Instances of the Meijer G and Fox H functions and of the distribution of the product of independent beta random variables with finite representations

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Abstract

In this paper we will show how we may, through the use of some extended product expressions for the Gamma function on the characteristic functions of the negative logarithm of the product of particular independent Beta random variables, whose second parameters are rational and follow some amenable rules, obtain in a single shot (i) simple expressions for both the probability density and cumulative distribution functions of this product, as well as, concomitantly (ii) very simple and easy to compute alternative finite form expressions for instances of the Meijer G and Fox H functions. These alternative expressions are based on the expressions for the probability density and cumulative distribution functions of the generalized integer Gamma distribution and are not used or recognized by the available softwares.

Keywords: characteristic function, distribution of likelihood ratio statistics, Fox H function, Generalized Integer Gamma (GIG) distribution, sum of independent Gamma random variables.

1. Introduction

By using two extended multiplication formulas for the Gamma function we will be able to show the equivalence of the distribution of two multiple products of independent Beta r.v.'s (random variables) with rational second parameters, which follow some amenable rules, to the distribution of products of positive powers of independent Beta r.v.'s with integer second parameters. Then, it will not be hard to establish the equivalence of the distribution of such products to that of the exponential of a GIG (Generalized Integer Gamma) distributed r.v. (Coelho, 1998, 1999). From this result we will then be able to establish the equivalence between some instances of the Meijer G function and the Fox H function and between these and the expressions for the p.d.f. (probability density function) and the c.d.f. (cumulative distribution function) of the exponential of GIG distributed r.v.'s.

The establishment of these results is useful in helping to recognize a number of situations, not previously identified, and as such, not able to be sorted out by the available softwares, in which the Meijer G and Fox H functions have an alternative finite and consequently much more manageable representations, through the p.d.f. and c.d.f. of a GIG distribution. This results in huge gains in computation time and precision.

A number of applications of the results obtained to the exact distribution of several l.r.t. (likelihood ratio test) statistics are brought to the reader's attention.

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2. Particular cases of the distribution of the product of independent Beta random variables with finite representation for the exact distribution

2.1. Some preliminary results

It is not hard to show that the exact distribution of any product of any positive powers of independent Beta distributed random variables with integer second parameters has a finite closed form representation.

Indeed, if

$$Z=\prod_{j=1}^{p} X_j^{c_j} \qquad \text{and} \qquad W=-\log Z\,, \tag{2.1}$$

where c_j are positive reals and $X_j \sim Beta(a_j, b_j)$ with b_j being positive integers, for all $j = 1, \dots, p$, then, using the fact that for every integer n and complex number a we have

$$\frac{\Gamma(a+n)}{\Gamma(a)}=\prod_{\ell=0}^{n-1}(a+\ell)\,,$$

we may write the c.f. (characteristic function) of $W = -\log Z$ as

$$\begin{aligned}\Phi_W(t) &= \prod_{j=1}^{p}\frac{\Gamma(a_j+b_j)}{\Gamma(a_j)}\,\frac{\Gamma(a_j-c_j\mathrm{i}t)}{\Gamma(a_j+b_j-c_j\mathrm{i}t)} = \prod_{j=1}^{p}\prod_{\ell=0}^{b_j-1}(a_j+\ell)\,(a_j+\ell-c_j\mathrm{i}t)^{-1}\\ &= \prod_{j=1}^{p}\prod_{\ell=0}^{b_j-1}\frac{a_j+\ell}{c_j}\left(\frac{a_j+\ell}{c_j}-\mathrm{i}t\right)^{-1},\end{aligned}$$

which is the c.f. of a sum of $\sum_{j=1}^{p} b_j$ independent Exponential r.v.'s with rate parameters $\frac{a_j+\ell}{c_j}$ ($\ell = 0, \dots, b_j - 1;$ $j = 1, \dots, p$).

From this result we may then write, for the Beta r.v.'s in (2.1),

$$\prod_{j=1}^{p} X_j^{c_j} \stackrel{\mathrm{d}}{\equiv} \prod_{j=1}^{p}\prod_{\ell=0}^{b_j-1} e^{-W_{j\ell}}\,, \qquad \text{or} \qquad W \stackrel{\mathrm{d}}{\equiv} \sum_{j=1}^{p}\sum_{\ell=0}^{b_j-1} W_{j\ell} \tag{2.2}$$

where $W_{j\ell} \sim Exp\left(\frac{a_j+\ell}{c_j}\right)$ ($\ell = 0, \dots, b_j - 1;\ j = 1, \dots, p$) are independent r.v.'s.

In case some of the parameters $\frac{a_j+\ell}{c_j}$ are equal, then if there are for example g of the $\sum_{j=1}^{p} b_j$ parameters $\frac{a_j+\ell}{c_j}$ which are different, let them be the elements of the set $\{d_h\ :\ h = 1, \dots, g\}$ and let then r_h be the number of times the parameter d_h occurs. Then the exact distribution of W is a Generalized Integer Gamma (GIG) distribution of depth g (Coelho, 1998, 1999), with rate parameters d_h and shape parameters r_h ($h = 1, \dots, g$). Consequently, the distribution of $Y = e^{-W}$ is what Arnold et al. (2012) call an EGIG (Exponentiated Generalized Integer Gamma) distribution of depth g, with the same shape and rate parameters of the corresponding GIG distribution. Both the p.d.f. and the c.d.f. of this distribution have finite form representations not involving any series or unsolved integrals and are highly manageable, allowing for an accurate and easy computation of quantiles and p-values.

Based on this result and on some extended gamma product expressions, it is possible to show that the exact distribution of a number of multiple products of independent Beta r.v.'s whose second parameter is rational have finite form representations of this type.

However, in order for this to happen, both parameters of the whole set of Beta r.v.'s involved in the product have to obey some amenable rules.

Such products are important since they not only allow us to devise finite form representations for particular instances of the Meijer G function, but also because particular cases of the more general forms presented here yield the exact distributions of a number of l.r.t. statistics used in Multivariate Analysis, which then have finite closed form representations for their p.d.f.'s and c.d.f.'s.

Two different kinds of such products are presented in subsection 2.2. In Section 4 we present a number of l.r.t. statistics whose distributions may be represented in the form of these products and that, as such, have closed finite forms for their p.d.f.'s and c.d.f.'s.

2.2. Two different multiple products of independent Beta r.v.'s which distributions have finite form representations

In this subsection we will show how by working on the c.f. of the negative logarithm of multiple products of independent Beta r.v.'s, whose second parameters are rational and follow some rules, we are able to establish that in these cases the distributions of these products have closed finite forms for their p.d.f.'s and c.d.f.'s, with quite simple and much manageable expressions, actually in the form of p.d.f.'s and c.d.f.'s of the exponential of GIG distributed r.v.'s.

In order to keep the notation simpler, in the following two Theorems we will denote the random variable with a given Beta distribution by the corresponding distribution itself.

Theorem 1. *For positive integers n_ν, $k_{\nu\ell}$ and $m_{\nu\ell}$ ($\nu = 1,\dots,m^*$; $\ell = 1,\dots,n_\nu$), and for $a_\nu > \frac{p_\nu}{\min\{k_{\nu\ell}\}}$, with $p_\nu = \sum_{\ell=1}^{n_\nu} k_{\nu\ell}$,*

$$Z = \prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\prod_{j=1}^{k_{\nu\ell}} Beta\left(a_\nu - \frac{j+\sum_{r=1}^{\ell-1}k_{\nu r}}{k_{\nu\ell}},\, \frac{m_{\nu\ell}}{k_{\nu\ell}}\right) \stackrel{\text{d}}{\equiv} \prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\prod_{j=1+\sum_{r=1}^{\ell-1}k_{\nu r}}^{\sum_{r=1}^{\ell}k_{\nu r}} Beta\left(a_\nu - \frac{j}{k_{\nu\ell}},\, \frac{m_{\nu\ell}}{k_{\nu\ell}}\right) \tag{2.3}$$

$$\stackrel{\text{d}}{\equiv} \prod_{\nu=1}^{m^*}\prod_{j=1}^{p_\nu} Beta\left(a_\nu - \frac{j}{k_{\nu j^*}},\, \frac{m_{\nu j^*}}{k_{\nu j^*}}\right) \stackrel{\text{d}}{\equiv} \prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\left(Beta\left(a_\nu k_{\nu\ell} - \sum_{r=1}^{\ell}k_{\nu r},\, m_{\nu\ell}\right)\right)^{k_{\nu\ell}} \stackrel{\text{d}}{\equiv} \prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\prod_{i=0}^{m_{\nu\ell}-1} e^{-W_{\nu\ell i}}, \tag{2.4}$$

or, for $W = -\log Z$,

$$W \stackrel{\text{d}}{\equiv} \sum_{\nu=1}^{m^*}\sum_{\ell=1}^{n_\nu}\sum_{i=0}^{m_{\nu\ell}-1} W_{\nu\ell i}\,,$$

where '$\stackrel{\text{d}}{\equiv}$' stands for 'is equivalent in distribution to', or 'is stochastically equivalent to', and

$$W_{\nu\ell i} \sim Exp\left(a_\nu + \frac{i - \sum_{r=1}^{\ell}k_{\nu r}}{k_{\nu\ell}}\right) \qquad \nu = 1,\dots,m^*;\ \ell = 1,\dots,n_\nu;\ i = 0,\dots,m_{\nu\ell}-1\,,$$

and where the Beta and exponential random variables involved in any product are all independent, with

$$j^* = 1 + \#\ \textit{of elements in}\ \{j - \underline{k}_\nu\} > 0\,, \qquad \underline{k}_\nu = \left\{\textstyle\sum_{r=1}^{h} k_{\nu r},\ h = 1,\ldots,n_\nu\right\}, \quad \nu = 1,\ldots,m^*\,.$$

Proof. The equivalence in (2.3) is obtained only by reversing the indexation in j, but while the first equivalence in (2.4) is obtained only by re-indexation, the second one needs the use of an extended version of the usual multiplication formula for the Gamma function which, for any $z, c \in \mathbb{C}$ and $n \in \mathbb{N}$, may be written as

$$\prod_{j=1}^{n} \Gamma\left(z + \frac{c-j}{n}\right) = \Gamma(nz + c - n)\,(2\pi)^{\frac{n-1}{2}}\ n^{\frac{1}{2}-nz-c+n}\,. \tag{2.5}$$

This extended multiplication formula may be very easily derived from the more common multiplication formula for the Gamma function, which uses $n \in \mathbb{N}$ in place of $c \in \mathbb{C}$ and which, for any $z \in \mathbb{C}$ and $n \in \mathbb{N}$, may be written as

$$\prod_{j=1}^{n} \Gamma\left(z + \frac{j-1}{n}\right) = \prod_{j=1}^{n} \Gamma\left(z + \frac{n-j}{n}\right) = \Gamma(nz)\,(2\pi)^{\frac{n-1}{2}}\ n^{\frac{1}{2}-nz}\,. \tag{2.6}$$

Indeed, (2.5) may be immediately derived from (2.6) since we may write

$$\prod_{j=1}^{n} \Gamma\left(z + \frac{c-j}{n}\right) = \prod_{j=1}^{n} \Gamma\left(z + \frac{c-n+n-j}{n}\right).$$

Then, from the first expression in (2.3), using (2.5) in the c.f. of $W = -\log Z$, this c.f. may be written as

$$\begin{aligned}
\Phi_W(t) &= \prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\prod_{j=1}^{k_{\nu\ell}} \frac{\Gamma\left(a_\nu - \frac{j+\sum_{r=1}^{\ell-1}k_{\nu r}}{k_{\nu\ell}} + \frac{m_{\nu\ell}}{k_{\nu\ell}}\right)}{\Gamma\left(a_\nu - \frac{j+\sum_{r=1}^{\ell-1}k_{\nu r}}{k_{\nu\ell}}\right)} \frac{\Gamma\left(a_\nu - \frac{j+\sum_{r=1}^{\ell-1}k_{\nu r}}{k_{\nu\ell}} - \mathrm{i}t\right)}{\Gamma\left(a_\nu - \frac{j+\sum_{r=1}^{\ell-1}k_{\nu r}}{k_{\nu\ell}} + \frac{m_{\nu\ell}}{k_{\nu\ell}} - \mathrm{i}t\right)} \\
&= \prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\prod_{j=1}^{k_{\nu\ell}} \frac{\Gamma\left(a_\nu + \frac{m_{\nu\ell}-\sum_{r=1}^{\ell}k_{\nu r}}{k_{\nu\ell}} + \frac{k_{\nu\ell}-j}{k_{\nu\ell}}\right)}{\Gamma\left(a_\nu - \frac{\sum_{r=1}^{\ell}k_{\nu r}}{k_{\nu\ell}} + \frac{k_{\nu\ell}-j}{k_{\nu\ell}}\right)} \frac{\Gamma\left(a_\nu - \frac{\sum_{r=1}^{\ell}k_{\nu r}}{k_{\nu\ell}} + \frac{k_{\nu\ell}-j}{k_{\nu\ell}} - \mathrm{i}t\right)}{\Gamma\left(a_\nu + \frac{m_{\nu\ell}-\sum_{r=1}^{\ell}k_{\nu r}}{k_{\nu\ell}} + \frac{k_{\nu\ell}-j}{k_{\nu\ell}} - \mathrm{i}t\right)} \\
&= \prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu} \frac{\Gamma\left(a_\nu k_{\nu\ell} + m_{\nu\ell} - \sum_{r=1}^{\ell}k_{\nu r}\right)}{\Gamma\left(a_\nu k_{\nu\ell} - \sum_{r=1}^{\ell}k_{\nu r}\right)} \frac{\Gamma\left(a_\nu k_{\nu\ell} - \sum_{r=1}^{\ell}k_{\nu r} - k_{\nu\ell}\mathrm{i}t\right)}{\Gamma\left(a_\nu k_{\nu\ell} + m_{\nu\ell} - \sum_{r=1}^{\ell}k_{\nu r} - k_{\nu\ell}\mathrm{i}t\right)}
\end{aligned}$$

which is the c.f. of the sum of the negative logarithm of the Beta r.v.'s in the second product in (2.4).

