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INSTANTANEOUS SHRINKING OF THE SUPPORT OF NONNEGATIVE SOLUTIONS TO CERTAIN NONLINEAR PARABOLIC EQUATIONS AND VARIATIONAL INEQUALITIES

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1. Introduction

This paper investigates the support of nonnegative solutions of the semilinear Cauchy problem

$$\begin{gathered} u_t(x, t) - \Delta u(x, t) + \beta(u(x, t)) = 0, \quad (x, t) \in \mathbf{R}^n \times (0, \infty), \\ u(x, 0) = u_0(x), \quad x \in \mathbf{R}^n \end{gathered} \tag{1.1}$$

(where $\beta: [0, \infty) \to [0, \infty)$ is continuous, nondecreasing, and $\beta(0) = 0$), of certain similar parabolic variational inequalities, and of various nonlinear degenerate parabolic problems in one spatial dimension. Our work is motivated by previous research of A. S. Kalashnikov [2] on problems of these types in one spatial dimension and of H. Brezis and A. Friedman [1] on variational inequalities.

Kalashnikov has discovered certain conditions on β which imply a remarkable contrast in the behavior of nonnegative solutions of (1.1) as compared to solutions of the heat equation with the same initial data. In particular (here we specialize Kalashnikov's results to the case at hand and restate them slightly), he proved that (a) if

$$\int_0^1 \frac{ds}{\beta(s)} < \infty, \tag{1.2}$$

then any bounded, nonegative solution of (1) must vanish identically after some finite time $T > 0$, and (b) if

$$\int_0^1 \frac{ds}{(s\beta(s))^{1/2}} < \infty, \tag{1.3}$$

then any solution with compact support initially has compact support at all later times $t > 0$. In this paper we demonstrate that (1.3) (which implies (1.2)) in fact gives rise to a somewhat more striking effect even in higher dimensions; namely, if $u_0(x)$ merely goes to zero uniformly as $|x| \to \infty$, then for each time $t > 0$, the support of the corresponding solution of (1.1) is

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bounded. The solution therefore experiences an "instantaneous shrinking" of its support. (L. Tartar in unpublished work has obtained this result for the case $\beta(s)=s^\gamma$, $0<\gamma<1$.) We show furthermore that (1.3) is necessary for this phenomenon (at least in one dimension), in the sense that should the integral in (1.3) diverge, then there exists some smooth function $u_0(x)$ for which $u_0(x)\to 0$ as $|x|\to\infty$, but the corresponding solution of (1.1) has unbounded support at some time $t_0>0$. These facts are proved in Section 2.

The instantaneous occurrence of compact support has been proved by Brezis and Friedman [1] for solution of certain parabolic variational inequalities. In Section 3 we employ our techniques to give very simple proofs of this and several other results from [1]; most notably, we completely avoid here the cumbersome construction of global comparison functions.

Section 4 comprises an extension of our techniques to one-dimensional, possibly degenerate equations of the form

$$u_t(x,t)-[\phi(u(x,t))]_{xx}+[\lambda(u(x,t))]_x+\beta(u(x,t))=0, \quad (x,t)\in \mathbf{R}\times(0,\infty),$$
$$u(x,0)=u_0(x), \quad x\in\mathbf{R}. \tag{1.4}$$

Here we identify conditions on ϕ, λ, and β again to insure the immediate onset of compact support of the solution, whenever $u_0(x)\to 0$ as $|x|\to\infty$.

2. Instantaneous shrinking of support

Our proof of Theorem 2.2 below depends upon the construction of certain nontrivial solutions to the differential equations (2.12) and (2.15). As we shall see (1.3) immediately implies the solvability of (2.15), but to solve (2.12) we must first verify that (1.2) holds:

LEMMA 2.1. *Let $\beta:[0,\infty)\to[0,\infty)$ be nondecreasing and continuous, with $0=\beta(0)$ and $\beta(x)>0$ for $x>0$. Assume that*

$$\int_0^1 \frac{ds}{(s\beta(s))^{1/2}}<\infty. \tag{2.1}$$

Then

$$\int_0^1 \frac{ds}{\beta(s)}<\infty. \tag{2.2}$$

Proof. First let us prove that (2.1) implies

$$\lim_{x\searrow 0}\frac{\beta(x)}{x}=+\infty. \tag{2.3}$$

If (2.3) were false, there would exist some $0<L<\infty$ and a sequence $x_n\searrow 0$ such that $1>x_n>0$ and

$$\beta(x_n)/x_n\le L, \quad n=1,2,\ldots. \tag{2.4}$$

We may in addition assume that the x_n's are selected so that

$$0 < x_{n+1} \leq x_n/4, \quad n = 1, 2, \ldots. \tag{2.5}$$

Then

$$\begin{aligned}\int_0^1 \frac{ds}{(s\beta(s))^{1/2}} &\geq \sum_{n=1}^{\infty} \int_{x_{n+1}}^{x_n} \frac{ds}{s^{1/2}} \frac{1}{\beta(x_n)^{1/2}} \quad \text{by the monotonicity of } \beta \\ &\geq \sum_{n=1}^{\infty} \frac{2}{L^{1/2}} \frac{x_n^{1/2} - x_{n+1}^{1/2}}{x_n^{1/2}} \quad \text{by (2.4)} \\ &\geq \sum_{n=1}^{\infty} \frac{1}{L^{1/2}} \quad \text{by (2.5).}\end{aligned}$$

