QUARTERLY OF APPLIED MATHEMATICS
JANUARY, 1971

INSTABILITY OF SYMMETRIC STRUCTURAL SYSTEMS WITH INDEPENDENT LOADING PARAMETERS*

BY

KONCAY HUSEYIN**

Middle East Technical University, Ankara, Turkey

Abstract. The general post-critical characteristics of a discrete structural system with independent loading parameters are studied. Attention is restricted to elastic conservative systems which satisfy certain analytical symmetry conditions and which lose their initial stability at a 'symmetric special' critical point. The results are compared with Koiter's 'stable symmetric' and 'unstable symmetric' bifurcation points, and three theorems are established.

A shallow circular arch under the action of a set of external loads which can be represented by two independent parameters is analysed to illustrate some aspects of the theory.

1. Introduction. In the development of the general theory of elastic stability, the problem of combined loading has not received much attention. Thus Poincaré [1], Koiter [2], Thompson [3] and others who have developed the basic nonlinear concepts of the elastic stability, normally restricted attention to situations in which the external loading of a structure could be represented by a single variable parameter.

It has recently been observed [4] that the buckling and post-buckling behaviour of structures under combined loading cannot be described adequately by the two well-known critical points, the limit and bifurcation points, and a reclassification of the critical conditions characterizing the buckling behaviour more aptly has been presented [4]. Thus, under combined loading, mainly two types of critical point arise, 'general' and 'special' critical points. The former is normally associated with a limit point at which the equilibrium surface (defined [4] in the load-deflection space as the entirety of the equilibrium points) is continuous; it is, however, shown [4] that under some circumstances the same point can also be regarded as a point of bifurcation. The latter, on the other hand, is a genuine bifurcation point at which a simple extremum is definitely ruled out.

The loss of stability at 'general' critical points and the associated problems have recently been discussed by the author [5] in detail. In the field of elastic stability, however, there exist a considerable number of stability problems associated with 'special' critical points. Frames and plates subjected to certain combinations of axial compression, shear, etc., for instance, will always lose their stability at 'special' critical points. It is, therefore, the aim of this paper to examine the initial post-critical characteristics of such systems in an effort to establish general results valid for the class of systems under consideration.

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* Received December 23, 1969.

** Presently at Solid Mechanics Division, University of Waterloo, Waterloo, Ontario, Canada.

The investigation will be restricted to discrete symmetric systems, symmetry being introduced by imposing certain analytical conditions on the potential energy function. Thus, we separate the generalized coordinates into two distinct groups and assume that the potential energy function is symmetric in one of the groups [6]. Such a system will, then, exhibit symmetry in buckling and post-buckling behaviour. A similar investigation concerning more general systems (not necessarily symmetric) is reported in [7].

In view of the increasing demands of weight economy, the significance of the post-critical behaviour of structures is clear, and it is felt that the assessment of the general post-critical characteristics under combined loading will provide an insight into many particular problems of this nature which are inherently nonlinear and complex.

Only elastic, conservative systems are considered.

2. Structural system. Consider a discrete conservative structural system characterized by a total potential energy function

$$V = V(Q_i, z_j, \Lambda^k) \tag{1}$$

which depends on the generalized coordinates

$$Q_i \ (i = 1, 2, \cdots, N), \qquad z_j \ (j = 1, 2, \cdots, K),$$

and the loading parameters Λ^k $(k = 1, 2, \cdots, M)$, and is assumed to be single-valued and well behaved at least in the region of interest. The division of the generalized coordinates into the two distinct groups, the Q_i and z_j, enables us to introduce the analytic symmetry conditions conveniently, and will consequently simplify the analysis. Thus we assume that the function V is symmetric in the generalized coordinates z_j in the sense that

$$V(Q_i, z_j, \Lambda^k) = V(Q_i, -z_j, \Lambda^k) \tag{2}$$

where the z_j change sign as a set and $i = 1, 2, \cdots, N$; $j = 1, 2, \cdots, K$ and $k = 1, 2, \cdots, M$. Hence in the expansion of the potential energy function about a point on the fundamental surface, terms such as

$$z_j, z_j^3, \cdots, Q_i z_j, Q_i, z_j^3, \cdots, \Lambda^k z_j, \cdots \tag{3}$$

cannot appear.

It can readily be shown (by expanding (2) into Taylor series) that the $N + K$ equilibrium equations $\partial V/\partial Q_i = \partial V/\partial z_j = 0$ can be solved simultaneously to yield a fundamental surface in the form

$$Q_i^F = Q_i^F(\Lambda^i), \qquad z_i^F = 0, \tag{4}$$

which indicates that the system initially deflects in the Q_i subspace without involving the z_i coordinates. In analogy with [7], we shall now assume that this fundamental surface is single-valued at least in the region of interest so that the correspondence between a set of Q_i^F and a set of Λ^i is unique. We can, then, refer the potential energy to the fundamental surface by setting

$$Q_i = Q_i^F(\Lambda^i) + q_i . \tag{5}$$

Two further changes of coordinates by means of the linear, nonsingular and orthogonal transformations

$$q_i = \alpha_{ij}(\Lambda^k)u_j, \qquad z_i = \beta_{ij}(\Lambda^k)y_j \tag{6}$$

will be introduced to diagonalize the quadratic forms of the energy in q_i and z_i respectively. Introducing (5) and (6) into the energy function (1) we get a new function

$$T(u_i, y_i, \Lambda^k) \equiv V[Q_i^F(\Lambda^k) + \alpha_{ij}(\Lambda^k)u_j, 0 + \beta_{ij}(\Lambda^k)y_j, \Lambda^k], \tag{7}$$

with the properties

$$\begin{aligned} T_{u_i}(0, 0, \Lambda^k) &= T_{u_i}^i(0, 0, \Lambda^k) = T_{u_i}^{ik}(0, 0, \Lambda^l) = \cdots = 0, \\ T_{y_i}(0, 0, \Lambda^k) &= T_{y_i}^i(0, 0, \Lambda^k) = \cdots = 0, \end{aligned} \tag{8}$$

and

$$\begin{aligned} T_{u_iu_j}(0, 0, \Lambda^k) &= T_{u_iu_j}^k(0, 0, \Lambda^l) = \cdots = 0, \quad \text{for} \quad i \neq j, \\ T_{y_iy_j}(0, 0, \Lambda^k) &= T_{y_iy_j}^k(0, 0, \Lambda^l) = \cdots = 0, \quad \text{for} \quad i \neq j, \end{aligned} \tag{9}$$

which follow immediately from the fact that $u_i = y_i = 0$ define the fundamental surface, and the quadratic forms of the energy are diagonalized at every point of this surface. Here and in the remainder of this paper suffix symbols on the T's indicate partial differentiation (e.g. $T_{u_iy_j}^k = (\partial^3 T/\partial u_i\, \partial y_j\, \partial \Lambda^k)$).