The last equivalence in (2.4) is then obtained by applying (2.2) and it shows that the exact distribution of the original product of independent Beta r.v.'s is an EGIG distribution of depth at most $\sum_{\nu=1}^{m^*}\sum_{\ell=1}^{n_\nu} m_{\nu\ell}$. □

The particular case for $k_{\nu\ell} = k_\nu$ and $m_{\nu\ell} = m_\nu$, for all $\ell = 1,\ldots,n_\nu$ yields the result in the following Theorem.

Theorem 2. *If in Theorem 1 we have $k_{\nu\ell} = k_\nu$ and $m_{\nu\ell} = m_\nu$, for all $\ell = 1,\ldots,n_\nu$, then with $p_\nu = n_\nu k_\nu$,*

$$Z = \prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\prod_{j=1}^{k_\nu} Beta\left(a_\nu + 1 - \ell - \frac{j}{k_\nu},\, \frac{m_\nu}{k_\nu}\right) \stackrel{d}{\equiv} \prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\prod_{j=1+k_\nu(\ell-1)}^{k_\nu \ell} Beta\left(a_\nu - \frac{j}{k_\nu},\, \frac{m_\nu}{k_\nu}\right) \tag{2.7}$$

$$\stackrel{d}{\equiv} \prod_{\nu=1}^{m^*}\prod_{j=1}^{p_\nu} Beta\left(a_\nu - \frac{j}{k_\nu},\, \frac{m_\nu}{k_\nu}\right) \stackrel{d}{\equiv} \prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\Big(Beta\big((a_\nu - \ell)k_\nu,\, m_\nu\big)\Big)^{k_\nu} \stackrel{d}{\equiv} \prod_{\nu=1}^{m^*}\prod_{j=1}^{m_\nu+k_\nu(n_\nu-1)} e^{-W_{\nu j}}, \tag{2.8}$$

or, for $W = -\log Z$*,*

$$W \stackrel{\mathrm{d}}{\equiv} \sum_{\nu=1}^{m^*} \sum_{j=1}^{m_\nu+k_\nu(n_\nu-1)} W_{\nu j}\,,$$

where

$$W_{\nu j} \sim \Gamma\left(r_{\nu j}, a_\nu - n_\nu + \frac{j-1}{k_\nu}\right), \quad \nu = 1,\ldots,m^*;\ j = 1,\ldots,m_\nu + k_\nu(n_\nu - 1)\,,$$

with

$$r_{\nu j} = \begin{cases} h_{\nu j} & j = 1,\ldots,k_\nu \\ h_{\nu j} + r_{\nu,j-k} & j = k_\nu + 1,\ldots,m_\nu + k_\nu(n_\nu - 1)\,, \end{cases} \tag{2.9}$$

where

$$h_{\nu j} = (\#\ \textit{of elements in}\ \{p_\nu, m_\nu\} \geq j) - 1\,, \quad \nu = 1,\ldots,m^*\,, \tag{2.10}$$

which shows that in this particular case the exact distribution is an EGIG distribution of depth at most $\sum_{\nu=1}^{m^*} m_\nu + k_\nu(n_\nu - 1)$*, with shape parameters* $r_{\nu j}$ *and rate parameters* $a_\nu - n_\nu + j - 1k_\nu$ $(\nu = 1,\ldots,m^*;$ $j = 1,\ldots,m_\nu + k_\nu(n_\nu - 1))$*.*

Proof. Because the result in this Theorem is a particular case of the one in Theorem 1, the main part of the proof follows similar lines. Then the expression for the shape parameters $r_{\nu j}$ is obtained either by identifying the different exponential c.f.'s that appear in the c.f. of W and counting how many of each of them occur, in a manner in all similar to the one used by Coelho (1998, 2006) and by Arnold et al. (2012), or, equivalently, by identifying the different poles through which the integration path used in the definition of the Meijer G function in Section 3, passes through and by counting the number of times the path of integration passes through each of these poles, as Wald and Brookner (1941) refer. □

A second type of multiple product of independent Beta r.v.'s with closed finite form representation for their p.d.f. and c.d.f. is presented in Theorem 3.

Theorem 3. *For positive integers* k_ν*,* n_ν*, non-negative integers* s_ν *and real* $a_\nu > n_\nu k_\nu$ $(\nu = 1,\ldots,m^*)$*,*

$$Z = \prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\prod_{j=1}^{k_\nu} Beta\left(a_\nu - \frac{(\ell-1)k_\nu + j}{n_\nu},\ \frac{j + (\ell-1)k_\nu + \ell + s_\nu - 1}{n_\nu}\right) \tag{2.11}$$

$$\stackrel{\mathrm{d}}{\equiv} \prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\prod_{j=1+k_\nu(\ell-1)}^{k_\nu \ell} Beta\left(a_\nu - \frac{j}{n_\nu},\ \frac{j + \ell + s_\nu - 1}{n_\nu}\right) \tag{2.12}$$

$$\stackrel{\mathrm{d}}{\equiv} \prod_{\nu=1}^{m^*}\prod_{j=1}^{n_\nu k_\nu} Beta\left(a_\nu - \frac{j}{n_\nu},\ \frac{j + s_\nu - 1 + Mod^*(j, n_\nu)}{n_\nu}\right) \stackrel{\mathrm{d}}{\equiv} \prod_{\nu=1}^{m^*}\prod_{j=1}^{k_\nu}\Big(Beta\big(a_\nu n_\nu - n_\nu j,\ n_\nu j + s_\nu\big)\Big)^{n_\nu} \tag{2.13}$$

$$\stackrel{\mathrm{d}}{\equiv} \prod_{\nu=1}^{m^*}\prod_{j=1}^{n_\nu k_\nu + s_\nu} e^{-W_{\nu j}}\,, \tag{2.14}$$

or, for $W = -\log Z$,

$$W \stackrel{d}{\equiv} \sum_{\nu=1}^{m^*} \sum_{j=1}^{n_\nu k_\nu + s_\nu} W_{\nu j}\,,$$

where the Beta random variables involved in any double or triple product are all independent, as well as the $W_{\nu j}$ *random variables , with*

$$Mod^*(j,n) = \begin{cases} Mod(j,n) & \text{if} \quad Mod(j,n) \neq 0 \\ n & \text{if} \quad Mod(j,n) = 0\,, \end{cases}$$

and

$$W_{\nu j} \sim \Gamma\left(r_{\nu j},\, a_\nu + \frac{s_\nu - j}{n_\nu}\right), \quad \nu = 1,\ldots,m^*;\ j = 1,\ldots,n_\nu k_\nu + s_\nu\,,$$

where, for $\nu = 1,\ldots,m^*$,

$$r_{\nu j} = \begin{cases} k_\nu & j = 1,\ldots,s_\nu \\ k_\nu + 1 + \left\lfloor \frac{s_\nu - j}{n_\nu} \right\rfloor & j = s_\nu + 1,\ldots,n_\nu k_\nu + s_\nu\,. \end{cases} \tag{2.15}$$

The exact distribution of Z *is thus in this case an EGIG distribution of depth at most* $\sum_{\nu=1}^{m^*} n_\nu k_\nu + s_\nu$, *with rate parameters* $a_\nu - \frac{s_\nu - j}{n_\nu}$ *and shape parameters* $r_{\nu j}$ $(j = 1,\ldots,n_\nu k_\nu + s_\nu; \nu = 1,\ldots,m^*)$.

Proof. The equivalence in (2.12) is obtained only by reversing the indexation in j, but while the first equivalence in (2.13) is obtained only by re-indexation, the second one needs the use of an extended product expression for the Gamma function.

This extended product expression for the Gamma function is

$$\begin{aligned} \prod_{\ell=1}^{n} \prod_{j=1}^{k} \Gamma\left(a - \frac{\ell - 1}{n} k - \frac{j}{n}\right) &= \prod_{j=1}^{k} \left\{ \Gamma(an - nj)\,(2\pi)^{\frac{n-1}{2}}\, n^{\frac{1}{2} - (an - nj)} \right\} \\ &= \left\{ \prod_{j=1}^{k} \Gamma(an - nj) \right\} (2\pi)^{\frac{n-1}{2} k}\, n^{\frac{k}{2} - kan + n\frac{k(k+1)}{2}}\,, \end{aligned}$$

which for $k = 1$ yields the usual product expression for the Gamma function and which when used on the expression for the c.f. of $W = -\log Z$ obtained from (2.11) yields

$$\begin{aligned} \Phi_W(t) &= \prod_{\nu=1}^{m^*} \prod_{\ell=1}^{n_\nu} \prod_{j=1}^{k_\nu} \frac{\Gamma\left(a_\nu + \frac{\ell-1}{n_\nu} + \frac{s_\nu}{n_\nu}\right)}{\Gamma\left(a_\nu - \frac{(\ell-1)k_\nu}{n_\nu} - \frac{j}{n_\nu}\right)} \frac{\Gamma\left(a_\nu - \frac{(\ell-1)k_\nu}{n_\nu} - \frac{j}{n_\nu} - \mathrm{i}t\right)}{\Gamma\left(a_\nu + \frac{\ell-1}{n_\nu} + \frac{s}{n_\nu} - \mathrm{i}t\right)} \\ &= \prod_{\nu=1}^{m^*} \prod_{j=1}^{k_\nu} \frac{\Gamma(a_\nu n_\nu + s_\nu)}{\Gamma(a_\nu n_\nu - n_\nu j)} \frac{\Gamma(a_\nu n_\nu - n_\nu j - n_\nu \mathrm{i}t)}{\Gamma(a_\nu n_\nu + s_\nu - n_\nu \mathrm{i}t)}\,. \end{aligned}$$

which is the c.f. of the negative logarithm of the distribution in the second expression in (2.13). Then, using the relation in (2.2) we obtain the distribution in (2.14).

The expression for the shape parameters $r_{\nu j}$ in (2.15) is, once again as in the previous subsection, obtained either by identifying the different exponential c.f.'s that appear in the c.f. of W and then counting how many of each of them occur, in a similar manner to the one used by Coelho (1998, 2006) and by Arnold et al. (2012), or, equivalently by identifying the different poles through which the integration path used in the definition of the Meijer G function in Section 3, passes through and then counting the number of times the path of integration passes through each of these poles, as Wald and Brookner (1941) refer. □

The particular case in Theorem 2 was studied by Coelho (2006) for $m^* = 1$ and by Coelho (1998) and Arnold et al. (2012) for $k_\nu = 2$, while the particular case for $m^* = 1$ in Theorem 3 was treated in Arnold et al. (2012).

3. Instances of Fox H and Meijer G functions with alternative finite representations

Although Meijer (1936a,b) first defined the Meijer G function as a series, related to the generalized hypergeometric function, a more encompassing definition, based on a Mellin-Barnes integral, was given later by the same author (Meijer, 1941, 1946), and a distilled version of this definition is provided by Erdélyi et al. (1953). We will use this latter definition.

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0$ denote the set of non-negative integers. For $m, n, p, q \in \mathbb{N}_0$, with $m \leq q$ and $n \leq p$, and for $a_k - b_j \notin \mathbb{N}$ ($k = 1, \ldots, n$; $j = 1, \ldots, m$) and $z \neq 0$, the Meijer G function with arguments $m, n, p, q \in \mathbb{N}_0$, $s \in \mathbb{C}$, and any real or complex $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$, is defined as

$$G^{m,n}_{p,q}\left(\begin{array}{c} \{a_j\}_{j=1,\ldots,p} \\ \{b_j\}_{j=1,\ldots,q} \end{array} \middle| \, z \right) = \frac{1}{2\pi \mathrm{i}} \oint_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s \, ds \,, \tag{3.1}$$

where an empty product is taken as evaluating to 1.

In (3.1) there are three different possible paths of integration L. These are listed in Section 5.3 of Erdélyi et al. (1953).

The definition in (3.1) is a Mellin-Barnes type integral and it allows us to view the Meijer G function as an inverse Mellin transform. Indeed, taking the Mellin transform of the r.v. X as $E(X^{s-1})$, for any $s \in \mathbb{C}$, for $p = q = m$ and $n = 0$, the integrand in (3.1), excluding the power of z, is, for $a_j, b_j > 0$, the Mellin transform of a product of p independent $Beta(a_j, b_j)$ random variables. Indeed, from the definition of the Meijer G function in (3.1) and from the inversion formula for the Mellin transform of the product of p independent $Beta(a_j, b_j)$ random variables we may immediately derive the result in Section 4.4.2 of Springer (1979), which states that the p.d.f. of the r.v. Z in (2.1), for all $c_j = 1$ and for any positive real a_j and b_j is given by

$$\left\{ \prod_{j=1}^p \frac{\Gamma(a_j + b_j)}{\Gamma(a_j)} \right\} G^{p,0}_{p,p}\left(\begin{array}{c} \{a_j + b_j - 1\}_{j=1,\ldots,p} \\ \{a_j - 1\}_{j=1,\ldots,p} \end{array} \middle| \, z \right), \tag{3.2}$$

where $z \in]0, 1]$ denotes the running value of the r.v. Z.