This contradicts assumption (2.1) and thereby proves (2.3). In view of (2.3) we have

$$\beta(x)/x \geq \delta > 0 \quad \text{for some } \delta > 0 \text{ and each } 0 < x < 1.$$

Hence

$$\int_0^1 \frac{ds}{\beta(s)} \leq \frac{1}{\delta^{1/2}} \int_0^1 \frac{ds}{(s\beta(s))^{1/2}} < \infty. \quad \blacksquare$$

Next we introduce some useful terminology. For a given function $u(x, t)\colon \mathbf{R}^n \times [0, \infty) \to [0, \infty)$ let us denote the support of $u(\cdot, t)$ by $S(t)$; that is,

$$S(t) \equiv \{x \in \mathbf{R}^n \mid u(x, t) > 0\}, \quad t \geq 0. \tag{2.6}$$

In addition, set

$$D(\eta) \equiv \{x \in \mathbf{R}^n \mid u_0(x) \geq \eta\}, \quad \eta \geq 0, \tag{2.7}$$

for a given function $u_0(x)\colon \mathbf{R}^n \to [0, \infty)$. Notice that if $u_0(x)$ satisfies (2.8) below, then $D(\eta)$ is a bounded set for each $\eta > 0$.

THEOREM 2.2. *Let $u_0(x)$ be a bounded, nonnegative function defined on $\mathbf{R}^n$; and suppose that*

$$\lim_{|x| \to \infty} u_0(x) = 0. \tag{2.8}$$

Assume that $\beta\colon [0, \infty) \to [0, \infty)$ is nondecreasing and continuous, with $0 = \beta(0)$, $\beta(x) > 0$ for $x > 0$, and

$$\int_0^1 \frac{ds}{(s\beta(s))^{1/2}} < \infty. \tag{2.9}$$

Then for each $t > 0$, the support $S(t)$ of the solution $u(x, t)$ of (1.1) is bounded. In addition,

$$\lim_{|x| \to \infty} u(x, t) = 0 \quad \textit{uniformly in } t \geq 0, \tag{2.10}$$

and there exists some $T > 0$ such that $u(x, t) \equiv 0$ for all $t \geq T$.

Remark 2.3. For simplicity of exposition we will assume that $u(x, t)$ is a *classical* solution of (1.1); but our methods extend easily to cover more general types of solutions, the existence of which follows, for example, from nonlinear semigroup theory.

Note also that our proof really requires only that u satisfy the inequalities $u \geq 0$ and

$$u_t(x, t) - \Delta u(x, t) + \beta(u(x, t)) \leq 0, \quad (x, t) \in \mathbf{R}^n \times (0, \infty). \tag{1.1$'$}$$

Proof. First of all we construct two auxiliary functions. For each $t \geq 0$, define $g(t)$ by the equation

$$\int_0^{g(t)} \frac{ds}{\beta(s)} = \frac{t}{n+1}; \tag{2.11}$$

Lemma 2.1 and (2.9) imply that the integral converges. (We may assume with no loss of generality that

$$\int_0^{\infty} \frac{ds}{\beta(s)} = \infty$$

and thus that $g(t)$ is defined for all $t \geq 0$. Indeed

$$\operatorname*{ess\,sup}_{\mathbf{R}^n \times [0,\infty)} u(x, t) = \operatorname*{ess\,sup}_{\mathbf{R}^n} u_0(x) \equiv M;$$

and so we may if necessary redefine $\beta(x)$ to be constant for $x \geq M+1$. This modification insures that the integral to infinity diverges, but does not otherwise affect the problem.) Upon differentiating (2.11) we discover that

$$\begin{aligned} &g'(t) = \frac{\beta(g(t))}{n+1}, \quad t \geq 0, \\ &g(0) = 0, \quad g(t) > 0 \quad \text{for} \quad t > 0, \\ &g(t) \to \infty \quad \text{as} \quad t \to \infty. \end{aligned} \tag{2.12}$$