It can further be shown that the symmetry properties (3) are now replaced by

$$T_{y_i} = T_{y_iy_jy_k} = \cdots = T_{u_iy_j} = \cdots = T_{y_i}^k = \cdots = 0. \tag{10}$$

Considering the $N + K$ stability coefficients of the system, $T_{u_iu_i}(0, 0, \Lambda^k)$ and $T_{y_iy_i}(0, 0, \Lambda^k)$, we focus attention on a discrete critical point, F, at which one of the latter coefficients, say $T_{y_1y_1}$, vanish. We now introduce a point transformation of the Λ^i coordinates which will provide a canonical representation of the linear form corresponding to $T_{y_1y_1}(0, 0, {}^*\Lambda_F^i)$ of the function T. Thus we choose a certain linear, nonsingular and orthogonal transformation

$$\Lambda^i = \gamma^{ij}\Phi^j \tag{11}$$

so that when this is substituted for Λ^i in the function T, the resulting function

$$\Psi(u_i, y_i, \Phi^k) \equiv T(u_i, y_i, \gamma^{kl}\Phi^l) \tag{12}$$

will have the following properties:

$$\Psi_{y_1y_1}^1(0, 0, {}^*\Phi_F^k) \neq 0, \qquad \Psi_{y_1y_1}^m(0, 0, {}^*\Phi_F^k) = 0 \tag{13}$$

where $m \neq 1$, and ${}^*\Phi_F^i$ are the critical values of Φ^i at the point F.

It can readily be shown that the properties (8), (9) and (10) are now replaced by

$$\begin{aligned} \Psi_{u_i}(0, 0, \Phi^k) &= \Psi_{u_i}^i(0, 0, \Phi^k) = \cdots = 0, \\ \Psi_{y_i}(0, 0, \Phi^k) &= \Psi_{y_i}^i(0, 0, \Phi^k) = \cdots = 0, \end{aligned} \tag{14}$$

$$\begin{aligned} \Psi_{u_iu_j}(0, 0, \Phi^k) &= \Psi_{u_iu_j}^l(0, 0, \Phi^k) = \cdots = 0, \quad \text{for} \quad i \neq j, \\ \Psi_{y_iy_j}(0, 0, \Phi^k) &= \Psi_{y_iy_j}^l(0, 0, \Phi^k) = \cdots = 0, \quad \text{for} \quad i \neq j, \end{aligned} \tag{15}$$

and

$$\Psi_{y_i} = \Psi_{y_iy_jy_k} = \cdots = \Psi_{u_iy_i} = \cdots = \Psi_{y_i}^i = \cdots = 0 \tag{16}$$

respectively.

We shall use the new function $\Psi(u_i, y_j, \Phi^k)$, the only necessary properties of which are given by (13), (14), (15) and (16), to explore the neighbourhood of the critical point F at which

$$\Psi_{y_1 y_1}(0, 0, {}^*\Phi_F^i) = 0, \qquad \Psi_{y_s y_s}(0, 0, {}^*\Phi_F^i) \neq 0 \quad \text{for all } s \neq 1$$

and

$$\Psi_{u_i u_i}(0, 0, {}^*\Phi_F^i) \neq 0.$$

3. The post-buckling equilibrium surface. In analogy with [7], instead of describing the $N + K$ equilibrium equations in parametric form, we start by choosing the independent variables as y_1 and Φ^m ($m \neq 1$) and have the functions in the form

$$u_i = u_i(y_1, \Phi^m), \qquad y_s = y_s(y_1\Phi^m), \qquad \Phi^1 = \Phi^1(y_1, \Phi^m), \tag{17}$$

which define the post-buckling equilibrium surface.

Substituting these functions back into the equilibrium equations $\Psi_{u_i} = \Psi_{y_j} = 0$ we have the identities

$$\Psi_{u_i}[u_j(y_1, \Phi^m), y_s(y_1, \Phi^m), \Phi^1(y_1, \Phi^m), y_1, \Phi^m] = 0 \tag{18a}$$

and

$$\Psi_{y_i}[u_j(y_1, \Phi^m), y_s(y_1, \Phi^m), \Phi^1(y_1, \Phi^m), y_1, \Phi^m] = 0. \tag{18b}$$

Differentiating (18a) once with respect to y_1 and once with respect to Φ^m ($m = 2, 3, \cdots, m$) we get

$$\begin{aligned} \Psi_{u_i u_j} u_{j,y_1} + \Psi_{u_i y_s} y_{s,y_1} + \Psi^1_{u_i} \Phi^1_{y_1} + \Psi_{u_i y_1} &= 0, \\ \Psi_{u_i u_j} u_j^m + \Psi_{u_i y_s} y_s^m + \Psi^1_{u_i} \Phi^{1,m} + \Psi^m_{u_i} &= 0. \end{aligned} \tag{19}$$

Here and elsewhere in the paper, summation convention is adopted.

Evaluating these equations at the critical point F (where $u_i = y_j = 0$, $\Phi^j = {}^*\Phi_F^j$) we have

$$u_{i,y_1} = 0 \quad \text{and} \quad u_i^m = 0. \tag{20}$$

Differentiating (18b) once with respect to y_1 and once with respect to Φ^m we get

$$\begin{aligned} \Psi_{y_i u_j} u_{j,y_1} + \Psi_{y_i y_s} y_{s,y_1} + \Psi^1_{y_i} \Phi^1_{y_1} + \Psi_{y_i y_1} &= 0, \\ \Psi_{y_i u_j} u_j^m + \Psi_{y_i y_s} y_s^m + \Psi^1_{y_i} \Phi^{1,m} + \Psi^m_{y_i} &= 0. \end{aligned} \tag{21}$$

Upon evaluation we obtain

$$y_{s,y_1} = 0, \qquad y_s^m = 0. \tag{22}$$

Differentiating (18a) with respect to y_1 for a second time we get

$$\begin{aligned} &[\cdots]u_{j,y_1} + \Psi_{u_i u_j} u_{j,y_1 y_1} + [\cdots] y_{s,y_1} + \Psi_{u_i y_s} y_{s,y_1 y_1} + [\Psi^1_{u_i u_j} u_{j,y_1} + \Psi^1_{u_i y_s} y_{s,y_1} + \Psi^{11}_{u_i} \Phi^1_{y_1} \\ &+ \Psi^1_{u_i y_1}]\Phi^1_{y_1} + \Psi^1_{u_i} \Phi^1_{y_1 y_1} + \Psi_{u_i y_1 u_j} u_{j,y_1} + \Psi_{u_i y_1 y_s} y_{s,y_1} + \Psi^1_{u_i y_1} \Phi^1_{y_1} + \Psi_{u_i y_1 y_1} = 0, \end{aligned} \tag{23}$$

which on evaluation yields

$$u_{i,y_1 y_1} = -\Psi_{u_i y_1 y_1} / \Psi_{u_i u_i}. \tag{24}$$

Differentiation of (18b) with respect to y_1 for a second time yields on evaluation

$$\Phi^1_{y_1} = 0, \qquad y_{s,y_1y_1} = 0. \tag{25}$$

We now differentiate (18a) first with respect to y_1 and for a second time with respect to Φ^m to get