Then, by simple integration, it is easy to see that the corresponding c.d.f. will be given by

$$\left\{ \prod_{j=1}^p \frac{\Gamma(a_j + b_j)}{\Gamma(a_j)} \right\} G^{p,1}_{p+1,p+1}\left(\begin{array}{c} \{1, a_j + b_j\}_{j=1,\ldots,p} \\ \{a_j, 0\}_{j=1,\ldots,p} \end{array} \middle| \, z \right),$$

The same author states then in Theorem 4.4.1 that there is a finite form representation for the p.d.f. in (3.2) when all a_j and b_j are integer. However, this is indeed true if just the b_j's are positive integers. In addition it is true when the a_j's and b_j's are rationals which satisfy certain relations. It is this latter case that we address here.

Although indeed in order to address the distribution of the r.v. Z in (2.1), for the general case where the a_j and b_j are just positive reals, we need to resort to the use of Fox H function, it is easy to show that whenever the b_j are positive integers then there is a simple finite representation for the Fox H function. Fox (1961) defined the H function, for $m, n, p, q,\ s,\ a_j$ and b_j as in (3.1), and positive reals α_j and β_j, as

$$H^{m,n}_{p,q}\left(\begin{array}{c}\{(a_j,\alpha_j)\}_{j=1,\dots,p}\\ \{(b_j,\beta_j)\}_{j=1,\dots,q}\end{array}\middle|\, z\right)=\frac{1}{2\pi \mathrm{i}}\oint_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\ \prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\ \prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}\, z^s\, ds\,. \tag{3.3}$$

It is then easy to see that the H function reduces to the G function if all α_j and β_j equal 1. Also, when all $\alpha_j = \beta_j = c$, it is easy to see that

$$H^{m,n}_{p,q}\left(\begin{array}{c}\{(a_j,c)\}_{j=1,\dots,p}\\ \{(b_j,c)\}_{j=1,\dots,q}\end{array}\middle|\, z\right)=\frac{1}{c}\,G^{m,n}_{p,q}\left(\begin{array}{c}\{a_j\}_{j=1,\dots,p}\\ \{b_j\}_{j=1,\dots,q}\end{array}\middle|\, z^{1/c}\right).$$

Yet, as Mathai (2007, Sec. 1.9) remarks, when all a_j and b_j are rationals it is always possible to reduce the Fox H function to a Meijer G function.

Then, for general positive real a_j, b_j and c_j, in (2.1), the p.d.f. of Z is given by

$$\left\{\prod_{j=1}^{p}\frac{\Gamma(a_j+b_j)}{\Gamma(a_j)}\right\}\ H^{p,0}_{p,p}\left(\begin{array}{c}\{(a_j+b_j-1,c_j)\}_{j=1,\dots,p}\\ \{(a_j-1,c_j)\}_{j=1,\dots,p}\end{array}\middle|\, z\right), \tag{3.4}$$

and the c.d.f. by

$$\left\{\prod_{j=1}^{p}\frac{\Gamma(a_j+b_j)}{\Gamma(a_j)}\right\}\ H^{p,1}_{p+1,p+1}\left(\begin{array}{c}\{(1,1),(a_j+b_j+c_j-1,c_j)\}_{j=1,\dots,p}\\ \{(a_j+c_j-1,c_j),(0,1)\}_{j=1,\dots,p}\end{array}\middle|\, z\right), \tag{3.5}$$

but there are a number of interesting situations where the b_j are positive integers, many of which yield the exact distribution of a number of l.r.t. statistics. These are exactly the situations described in the previous section.

Indeed, although the Fox H and Meijer G functions yield very handy representations for the p.d.f.'s and c.d.f.'s of several distributions and distributions of products of independent r.v.'s, they are not computationally efficient and most of the time not even utilizable in practice because of serious difficulties found in their efficient computational implementation, even when using the most up-to-date software. This way, looking for alternative representations, namely finite form ones, is a most useful and desirable goal.

It happens that, based on the results in the previous section, this aim is rendered easy for a number of situations in which the Fox H function always has a simple finite form representation and which seems to have been overlooked by other authors. This is when all b_j in (3.3) are positive integers. Indeed, based on the result in (2.2), using the notation in Appendix A, we may write, for positive integer b_j,

$$H^{p,0}_{p,p}\left(\begin{array}{c}\{(a_j+b_j-1,c_j)\}_{j=1,\dots,p}\\ \{(a_j-1,c_j)\}_{j=1,\dots,p}\end{array}\middle|\, z\right)=$$
$$\left\{\prod_{j=1}^{p}\frac{\Gamma(a_j)}{\Gamma(a_j+b_j)}\right\} f^{GIG}\!\left(-\log z\,\middle|\,\{\{1\}\}_{\substack{j=1:p\\ \ell=0:b_j-1}}\,;\left\{\left\{\frac{a_j+\ell}{c_j}\right\}\right\}_{\substack{j=1:p\\ \ell=0:b_j-1}}\,;g\le \textstyle\sum_{j=1}^{p} b_j\right)\frac{1}{z}\,, \tag{3.6}$$

and

$$H^{p,1}_{p+1,p+1}\left(\begin{array}{l}\{(1,1),(a_j+b_j-1+c_j,c_j)\}_{j=1,\dots,p}\\ \{(a_j-1+c_j,c_j),(0,1)\}_{j=1,\dots,p}\end{array}\middle|\, z\right)=$$

$$\left\{\prod_{j=1}^{p}\frac{\Gamma(a_j)}{\Gamma(a_j+b_j)}\right\}F^{GIG}\left(-\log z\,\middle|\,\{\{1\}\}_{\substack{j=1:p\\ \ell=0:b_j-1}};\left\{\left\{\frac{a_j+\ell}{c_j}\right\}\right\}_{\substack{j=1:p\\ \ell=0:b_j-1}};g\leq \textstyle\sum_{j=1}^{p}b_j\right), \tag{3.7}$$

where

$$\left\{\left\{\frac{a_j+\ell}{c_j}\right\}\right\}_{\substack{j=1:p\\ \ell=0:b_j-1}}=\{d_h\}_{h=1:g} \tag{3.8}$$

denotes the 'contraction' of the set on the left, that is, the set of the $g\leq\sum_{j=1}^{p}b_j$ different rate parameters $\frac{a_j+\ell}{c_j}$ ($\ell=0,\dots,b_j-1;\ j=1,\dots,p$) and

$$\{\{1\}\}_{\substack{j=1:p\\ \ell=0:b_j-1}}=\{r_h\}_{h=1:g}$$

denotes the 'contraction' of the set on the left, corresponding to the contraction of the set on the left hand side of (3.8), that is, the set of the corresponding shape parameters r_h corresponding to the rate parameters d_h, with $r_h=\sum_{k=1}^{r_h}1$, being the number of times the value d_h in (3.8) occurs.

It also happens that, based on the results in the previous section, obtaining a finite closed form for the Meijer G function is also rendered easy for a number of situations. These situations are when the a_j and b_j are rational and comply with some rules, which are the ones described in Subsections 2.2 and 2.3 of the previous section, situations which, as it may be seen from the results in these subsections, always correspond to situations which may also be represented by a H function with all integer b_j parameters (which are different from the b_j parameters in the corresponding representation of the Meijer G function).

These situations are summarized in the next two Corollaries. In these two Corollaries we use the notation in (3.1) and (3.3) for the Meijer G and Fox H functions and the notation in (3.6) and (3.7) for the p.d.f. and c.d.f. of the EGIG distribution.

Corollary 1. *For the more general case in Theorem 1 we may write the p.d.f. of Z as,*

$$\left\{\prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\prod_{j=1}^{k_{\nu\ell}}\frac{\Gamma\left(a_\nu-\frac{j-m_{\nu\ell}+\sum_{r=1}^{\ell-1}k_{\nu r}}{k_{\nu\ell}}\right)}{\Gamma\left(a_\nu-\frac{j+\sum_{r=1}^{\ell-1}k_{\nu r}}{k_{\nu\ell}}\right)}\right\}G^{p^*,0}_{p^*,p^*}\left(\begin{array}{l}\left\{a_\nu-\frac{j-m_{\nu\ell}+\sum_{r=1}^{\ell-1}k_{\nu r}}{k_{\nu\ell}}-1\right\}_{\substack{\nu=1,\dots,m^*\\ \ell=1,\dots,n_\nu\\ j=1,\dots,k_{\nu\ell}}}\\ \left\{a_\nu-\frac{j+\sum_{r=1}^{\ell-1}k_{\nu r}}{k_{\nu\ell}}-1\right\}_{\substack{\nu=1,\dots,m^*\\ \ell=1,\dots,n_\nu\\ j=1,\dots,k_{\nu\ell}}}\end{array}\middle|\, z\right)$$

$$=\left\{\prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\frac{\Gamma\left(a_\nu k_{\nu\ell}+m_{\nu\ell}-\sum_{r=1}^{\ell}k_{\nu r}\right)}{\Gamma\left(a_\nu k_{\nu\ell}-\sum_{r=1}^{\ell}k_{\nu r}\right)}\right\}H^{p^{**},0}_{p^{**},p^{**}}\left(\begin{array}{l}\left\{(a_\nu k_{\nu\ell}+m_{\nu\ell}-\sum_{r=1}^{\ell}k_{\nu r}-1,k_{\nu\ell})\right\}_{\substack{\nu=1,\dots,m^*\\ \ell=1,\dots,n_\nu}}\\ \left\{(a_\nu k_{\nu\ell}-\sum_{r=1}^{\ell}k_{\nu r}-1,k_{\nu\ell})\right\}_{\substack{\nu=1,\dots,m^*\\ \ell=1,\dots,n_\nu}}\end{array}\middle|\, z\right)$$

$$= f^{GIG}\left(-\log z \,\middle|\, \{\{1\}\}_{\substack{\nu=1:m^* \\ \ell=1:n_\nu \\ i=0:m_{\nu\ell}-1}} ; \left\{\left\{a_\nu + \frac{i-\sum_{r=1}^{\ell} k_{\nu r}}{k_{\nu\ell}}\right\}\right\}_{\substack{\nu=1:m^* \\ \ell=1:n_\nu \\ i=0:m_{\nu\ell}-1}} ; g \leq \sum_{\nu=1}^{m^*}\sum_{\ell=1}^{n_\nu} m_{\nu\ell}\right)\frac{1}{z},$$

where

$$p^* = \sum_{\nu=1}^{m^*}\sum_{\ell=1}^{n_\nu} k_{\nu\ell}, \qquad \text{and} \qquad p^{**} = \sum_{\nu=1}^{m^*} n_\nu,$$

and the c.d.f. by

$$\left\{\prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\prod_{j=1}^{k_{\nu\ell}} \frac{\Gamma\left(a_\nu - \frac{j-m_{\nu\ell}+\sum_{r=1}^{\ell-1} k_{\nu r}}{k_{\nu\ell}}\right)}{\Gamma\left(a_\nu - \frac{j+\sum_{r=1}^{\ell-1} k_{\nu r}}{k_{\nu\ell}}\right)}\right\} G^{p^*,1}_{p^*+1,p^*+1}\left(\begin{array}{c} \left\{1, a_\nu - \frac{j-m_{\nu\ell}+\sum_{r=1}^{\ell-1} k_{\nu r}}{k_{\nu\ell}}\right\}_{\substack{\nu=1,\dots,m^* \\ \ell=1,\dots,n_\nu \\ j=1,\dots,k_{\nu\ell}}} \\ \left\{a_\nu - \frac{j+\sum_{r=1}^{\ell-1} k_{\nu r}}{k_{\nu\ell}}, 0\right\}_{\substack{\nu=1,\dots,m^* \\ \ell=1,\dots,n_\nu \\ j=1,\dots,k_{\nu\ell}}} \end{array} \middle|\, z\right)$$

$$= \left\{\prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu} \frac{\Gamma\left(a_\nu k_{\nu\ell} + m_{\nu\ell} - \sum_{r=1}^{\ell} k_{\nu r}\right)}{\Gamma\left(a_\nu k_{\nu\ell} - \sum_{r=1}^{\ell} k_{\nu r}\right)}\right\} H^{p^{**},1}_{p^{**}+1,p^{**}+1}\left(\begin{array}{c} \left\{(1,1), (a_\nu k_{\nu\ell} + m_{\nu\ell} - \sum_{r=1}^{\ell} k_{\nu r} + k_{\nu\ell} - 1, k_{\nu\ell})\right\}_{\substack{\nu=1,\dots,m^* \\ \ell=1,\dots,n_\nu}} \\ \left\{(a_\nu k_{\nu\ell} - \sum_{r=1}^{\ell} k_{\nu r} - 1 + k_{\nu\ell}, k_{\nu\ell}), (0,1)\right\}_{\substack{\nu=1,\dots,m^* \\ \ell=1,\dots,n_\nu}} \end{array} \middle|\, z\right)$$

$$= 1 - F^{GIG}\left(-\log z \,\middle|\, \{\{1\}\}_{\substack{\nu=1:m^* \\ \ell=1:n_\nu \\ i=0:m_{\nu\ell}-1}} ; \left\{\left\{a_\nu + \frac{i-\sum_{r=1}^{\ell} k_{\nu r}}{k_{\nu\ell}}\right\}\right\}_{\substack{\nu=1:m^* \\ \ell=1:n_\nu \\ i=0:m_{\nu\ell}-1}} ; g \leq \sum_{\nu=1}^{m^*}\sum_{\ell=1}^{n_\nu} m_{\nu\ell}\right),$$