Next define $f(x)$: $(-\infty, \infty) \to [0, \infty)$ by

$$\int_0^{f(\pm x)} \left[\frac{2}{n+1} \int_0^s \beta(\xi)\, d\xi\right]^{-1/2} ds = x, \quad x \geq 0; \tag{2.13}$$

in view of the inequalities

$$\frac{s}{2}\beta\left(\frac{s}{2}\right) \leq \int_0^s \beta(\xi)\, d\xi \leq s\beta(s), \tag{2.14}$$

(2.9) is a necessary and sufficient condition for the integral in (2.13) to converge. As before there is no loss of generality in assuming that f is defined for all $x \in \mathbf{R}$. We twice differentiate (2.13) to find

$$\begin{aligned} &f''(x) = \frac{\beta(f(x))}{n+1}, \quad x \in \mathbf{R}, \\ &f(0) = f'(0) = 0, \quad f(x) > 0 \quad \text{for} \quad x > 0, \\ &f(x) \to \infty \quad \text{as} \quad |x| \to \infty. \end{aligned} \tag{2.15}$$

Now fix some $t_0>0$ and choose $x_0=(x_0^1, x_0^2, \ldots, x_0^n)\in S(t_0)$. Define

$$w(x, t)\equiv g(t_0-t)+\sum_{i=1}^{n} f(x^i-x_0^i), \quad 0\le t\le t_0, x=(x^1, x^2, \ldots, x^n)\in \mathbf{R}^n.$$

Then for $x\in\mathbf{R}^n$ and $0<t\le t_0$,

$$\begin{aligned} w_t(x, t)&-\Delta w(x, t)+\beta(w(x, t))\\ &=-g'(t_0-t)-\sum_{i=1}^{n} f''(x^i-x_0^i)+\beta\left(g(t_0-t)+\sum_{i=1}^{n} f(x^i-x_0^i)\right)\\ &\ge -g'(t_0-t)+\frac{\beta(g(t_0-t))}{n+1}+\sum_{i=1}^{n}\left[-f''(x^i-x_0^i)+\frac{\beta(f(x^i-x_0^i))}{n+1}\right]\\ &=0, \end{aligned} \tag{2.16}$$

by the monotonicity of β, (2.12), and (2.15). In addition, since $f(x)\to\infty$ as $|x|\to\infty$ and since $u(x, t)$ is bounded, there is some ball B, centered at x_0, such that $u(x, t)\le w(x, t)$, $x\in\partial B$, $0\le t\le t_0$. Now, from (1.1), (2.16), and standard comparison theorems, the difference $u-w$ must attain maximum on the parabolic boundary of the cylinder $B\times[0, t_0]$. Since $u-w<0$ on $\partial B\times[0, t_0]$, but

$$0<u(x_0, t_0)=u(x_0, t_0)-w(x_0, t_0),$$

there must exist some point $\bar{x}=(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)\in B$ for which

$$u(x_0, t_0)+g(t_0)+\sum_{i=1}^{n} f(\bar{x}^i-x_0^i)\le u(\bar{x}, 0). \tag{2.17}$$

Several consequences follow from this inequality. First, we note that $u(\bar{x}, 0)\ge g(t_0)>0$; and so

$$\bar{x}\in D(g(t_0)).$$

(Recall the definition (2.7) of the sets $D(\eta)$, $\eta\ge 0$). Owing to assumption (2.8), $D(g(t_0))$ is a bounded subset of $\mathbf{R}^n$. Second, inequality (2.17) implies that

$$\max_{1\le i\le n} f(\bar{x}^i-x_0^i)\le u(\bar{x}, 0)\le M.$$

Since $f(x)\to\infty$ as $|x|\to\infty$, this estimate provides a bound, solely in terms of β and M, on the distance $|x_0-\bar{x}|$.

Hence we have proved that an arbitrary point $x_0\in S(t_0)$ is at most a bounded distance away from the bounded set $D(g(t_0))$: $S(t_0)$ is therefore itself bounded.

Next we prove (2.10). Inequality (2.17) implies that $\bar{x}\in D(u(x_0, t_0))$ and, as we have seen, that $|\bar{x}-x_0|$ is bounded, independently of $t_0>0$ and $u(x_0, t_0)$. Hence for each $\epsilon>0$, the set

$$D(\epsilon, t_0)\equiv\{x\in\mathbf{R}^n \mid u(x, t_0)\ge\epsilon\}$$

is within a uniformly bounded (in ϵ and t_0) distance from the set $D(\epsilon)$.