$$[\cdots]u_{j,y_1} + \Psi_{u_iu_j}u^m_{j,y_1} + [\cdots]y_{s,y_1} + \Psi_{u_iy_s}y^m_{s,y_1} + [\cdots]\Phi^1_{y_1} + \Psi^1_{u_i}\Phi^{1,m}_{y_1} + \Psi_{u_iy_1u_j}u^m_j$$
$$+ \Psi_{u_iy_sy_1}y^m_s + \Psi^1_{u_iy_1}\Phi^{1,m} + \Psi^m_{u_iy_1} = 0, \tag{26}$$

which gives on evaluation

$$u^m_{i,y_1} = 0. \tag{27}$$

Similarly the differentiation of (18a) and (18b) with respect to certain independent variables twice yields

$$\Phi^{1,m} = 0, \qquad y^m_{s,y_1} = 0, \tag{28}$$

$$u^{mn}_i = 0, \tag{29}$$

$$y^{mn}_s = 0. \tag{30}$$

Proceeding in the same manner the third perturbation of the equations (18a) and (18b) yields the following derivatives of the post-buckling equilibrium surface:

$$\begin{aligned}
\Phi^1_{y_1y_1} &= -\frac{1}{3\Psi^1_{y_1y_1}}\left[\Psi_{y_1y_1y_1y_1} - 3\sum_{i=1}^{N}\frac{(\Psi_{u_iy_1y_1})^2}{\Psi_{u_iu_i}}\right],\\
\Phi^{1,mn} &= -\Psi^{mn}_{y_1y_1}/\Psi^1_{y_1y_1},\\
y^{mn}_{s,y_1} &= 0, \qquad \Phi^{1,m}_{y_1} = 0,\\
y_{s,y_1y_1y_1} &= \frac{1}{\Psi_{y_sy_s}}\left[\Psi_{y_sy_1y_1y_1} - 3\,\frac{\Psi_{u_iy_sy_1}\Psi_{u_iy_1y_1}}{\Psi_{u_iu_i}}\right],\\
y^m_{s,y_1y_1} &= 0, \qquad y^{mnr}_s = 0,\\
u^{mn}_{i,y_1} &= 0, \qquad u^{mnr}_i = 0.
\end{aligned} \tag{31}$$

We are now in a position to construct the asymptotic relationships $\Phi^1 = \Phi^1(u_1, \Phi^m)$, $u_s = u_s(u_1, \Phi^m)$ and $u_i = u_i(u_1, \Phi^m)$. Thus using the derivatives (20), (22), (24), (25), (27), (28), (29), (30) and (31) together with the translation $\varphi^i = \Phi^i - {}^*\Phi^i_F$ we have

$$\begin{aligned}
\varphi^1 &= -\frac{1}{6\Psi^1_{y_1y_1}}\left[\Psi_{y_1y_1y_1y_1} - 3\sum\frac{(\Psi_{u_iy_1y_1})^2}{\Psi_{u_iu_i}}\right]y_1^2 - \frac{\Psi^{mn}_{y_1y_1}}{2\Psi^1_{y_1y_1}}\varphi^m\varphi^n\\
&= ay_1^2 + b^{mn}\varphi^m\varphi^n,
\end{aligned} \tag{32}$$

$$u_i = -\tfrac{1}{2}(\Psi_{u_iy_1y_1}/\Psi_{u_iu_i})y_1^2, \tag{33}$$

and

$$y_s = -\frac{1}{3!}\frac{1}{\Psi_{y_sy_s}}\left[\Psi_{y_sy_1y_1y_1} - 3\,\frac{\Psi_{u_iy_sy_1}\Psi_{u_iy_1y_1}}{\Psi_{u_iu_i}}\right]y_1^3 \tag{34}$$

which define the post-buckling equilibrium surface in the vicinity of the critical point F.

Suppose we take a ray defined by $\varphi^i = l^i\xi$ where $l^1 \neq 0$; then the Eq. (32) takes the form

$$\xi = (a/l^1)y_1^2 , \tag{35}$$

which indicates a "symmetric point of bifurcation" (as Koiter [2] and Thompson [9] define it) on a plot of ξ against y_1 . Figs. 1a and 2a are drawn for $a/l^1 > 0$ and $a/l^1 < 0$ respectively. Here, the l^i are the direction cosines and ξ is a variable loading parameter.

On the other hand if we specify $l^1 = 0$, (32) yields

$$ay_1^2 + b^{mn}l^m l^n \xi^2 = 0, \tag{36}$$

which defines either the point $y_1 = \xi = 0$ or two intersecting straight lines depending on the signs of the coefficients. We cannot now consider the critical point F as a "symmetric point of bifurcation" due to the fact that post-buckling paths now have a finite slope; it should, therefore, be regarded (in Koiter's terminology) as a asymmetric point of bifurcation. Figs. 1 and 2 illustrate these phenomena and various equilibrium paths in the vicinity of the critical point F. We observe that the system exhibits symmetry with regard to the post-buckling behaviour, and the special critical point F can, therefore, be termed as 'symmetric'.

It is thus demonstrated that although a 'symmetric special' critical point is normally associated with Koiter's 'symmetric bifurcation' point, under some circumstances the post-buckling paths can have a finite slope at the same critical point, this being dependent on the shape of the post-buckling surface. We shall return to this point later for a full discussion of stability of the equilibrium paths involved.

4. Stability boundary. Since we are dealing with a special critical point (whether it is symmetric or asymmetric), the stability boundary of the system can be obtained by setting $u_i = y_i = 0$ in the equilibrium equations (32), (33) and (34) as was shown in [7]. Thus

$$^*\varphi^1 = -(\Psi^{mn}_{v_1v_1}/2\Psi^1_{v_1v_1})^*\varphi^{m*}\varphi^n \tag{37}$$

defines the stability boundary in the vicinity of the critical point F provided F is primary (i.e. $\Psi_{v_1v_1} = 0$, $\Psi_{v_sv_s} > 0$ for all $s \neq 1$, and $\Psi_{u_iu_i} > 0$). If F is not primary, then (37) becomes the equation of a critical surface not associated with an initial loss of stability.

The condition ensuring that the stability boundary is synclastic is the positive (negative) definiteness of the matrix $[\Psi^{mn}_{v_1v_1}]$.

5. Stability of equilibrium. Assuming that the critical point F is primary, we shall now examine the stability of the neighbouring equilibrium states. Following the same procedure as in [7], we introduce the stability determinant

$$\Delta(u_i , y_i , \varphi^k) = |\Psi_{ij}(u_i , y_i , \varphi^k)| , \tag{38}$$

where the unspecified subscripts on Ψ denote partial differentiation with respect to u_i and/or y_i .