while for the particular case in Theorem 2 we may write the p.d.f. of Z as

$$\left\{\prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\prod_{j=1}^{k_\nu} \frac{\Gamma\left(a_\nu + 1 - \ell + \frac{m_\nu - j}{k_\nu}\right)}{\Gamma\left(a_\nu + 1 - \ell - \frac{j}{k_\nu}\right)}\right\} G^{p^*,0}_{p^*,p^*}\left(\begin{array}{c} \left\{a_\nu + 1 - \ell + \frac{m_\nu - j}{k_\nu} - 1\right\}_{\substack{\nu=1,\dots,m^* \\ \ell=1,\dots,n_\nu \\ j=1,\dots,k_\nu}} \\ \left\{a_\nu + 1 - \ell - \frac{j}{k_\nu} - 1\right\}_{\substack{\nu=1,\dots,m^* \\ \ell=1,\dots,n_\nu \\ j=1,\dots,k_\nu}} \end{array} \middle|\, z\right)$$

$$= \left\{\prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu} \frac{\Gamma\left((a_\nu - \ell)k_\nu + m_\nu\right)}{\Gamma\left((a_\nu - \ell)k_\nu\right)}\right\} H^{p^{**},0}_{p^{**},p^{**}}\left(\begin{array}{c} \left\{((a_\nu - \ell)k_\nu + m_\nu - 1, k_\nu)\right\}_{\substack{\nu=1,\dots,m^* \\ \ell=1,\dots,n_\nu}} \\ \left\{((a_\nu - \ell)k_\nu - 1, k_\nu)\right\}_{\substack{\nu=1,\dots,m^* \\ \ell=1,\dots,n_\nu}} \end{array} \middle|\, z\right)$$

$$= f^{GIG}\left(-\log z \,\middle|\, \{\{r_{\nu j}\}\}_{\substack{\nu=1:m^* \\ j=1:m_\nu+k_\nu(n_\nu-1)}} ; \left\{\left\{a_\nu - n_\nu + \frac{j-1}{k_\nu}\right\}\right\}_{\substack{\nu=1:m^* \\ j=1:m_\nu+k_\nu(n_\nu-1)}} ; g \leq \sum_{\nu=1}^{m^*} m_\nu + k_\nu(n_\nu - 1)\right)\frac{1}{z},$$

and the c.d.f. as

$$\left\{\prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\prod_{j=1}^{k_\nu}\frac{\Gamma\left(a_\nu+1-\ell+\frac{m_\nu-j}{k_\nu}\right)}{\Gamma\left(a_\nu+1-\ell-\frac{j}{k_\nu}\right)}\right\} G^{p^*,1}_{p^*+1,p^*+1}\left(\begin{array}{c|c} \left\{1, a_\nu+1-\ell+\frac{m_\nu-j}{k_\nu}\right\}_{\substack{\nu=1,\dots,m^*\\ \ell=1,\dots,n_\nu\\ j=1,\dots,k_\nu}} & \\ & z \\ \left\{a_\nu+1-\ell-\frac{j}{k_\nu}, 0\right\}_{\substack{\nu=1,\dots,m^*\\ \ell=1,\dots,n_\nu\\ j=1,\dots,k_\nu}} & \end{array}\right)$$

$$= \left\{\prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\frac{\Gamma\left((a_\nu-\ell)k_\nu+m_\nu\right)}{\Gamma\left((a_\nu-\ell)k_\nu\right)}\right\} H^{p^{**},1}_{p^{**}+1,p^{**}+1}\left(\begin{array}{c|c} \left\{(1,1),((a_\nu-\ell)k_\nu+m_\nu-1+k_\nu,k_\nu)\right\}_{\substack{\nu=1,\dots,m^*\\ \ell=1,\dots,n_\nu}} & \\ & z \\ \left\{((a_\nu-\ell)k_\nu-1+k_\nu,k_\nu),(0,1)\right\}_{\substack{\nu=1,\dots,m^*\\ \ell=1,\dots,n_\nu}} & \end{array}\right)$$

$$= 1-F^{GIG}\left(-\log z\,\middle|\,\{\{r_{\nu j}\}\}_{\substack{\nu=1:m^*\\ j=1:m_\nu+k_\nu(n_\nu-1)}};\left\{\left\{a_\nu-n_\nu+\frac{j-1}{k_\nu}\right\}\right\}_{\substack{\nu=1:m^*\\ j=1:m_\nu+k_\nu(n_\nu-1)}};g\le\sum_{\nu=1}^{m^*}m_\nu+k_\nu(n_\nu-1)\right),$$

for

$$p^*=\sum_{\nu=1}^{m^*}n_\nu k_\nu\,,\qquad\text{and}\qquad p^{**}=\sum_{\nu=1}^{m^*}n_\nu\,,$$

and where $0<z\le 1$ *represents the running value of the r.v. Z,* $r_{\nu j}$ *is given by* (2.9)-(2.10)*, and*

$$\{\{r_{\nu j}\}\}_{\substack{\nu=1:m^*\\ j=1:m_\nu+k_\nu(n_\nu-1)}}$$

is the vector which h-th component ($h=1,\dots,g\le\sum_{\nu=1}^{m^*}m_\nu+k_\nu(n_\nu-1)$) *is the sum of all* $r_{\nu j}$ *associated with rate parameters* $a_\nu-n_\nu+\frac{j-1}{k_\nu}$ *with the same value as the h-th component of the vector*

$$\left\{\left\{a_\nu-n_\nu+\tfrac{j-1}{k_\nu}\right\}\right\}_{\substack{\nu=1:m^*\\ j=1:m_\nu+k_\nu(n_\nu-1)}}.$$

Proof. Since this is a Corollary of Theorems 1 and 2 in the previous section, there is indeed no need for a formal proof. One only has to keep in mind the results in those two Theorems and the definition of the Meijer G function in (3.1), or rather, its relation with the expression for the p.d.f. of the product of independent Beta r.v.'s in (3.2), together with the definition of Fox H function in (3.3), or rather, its relation with the expression for the p.d.f. of the product of independent Beta r.v.'s in (3.4), and the definition of the p.d.f. for the GIG distribution in (A.1) in Appendix A together with the notation used in (3.6). ◻

Corollary 2. *For the case in Theorem 3 we may write the p.d.f. of Z, for* $0<z\le 1$*, as*

$$\left\{\prod_{\nu=1}^{m^*}\prod_{\ell=1}^{n_\nu}\prod_{j=1}^{k_\nu}\frac{\Gamma\left(a_\nu+\frac{\ell+s_\nu-1}{n_\nu}\right)}{\Gamma\left(a_\nu-\frac{(\ell-1)k_\nu+j}{n_\nu}\right)}\right\} G^{p^*,0}_{p^*,p^*}\left(\begin{array}{c|c} \left\{a_\nu+\frac{\ell+s_\nu-1}{n_\nu}-1\right\}_{\substack{\nu=1,\dots,m^*\\ \ell=1,\dots,n_\nu\\ j=1,\dots,k_\nu}} & \\ & z \\ \left\{a_\nu-\frac{(\ell-1)k_\nu+j}{n_\nu}-1\right\}_{\substack{\nu=1,\dots,m^*\\ \ell=1,\dots,n_\nu\\ j=1,\dots,k_\nu}} & \end{array}\right)$$

$$= \left\{ \prod_{\nu=1}^{m^*} \prod_{\ell=1}^{n_\nu} \frac{\Gamma(a_\nu n_\nu + s_\nu)}{\Gamma(a_\nu n_\nu - n_\nu j)} \right\} H^{p^{**},0}_{p^{**},p^{**}} \left(\begin{array}{c|c} \left\{(a_\nu n_\nu + s_\nu - 1, k_\nu)\right\}_{\substack{\nu=1,\dots,m^* \\ j=1,\dots,k_\nu}} & \\ & z \\ \left\{(a_\nu n_\nu - n_\nu j - 1, k_\nu)\right\}_{\substack{\nu=1,\dots,m^* \\ j=1,\dots,k_\nu}} & \end{array} \right)$$

$$= f^{GIG}\left(-\log z \,\middle|\, \{\{r_{\nu j}\}\}_{\substack{\nu=1:m^* \\ j=1:n_\nu k_\nu + s_\nu}} ; \left\{\left\{ a_\nu + \frac{s_\nu - j}{n_\nu} \right\}\right\}_{\substack{\nu=1:m^* \\ j=1:n_\nu k_\nu + s_\nu}} ; g \leq \sum_{\nu=1}^{m^*} m_\nu + k_\nu(n_\nu - 1) \right) \frac{1}{z} \,,$$

and the c.d.f. as

$$\left\{ \prod_{\nu=1}^{m^*} \prod_{\ell=1}^{n_\nu} \prod_{j=1}^{k_\nu} \frac{\Gamma\left(a_\nu + \frac{\ell + s_\nu - 1}{n_\nu}\right)}{\Gamma\left(a_\nu - \frac{(\ell-1)k_\nu + j}{n_\nu}\right)} \right\} G^{p^*,1}_{p^*+1,p^*+1} \left(\begin{array}{c|c} \left\{1, a_\nu + \frac{\ell + s_\nu - 1}{n_\nu}\right\}_{\substack{\nu=1,\dots,m^* \\ \ell=1,\dots,n_\nu \\ j=1,\dots,k_\nu}} & \\ & z \\ \left\{a_\nu - \frac{(\ell-1)k_\nu + j}{n_\nu}, 0\right\}_{\substack{\nu=1,\dots,m^* \\ \ell=1,\dots,n_\nu \\ j=1,\dots,k_\nu}} & \end{array} \right)$$

$$= \left\{ \prod_{\nu=1}^{m^*} \prod_{\ell=1}^{n_\nu} \frac{\Gamma(a_\nu n_\nu + s_\nu)}{\Gamma(a_\nu n_\nu - n_\nu j)} \right\} H^{p^{**},1}_{p^{**}+1,p^{**}+1} \left(\begin{array}{c|c} \left\{(1,1)(a_\nu n_\nu + s_\nu - 1 + k_\nu, k_\nu)\right\}_{\substack{\nu=1,\dots,m^* \\ j=1,\dots,k_\nu}} & \\ & z \\ \left\{(a_\nu n_\nu - n_\nu j - 1 + k_\nu, k_\nu), (0,1)\right\}_{\substack{\nu=1,\dots,m^* \\ j=1,\dots,k_\nu}} & \end{array} \right)$$

$$= 1 - F^{GIG}\left(-\log z \,\middle|\, \{\{r_{\nu j}\}\}_{\substack{\nu=1:m^* \\ j=1:n_\nu k_\nu + s_\nu}} ; \left\{\left\{ a_\nu + \frac{s_\nu - j}{n_\nu} \right\}\right\}_{\substack{\nu=1:m^* \\ j=1:n_\nu k_\nu + s_\nu}} ; g \leq \sum_{\nu=1}^{m^*} m_\nu + k_\nu(n_\nu - 1) \right),$$

for

$$p^* = \sum_{\nu=1}^{m^*} n_\nu k_\nu \,, \qquad \text{and} \qquad p^{**} = \sum_{\nu=1}^{m^*} k_\nu \,,$$

$r_{\nu j}$ *given by* (2.15) *and where*

$$\{\{r_{\nu j}\}\}_{\substack{\nu=1:m^* \\ j=1:m_\nu + k_\nu(n_\nu - 1)}}$$

has a similar definition to the corresponding vector in Corollary 1.

Proof. As for Corollary 1, there is no need for a formal proof. It is enough to keep in mind the same remarks made in the proof of Corollary 1. □

4. Examples of l.r.t. statistics whose distributions are represented by the products in Section 2

Some particular cases of the products of independent Beta r.v.'s in Theorems 1-3 in Section 2 yield the exact distribution of a number of l.r.t. statistics used in Multivariate Analysis. In this section we present some of these statistics.