The last assertion of the theorem is also a simple consequence of (2.17): if $t_0>0$ is selected so large that $g(t_0)\geq M+1$, then from (2.17) follows the contradiction

$$M+1\leq g(t_0)\leq u(\bar{x},0)\leq M.$$

Hence the set $S(t_0)$ must be empty in this case. ■

Remark 2.4. The following example demonstrates that hypothesis (2.9) is in a certain sense a necessary condition for Theorem 2.2. Suppose

$$\int_0^1 \frac{ds}{(s\beta(s))^{1/2}}=\infty, \tag{2.18}$$

and define $f(x)\colon [0,\infty)\to[0,\infty)$ by

$$\int_{f(x)}^1 \left[2\int_0^s \beta(\xi)\,d\xi\right]^{-1/2} ds = x.$$

Differentiating we find that

$$f''(x)=\beta(f(x)) \quad x\geq 0. \tag{2.19}$$

According to the inequalities (2.14) and to (2.18),

$$f(0)=1,\quad f(x)>0 \quad \text{for all} \quad x>0,$$
$$f(x)\to 0 \quad \text{as} \quad x\to\infty.$$

Now choose any smooth, bounded function $u_0(x)\colon \mathbf{R}^1\to(0,\infty)$ such that

$$u_0(0)=2,\quad \lim_{|x|\to\infty} u_0(x)=0 \quad \text{and} \quad u_0(x)\geq f(x) \quad \text{for} \quad x>0.$$

By continuity of the solution $u(x,t)$ of

$$u_t(x,t)-u_{xx}(x,t)+\beta(u(x,t)),\quad (x,t)\in\mathbf{R}\times(0,\infty),$$
$$u(x,0)=u_0(x),\quad x\in\mathbf{R}^1,$$

we have $u(0,t)\geq f(0)$ for some $0\leq t\leq t_0$. Then (2.19) and (2.21) imply that $u(x,t)\geq f(x)$, $0\leq x<\infty$, $0\leq t<t_0$; and so, by (2.20), we see that $S(t)$ is unbounded for $0\leq t\leq t_0$.

Notice that this argument does not preclude the possibility that there may be *certain* initial functions $u_0(x)$ satisfying (2.8) for which the conclusions of Theorem 2.2 are valid, even if (2.9) fails.

3. Estimates on the support of variational inequalities

In [1] Brezis and Friedman proved an instantaneous shrinking of support assertion for the solutions of the parabolic variational inequality

$$\begin{aligned} &u(x, t) \geq 0, \quad \text{a.e. } (x, t) \in \mathbf{R}^n \times [0, \infty), \\ &(u_t(x, t) - \Delta u(x, t))(v(x, t) - u(x, t)) \geq (f(x, t))(v(x, t) - u(x, t)) \text{ a.e.} \\ &\quad \text{for all} \quad v \geq 0 \text{ a.e.}, \\ &u(x, 0) = u_0(x),\ x \in \mathbf{R}^n, \end{aligned} \tag{3.1}$$

whenever $u_0(x)$ satisfies (2.6) and

$$f(x, t) \leq -\theta < 0, \quad (x, t) \in \mathbf{R}^n \times [0, \infty), \tag{3.2}$$

for some constant $\theta > 0$. The demonstration of this result in [1] depends upon the clever construction of a global comparison function with the required properties, but in fact a simple proof follows as well from our methods:

THEOREM 3.1 (Brezis-Friedman). *Suppose that $f \in L^\infty(\mathbf{R}^n \times (0, \infty))$ satisfies* (3.2) *and that*

$$u_0 \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n)$$

satisfies (2.6). *Let $u(x, t)$ denote the corresponding solution of the variational inequality* (3.1).

Then for each $t > 0$, the set $S(t)$ is bounded. Furthermore $\lim_{|x| \to \infty} u(x, t) = 0$ uniformly in $t > 0$, and there exists some $T > 0$ such that $u(x, t) \equiv 0$ for all $t \geq T$.

Proof. Let Q denote the support of u in $\mathbf{R}^n \times (0, T)$; since u is a continuous function in $\mathbf{R}^n \times (0, T)$, Q is open and

$$u_t(x, t) - \Delta u(x, t) = f(x, t), \quad (x, t) \in Q. \tag{3.3}$$

Now fix $t_0 > 0$ and choose $x_0 \in S(t_0)$. Define

$$w(x, t) \equiv \frac{\theta(t_0 - t)}{2} + \frac{\theta}{4n} |x - x_0|^2, \quad x \in \mathbf{R}^n, 0 \leq t \leq t_0.$$

We have

$$w_t - \Delta w = -\theta \geq f \quad \text{in } Q. \tag{3.4}$$

The solution $u(x, t)$ is bounded; and so there is some ball B, centered at x_0, with radius so large that $u(x, t) \leq w(x, t)$ for $x \in \partial B$, $0 \leq t \leq t_0$.