Differentiating this determinant by rows once with respect to u_i and once with respect to φ^1, and evaluating at the critical point F we have

$$\Delta_{u_i} = \Psi_{v_1v_1u_i} \prod_{i=1}^{N} \Psi_{u_iu_i} \prod_{s=2}^{K} \Psi_{v_sv_s} \tag{39}$$

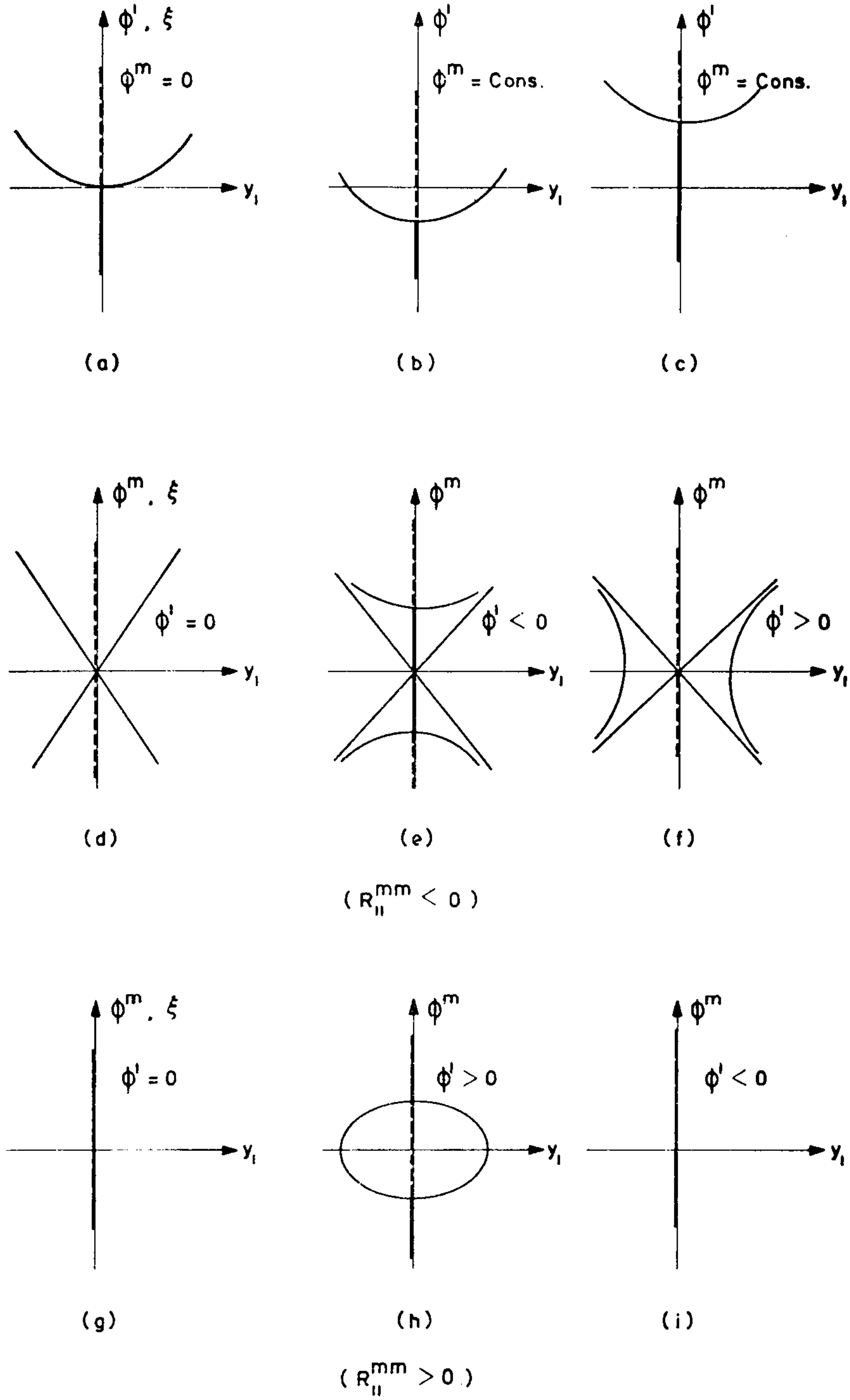
φ', ξ
φ^m = 0
y₁
(a)
φ'
φ^m = Cons.
y₁
(b)
φ'
φ^m = Cons.
y₁
(c)
φ^m, ξ
φ' = 0
y₁
(d)
φ^m
φ' < 0
y₁
(e)
φ^m
φ' > 0
y₁
(f)
(R₁₁^mm < 0)
φ^m, ξ
φ' = 0
y₁
(g)
φ^m
φ' > 0
y₁
(h)
φ^m
φ' < 0
y₁
(i)
(R₁₁^mm > 0)

Fig. 1.

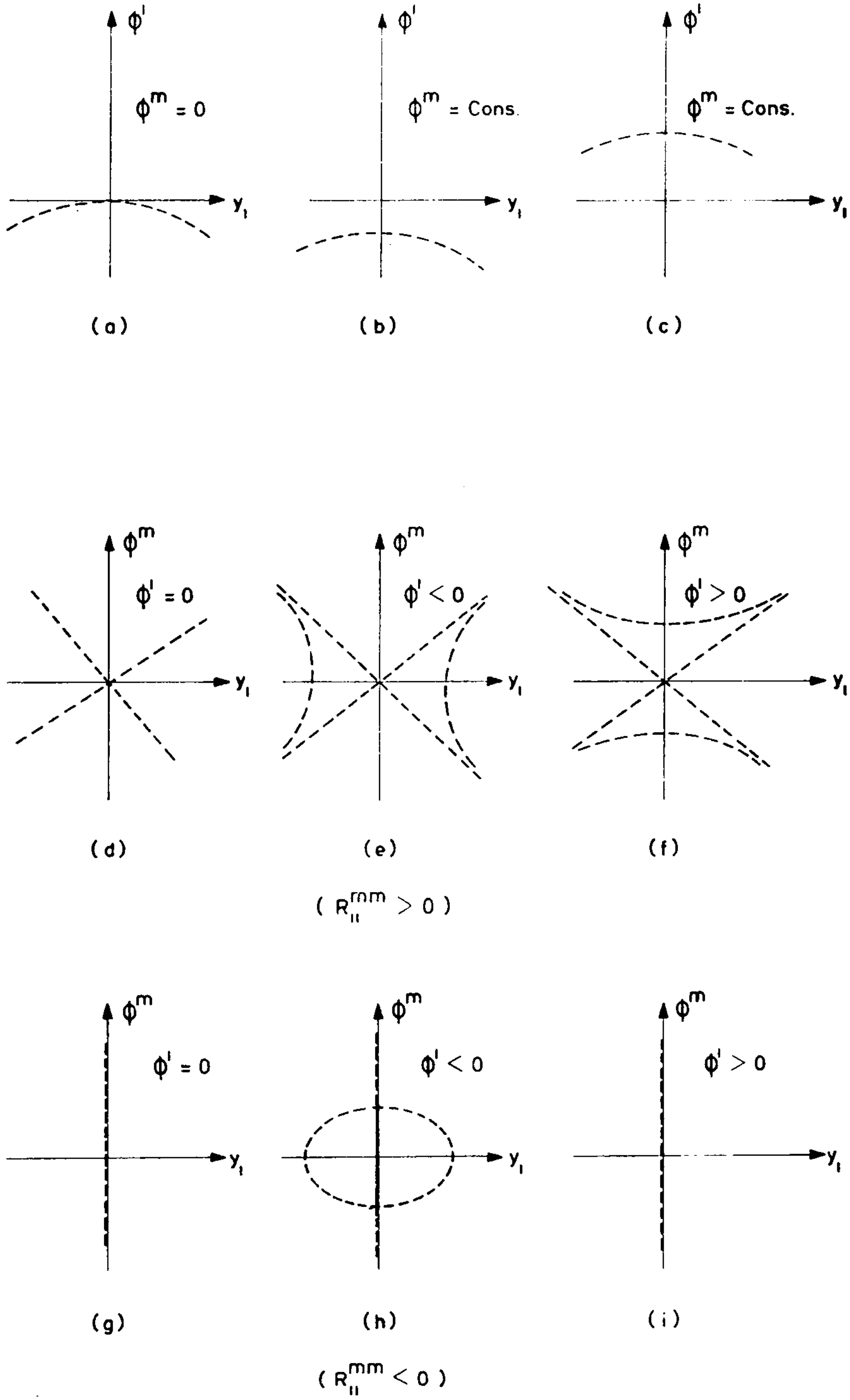
φ^l
φ^m = 0
y_1
(a)
φ^l
φ^m = Cons.
y_1
(b)
φ^l
φ^m = Cons.
y_1
(c)
φ^m
φ^l = 0
y_1
(d)
φ^m
φ^l < 0
y_1
(e)
φ^m
φ^l > 0
y_1
(f)
(R_11^mm > 0)
φ^m
φ^l = 0
y_1
(g)
φ^m
φ^l < 0
y_1
(h)
φ^m
φ^l > 0
y_1
(i)
(R_11^mm < 0)