4.1. Likelihood ratio test statistics whose distributions correspond to the products in Theorems 1 and 2 and that have p.d.f. and c.d.f. given by Corollary 1

The products of independent Beta r.v.'s in Theorems 1 and 2, yield,

- for $m^* = 1$, $k_1 = 2$, and $a_1 = \frac{n-p_2}{2}$, for some positive integer $n > p_2 + 2n_1$, the exact distribution of the $(2/n)$-th power of the l.r.t. statistic to test the independence of two groups of variables, with $p_1 = 2n_1$ and $p_2 = m_1$ variables, based on a sample of size n from a multivariate Normal, elliptically contoured or left orthogonal-invariant distribution, or the distribution of the $(2/n)$-th power of the l.r.t. statistic to test the equality of $p_2 + 1$ mean vectors of dimension p_1, from a multivariate Normal, based on a set of $p_2 + 1$ independent samples with a combined size of n, or, equivalently, the $(2/n)$-th power of the l.r.t. statistic to test the equality of $p_1 + 1$ mean vectors of dimension p_2, based on a set of $p_1 + 1$ independent samples with a combined size of n from a set of multivariate Normal, elliptically contoured or left orthogonal-invariant distributions, assuming the equality of the population covariance matrices, since any of these statistics has, under the corresponding null hypothesis, the same distribution as (using the same 'abuse of notation' used in Theorems 1-3)

$$\prod_{j=1}^{p_1} Beta\left(\frac{n-p_2-j}{2}, \frac{p_2}{2}\right) \stackrel{\text{d}}{\equiv} \prod_{j=1}^{p_2} Beta\left(\frac{n-p_1-j}{2}, \frac{p_1}{2}\right),$$

  where all the Beta distributed r.v.'s are independent (see Anderson (2003, Chap. 8,9); Anderson et al. (1986); Anderson and Fang (1990); Jensen and Good (1981); Kariya (1981); Marques et al. (2011); Arnold et al. (2012)); the p.d.f. and c.d.f. are thus given by Corollary 1 (see Coelho (1998, 1999));

- for $k_\nu = 2$, $m_\nu = 2(n_{\nu+1} + \ldots + n_{m^*}) + p_{m^*+1}$ and $a_\nu = \frac{n-m_\nu}{2}$, for $\nu = 1, \ldots, m^*$, where p_{m^*+1} and n are positive integers, with $n > p_{m^*+1} + 2\sum_{\nu=1}^{m^*} n_\nu$, the exact distribution of the $(2/n)$-th power of the likelihood ratio test statistic used to test the independence of $m^* + 1$ sets of variables, the ν-th of which has $p_\nu = 2n_\nu$ variables ($\nu = 1, \ldots, m^*$) and the last one p_{m^*+1} variables, based on a sample of size n from a multivariate Normal, elliptically contoured or left orthogonal-invariant distribution, since this statistic has, under the null hypothesis of independence of the $m^* + 1$ sets of variables, the same distribution as (using the same 'abuse of notation' used in Theorems 1-3)

$$\prod_{\nu=1}^{m^*} \prod_{j=1}^{p_\nu} Beta\left(\frac{n-m_\nu-j}{2}, \frac{m_\nu}{2}\right),$$

  where all the Beta distributed r.v.'s are independent (see Anderson (2003, Chap. 9); Anderson et al. (1986); Anderson and Fang (1990); Jensen and Good (1981); Kariya (1981); Marques et al. (2011); Arnold et al. (2012)); the p.d.f. and c.d.f. are thus given by Corollary 1; in this case, given the relations that exist between a_ν, m_ν and n_ν, it is even possible to obtain simpler expressions for the rate and shape parameters, with the rate parameters given by

$$\left\{\left\{a_\nu - n_\nu + \frac{j-1}{k_\nu}\right\}\right\}_{\substack{\nu=1:m^* \\ j=1:m_\nu+k_\nu(n_\nu-1)}} = \left\{\frac{n-j-2}{2}\right\}_{j=1:p-2}$$

  and shape parameters

$$r_j = \begin{cases} h_j + (-1)^j & j = 1, 2 \\ r_{j-2} + h_j & j = 3, \ldots, p-2 \end{cases}$$

with

$$h_j = (\# \text{ of } p_\nu \, (\nu = 1, \dots, m^* + 1) \geq j) - 1\,, \qquad j = 1, \dots, p-2\,,$$

where $p = \sum_{\nu=1}^{m^*+1} p_\nu$ (see Coelho (1998, 1999); Marques et al. (2011));

4.2. Likelihood ratio test statistics whose distributions correspond to the product in Theorem 3 and that have p.d.f. and c.d.f. given by Corollary 2

The product of independent Beta r.v.'s in Theorem 3 yields,

- for $m^* = 1$, $n_1 = 2$, $k_1 = \frac{p-1}{2}$, $a_1 = \frac{n-1}{2}$ and $s_1 = 0$, the exact distribution of the $(2/n)$-th power of the l.r.t. statistic to test circularity of the covariance matrix, based on a sample of size n from a p-multivariate Normal distribution, for the case of odd p (see Sec. 3.3 of Olkin and Press (1961));
- for $m^* = 1$, $n_1 = 2$, $k_1 = \frac{p-1}{2}$, $a_1 = \frac{n-1}{2}$ and $s_1 = 1$, the exact distribution of the $(2/n)$-th power of the l.r.t. statistic to test simultaneously the circularity of the covariance matrix and the equality of the p means, based on a sample of size n from a p-multivariate Normal distribution, for the case of odd p (see Sec. 5.2 of Olkin and Press (1961));
- for $n_\nu = 2$, $k_\nu = \frac{p_\nu-1}{2}$, $a_\nu = \frac{n-1}{2}$ and $s_\nu = 0$ $(\nu = 1, \dots, m^*)$, the exact distribution of the $(2/n)$-th power of the l.r.t. statistic to test circularity of the m^* $p_\nu \times p_\nu$ covariance matrices in a set of m^* subsets of variables with a joint p-multivariate Normal distribution ($p = \sum_{\nu=1}^{m^*} p_\nu$), the ν-th subset with p_ν variables, assuming the independence of these m^* subsets of variables, with all p_ν odd, and based on a sample of size n;
- for $n_\nu = 2$, $k_\nu = \frac{p_\nu-1}{2}$, $a_\nu = \frac{n-1}{2}$ and $s_\nu = 1$ $(\nu = 1, \dots, m^*)$, the exact distribution of the $(2/n)$-th power of the l.r.t. statistic to test simultaneously the circularity of the m^* $p_\nu \times p_\nu$ covariance matrices and the equality of the p_ν means in each subset of a set of m^* subsets of variables with a joint p-multivariate Normal distribution ($p = \sum_{\nu=1}^{m^*} p_\nu$), the ν-th subset with p_ν variables, assuming the independence of these m^* subsets of variables, with all p_ν odd, and based on a sample of size n.

4.3. Likelihood ratio test statistics whose distributions correspond to a multiplication of the products in Theorems 1 or 2 and in Theorem 3 and that have p.d.f. and c.d.f. given by a combination of the expressions in Corollaries 1 and 2

There are several likelihood ratio test statistics, whose exact distribution may be obtained from the product of the products of independent Beta r.v.'s in Theorems 1 or 2 and the product in Theorem 3. The exact p.d.f. and c.d.f. of these statistics is still the p.d.f. and c.d.f. of the exponential of a GIG distributed r.v., given by the adequate combination of the p.d.f.'s and c.d.f.'s given in Corollaries 1 and 2 in Section 3, that is, by building one only p.d.f. or c.d.f. by adequately merging the vectors of rate parameters and then adding the corresponding shape parameters. Some of these statistics are:

- the $(2/n)$-th power of the l.r.t. statistic to test, for even p, simultaneously the circularity of the covariance matrix and equality of the p means, based on a sample of size n from a p-multivariate Normal distribution, whose exact distribution is obtained from the multiplication of the product in Theorem 3 for $m^* = 1$, $n_1 = 2$, $k_1 = \frac{p}{2} - 1$, $a_1 = \frac{n-1}{2}$ and $s_1 = 2$ by the product in Theorems 1 or 2 for $m^* = 1$, $n_1 = 1$, $k_1 = 2$, $a_1 = \frac{n+1}{2}$ and $m_1 = 1$, with p.d.f. and c.d.f. respectively given by

$$f^{GIG}\left(-\log z \,\middle|\, \{r_j\}_{j=1:p}; \left\{\frac{n+1-j}{2}\right\}_{j=1:p}; p\right)\frac{1}{z}$$

and

$$1-F^{GIG}\left(-\log z\,\middle|\,\{r_j\}_{j=1:p};\left\{\frac{n+1-j}{2}\right\}_{j=1:p};p\right),$$

where

$$r_j=\begin{cases}\frac{p}{2}-1 & j=1\\ \frac{p}{2}+\left\lfloor\frac{2-j}{2}\right\rfloor & j=2,\dots,p\,;\end{cases}$$

- the $(2/n)$-th power of the l.r.t. statistic, based on a sample of size n, used to test simultaneously the circularity of the $p_\nu\times p_\nu$ covariance matrices and the equality of the p_ν means from each of m^* subsets, the ν-th of which with p_ν variables ($\nu=1,\dots,m^*$), of a set with a joint $(\sum_{\nu=1}^{m^*}p_\nu)$-multivariate Normal distribution, where we assume all p_ν even and the independence of all these m^* subsets of variables; we obtain the exact distribution of this statistic from the multiplication of the product in Theorem 3, for $n_\nu=2$, $k_\nu=\frac{p_\nu}{2}-1$, $a_\nu=\frac{n-1}{2}$ and $s_\nu=2$ ($\nu=1,\dots,m^*$), and the product in Theorem 2 for $n_\nu=1$, $k_\nu=2$, $a_\nu=\frac{n+1}{2}$ and $m_\nu=1$ ($\nu=1,\dots,m^*$), with p.d.f. and c.d.f. respectively given by

$$f^{GIG}\left(-\log z\,\middle|\,\{r_j\}_{j=1:\max\limits_{1\le\nu\le m^*}p_\nu};\left\{\frac{n+1-j}{2}\right\}_{j=1:\max\limits_{1\le\nu\le m^*}p_\nu};\max_{1\le\nu\le m^*}p_\nu\right)\frac{1}{z} \tag{4.1}$$

and

$$1-F^{GIG}\left(-\log z\,\middle|\,\{r_j\}_{j=1:\max\limits_{1\le\nu\le m^*}p_\nu};\left\{\frac{n+1-j}{2}\right\}_{j=1:\max\limits_{1\le\nu\le m^*}p_\nu};\max_{1\le\nu\le m^*}p_\nu\right), \tag{4.2}$$

where

$$r_j=\sum_{\nu=1}^{m^*}r_{\nu j}\qquad (j=1,\dots,\max_{1\le\nu\le m^*}p_\nu)\,,$$

with,

$$r_{\nu j}=\begin{cases}\frac{p_\nu}{2}-1 & j=1\\ \frac{p_\nu}{2}+\left\lfloor\frac{2-j}{2}\right\rfloor & j=2,\dots,p_\nu\\ 0 & j=p_\nu+1,\dots,\max_{1\le\nu\le m^*}p_\nu\,;\end{cases} \tag{4.3}$$

- the $(2/n)$-th power of the l.r.t. statistic, based on a sample of size n, used to test simultaneously the circularity of the $p_\nu\times p_\nu$ covariance matrices and the equality of the p_ν means from each of $q=q_1+q_2$ subsets, the ν-th of which with p_ν variables ($\nu=1,\dots,q$), of a set with a joint $(\sum_{\nu=1}^{q}p_\nu)$-multivariate Normal distribution, where we assume, without any loss of generality, the first q_1 of the p_ν's even and the last q_2 of them odd, and the independence of all these q subsets of variables; we obtain the exact distribution of this statistic from the multiplication of a first product of the type of the one in Theorem

3, with $m^* = q_1$, $n_\nu = 2$, $k_\nu = \frac{p_\nu}{2}$, $a_\nu = \frac{n-2}{2}$ and $s_\nu = 2$ ($\nu = 1, \ldots, q_1$), a second product of this type, with $m^* = q_2$, $n_\nu = 2$, $k_\nu = \frac{p_{\nu+q_1}-1}{2}$, $a_\nu = \frac{n-1}{2}$ and $s_\nu = 1$ ($\nu = 1, \ldots, q_2$), and also a third product of the type of the one in Theorem 2, with $m^* = q_2$, $n_\nu = 1$, $k_\nu = 2$, $a_\nu = \frac{n}{2}$ and $m_\nu = 1$ ($\nu = 1, \ldots, q_2$), with p.d.f. and c.d.f. respectively given by (4.1) and (4.2), with depth $\max_{1\leq\nu\leq q} p_\nu$ and

$$r_j = \sum_{\nu=1}^{q_1} r_{\nu j} + \sum_{\nu=1}^{q_2} r^*_{\nu j} \qquad (j = 1, \ldots, \max_{1\leq\nu\leq q} p_\nu)\,,$$

where $r_{\nu j}$ ($j = 1, \ldots, \max_{1\leq\nu\leq q} p_\nu$) are given by (4.3), with m^* replaced by q, and

$$r^*_{\nu j} = \begin{cases} \frac{p_\nu-1}{2} - \left\lfloor \frac{|2-j|}{2} \right\rfloor & j = 1, \ldots, p_\nu \\ 0 & j = p_\nu + 1, \ldots, \max_{1\leq\nu\leq q} p_\nu\,; \end{cases} \qquad (\nu = 1, \ldots, q_2)\,;$$

- the $(2/n)$-th power of the l.r.t. statistic used to test, based on a sample of size n, simultaneously: (i) the independence of q sets of variables, the ν-th of which with p_ν variables ($\nu = 1, \ldots, q$), (ii) the circularity of each of the q covariance matrices of dimension $p_\nu \times p_\nu$, and (iii) the equality of the p_ν means in the mean vector of each of the q subsets of a set with a joint $\left(\sum_{\nu=1}^{q} p_\nu\right)$-multivariate Normal distribution, assuming all the p_ν even and $n > \sum_{\nu=1}^{q} p_\nu$, whose exact distribution is obtained from the product of two instances of the product of independent Beta r.v.'s in Theorem 2, one with $m^* = q-1$, $k_\nu = 2$, $n_\nu = p_\nu/2$, $m_\nu = p_{\nu+1} + \ldots + p_q$, $a_\nu = \frac{n-m_\nu}{2}$ for $\nu = 1, \ldots, q-1$, and the other one with $m^* = q$, $k_\nu = 2$, $n_\nu = 1$, $m_\nu = 1$ and $a_\nu = \frac{n+1}{2}$ for $\nu = 1, \ldots, q$, multiplied by one instance of the product in Theorem 3 with $m^* = q$, $n_\nu = 2$, $k_\nu = \frac{p_\nu}{2} - 1$, $s_\nu = 2$ and $a_\nu = \frac{n-1}{2}$, for $\nu = 1, \ldots, q$;
- the $(2/n)$-th power of the l.r.t. statistic used for a similar test to the one described above, in a situation where all but one of the q subsets of variables have an even number of variables, assuming, without any loss of generality, that the subset with an odd number of variables is the q-th subset, whose exact distribution is given by the product of two instances of the product in Theorem 2, the first one with the same parameters as the corresponding instance above, and a second instance, also with the same parameters as the corresponding instance above, but now only for $\nu = 1, \ldots, q-1$, and two instances of the product in Theorem 3, the first one being similar to the instance of the corresponding product considered above, now for $\nu = 1, \ldots, q-1$ and another instance with $m^* = 1$, $k_1 = \frac{p_q-1}{2}$, $n_1 = 2$, $s_1 = 1$ and $a_1 = (n-1)/2$.