Set $Q^* = Q \cap B \times [0, t_0]$. By (3.3) and (3.4) the maximum of $u - w$ must occur on the parabolic boundary $\partial p Q^*$ of Q^*. But $u - w \leq 0$ on

$\partial pQ^* \cap \{t>0\}$, whereas

$$u(x_0, t_0) - w(x_0, t_0) = u(x_0, t_0) > 0.$$

Hence there exists some $\bar{x} \in B$ such that

$$u(x_0, t_0) + \frac{\theta t_0}{2} + \frac{\theta}{4n} |\bar{x} - x_0|^2 = w(\bar{x}, 0) \leq u(\bar{x}, 0). \tag{3.5}$$

Therefore $\bar{x} \in D(\theta t_0/2)$, and by assumption this is a bounded subset of $\mathbf{R}^n$. Estimate (3.5) also implies that x_0 is at most a bounded distance from $D(\theta t_0/2)$, and so $S(t_0)$ is itself a bounded set.

The last two assertions of the theorem follow as in the proof of Theorem 2.2. ■

Remark 3.2. The conclusions of Theorem 3.1 can in fact be proved under somewhat a weaker assumption than (3.2). Suppose instead that

$$\limsup_{|x| \to \infty} f(x, t) < 0, \text{ uniformly in } t. \tag{3.6}$$

In this case we first choose a ball $\tilde{B}$, centered at the origin, such that $f(x, t) \leq -\theta < 0$ for $x \notin \tilde{B}$ and some $\theta > 0$. Again fix $t_0 > 0$ and choose $x_0 \in S(t_0) \backslash \tilde{B}$. Then as in the previous proof, $u - w$ must attain its (positive) maximum on the parabolic boundary $\partial p\bar{Q}$ of the open set $\bar{Q} \equiv \{Q \cap B \times [0, t_0]\} \backslash \tilde{B}$. If a maximum is attained at $(\bar{x}, \bar{t}) \in \partial \tilde{B} \times (0, t_0]$, then

$$\frac{\theta(t_0 - \bar{t})}{2} + \frac{\theta}{4n} |\bar{x} - x_0|^2 = w(\bar{x}, \bar{t}) \leq u(\bar{x}, \bar{t}) \leq M;$$

and this again implies a bound on the distance from x_0 to $\tilde{B}$. More generally, what we require of f is that there exist some function $w(x, t)$ with $w(x_0, t_0) = 0$, $\min_{x \in \mathbf{R}^n} w(x, t) > 0$ for each $0 < t < t_0$, $\lim_{|x| \to \infty} w(x, t) = \infty$ uniformly in t, and $w_t - \Delta w \geq f$.

We now modify slightly the techniques of the previous proof to give simple demonstrations of some other assertions in [1], concerning the propagation of the support of solutions of (3.1) when u_0 has compact support initially.

THEOREM 3.2 (Brezis-Friedman). *Let f satisfy* (3.2) *and suppose that $u(x, t)$ is the solution to the variational inequality* (3.1).

(i) *If $u_0 \in W^{2,\infty}(\mathbf{R}^n)$ has compact support S, and f, $f_t \in L^\infty(\mathbf{R}^n \times (0, T))$ for some $T > 0$, then there exists a constant C_1 such that*

$$S(t) \subset S + B(C_1\sqrt{t}) \tag{3.7}$$

for all $0 < t < T$. (Here $B(r)$ is ball centered at zero with radius r and "+" denotes the vector sum.)

(ii) *If* $u_0 \in L^\infty(\mathbf{R}^n)$ *has compact support and* $f \in L^\infty(\mathbf{R}^n \times (0, T))$ *for some* $T > 0$, *then there exists a constant* C_2 *such that*

$$S(t) \subset S + B(C_2\sqrt{t}\,|\log t|) \tag{3.8}$$

for all sufficiently small $t > 0$.

Proof of (i). Under the assumption on the smoothness of u_0 and f, it follows from the maximum principle that u_t is bounded, say

$$\|u_t\|_{L^\infty(\mathbf{R}^n \times (0,T))} \le C_3. \tag{3.9}$$

Let $t_0 > 0$ be given and $x_0 \in S(t_0) \setminus S$. Define the open set

$$Q = \{(x, t) \mid 0 < t < t_0,\ u(x, t) > 0,\ x \notin S\},$$

and the function

$$w(x) \equiv \frac{\theta\,|x - x_0|^2}{2n}, \quad x \in \mathbf{R}^n. \tag{3.10}$$

In Q we have $u_t - \Delta u = f$ and $w_t - \Delta w = -\theta \ge f$. Since $u(x) - w(x) \to -\infty$, the maximum of $u - w$ is attained, and this maximum occurs on the parabolic boundary ∂pQ of Q. But

$$u - w \le 0 \text{ on } \partial pQ \setminus \partial S \times (0, t_0) \quad \text{and} \quad u(x_0, t_0) - w(x_0) = u(x_0, t_0) > 0.$$