FIG. 2.

and

$$\Delta^1 = \Psi^1_{v_1v_1} \prod_{i=1}^{N} \Psi_{u_iu_i} \prod_{s=2}^{K} \Psi_{v_sv_s} . \tag{40}$$

Differentiating (38) once with respect to y_i ($i = 1, 2, \cdots, K$) and once with respect to φ^m ($m = 2, 3, \cdots, M$) and evaluating at the critical point, we see that $\Delta_{v_i} = \Delta^m = 0$. A second differentiation yields on evaluation

$$\begin{aligned} \Delta^{mn} &= \Psi^{mn}_{v_1v_1} \prod_1^N \Psi_{u_iu_i} \prod_2^K \Psi_{v_sv_s} , \\ \Delta_{v_jv_k} &= \left[\Psi_{v_1v_1v_jv_k} - 2\,\frac{\Psi_{u_iv_1v_j}\Psi_{u_iv_1v_k}}{\Psi_{u_iu_i}}\right] \prod_1^N \Psi_{u_iu_i} \prod_2^K \Psi_{v_sv_s} , \\ \Delta^m_{v_i} &= 0. \end{aligned} \tag{41}$$

Using the Taylor's expansion and these derivatives we get

$$\Delta = \left[\Psi_{u_iv_1v_1}u_i + \Psi^1_{v_1v_1}\varphi^1 + \frac{1}{2!}\Psi^{mn}_{v_1v_1}\varphi^m\varphi^n + \frac{1}{2!}\left(\Psi_{v_1v_1v_jv_k} - 2\,\frac{\Psi_{u_iv_1v_j}\Psi_{u_iv_1v_k}}{\Psi_{u_iu_i}}\right)y_jy_k\right] \cdot \prod_1^N \Psi_{u_iu_i} \prod_2^K \Psi_{v_sv_s} + \frac{1}{3!}(\cdots) + . \tag{42}$$

Evaluating the stability determinant (42) at an arbitrary state on the fundamental surface (i.e. setting $u_i = y_j = 0$) we have to a first approximation

$$\Delta = (\Psi^1_{v_1v_1}\varphi^1 + \tfrac{1}{2}\Psi^{mn}_{v_1v_1}\varphi^m\varphi^n) \prod_1^N \Psi_{u_iu_i} \prod_2^K \Psi_{v_sv_s} , \tag{43}$$

which yields the following stability criterion:

$$\Psi^1_{v_1v_1}\varphi^1 + \Psi^{mn}_{v_1v_1}\varphi^m\varphi^n \begin{array}{ll} > 0 & \text{for stable equilibrium,} \\ = 0 & \text{for critical equilibrium,} \\ < 0 & \text{for unstable equilibrium.} \end{array} \tag{44}$$

Considering an arbitrary point A on the stability boundary, we can examine the stability of the neighbouring states by keeping $\varphi^m = {}^*\varphi^m_A =$ const and giving a small but finite increment ϵ to ${}^*\varphi^1_A$. Thus, for the points defined by

$$\varphi^1 = {}^*\varphi^1_A + \epsilon, \qquad \varphi^m = {}^*\varphi^m_A , \tag{45}$$

the stability criterion takes the simple form

$$\Psi^1_{v_1v_1}\epsilon. \tag{46}$$

If, for instance, $\Psi^1_{v_1v_1} < 0$, then, $\epsilon < 0$ defines *the region of stability* and $\epsilon > 0$ *the region of instability*. It is seen that the stability boundary divides the fundamental surface into the *stable* and *unstable domains.*

In order to examine the stability of the states lying on the post-buckling surface we evaluate the determinant (42) on this surface which is defined by Eqs. (32), (33) and (34). Substituting for u_i, y_s and φ^i in the determinant (42) we have to a first approximation

$$\Delta = \frac{1}{3}\left[\Psi_{v_1v_1v_1v_1} - 3\sum_1^N \frac{(\Psi_{u_iv_1v_1})^2}{\Psi_{u_iu_i}}\right] y_1^2 \prod_1^N \Psi_{u_iu_i} \prod_2^K \Psi_{v_sv_s} . \tag{47}$$

Evidently, the stability of the neighbouring equilibrium states is not dependent on the coordinate y_1. In fact, the sign of the expression in the brackets determines the stability of the post-buckling surface as a whole so that if this expression is positive (negative) the surface is totally stable (unstable).

The stability determinant evaluated on the post-buckling surface can alternatively be expressed as

$$\Delta = (-2\Psi^1_{\nu_1\nu_1}\varphi^1 - \Psi^{mn}_{\nu_1\nu_1}\varphi^m\varphi^n) \prod_1^N \Psi_{u_iu_i} \prod_2^K \Psi_{\nu_s\nu_s}, \tag{48}$$

in which case the stability of the states defined by (45) can be studied by examining the sign of the expression $-\Psi^1_{\nu_1\nu_1}\epsilon$ which obviously indicates that the region of stability can only correspond to an unstable post-buckling surface and vice versa. It can be shown, however, that if the post-buckling surface is stable (unstable) corresponding to the points of the region of stability (the region of instability) there exists no post-buckling equilibrium states. Thus, considering again an arbitrary point A on the stability boundary, the post-buckling states corresponding to (45) can be obtained by substituting for φ^1 and φ^m in the equilibrium Eq. (32) as

$$y_1 = \pm(\epsilon/a)^{1/2}. \tag{49}$$

If $a > 0$ ($a < 0$), then, only for $\epsilon > 0$ ($\epsilon < 0$) real equilibrium states can exist. Supposing that $\Psi^1_{\nu_1\nu_1} < 0$, $a > 0$ will, then, correspond to a stable post-buckling surface, in which case we clearly see that for $\epsilon < 0$ the post-buckling states are not real. Figs. 1 and 2 show various stable and unstable equilibrium paths in the vicinity of the critical point F. On the basis of the foregoing theory, the following theorems are proved:

THEOREM 1. *The initial post-buckling equilibrium surface is either totally stable or unstable.*

THEOREM 2. *The stability boundary constitutes an existence boundary with regard to the post-buckling surface so that if this surface is stable (unstable), no post-buckling equilibrium states can correspond to the points of the region of stability (region of instability).*

6. Stability of the critical point F. Finally we shall discuss the stability of the critical point F itself at which the determinant Δ vanishes and higher order variations of the energy are required.

