$ , giving an exponent for the probability distributions of  $\tau + 1 = 1.12 \pm 0.03$ . As the threshold distribution gets more homogeneous (*i.e.* bigger  $\rho$ ), the avalanche size distribution becomes a strongly peaked function around a large mean value. This occurs because, as we move to the region of large critical points values (increasing  $\rho$ ), there is a very poor precursory activity and almost all the nodes forming the network break down in only one time step. Above the critical point, the probability distribution of avalanches splits into two parts and is characterized by the excess of large avalanches signaling that we are in the supercritical region. Without providing a figure, we also want to report that in scale-free networks the value of the critical stress, for the same  $\rho$ , is independent of the size of the system. This

differs from what is obtained in regular lattices simulating non-mean-field fracture models: there a bigger system implies a lower critical stress [15].

During the completion of this work, we have become aware of a similar study by Watts [20]. He presents an interesting analytical approach to this kind of problems and solves the small-world case. Both models and the results reported, however, are different. In our model the initial distribution of intact and broken nodes after the first casualty has no constraints as in Watts' model. Besides, and more important, the relationship between heterogeneity level, the stability of the system and the precursory activity is not accounted for in [20]. We refer the reader to [20] for more details.

To sum up, we have introduced a model that accounts for the cascading events observed in many complex networks. By imposing an external pressure over the system, several magnitudes have been recorded and the system has been shown to exhibit a sort of critical point. The results point out that, in order to prevent the breakdown of scale-free networks, one has to find an optimal criterion that takes into account two factors: the robustness of the system itself under repeated failures and the possibility of knowing in advance that the collapse of the system is approaching.

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