Therefore there is some point $(\bar{x}, \bar{t}) \in \partial S \times (0, t_0)$ such that

$$\theta\,\frac{|\bar{x} - x_0|^2}{2n} = w(\bar{x}) \le u(\bar{x}, \bar{t}).$$

Hence

$$\begin{aligned} \operatorname{dist}(x_0, S) &\le |\bar{x} - x_0| \\ &\le \left(\frac{2n}{\theta}\,u(\bar{x}, \bar{t})\right)^{1/2} \\ &= \left(\frac{2n}{\theta}\,(u(\bar{x}, \bar{t}) - u(\bar{x}, 0))\right)^{1/2} \\ &\le \left(\frac{2n}{\theta}\,C_3 \bar{t}\right)^{1/2} \quad \text{by (3.9)} \\ &\le C_1 t_0^{1/2}. \end{aligned}$$

Since x_0 is an arbitrary point in $S(t_0)$, part (i) of Theorem 3.2 is proved.

For the proof of part (ii) we will need an auxillary estimate:

LEMMA 3.4. *Let* v *be a bounded solution of the heat equation in* $\mathbf{R}^n \times (0, \infty)$, *with bounded initial data* $v_0(x)$. *If for some* $x \in \mathbf{R}^n$, $\operatorname{dist}(x, \operatorname{supt} v_0) \ge \epsilon > 0$, *then there is a constant* C_4 *such that*

$$|v(x, t)| \le C_4 e^{-\epsilon^2/8t} \quad \text{for all } t > 0. \tag{3.11}$$

Proof.

$$
\begin{aligned}
|v(x,t)| &= C\left|\int_{\mathbf{R}^n} v_0(\xi)\frac{e^{-|x-\xi|^2/4t}}{t^{n/2}}\,d\xi\right| \\
&\le C\int_{\text{supt}\,v_0}\frac{e^{-|x-\xi|^2}}{t^{n/2}}\,d\xi \\
&\le C\int_\epsilon^\infty \frac{e^{-r^2/4t}}{t^{n/2}}\,r^{n-1}\,dr \\
&= C\int_{\epsilon^2/4t}^\infty e^{-s}s^{(n-2)/2}\,ds \le Ce^{-\epsilon^2/8t}. \quad \blacksquare
\end{aligned}
$$

Proof of (ii). Again choose $t_0>0$, $x_0\in S(t_0)$. Set $\epsilon\equiv(8t_0\,|\log t_0|)^{1/2}$ and $S_\epsilon\equiv S+B(\epsilon)$. Now if $x_0\in S_\epsilon$, we are done. If not, consider the open set

$$Q_\epsilon\equiv\{(x,t)\mid 0<t<t_0,\ u(x,t)>0,\ x\notin S_\epsilon\}.$$

Reasoning as in the proof of (i) there must exist a point $(\bar{x},\bar{t})\in\partial S_\epsilon\times(0,t_0)$ such that

$$\theta\frac{|\bar{x}-x_0|^2}{2n}=w(\bar{x})\le u(\bar{x},\bar{t}),$$

for w defined by (3.10). Hence

$$\text{dist}\,(x_0,S)\le\text{dist}\,(x,S)+|x-x_0|\le\epsilon+\frac{2n}{\theta}|u(\bar{x},\bar{t})|^{1/2}. \tag{3.12}$$

Under the assumptions of (ii), there exists a supremum norm bound on $u_t-\Delta u$,

$$\|u_t-\Delta u\|_{L^\infty(\mathbf{R}^n\times(0,T))}\le 2\,\|f\|_{L^\infty(\mathbf{R}^n\times(0,T))}\equiv C_5.$$

Hence

$$u(\bar{x},\bar{t})\le\bar{t}C_5+v(\bar{x},\bar{t}), \tag{3.13}$$

where

$$
\begin{aligned}
&v_t(x,t)-\Delta v(x,t)=0, \quad (x,t)\in\mathbf{R}^n\times(0,\infty),\\
&v(x,0)=u_0(x).
\end{aligned}
$$

By Lemma 3.4,

$$v(\bar{x},\bar{t})\le C_4e^{-\epsilon^2/8t}\le C_4e^{-\epsilon^2/8t_0}. \tag{3.14}$$

Therefore from (3.12)–(3.14) we may calculate that

$$
\begin{aligned}
\text{dist}\,(x_0,S) &\le\epsilon+(C_5t_0+C_4e^{-\epsilon^2/8t_0})^{1/2}\\
&=(8t_0\,|\log t_0|)^{1/2}\\
&\qquad +((C_5+C_4)t_0)^{1/2} \quad \text{by the definition of } \epsilon\\
&\le C_2(t_0\,|\log t_0|)^{1/2} \quad \text{for} \quad 0<t_0<1/e.
\end{aligned}
$$

The constant does not depend on the choice of $x_0 \in S(t_0)$; and so

$$\text{dist}\,(S(t_0), S) \leq C_2(t_0\,|\log t_0|)^{1/2}. \quad \blacksquare$$

Remark 3.5. Inequality (3.8) is shown in [1] to be the best possible estimate under the given hypotheses.