This is simply a problem of finding whether or not the energy function $\Psi(u_i, y_i)$ has a minimum at the point F. It is important to note that, since we are no longer dealing with a quadratic form, $\Psi(u_i, y_i)$ might not have a minimum even though the partial second derivatives with respect to u_i and y_s, and the fourth derivative with respect to y_1 (notice that third derivative is zero due to symmetry) are all positive.

In order to determine whether the energy has a minimum we examine the variation of the energy function $\Psi(u_i, y_i)$ with respect to an arbitrarily chosen path defined by

$$u_i = u_i(\eta), \qquad y_i = y_i(\eta), \tag{50}$$

where η is a path parameter, $\eta = 0$ giving the critical point F. Thus writing

$$\Psi(\eta) \equiv \Psi[u_i(\eta), y_i(\eta)] \tag{51}$$

and differentiating this with respect to η we get

$$d\Psi/d\eta = \Psi_{u_i}u_{i,\eta} + \Psi_{y_i}y_{i,\eta} \tag{52}$$

where the subscript η on the variables denotes differentiation with respect to η.

Evaluating (52) at the critical point F we get

$$(d\Psi/d\eta)|_F = 0. \tag{53}$$

Differentiating (51) for a second time we have

$$d^2\Psi/d\eta^2 = (\Psi_{u_iu_j}u_{j,\eta} + \Psi_{u_iy_j}y_{j,\eta})u_{i,\eta} + \Psi_{u_i}u_{i,\eta\eta} + (\Psi_{y_iu_j}u_{j,\eta} + \Psi_{y_iy_j}y_{j,\eta})y_{i,\eta} + \Psi_{y_i}y_{i,\eta\eta}, \tag{54}$$

giving on evaluation

$$(d^2\Psi/d\eta^2)|_F = \Psi_{u_iu_i}(u_{i,\eta})^2 + \Psi_{y_sy_s}(y_{s,\eta})^2. \tag{55}$$

(55) indicates that the second variation of the energy is positive for all paths provided $u_{i,\eta} \neq 0$ and $y_{s,\eta} \neq 0$. If

$$u_{i,\eta} = y_{s,\eta} = 0, \tag{56}$$

the higher-order variations of the energy are required. (56) implies that the only candidate for η among the $N + K$ variables is y_1, the critical coordinate. In other words, the higher variations of energy must only be determined for the path which is initially given by

$$u_{i,y_1} = y_{s,y_1} = 0. \tag{57}$$

Differentiating (5) for a third time with respect to η $(= y_1)$, and evaluating at the critical point, we see that

$$d^3\Psi/dy_1^3)|_F = 0. \tag{58}$$

Proceeding in the same manner as before the fourth differentiation yields on evaluation

$$d^4\Psi/dy_1^4 = \Psi_{y_1y_1y_1y_1} + 6\Psi_{u_iy_1y_1}u_{i,y_1y_1} + 3\Psi_{u_iu_i}(u_{i,y_1y_1})^2. \tag{59}$$

We now see that the curvature u_{i,y_1y_1} is unknown and has to be determined. But we recall from the theory of maxima and minima that the necessary condition for a relative extremum

$$\Psi_{u_i}(u_j, y_k) = 0, \qquad \Psi_{y_i}(u_j, y_k) = 0, \tag{60}$$

must also be satisfied.

Thus, for the path under consideration we can write

$$\Psi_{u_i}[u_j(y_1), y_k(y_1)] = 0, \qquad \Psi_{y_i}[u_j(y_1), y_k(y_1)] = 0. \tag{61}$$

Differentiating these functions with respect to y_1 and evaluating at the critical point, we have

$$u_{i,y_1} = 0, \qquad y_{s,y_1} = 0. \tag{62}$$

Differentiating the first of the equations (61) for a second time and evaluating, we get

$$u_{i,y_1y_1} = \Psi_{u_i,y_1y_1}/\Psi_{u_iu_i}. \tag{63}$$

Using (63), Eq. (59) yields

$$\frac{d^4\Psi}{dy_1^4} = \Psi_{y_1y_1y_1y_1} - 3\sum_{i=1}^{N}\frac{(\Psi_{u_iy_1y_1})^2}{\Psi_{u_iu_i}}, \tag{64}$$

which can take positive or negative values even if $\Psi_{v_1v_1v_1v_1}$ is positive. Thus for the critical point F we have the following stability criterion:

$$\Psi_{v_1v_1v_1v_1} - 3 \sum_{i=1}^{N} \frac{(\Psi_{u_iv_1v_1})^2}{\Psi_{u_iu_i}} \begin{array}{ll} > 0 & \text{for stable equilibrium,} \\ = 0 & \text{for critical equilibrium,} \\ < 0 & \text{for unstable equilibrium.} \end{array} \tag{65}$$

If (64) is zero we have to determine the higher order variations of the energy, and this can readily be done by following the same procedure as before.

It is interesting to note that (64) is exactly the same quantity which determines the stability of the post-buckling surface (see Eq. 47). Hence the following theorem is proved.

THEOREM 3. *The initial post-buckling surface is stable or unstable according to whether the critical point itself is stable or unstable.*

7. Example: instability of a shallow circular arch. The buckling and post-buckling behaviour of an arch has been investigated by several authors [9], [10]. Our interest here will be focused on the combined loading with a view to illustrating some aspects of the theory presented in the preceding sections.