5. Conclusions

Through a technique that uses some detailed work on the c.f. of the negative logarithm of two multiple products of independent Beta r.v.'s whose parameters are all rational and follow some rules we were able to establish a number of situations where the Meijer G function has finite closed form representations through the expressions for the p.d.f. and c.d.f. of the GIG distribution, or, more precisely, through the p.d.f. and c.d.f. of what Arnold et al. (2012) call an EGIG (Exponentiated GIG) distribution.

By using two extended multiplication formulas for the Gamma function the authors also show that these instances of the Meijer G function are indeed equivalent to Fox H functions with integer b_j parameters. Such instances of Fox's H function are shown to have always finite closed form representations through the p.d.f. or the c.d.f. of the EGIG distribution.

Particular cases of the multiple products of independent Beta r.v.'s in Theorems 1-3 in Section 2 yield the exact distribution of a number of l.r.t. statistics whose p.d.f.'s and c.d.f.'s can thus be expressed in finite closed forms, through the p.d.f. and c.d.f. of EGIG distributions.

These finite closed form representations are much more manageable than the representations obtained through the use of Meijer G or Fox H functions and entail enormous gains in terms of computation time and sometimes also in precision, as it may be seen by analyzing the tables in Appendix B. As such, these representations for the c.d.f.'s enable a quick, easy and accurate computation of exact quantiles and p-values. Mathematica® and R modules to implement the GIG/EGIG p.d.f. and c.d.f. are provided in Appendix C, together with an example.

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Appendix A. Notation and expressions for the probability density and cumulative distribution functions of the Gamma, GIG and GNIG distributions

This appendix is used to settle the notation used concerning the p.d.f. of the GIG (Generalized Integer Gamma) distribution.

We say that the r.v. (random variable) X has a Gamma distribution with shape parameter $r\,(>0)$ and rate parameter $\lambda\,(>0)$, and we will denote this fact by $X \sim \Gamma(r,\lambda)$, if the probability density function of X is

$$f_X(x) = \frac{\lambda^r}{\Gamma(r)}\, e^{-\lambda x}\, x^{r-1} \quad (x>0)\,.$$

Let $X_j \sim \Gamma(r_j,\lambda_j)$ $(j=1,\dots,p)$ be a set of p independent r.v.'s and consider the r.v.

$$Z = \sum_{j=1}^{p} X_j\,.$$

In case all the $r_j \in \mathbb{N}$, the distribution of Z is what we call a GIG distribution (Coelho, 1998, 1999). If all the λ_j are different, Z has a GIG distribution of depth p, with shape parameters r_j and rate parameters λ_j, with p.d.f.

$$f_Z(z) = f^{GIG}(z\,|\,\{r_j\}_{j=1:p};\{\lambda_j\}_{j=1:p};p) = K\sum_{j=1}^{p} P_j(z)\, e^{-\lambda_j z}\,, \quad (z>0) \tag{A.1}$$

and c.d.f.

$$F_Z(z) = F^{GIG}(z\,|\,\{r_j\}_{j=1:p};\{\lambda_j\}_{j=1:p};p) = 1 - K\sum_{j=1}^{p} P_j^*(z)\, e^{-\lambda_j z}\,, \quad (z>0)$$

where

$$K = \prod_{j=1}^{p} \lambda_j^{r_j}\,, \qquad P_j(z) = \sum_{k=1}^{r_j} c_{j,k}\, z^{k-1}$$

and

$$P_j^*(z) = \sum_{k=1}^{r_j} c_{j,k}(k-1)! \sum_{i=0}^{k-1} \frac{z^i}{i!\,\lambda_j^{k-i}}\,,$$

with

$$c_{j,r_j} = \frac{1}{(r_j-1)!} \prod_{\substack{i=1\\i\neq j}}^{p} (\lambda_i - \lambda_j)^{-r_i}\,, \quad j=1,\ldots,p\,, \tag{A.2}$$

and

$$c_{j,r_j-k} = \frac{1}{k} \sum_{i=1}^{k} \frac{(r_j-k+i-1)!}{(r_j-k-1)!} R(i,j,p) c_{j,r_j-(k-i)}\,, \quad (k=1,\ldots,r_j-1;\; j=1,\ldots,p) \tag{A.3}$$

where

$$R(i,j,p) = \sum_{\substack{k=1\\k\neq j}}^{p} r_k(\lambda_j-\lambda_k)^{-i} \quad (i=1,\ldots,r_j-1)\,. \tag{A.4}$$

In case some of the λ_j assume the same value as other λ_j's, the distribution of Z still is a GIG distribution, but in this case with a reduced depth. In this more general case, let $\{\lambda_\ell; \ell=1,\ldots,g(\leq p)\}$ be the set of different λ_j's and let $\{r_\ell; \ell=1,\ldots,g(\leq p)\}$ be the set of the corresponding shape parameters, with r_ℓ being the sum of all r_j ($j\in\{1,\ldots,p\}$) which correspond to the λ_j assuming the value λ_ℓ. In this case Z will have a GIG distribution of depth g, with shape parameters r_ℓ and rate parameters λ_ℓ ($\ell=1,\ldots,g$).

Appendix B. Tables with computation times and values for the Meijer G function and the EGIG p.d.f. for the results in Corollaries 1 and 2

All computations were done with Mathematica®, version 7.0.0, as the only software running on an Intel® Core(TM)2 Duo CPU P8700 at 2.53GHz, with 4GB of RAM, running the Windows 7 Home Premium 64-bit operating system.

All computing times are reported in seconds and all computing times greater than 1 hour are reported as >3600 (seconds).

In Tables B.2 and B.4 the following abbreviations were used:

Scen.	–	Scenario
p.d.	–	precision digits
c. time	–	computing time
(b)/(a)	–	the ratio of the computing times for the Mathematica implementations of the Meijer G function and the GIG/EGIG p.d.f. (it is not computed when the result obtained with the Meijer G function implementation is clearly wrong)

We may easily see the huge gains in computing time obtained with the GIG/EGIG implementation and also the gain in precision for some of the scenarios in Tables B.1 and B.2.

Table B.1 – Description of the scenarios used for the result in Corollary 1

Scen.	m^*	n_ν and a_ν	$k_{\nu\ell}$ and $m_{\nu\ell}$	z
I.1	3	$n_\nu = \{3, 5, 4\}$	$k_{\nu\ell} = \{\{3, 1, 4\}, \{2, 5, 3, 2, 2\}, \{4, 1, 2, 5\}\}$	2.0
		$a_\nu = \{16.3, 18.6, 15.3\}$	$m_{\nu\ell} = \{\{2, 2, 3\}, \{1, 5, 2, 4, 3\}, \{1, 3, 4, 5\}\}$	
I.2	1	$n_\nu = \{3\}$	$k_{\nu\ell} = \{\{3, 1, 4\}\}$	1.1
		$a_\nu = \{16.3\}$	$m_{\nu\ell} == \{\{2, 2, 3\}\}$	
I.3	3	$n_\nu = \{3, 4, 2\}$	$k_{\nu\ell} = \{\{2, 2, 2\}, \{2, 2, 2, 2\}, \{2, 2\}\}$	70.5
		$a_\nu = \{4, 8, 10\}$	$m_{\nu\ell} = \{\{17, 17, 17\}, \{9, 9, 9, 9\}, \{5, 5\}\}$	
I.4	2	$n_\nu = \{3, 5\}$	$k_{\nu\ell} = \{\{6, 6, 6\}, \{3, 4, 5, 6, 7\}\}$	0.4
		$a_\nu = \{45.3, 25.6\}$	$m_{\nu\ell} = \{\{2, 2, 3\}, \{2, 2, 3, 3, 4\}\}$	
I.5	2	$n_\nu = \{3, 4\}$	$k_{\nu\ell} = \{\{6, 6, 6\}, \{3, 3, 3, 4\}\}$	0.4
		$a_\nu = \{45.3, 12.3\}$	$m_{\nu\ell} = \{\{2, 2, 3\}, \{4, 4, 5, 3\}\}$	
I.6	3	$n_\nu = \{3, 2, 3\}$	$k_{\nu\ell} = \{\{6, 6, 6\}, \{3, 3\}, \{3, 4, 5\}\}$	0.4
		$a_\nu = \{45.3, 12.3, 22.3\}$	$m_{\nu\ell} = \{2, 2, 3\}, \{4, 4\}, \{2, 1, 3\}\}$	
I.7	3	$n_\nu = \{3, 5, 4\}$	$k_{\nu\ell} = \{\{3, 1, 4\}, \{2, 5, 3, 2, 2\}, \{4, 1, 2, 5\}\}$	0.2
		$a_\nu = \{16.3, 18.6, 15.3\}$	$m_{\nu\ell} = \{\{2, 2, 3\}, \{1, 5, 2, 4, 3\}, \{1, 3, 4, 5\}\}$	
I.8	3	$n_\nu = \{3, 2, 2\}$	$k_{\nu\ell} = \{\{3, 1, 4\}, \{2, 5\}, \{4, 1\}\}$	0.2
		$a_\nu = \{12.3, 8.6, 5.3\}$	$m_{\nu\ell} = \{\{2, 2, 3\}, \{1, 5\}, \{1, 3\}\}$	
I.9	3	$n_\nu = \{3, 2, 2\}$	$k_{\nu\ell} = \{\{6, 6, 6\}, \{3, 3\}, \{3, 2\}\}$	0.4
		$a_\nu = \{45.3, 12.3, 22.3\}$	$m_{\nu\ell} = \{\{2, 2, 3\}, \{4, 4\}, \{5, 3\}\}$	
I.10	3	$n_\nu = \{3, 2, 3\}$	$k_{\nu\ell} = \{\{6, 6, 6\}, \{3, 3\}, \{1, 2, 3\}\}$	0.4
		$a_\nu = \{45.3, 12.3, 22.3\}$	$m_{\nu\ell} = \{\{2, 2, 3\}, \{4, 4\}, \{2, 3, 4\}\}$	
I.11	3	$n_\nu = \{3, 2, 3\}$	$k_{\nu\ell} = \{\{3, 4, 6\}, \{2, 3\}, \{1, 2, 3\}\}$	0.4
		$a_\nu = \{45.3, 12.3, 22.3\}$	$m_{\nu\ell} = \{\{4, 2, 3\}, \{4, 5\}, \{2, 3, 4\}\}$	
I.12	4	$n_\nu = \{3, 4, 3, 5\}$	$k_{\nu\ell} = \{\{3, 4, 6\}, \{2, 3, 3, 5\}, \{1, 2, 3\}, \{2, 3, 4, 5, 2\}\}$	0.4
		$a_\nu = \{45.3, 12.3, 22.3, 67.3\}$	$m_{\nu\ell} = \{\{4, 2, 3\}, \{4, 5, 2, 3\}, \{2, 3, 4\}, \{3, 4, 7, 2, 3\}\}$	
I.13	5	$n_\nu = \{3, 4, 3, 5, 4\}$	$k_{\nu\ell} = \{\{3, 4, 6\}, \{2, 3, 3, 5\}, \{1, 2, 3\}, \{2, 3, 4, 5, 2\}, \{1, 2, 3, 4\}\}$	0.2
		$a_\nu = \{45.3, 12.3, 22.3, 67.3, 34.5\}$	$m_{\nu\ell} = \{\{4, 2, 3\}, \{4, 5, 2, 3\}, \{2, 3, 4\}, \{3, 4, 7, 2, 3\}, \{4, 3, 2, 1\}\}$	

Table B.2 – Computing times (in seconds) for the Meijer G function and the EGIG p.d.f. implementations in Mathematica, for the scenarios in Table B.1, relative to the results in Corollary 1

	EGIG			Meijer G			
Scen.	p.d.	c. time (a)	result	p.d.	c. time (b)	result	(b)/(a)
I.1	60	0.016	$1.1359433007676998\times10^{+00}$	1000	5.522	0.0×10^{-960}	——
I.2	27	< 0.010	$4.4643963452245869\times10^{-01}$	1000	0.047	0.0×10^{-992}	——
I.3	92	0.047	$2.5922538594634323\times10^{-01}$	92	737.182	0.0×10^{-15}	——
I.4	33	0.016	$3.4123807297252911\times10^{+00}$	34	915.101	$3.4123807297252911\times10^{+00}$	57,193.8
I.5	35	0.015	$2.3420495104295541\times10^{-01}$	35	103.507	$2.3420495104295541\times10^{-01}$	6,900.5
I.6	26	0.016	$2.9172600291608775\times10^{+00}$	27	2945.330	$2.9172600291608775\times10^{+00}$	184,083.1
I.7	49	0.015	$3.2209740218339274\times10^{-01}$	49	(> 3600)	———	(> 240,000.0)
I.8	25	< 0.010	$7.2492494683426487\times10^{-02}$	25	2538.480	$7.2492494683426487\times10^{-02}$	> 253,848.0
I.9	26	< 0.010	$2.1427071831326017\times10^{+00}$	26	15.522	$2.1427071831326017\times10^{+00}$	> 1,552.2
I.10	27	< 0.010	$1.6978283154473830\times10^{+00}$	28	21.325	$1.6978283154473830\times10^{+00}$	> 2,132.5
I.11	30	0.015	$8.9124157730052532\times10^{-01}$	31	941.154	$8.9124157730052532\times10^{-01}$	62,743.6
I.12	39	0.031	$1.2110730587266316\times10^{-04}$	39	(> 3600)	———	(> 116,129.0)
I.13	35	0.047	$2.2636145605264793\times10^{-01}$	35	(> 3600)	———	(> 76,595.7)