4. One-dimensional equations with higher order nonlinear terms and possible degeneracies

In this section we restrict our attention to equations in one space variable, but now consider more general and possibly degenerate Cauchy problems of the form:

$$\begin{aligned} &u_t(x,t) - [\phi(u(x,t))]_{xx} + \beta(u(x,t)) = 0, \quad (x,t) \in \mathbf{R} \times (0,\infty), \\ &u(x,0) = u_0(x), \quad x \in \mathbf{R}, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &u_t(x,t) - [\phi(u(x,t))]_{xx} + [\lambda(u)]_x + \beta(u(x,t)), (x,t) \in \mathbf{R} \times (0,\infty), \\ &u(x,0) = u_0(x), \quad x \in \mathbf{R}. \end{aligned} \tag{4.2}$$

Here ϕ, λ and β are given nonlinear functions, various properties of which we shall identify as forcing instantaneous shrinking of support phenomena. First of all, we will henceforth assume

$$\begin{aligned} &\beta : [0,\infty) \to [0,\infty) \text{ is nondecreasing, } 0 = \beta(0), \\ &\beta(x) > 0 \quad \text{for} \quad x > 0; \\ &\phi : [0,\infty) \to [0,\infty) \text{ is continuously differentiable, strictly} \\ &\quad \text{increasing, and convex, } 0 = \phi(0). \end{aligned} \tag{4.3}$$

Kalashnikov proved in [2] that the convergence of the integral (1.2) implies (assuming also (4.3)) that each bounded, nonnegative solution of (4.1) vanishes identically after some finite time $T > 0$. Additionally, if

$$\int_0^1 \frac{ds}{(s\beta(\Phi(s)))^{1/2}} < \infty \quad \text{where} \quad \Phi(x) \equiv \phi(x)^{-1} \text{ for } x \geq 0, \tag{4.4}$$

then spatial compact support is *maintained* for solutions of (4.1); that is, if $u_0(x)$ has compact support, then the support of $u(x,t)$ is for all time contained in some finite interval. In his recent dissertation [3] Kershner has established similar results for equation (4.2) (and has discovered also some interesting "one-sided compact support" effects.)

In Theorem 4.1 we prove that assumptions (1.2) and (4.4) together imply the instantaneous onset of compact support for solutions of (4.1), if only $u_0(x) \to 0$ as $|x| \to \infty$. Theorem 4.3 is the same result for solutions of (4.2), under an appropriate hypothesis on the behaviour of $\lambda(x)$ and a stronger assumption on $\beta(\Phi(s))$ for $0 < s < 1$.

THEOREM 4.1. *Suppose that ϕ and β satisfy assumptions* (4.3), (1.2), *and* (4.4). *Assume that $u_0(x)$ is a bounded, nonnegative function defined on* $\mathbf{R}$; and that

$$u_0(x) \to 0 \quad as \quad |x| \to \infty. \tag{4.5}$$

Then for each $t>0$, the support $S(t)$ of the solution $u(x, t)$ of (4.1) *is bounded. In addition*

$$\lim_{|x|\to\infty} u(x, t) = 0 \quad \textit{uniformly in } t \geq 0;$$

and there exists some $T>0$ such that $u(x, t) \equiv 0$ for all $t \geq T$.

Remark 4.2. By a "solution $u(x, t)$ of (4.1)" we mean a bounded, nonnegative

$$u \in C(\mathbf{R}^1 \times [0, \infty)) \cap C^{2,1}(\{u>0\}),$$

which solves (4.1) in a classical sense in the open set $\{u>0\}$. Although unique generalized solutions with even more regularity are known to exist for a fairly wide class of functions ϕ and β (see Kalashnikov [2]), no global smoothness beyond continuity is needed for our proofs.