Consider a simply supported (pinned) shallow circular arch of radius R and with a central angle $2\theta_0$. The arch is subjected to the combined action of five symmetrically located radial concentrated loads described by two independent parameters Λ^1 and Λ^2 (Fig. 3). It is assumed that the arch has a constant cross-section with an area A and moment of intertia I. Using u and ω to denote the tangential and radial displacements respectively, we note that for a shallow arch

$$u \ll R, \qquad \omega \ll R, \qquad (\omega_\theta/R)^2 \ll 1 \tag{66}$$

where the subscript θ denotes differentiation with respect to θ. The total potential energy

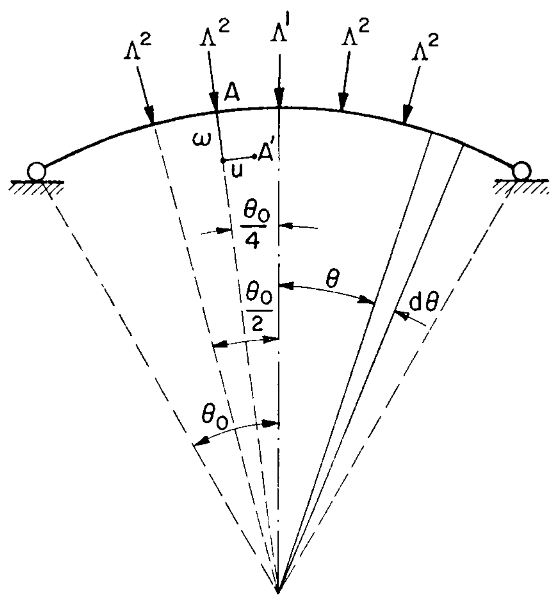

FIG. 3.

of the system, nondimensionalized by dividing by EAR, is

$$V = \frac{1}{2}\int_{-\theta_0}^{+\theta_0} \epsilon^2 \, d\theta + \frac{I}{2A}\int_{-\theta_0}^{+\theta_0} \chi^2 \, d\theta$$
$$- \Lambda_1\omega' \,|_{\theta=0} - \Lambda_2[\omega' \,|_{\theta=+\theta_0/2} + \omega' \,|_{\theta=-\theta_0/2} + \omega' \,|_{\theta=+\theta_0/4} + \omega' \,|_{\theta=-\theta_0/4}] \quad (67)$$

where ϵ and χ denote the axial strain and change in curvature respectively and $\Lambda_1 = \Lambda^1/EA$, $\Lambda_2 = \Lambda^2/EA$ and $\omega' = \omega/R$.

On the basis of the assumptions (66) the axial strain can be approximately expressed as

$$\epsilon = (1/R)(u_\theta - \omega) + (1/2R^2)(\omega_\theta)^2, \quad (68)$$

and the change in curvature as

$$\chi = (1/R^2)\omega_{\theta\theta} \, . \quad (69)$$

In the energy expression (67) u appears only in ϵ and can, therefore, readily be eliminated. Thus, integrating the strain (68) between $\theta = -\theta_0$ and $\theta = +\theta_0$ we have

$$\epsilon\theta\Big]_{-\theta_0}^{+\theta_0} = \frac{1}{R}u\Big]_{-\theta_0}^{+\theta_0} - \frac{1}{R}\int_{-\theta_0}^{+\theta_0} \omega \, d\theta + \frac{1}{2R^2}\int_{-\theta_0}^{+\theta_0} \omega_\theta^2 \, d\theta \cdots . \quad (70)$$

Using the boundary conditions $u = 0$ at $\theta = \pm\theta_0$, Eq. (70) yields

$$\epsilon = \frac{1}{2\theta_0}\int_{-\theta_0}^{+\theta_0} \left(-\frac{\omega}{R} + \frac{\omega_\theta^2}{2R^2}\right) d\theta. \quad (71)$$

Substituting for ϵ into the energy function (67) we finally get

$$V = \frac{1}{4\theta_0}\left[\int_{-\theta_0}^{+\theta_0} (-\omega' + \tfrac{1}{2}\omega'^2_\theta) \, d\theta\right]^2 + \frac{I}{2R^2A}\int_{-\theta_0}^{+\theta_0} (\omega'_{\theta\theta})^2 - \Lambda_1\omega'\Big|_{\theta=0} - \Lambda_2[\cdots] \quad (72)$$

which is independent of u, and where the generalized deflection corresponding to Λ_2 is the same as in Eq. (67).

We now assume that the radial displacements are approximately represented by

$$\omega' = Q_1 \cos\frac{\Pi\theta}{2\theta_0} + z_1 \sin\frac{\Pi\theta}{\theta_0}. \quad (73)$$

which satisfies the statical as well as the geometrical boundary conditions.

Substituting for ω', performing the integrals and dividing both sides by θ_0, we have

$$V' = \tfrac{1}{4}[\tfrac{1}{2}c^2Q_1^2 + 2c^2z_1^2 - (4/\Pi)Q_1]^2 + d[\tfrac{1}{2}c^4Q_1^2 + 8c^4z_1^2] - \Lambda_1'Q_1 - \Lambda_2'3.26Q_1 \quad (74)$$

where $V' = V/\theta_0$, $c = \Pi/2\theta_0$, $d = 1/R^2A$, $\Lambda_1' = \Lambda_1/\theta_0$ and $\Lambda_2' = \Lambda_2/\theta_0$.

We immediately note that the system under consideration is reduced to a two-degree-of-freedom one, and that odd terms in z_1 do not appear. In other words the system satisfies the symmetry conditions introduced in Sec. 2, and should therefore comply with the theory presented in the preceding sections.

The equilibrium equations $V_i = 0$ can be solved to yield the fundamental equilibrium surface in the form

$$z_1 = 0,$$
$$\tfrac{1}{2}[\tfrac{1}{2}c^2Q_1^2 - (4/\Pi)Q_1][c^2Q_1 - (4/\Pi)] + c^4 \, dQ_1 - \Lambda_1' - 3.26\Lambda_2' = 0, \quad (75)$$

which indicates that the deflections will initially take place in $Q_1 - \Lambda_i'$ subspace. Assuming that the geometrical properties of the system are so that the initial loss of stability occurs at special critical points when $V_{z_1 z_1} = 0$ (it can be shown that the condition for this is given by $8/11\ \Pi^2 > c^4 d$) we obtain

$$Q_{1\,cr} = (4/c^2)[(1/\Pi) \pm ((1/\Pi^2) - c^4 d)^{1/2}]. \tag{76}$$

Using this result and the equilibrium equations $V_i = 0$ we get the stability boundary in the form

$$\Lambda_1' + 3.26\Lambda_2' = b \tag{77}$$

where $b = 4c^2 d/\Pi + 12c^2 d(1/\Pi^2 - c^4 d)^{1/2}$.

Evidently Eq. (77) defines a straight line.

The Post-Critical Behaviour. Eq. (75) shows clearly that the fundamental surface is a highly nonlinear curved surface. To examine the post-critical characteristics in the neighbourhood of a special critical point on this surface we now introduce the incremental variable q_1,

$$Q_1 = Q_{1\,cr} + q_1\,, \tag{78}$$

and linearize the fundamental surface in the vicinity of this point (Fig. 4). Thus, after simplification, the potential energy takes the form

$$V' = \tfrac{1}{2}F_1 q_1^2 + \tfrac{1}{2}F_2 z_1^2 + \tfrac{1}{2}c q_1 z_1^2 + \tfrac{1}{4}!\, d z_1^4 - \Lambda_1' q_1 - 3.26\Lambda_2' q_1 \tag{79}$$

where

$$F_1 = 2(3(c^4/8)Q_{1\,cr}^2 + (4/\Pi^2) - (3/\Pi)c^2 Q_{1\,cr} + dc^4/2)$$

and

$$F_2 = 2(8c^4 - 16/\Pi c^2 Q_{1\,cr}^2). \tag{80}$$

It is understood that here higher-order terms in q_1 as well as irrelevant terms such as constants, etc. are ignored.