Table B.3 – Table with the scenarios used for the result in Corollary 2

Scenario	m^*	n_ν	a_ν	k_ν	s_ν	z
II.1	2	{2,2}	{45.3,12.3}	{3,2}	{2,3}	0.40
II.2	2	{4,2}	{45.3,12.3}	{3,5}	{2,3}	0.10
II.3	3	{4,2,2}	{45.3,12.3,22.3}	{3,4,5}	{2,3,3}	0.05
II.4	3	{3,3,3}	{45.3,12.3,22.3}	{4,4,4}	{2,2,2}	0.05
II.5	3	{3,2,2}	{45.3,12.3,22.3}	{2,3,2}	{2,2,2}	0.40
II.6	3	{3,2,2}	{45.3,12.3,22.3}	{2,3,2}	{2,4,2}	0.40
II.7	3	{3,2,3}	{45.3,12.3,22.3}	{2,3,2}	{2,4,2}	0.40
II.8	3	{3,2,4}	{45.3,12.3,22.3}	{2,3,2}	{2,4,2}	0.40
II.9	3	{3,2,5}	{45.3,12.3,22.3}	{2,3,2}	{2,4,2}	0.40
II.10	4	{3,2,5,6}	{45.3,12.3,22.3,34.2}	{5,4,5,6}	{2,4,3,6}	10^{-5}
II.11	4	{3,2,2,3}	{45.3,12.3,22.3,34.2}	{5,4,3,2}	{2,4,3,6}	10^{-4}
II.12	4	{3,2,2,3}	{45.3,12.3,22.3,34.2}	{2,3,3,2}	{2,3,3,2}	0.01
II.13	4	{3,2,4,3}	{45.3,12.3,22.3,34.2}	{2,3,3,4}	{2,3,3,4}	0.01
II.14	4	{3,2,4,3}	{45.3,12.3,22.3,34.2}	{2,3,3,2}	{2,3,3,2}	0.01
II.15	5	{3,2,5,6,7}	{45.3,12.3,22.3,34.2,76.8}	{5,4,5,6,7}	{2,4,3,6,7}	10^{-5}
II.16	3	{3,2,3}	{45.3,12.3,12.3}	{2,3,2}	{2,4,2}	0.10
II.17	3	{3,2,3}	{45.3,12.3,12.3}	{2,3,3}	{2,4,2}	0.10
II.18	3	{3,4,3}	{45.3,12.3,12.3}	{2,3,2}	{2,4,2}	0.10

Table B.4 – Computing times (in seconds) for the Meijer G function and the EGIG p.d.f. implementations in Mathematica, for the scenarios in Table B.3, relative to the results in Corollary 2

	EGIG			Meijer G			
Scen.	p.d.	c. time (a)	result	p.d.	c. time (b)	result	(b)/(a)
II.1	30	0.016	$8.7930569156048830\times10^{-01}$	31	0.561	$8.7930569156048830\times10^{-01}$	35.1
II.2	61	0.031	$2.6185313989828837\times10^{-05}$	63	380.845	$2.6185313989828837\times10^{-05}$	12,285.3
II.3	66	0.063	$6.7325883118341056\times10^{-06}$	66	1298.900	$6.7325883118341056\times10^{-06}$	20,617.5
II.4	73	0.078	$7.4514940226549799\times10^{-08}$	74	11.154	$7.4514940226549799\times10^{-08}$	143.0
II.5	45	0.016	$7.0684660196762765\times10^{-05}$	45	19.297	$2.3442049510429554\times10^{-01}$	1,206.1
II.6	53	0.016	$9.7772597037140489\times10^{-08}$	53	16.910	$9.7772597037140489\times10^{-08}$	1,056.9
II.7	55	0.016	$6.2226599844952503\times10^{-09}$	56	46.520	$6.2226599844952503\times10^{-09}$	2,907.5
II.8	57	0.031	$3.3607577941794188\times10^{-10}$	58	117.827	$3.3607577941794188\times10^{-10}$	3,800.9
II.9	60	0.031	$1.5483311810263276\times10^{-11}$	60	380.315	$1.5483311810263276\times10^{-11}$	12,268.2
II.10	145	0.608	$8.0216912447426785\times10^{+02}$	145	(> 3600)	———	(> 5,921.1)
II.11	32	0.109	$2.4354016174353771\times10^{-01}$	32	(> 3600)	———	(> 33,027.5)
II.12	32	0.031	$5.7301671577955366\times10^{+00}$	32	367.258	$5.7301671577955366\times10^{+00}$	11,847.0
II.13	52	0.078	$5.9256771376197300\times10^{+01}$	52	(> 3600)	———	(> 46,153.8)
II.14	37	0.062	$3.7925500531027583\times10^{+01}$	37	1856.320	$3.7925500531027583\times10^{+01}$	29,940.6
II.15	216	2.012	$9.8955059987453691\times10^{-09}$	216	(> 3600)	———	(> 1,789.3)
II.16	55	0.016	$5.3493904518838389\times10^{-01}$	55	106.252	$5.3493904518838389\times10^{-01}$	6,640.8
II.17	71	0.031	$8.4934555803875667\times10^{-04}$	71	(> 3600)	———	(> 116,129.0)
II.18	75	0.015	$3.9150062308329812\times10^{-04}$	75	3268.390	$3.9150062308329812\times10^{-04}$	217,892.7

Appendix C. Modules for the computation of the GIG and EGIG p.d.f. and c.d.f.

Appendix C.1. Mathematica® modules for the computation of the GIG/EGIG p.d.f. and c.d.f.

In this part of the Appendix we present the Mathematica® modules used to compute the GIG/EGIG p.d.f. and c.d.f. and the set of commands used to compute the values for scenario I.6 in Tables B.1 and B.2.

```
Makec[r_,l_,p_]:=Module[{c},
  c=Table[Table[1,{j,1,Max[r]}],{i,1,p}];
  Table[c=ReplacePart[c,(Product[(l[[j]]-l[[i]])^(-r[[j]]),{j,1,i-1}]*
        Product[(l[[j]]-l[[i]])^(-r[[j]]),{j,i+1,p}])/(r[[i]]-1)!,{i,r[[i]]}],
                                                                      {i,1,p}];
  Table[Table[c=ReplacePart[c,Sum[((r[[i]]-k+j-1)!*(Sum[r[[h]]/(l[[i]]-l[[h]])^j,
                {h,1,i-1}] + Sum[r[[h]]/(l[[i]]-l[[h]])^j,{h,i+1,p}])*
                c[[i]][[r[[i]]-(k-j)]])/(r[[i]]-k-1)!,{j,1,k}]/k,{i,r[[i]]-k}],
                                                  {k,1,r[[i]]-1}],{i,1,p}];
  c  ]
```

Figure C.1 – Mathematica® module used to compute the coefficients $c_{j,k}$ ($k = 1, \ldots, r_j$; $j = 1, \ldots, p$) in the GIG p.d.f. and c.d.f.

```
GIGpdf[r_,li_,z_]:=Module[{p,l,c},
  If[Count[r,_Integer]==Length[r]&&And@@NonNegative[r]&&And@@Positive[li]&&
                                                      Length[r]==Length[li],
   p=Length[r]; l=Rationalize[li,0]; c=Makec[r,l,p];
   P[w_]:=Table[Sum[c[[j]][[k]]*w^(k-1),{k,1,r[[j]]}],{j,1,p}];
   Product[l[[j]]^r[[j]],{j,1,p}]*Sum[P[z][[j]]*Exp[-l[[j]]*z],{j,1,p}]  ]]


GIGcdf[r_,li_,z_]:=Module[{p,l,c},
  If[Count[r,_Integer]==Length[r]&&And@@Positive[r]&&And@@Positive[li]&&
                                                      Length[r]==Length[li],
   p=Length[r]; l=Rationalize[li,0]; c=Makec[r,l,p];
   P[w_]:=Table[Sum[c[[j]][[k]]*(k-1)!*Sum[w^i/(i!*l[[j]]^(k-i)),{i,0,k-1}],
                                                     {k,1,r[[j]]}],{j,1,p}];
   1-Product[l[[j]]^r[[j]],{j,1,p}]*Sum[P[z][[j]]*Exp[-l[[j]]*z],{j,1,p}]  ]]
```

Figure C.2 – Mathematica® modules used to compute the GIG p.d.f. and c.d.f.

The module `Makec` in Figure C.1 is used to compute the coefficients $c_{j,k}$ ($k = 1, \ldots, r_j$; $j = 1, \ldots, p$) in the GIG p.d.f. and c.d.f. and, as such, is called by the modules `GIGpdf` and `GIGcdf` in Figure C.2, which are the final modules used to compute the values for the GIG p.d.f. and c.d.f..

Both these modules have three arguments which are, in this order, a list with the shape parameters, a list with the rate parameters and the value for the running variable.

In Figure C.3 we have the set of commands used to compute the values for scenario I.6. The command right before the last two which compute the Mathematica® implementation of the Meijer G function and of the GIG/EGIG p.d.f. uses a call to the module `Screen2` in Figure C.4, which is used to scan the vector of rate parameters for the GIG/EGIG p.d.f. and c.d.f., looking for repeated values in the rate parameters list.

```
ms=3;
z=40/100;
a={453/10,123/10,223/10};
n={3,2,3};
k={{6,6,6},{3,3},{3,4,5}};
m={{2,2,3},{4,4},{2,1,3}};
av=Table[Table[Table[a[[nu]]-(j+Sum[k[[nu,r]],{r,1,ell-1}])/k[[nu,ell]],
                                    {j,1,k[[nu,ell]]}],{ell,1,n[[nu]]}],{nu,1,ms}];
bv=Table[Table[Table[m[[nu,ell]]/k[[nu,ell]],{j,1,k[[nu,ell]]}],{ell,1,n[[nu]]}],
                                                                    {nu,1,ms}];
K1=Product[Product[Product[Gamma[av[[nu,ell,j]]+bv[[nu,ell,j]]]/Gamma[av[[nu,ell,j]]],
                                    {j,1,k[[nu,ell]]}],{ell,1,n[[nu]]}],{nu,1,ms}];
lvf=Flatten[Table[Table[Table[a[[nu]]+(-Sum[k[[nu,r]],{r,1,ell}]+i)/k[[nu,ell]],
                                 {i,0,m[[nu,ell]]-1}],{ell,1,n[[nu]]}],{nu,1,ms}]];
rvf=Table[1,{j,1,Length[lvf]}]; vec=Screen2[lvf,rvf]
Timing[SetPrecision[K1*MeijerG[{{},Flatten[av+bv]-1},{Flatten[av]-1,{}},z],26]]
Timing[SetPrecision[GIGpdf[vec[[2]], vec[[1]], -Log[z]]*1/z, 26]]
```

Figure C.3 – Mathematica® instructions used to compute the Meijer G function and the GIG/EGIG p.d.f. implementation for scenario I.6.

```
Screen2[v1_,v2_]:=Module[{v2s,v1s,n,nv1,nv2},
  v2s=v2[[Ordering[v1]]]; v1s=Sort[v1];
  n=Length[v1];
  nv1={v1s[[1]]}; nv2={v2s[[1]]}; isc=1;
  Do[If[v1s[[i]]==v1s[[i-1]],{nv2[[isc]]=nv2[[isc]]+v2s[[i]]},
                  {isc=isc+1,nv1=Append[nv1,v1s[[i]]],nv2=Append[nv2,v2s[[i]]]}],
                                                                      {i,2,n}];
  {nv1,nv2}  ]
```

Figure C.4 – Mathematica® module used to scan the vectors of rate parameters for the GIG/EGIG p.d.f. and c.d.f., looking for repeated values.

Appendix C.2. R functions for the computation of the GIG/EGIG p.d.f. and c.d.f.

In this part of the Appendix we present a set of functions programed in *R*, which may be used, alternatively to the Mathematica® modules, to compute the GIG/EGIG p.d.f. and c.d.f. and the values for the p.d.f. and c.d.f. in Corollaries 1 and 2.