Proof. Define the auxiliary functions $g(t)$ and $f(x)$ according to equations (2.11) and (2.13), except that in (2.13) we replace β by $\beta \circ \Phi$ and that we set $n=1$. Next choose $t_0>0$, $x_0 \in S(t_0)$; and then define

$$w(x, t) \equiv \Phi(f(x-x_0)+h(t_0-t)), \quad 0 \leq t \leq t_0, \quad x \in \mathbf{R},$$

for

$$h(t) \equiv \phi(g(t)). \tag{4.6}$$

We now calculate

$$\begin{aligned} & w_t - [\phi(w)]_{xx} + \beta(w) \\ & \quad = -\Phi'(f+h)h'(t_0-t) - f''(x-x_0) + (\beta \circ \Phi)(f+h) \\ & \quad \geq \left(-\Phi'(h)h' + \frac{\beta \circ \Phi}{2}(h)\right) + \left(-f'' + \frac{\beta \circ \Phi}{2}(f)\right) \quad \text{since } h \text{ and } \beta \circ \Phi \\ & \qquad \text{are increasing and } \Phi' \text{ is decreasing (by (4.3))} \\ & \quad = 0 \quad \text{by (4.6), (2.12), and (2.15) (with } \beta \circ \Phi \text{ in place of } \beta). \end{aligned} \tag{4.7}$$

Having now proved estimate (4.7), we may finish by reasoning as in the proof of Theorem 2.2. ∎

THEOREM 4.3. *Suppose that ϕ and β satisfy* (4.3), (1.2), *and that there exist positive constants C_6 and ω for which*

$$\beta \circ \Phi(x) \geq C_6 x^{1-2/\omega}, \quad 0 \leq x \leq 1. \tag{4.8}$$

Suppose also that $\lambda: [0, \infty) \to [0, \infty)$ *is continuously differentiable and*

$$(\lambda \circ \Phi) \text{ is bounded on compact subsets of } [0, \infty). \tag{4.9}$$

Let u_0 *be nonnegative and bounded,* $u_0(x) \to 0$ *as* $|x| \to \infty$.

Then the conclusions of Theorem 4.1 *are valid for a solution* $u(x, t)$ *of* (4.2).

Proof. From (1.2) it follows as before that $u(x, t) \equiv 0$ for all t greater than or equal to some finite time T (depending only on β and $M \equiv \text{ess}_{\mathbf{R}} \sup u(x, t)$).

Select $0' < t_0 \leq T$, $x_0 \in S(t_0)$, and then choose $K > 0$ so large that

$$\Phi(K) \geq M + 1. \tag{4.10}$$

Next define

$$A(x) \equiv \left(\frac{x - x_0}{L}\right)^2 \leq 1 \quad \text{for} \quad |x - x_0| \leq L \text{ (L to be selected)}, \tag{4.11}$$

and finally set

$$w(x, t) \equiv \Phi(KA(x)^{\omega/2} + (\phi \circ g)(t_0 - t)) \quad \text{for} \quad |x - x_0| \leq L, 0 \leq t \leq t_0, \tag{4.12}$$

where g is given by equation (2.11).

Let us now derive a differential inequality for w in the rectangle $|x - x_0| \leq L$, $0 \leq t \leq t_0$. We calculate as in the previous proof that

$$\begin{aligned} \mathscr{L}w &\equiv w_t - [\phi(w)]_{xx} + [\lambda(w)]_x + \beta(w) \\ &\geq -K(A^{\omega/2})_{xx} + \frac{\partial}{\partial x}[\lambda \circ \Phi(KA^{\omega/2} + (\phi \circ g)(t_0 - t))] + \frac{\beta \circ \Phi}{2}(KA^{\omega/2}). \end{aligned}$$

Let $D \equiv \sup\{(\lambda \circ \Phi)'(x) \mid 0 \leq x \leq K + (\phi \circ g)(T)\}$; then by direct computation we obtain, for $|x - x_0| \leq L$,

$$\begin{aligned} \mathscr{L}w &\geq \frac{\beta \circ \Phi}{2}(KA^{\omega/2})\left\{1 - \frac{2K\omega(\omega - 1)A^{\omega/2-1}}{L^2 \beta \circ \Phi(KA^{\omega/2})} - \frac{2DK\omega A^{\omega/2-1/2}}{L\beta \circ \Phi(KA^{\omega/2})}\right\} \\ &\geq \frac{\beta \circ \Phi}{2}(KA^{\omega/2})\left\{1 - \frac{2K\omega(\omega - 1)}{L^2 C_6} - \frac{2DK\omega}{LC_6}\right\} \quad \text{by (4.6) and (4.11)} \\ &\geq 0 \quad \text{for } L \text{ large enough.} \end{aligned}$$

Since $w(x_0, t_0) = 0 < u(x_0, t_0)$, but

$$w(x, t) - u(x, t) \geq \Phi(K) - M > 0 \quad \text{for} \quad x = x_0 \pm L, 0 \leq t \leq t_0,$$

by (4.10), there must exist some $\bar{x} \in (x_0 - L, x_0 + L)$ for which

$$u(x_0, t_0) + \Phi(KA(\bar{x})^{\omega/2} + (\phi \circ g)(t_0)) \leq u(\bar{x}, 0);$$

this estimate implies each conclusion of the theorem. ∎

Remark 4.4. Our choice of the function $A(x)^{\omega/2}$ is inspired by a similar construction of Kershner [3].

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