The equilibrium equations $V_i' = 0$ now yield the fundamental surface

$$z_1 = 0, \qquad q_1 = (1/F_1)(\Lambda_1' + 3.26\Lambda_2') \tag{81}$$

which is in the form of a plane. It must be remarked here that in many other problems

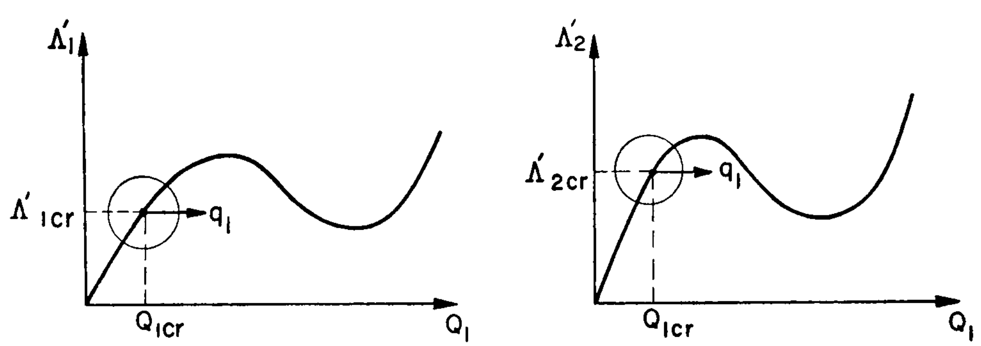

FIG. 4.

the fundamental surface will be a plane to start with and that the whole step described by Eqs. (78), (79), etc., will then be unnecessary.

The sliding coordinate q_1' can now be introduced,

$$q_1 = Q_1^F(\Lambda_i) + q_1' \equiv (1/F_1)(\Lambda_1' + 3.26\Lambda_2') + q_1' , \tag{82}$$

to obtain the function $T = T(u_1, y_1, \Lambda^i)$ as

$$T = \tfrac{1}{2}F_1u_1^2 + \tfrac{1}{2}[F_2 + (c/F_1)(\Lambda_1'' + 3.26\Lambda_2'')]y_1^2 + \tfrac{1}{2}cu_1y_1^2 + (1/4!)\, dy_1^4 \tag{83}$$

where z_1 and q_1' are directly equated to y_1 and u_1 respectively, the diagonalizing transformations (6) not being required.

Finally, the canonical representation of the linear form $(\Lambda_1' + 3.26\ \Lambda_2')$ is achieved by introducing the rotational transformation

$$\Lambda_1' = 0.293\Phi^1 - 0.956\Phi^2, \qquad \Lambda_2' = 0.956\Phi^1 + 0.293\Phi^2, \tag{84}$$

into the above energy function T which, then, yields the function

$$\Psi = \Psi(u_1, y_1, \Phi^i) = \tfrac{1}{2}F_1u_1^2 + \tfrac{1}{2}[F_2 + (c/F_1)3.41\Phi^1]y_1^2 + (1/2!)cu_1y_1^2 + (1/4!)\, dy_1^4 . \tag{85}$$

We are now in a position to derive the results of the theory directly from this energy function.

The following derivatives evaluated at an arbitrary critical point $\Phi^1 = \Phi^1_{cr}$ and $\Phi^2 = \Phi^2_{cr}$ are immediately obtainable:

$$\begin{aligned} &\Psi_{y_1y_1y_1y_1} = d, \qquad \Psi_{u_1y_1y_1} = c, \qquad \Psi_{u_1u_1} = F_1 \\ &\Psi_{y_1y_1} = F_2 + \frac{c}{F_1}\, 3.41 \times \varphi^1_{cr} , \qquad \Psi^{mn}_{y_1y_1} = 0, \qquad \Psi^1_{y_1y_1} = 3.41\,\frac{c}{F_2} \end{aligned} \tag{86}$$

which yields, for example, the post-buckling equilibrium surface

$$\Phi^1 = \Phi^1_{cr} - \frac{1}{6 \times 3.41(c/F_1)}\left[d - 3\,\frac{c^2}{F_1}\right]y_1^2 , \qquad u_1 = -\frac{1}{2}\,\frac{c}{F_1}\,y_1^2 \tag{87}$$

and the stability boundary

$$^*\Phi^1 = \Phi^1_{cr} \quad \text{or} \quad {}^*\varphi \equiv {}^*\Phi^1 - {}^*\Phi^1_{cr} = 0. \tag{88}$$

Similarly, the stability determinant, stability criterion and other results of the preceding sections can readily be constructed. If numerical data is introduced it will be seen that normally $\partial^2\ \Phi^1/\partial\ (y_1)^2 < 0$ and hence the loss of stability will generally be associated with unstable special points.

It is interesting to note that the post-critical analysis can readily be performed in the vicinity of *any* critical point on the stability boundary by simply computing the value of desired Φ^1_{cr} through Eqs. (77) and (84).

Discussion and conclusions. An intrinsic nonlinear analysis concerning the post-buckling characteristics of a symmetric conservative system is presented.

It is demonstrated that a 'symmetric special' critical point is normally associated with Koiter's 'symmetric point of bifurcation,' but under some circumstances, two symmetric equilibrium paths with *finite* slopes are obtained, in which case the critical point F can no longer be considered as a 'symmetric point of bifurcation' in the terms of the general theory of Koiter. Even then, however, the critical point F can be stable (see Fig. 1) while 'asymmetric points of bifurcation' discussed by Poincaré [1], Koiter [2] and Thompson [8] are always unstable. If no such particular phenomena exist, i.e. if no equilibrium paths with finite slopes are obtainable under any combination of loads, then the post-buckling equilibrium surface and consequently the stability boundary are synclastic.

The following theorems concerning the stability of the fundamental and post-buckling equilibrium surfaces of the system under consideration are established:

THEOREM 1. *The initial post-buckling equilibrium surface is either totally stable or unstable.*

THEOREM 2. *The stability boundary constitutes an existence boundary with regard to the post-buckling equilibrium surface so that if this surface is stable (unstable), no post-buckling equilibrium states can correspond to the points of the region of stability (region of instability).*

THEOREM 3. *The initial post-buckling equilibrium surface is stable or unstable according to whether the critical point itself is stable or unstable respectively.*

The method introduced in Sec. 6 is purely mathematical, and can be used in determining whether a function of several variables has a minimum or maximum.

Finally, a shallow circular arch under the combined action of two independent sets of external loads is analysed to illustrate some aspects of the theory. The system is first reduced to a two-degree-of-freedom one by assuming a certain shape for deflections, and then the general theory is applied. In spite of the fact that this example exhibits highly nonlinear pre-buckling as well as post-buckling characteristics, the theory is well illustrated. It is understood, however, that the application of the theory would have become much simpler if the fundamental surface was not nonlinear (as in many other particular problems). As a matter of fact, it should be once more emphasized that the purpose of this paper is to establish general results valid for the class of systems under consideration rather than propose a method of analysis.

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