This set of function requires the use of the `Rmpfr` package, in order to allow for computations with arbitrary precision, since quite often we need to use more than double precision in the computations of the GIG/EGIG p.d.f. and c.d.f., as it is the case with the computations in Appendix B.

```
makec<-function(r,ll,p,prec)
{rr<-mpfr(r,prec)
 cv<-array(mpfr(1,prec),c(p,max(r)))
 for (j in 1:p) cv[j,r[j]]<-prod(((ll-ll[j])^(-rr))[-j])/factorial(rr[j]-1)
 rrr<-array(mpfr(0,prec),c(p,max(r)-1))
 for (j in 1:p) {if (r[j]>1) for (i in 1:(r[j]-1))
                                          rrr[j,i]<-sum((rr*((ll[j]-ll)^(-i)))[-j])}
 for (j in 1:p) {if (r[j]>1) for (k in 1:(r[j]-1)) cv[j,r[j]-k]<-1/mpfr(k,prec)*
          sum(factorial(rr[j]-k-1+(1:k))/factorial(rr[j]-k-1)*cv[j,r[j]-(k-(1:k))]
                                                                  *rrr[j,1:k])}
 cv}
```

Figure C.5 – R function used to compute the coefficients $c_{j,k}$ ($k = 1, \ldots, r_j$; $j = 1, \ldots, p$) in the GIG p.d.f. and c.d.f.

```
GIGpdf<-function(r,l,z,prec)
{p<-length(r)
 rr<-mpfr(r,prec)
 if (is.real(l)) ll<-mpfr(l*10^7,prec)/10^7 else ll<-l
 if (is.real(z)) zz<-mpfr(z*10^7,prec)/10^7 else zz<-z
 cvc<-makec(r,ll,p,prec)
 pj<-rep(mpfr(0,100),p)
 for (j in 1:p) pj[j]<-sum(cvc[j,1:r[j]]*zz^((1:r[j])-1))
 prod(ll^rr)*sum(pj*exp(-ll*zz))}

GIGpdfn<-function(r,ln,ld,z,prec)
{p<-length(r)
 rr<-mpfr(r,prec)
 ll<-mpfr(ln,prec)/ld
 if (is.real(z)) zz<-mpfr(z*10^7,prec)/10^7 else zz<-z
 cvc<-makec(r,ll,p,prec)
 pj<-rep(mpfr(0,100),p)
 for (j in 1:p) pj[j]<-sum(cvc[j,1:r[j]]*zz^((1:r[j])-1))
 prod(ll^rr)*sum(pj*exp(-ll*zz))}
```

Figure C.6 – R functions used to compute the GIG p.d.f..

The function makec in Figure C.5 is used to compute the coefficients $c_{j,k}$ ($k = 1, \ldots, r_j$; $j = 1, \ldots, p$) in the GIG p.d.f. and c.d.f. and, as such, is called by the functions GIGpdf, GIGpdfn, GIGcdf and GIGcdfn in Figures C.6 and C.7, which are the final modules used to compute the values for the GIG p.d.f. and c.d.f..

The functions GIGpff and GIGcdf have four arguments each, which are, in this order, a list with the shape parameters, a list with the rate parameters, the value for the running variable and the number of precision bits. For example if one wants a precision of 26 digits, since $26 * Log_{10}2 \approx 86.37$, a value of at least 87 or 88 should be given for this last argument.

The functions GIGpdf and GIGcdf are to be used when we give the values for the rate parameters as reals. However, in this case, given the way the programming is done, the rate parameters should note have more than seven decimal places, in order to assure that precision is kept.

```
GIGcdf<-function(r,l,z,prec)
{p<-length(r)
 rr<-mpfr(r,prec)
 ll<-mpfr(l*10^7,prec)/10^7
 zz<-mpfr(z*10^7,prec)/10^7
 cvc<-makec(r,ll,p,prec)
 pjk<-array(mpfr(0,prec),c(p,max(r)))
 for (j in 1:p) {for (k in 1:r[j]) pjk[j,k]<-sum(zz^(0:(k-1))/(ll[j]^(k-0:(k-1))
                                                   *factorial(mpfr(0:(k-1),prec))))}
 pj<-rep(mpfr(0,prec),p)
 for (j in 1:p) pj[j]<-sum(cvc[j,1:r[j]]*factorial(mpfr((1:r[j])-1,prec))
                                                                  *pjk[j,1:r[j]])
 mpfr(1,prec)-prod(ll^rr)*sum(pj*exp(-ll*zz))}

GIGcdfn2<-function(r,ln,ld,z,prec)
{p<-length(r)
 rr<-mpfr(r,prec)
 ll<-mpfr(ln,prec)/ld
 zz<-mpfr(z*10^7,prec)/10^7
 cvc<-makec4n(r,ll,p,prec)
 pjk<-array(mpfr(0,prec),c(p,max(r)))
 for (j in 1:p) {for (k in 1:r[j]) pjk[j,k]<-sum(zz^(0:(k-1))/(ll[j]^(k-0:(k-1))
                                                   *factorial(mpfr(0:(k-1),prec))))}
 pj<-rep(mpfr(0,prec),p)
 for (j in 1:p) pj[j]<-sum(cvc[j,1:r[j]]*factorial(mpfr((1:r[j])-1,prec))
                                                                  *pjk[j,1:r[j]])
 mpfr(1,prec)-prod(ll^rr)*sum(pj*exp(-ll*zz))}
```

Figure C.7 – R functions used to compute the GIG c.d.f..

In case we need to use rate parameters with more decimal places or in case we only want to give them as rationals, then due to the use of the arbitrary precision package Rmpfr for the computations, we have to use functions GIGpdfn and GIGcdfn. Both of these functions have a set of five arguments, the first one being a list with the shape parameters, and the second and third, two lists, the first one with the numerators of the rate parameters and the second one with the denominators. Both these lists have to be lists of integers. The two last argumnets are the same as the two last arguments of GIGpdf or GIGcdf.

In Figure C.8 we have the function scenI, which once given the values for the parameters n_ν, $m_{\nu\ell}$, $k_{\nu\ell}$ and a_ν in Theorems 1 or 2, it computes the lists of shape and rate parameters for the corresponding GIG/EGIG p.d.f. or c.d.f. in Corollary 1. This function calls the function screen2, also in Figure C.8, which is used to scan the vector of rate parameters for the GIG/EGIG p.d.f. and c.d.f., looking for repeated values in the rate parameters list.

In Figure C.9 we have the function pdfscenI and cdfscenI, which, being given the values for the parameters n_ν, $m_{\nu\ell}$, $k_{\nu\ell}$ and a_ν in Theorems 1 or 2, compute the p.d.f. or the c.d.f. corresponding to Corollary 1.

In Figure C.10 we may see an example of a set of commands which may be used in R to obtain, with the help of function pdfscenI in Figure C.9, the result in Table B.2 in Appendix B for the GIG/EGIG p.d.f. for scenario I.6. Note that we actually need to use a precision of 88 bits in order to obtain a result which is correct to rounding at the 16th decimal place. This actually closely agrees with the need to use a precision of 26 decimal places in the computation of the GIG/EGIG p.d.f. in Mathematica, as indicated in Table B.2, since as already remarked above, $26 * Log_{10}2 \approx 86.37$.

```
scenI3<-function(k,m,a,z,prec)
{ms<-length(k); ll<-rep(0,ms)
 for (i in 1:ms) ll[i]<-length(k[[i]])
 llm<-max(ll); n<-rep(0,ms)
 for (i in 1:ms) n[i]<-length(k[[i]])
 zz<-mpfr(z*10^7,prec)/10^7
 kk<-cbind(append(k[[1]],rep(0,llm-ll[1])))
 if (ms>1) for (i in 2:ms) kk<-cbind(kk,append(k[[i]],rep(0,llm-ll[i])))
 kk<-mpfr(kk,prec)
 mm<-cbind(append(m[[1]],rep(0,llm-ll[1])))
 if (ms>1) for (i in 2:ms) mm<-cbind(mm,append(m[[i]],rep(0,llm-ll[i])))
 aa<-mpfr(a*10^7,prec)/10^7
 vv<-c(mpfr(1,prec))
 for (nu in 1:ms) {for (ell in 1:n[nu]) {for (i in 0:(mm[ell,nu]-1))
                         vv<-append(vv,aa[nu]+(i-sum(kk[1:ell,nu]))/kk[ell,nu])}}
 vv[-1]}

screen2<-function(v1,v2)
{n<-length(v1); v2s<-v2[order(v1)]; v1s<-sort(v1)
 nv1<-v1s[1]; nv2<-v2s[1]; isc<-1;
 for (i in (2:n)) if (v1s[i]==v1s[i-1]) nv2[isc]<-nv2[isc]+v2s[i] else
                     {isc<-isc+1;nv1<-append(nv1,v1s[i]);nv2<-append(nv2,v2s[i])}
 cbind(nv1,nv2)}
```

Figure C.8 – R functions scenI and screen2 used to compute the parameters for the GIG/EGIG p.d.f. and c.d.f. implementations corresponding to Corollary 1.

```
pdfscenI<-function(k,m,a,z,prec)
{vl<-scenI3(k,m,a,z,prec)
 vr<-rep(1,length(vl))
 mm<-screen2(vl,vr)
 zz<-mpfr(z*10^7,prec)/10^7
 GIGpdfn(as.real(mm[,2]),mm[,1],-log(zz),prec)*1/zz}

cdfscenI<-function(k,m,a,z,prec)
{vl<-scenI3(k,m,a,z,prec)
 vr<-rep(1,length(vl))
 mm<-screen2(vl,vr)
 zz<-mpfr(z*10^7,prec)/10^7
 GIGcdfn(as.real(mm[,2]),mm[,1],-log(zz),prec)*1/zz}
```

Figure C.9 – R functions used to compute the GIG/EGIG p.d.f. and c.d.f. implementations corresponding to Corollary 1.

```
> k<-list(c(6,6,6),c(3,3),c(3,4,5))
> m<-list(c(2,2,3),c(4,4),c(2,1,3))
> a<-c(45.3,12.3,22.3)
> pdfscenI(k,m,a,.4,88)
1 'mpfr' number of precision  88   bits
[1] 2.917260029160877465447199 32
```

Figure C.10 – Example of a set of R commands used to obtain the value for the GIG/EGIG p.d.f. for scenario I.6 in Appendix B and corresponding result obtained.

References

Anderson, T. W. (2003). *An Introduction to Multivariate Statistical Analysis*, 3rd ed. (2004), J. Wiley & Sons, New York.

Anderson, T. W., Fang, K.-T. (1990). Inference in multivariate elliptically contoured distributions based on maximum likelihood. In: Fang K-T, Anderson TW (eds) *Statistical inference in elliptically contoured and related distributions*, 201216.

Anderson, T. W., Fang, K.-T., Hsu, H. (1986). Maximum-likelihood estimates and likelihood-ratio criteria for multivariate elliptically contoured distributions. *Canad. J. Statist.*, **14**, 55–59.

Arnold, B. C., Coelho, C. A., Marques, F. J. (2012). The distribution of the product of powers of independent uniform random variables – a simple but useful tool to address and better understand the structure of some distributions. *J. Multiv. Analysis* (in print, DOI: 10.1016/jmva.2011.04.006).

Coelho, C. A. (1998). The generalized integer Gamma distribution – a basis for distributions in multivariate statistics. *J. Multiv. Analysis*, **64** 86-102.

Coelho, C. A. (1999). Addendum to the paper "The generalized integer Gamma distribution – a basis for distributions in multivariate statistics". *J. Multiv. Analysis*, **69** 281-285.

Coelho, C. A. (2006). The exact and near-exact distributions of the product of independent Beta random variables whose second parameter is rational. *J. of Combinatorics, Information & System Sciences*, **31** 21-44.

Erdélyi, A. (Magnus, W., Oberhettinger, F. & Tricomi, F. G.) (eds.) (1953). *Higher Transcendental Functions (The Bateman Manuscript Project)*, Vol. I, McGraw-Hill, New York.

Fox, C. (1961). The G and H functions as symmetrical kernels, *Transactions of the American Mathematical Society* **98**, 395-429.

Jensen, D. R., Good, I. J. (1981). Invariant distributions associated with matrix laws under structural symmetry. *J. Roy. Statist. Soc.*, Ser B 43:327332.

Kariya, T. (1981). Robustness of multivariate tests. *Ann. Statist.* 9:12671275.

Marques, F. J., Coelho, C. A., Arnold, B. C. (2011). A general near-exact distribution theory for the most common likelihood ratio test statistics used in Multivariate Analysis. *TEST* **20**, 180-203.

Mathai, A. M. (ed.) (2007). *Special functions and functions of matrix argument: recent advances and applications in stochastic processes, statistics, wavelet analysis and astrophysics*, Lecture notes of the 5th S.E.R.C. School on Special functions and functions of matrix argument, Publication No- 34, Centre for Mathematical Sciences, Pala Campus, St. Thomas College, Arunapuram P.O., Pala, Kerala 686 574, India.

Meijer, C. S. (1936a). Uber Whittakersche bzw. Besselsche Funktionen und deren Produkte, *Nieuw Archief voor Wiskunde*, **18** (2), 10–29.

Meijer, C. S. (1936b). Neue Integraldarstellungen aus der Theorie der Whittakerschen und Hankelschen Funktionen, *Math. Annalen*, **112** (2), 469–489.

Meijer, C. S. (1941). Multiplikationstheoreme für die Funktion $G^{m,n}_{p,q}(z)$, *Proc. Koninklijk Nederlandse Akademie van Weteenschappen*, **44**, 1062-1070.

Meijer, C. S. (1946). On the G-function, *Proc. Koninklijk Nederlandse Akademie van Weteenschappen*, **49**, 227-237, 344-356, 457-469, 632-641, 765-772, 936-943, 1063-1072, 1165-1175.

Olkin, I., Press, S. J. (1961). Testing and estimation for a circular stationary model. *Ann. Math. Statist.* **40**, 1358-1373.

Springer, M. D. (1979). *The Algebra of Random Variables*, J. Wiley & Sons, New York.

Wald, A., Brookner, R. J. (1941). On the distribution of Wilks' statistic for testing the independence of several groups of variates. *Ann. Math. Statist.*, **12**, 137–152.