Instanton corrections and Emergent Strings

Florent Baume,[a] **Fernando Marchesano**[a] **and Max Wiesner**[a,b]

[a]*Instituto de Física Teórica UAM-CSIC,*
Cantoblanco, 28049 Madrid, Spain

[b]*Departamento de Física Teórica, Universidad Autónoma de Madrid,*
28049 Madrid, Spain

E-mail: florent.baume@uam.es, fernando.marchesano@csic.es,
max.wiesner@uam.es

ABSTRACT: We study limits of infinite distance in the moduli space of 4d $\mathcal{N} = 2$ string compactifications, in which instanton effects dominate. We first consider trajectories in the hypermultiplet moduli space of type IIB Calabi-Yau compactifications. We observe a correspondence between towers of D-brane instantons and D-brane 4d strings, such that the lighter the string the more relevant the instanton effects are. The dominant instantons modify the classical trajectory such that the lightest D-brane string becomes tensionless even faster, while the other strings are prevented to go below the fundamental string tension. This lightest string is dual to a fundamental type IIB string and realises the Emergent String Conjecture. We also consider the vector multiplet moduli space of type I string theory on $K3 \times T^2$, where quantum corrections can also become significant. Naively, we only find trajectories that correspond to decompactification limits, in apparent contradiction with the picture obtained in some dual setup.

Contents

1   Introduction

Recently, the swampland programme has stirred the field of string phenomenology, by aiming at drawing a clear line between effective field theories (EFTs) that have a consistent UV completion including quantum gravity from those that do not. While the landscape of string compactifications is expected to either cover or provide a representative subset of the former set of theories, the latter set is dubbed to form the *swampland* [1].

To differentiate between what belongs to the swampland and what does not, a series of swampland conjectures has been proposed, see [2, 3] for recent reviews. In this context, the Swampland Distance Conjecture (SDC) [4] has played a central rôle, not only because it describes inherent limitations of EFTs at long distances in field space, but also because non-trivial connections between several swampland conjectures have been drawn through its study. The SDC has been tested in significant detail for string compactifications with eight supercharges [5–19], where there is an actual moduli space, and also analysed in the context of four supercharges [20–31], where the field space is endowed with a potential.

As a by-product of these analyses, the notion of emergence has been established, stating that the infinite distances occurring at boundaries of moduli spaces are in fact a consequence of illegally integrating out the states that become massless in this limit [3, 6, 7, 32, 33]. Recently, this notion has been refined in terms of the Emergent String Conjecture [19], which states that in fact at all equi-dimensional infinite distance limits, the theory reduces

to an asymptotically tensionless, weakly-coupled string theory. As such, the tower of light modes required by the SDC are given by the oscillations of a critical heterotic or Type II fundamental string. The refinement of specialising to equi-dimensional limits excludes trajectories of infinite distance that are partial decompactification limits due to the appearance of a KK-like spectrum dominating the EFT breakdown.

In [19] the Emergent String Conjecture was tested in the Kähler moduli space of Calabi-Yau compactifications of M-theory to 5d and type IIA to 4d. The latter describes the vector multiplet moduli space of type II Calabi-Yau compactifications, on which the masses of BPS particles depend and whose metric is known exactly. These two features provide powerful tools to classify the different infinite distance trajectories, and in particular determine whether they are equi-dimensional or not.

The purpose of this work is to further test the Emergent String Conjecture, considering cases where instanton corrections significantly modify the moduli space metric. In particular we consider infinite distance trajectories in the hypermultiplet space of type IIB Calabi-Yau compactifications to 4d, making use of the analysis performed in [15]. This moduli space contains the Calabi-Yau Kähler moduli and so, classically, looks very similar to a type IIA CY vector multiplet moduli space. Nevertheless, there are two important differences between these two cases that directly affect the analysis of infinite distance trajectories. First, the masses of charged, BPS particles of the compactification do not depend on the hypermultiplets. As a result, when going to the boundary of the HM moduli space the only candidates to drive a decompactification limit are standard Kaluza-Klein modes, in stark contrast with the plethora of possibilities observed in [19]. Instead, the hypermultiplets control the action of 4d instantons, which non-trivially correct the classical metric. As pointed out in [15] (see also [17]), the trajectories in vector moduli space where towers of particles become asymptotically massless correspond, via the c-map, to hypermultiplet trajectories in which towers of towers of instantons develop a vanishing action. In this regime the instanton effects become very large and dominate over the classical contribution to the hypermultiplet moduli space metric, again in dramatic contrast to the vector multiplet case.

It follows from these remarks that CY hypermultiplet spaces provide a very interesting arena to test the Emergent String Conjecture. On the one hand, classically one may apply the results of [6, 19] to classify potential infinite distance limits. On the other hand, the physics in such limits will be very different from their vector multiplet counterpart, because $i)$ there are less candidates to drive a decompactification limit and $ii)$ instantons will significantly modify the EFT when proceeding along such trajectories. In particular, as we will see they will modify the tensions of the 4d strings, which are the second key ingredient of the Emergent String Conjecture.

As a matter of fact, considering the spectrum of 4d strings that are sensitive to the hypermultiplet sector gives a new perspective on the whole analysis. Note that for each 4d string made up from a $D(p+2)$-brane wrapping an internal $(p+1)$-cycle of X there will be a D-brane instanton tower constructed by wrapping m Dp-branes on the same cycle. A limit in which the instanton action asymptotically vanishes corresponds to a similar behaviour for the string tension. As a result, in those regions of the hypermultiplet space in which

some strings become light certain instanton sectors become very relevant. This poses an interesting challenge for the Emergent String Conjecture, which states that there should be a unique string dominating the EFT breakdown. In the infinite distance trajectories which we analyse several strings become asymptotically tensionless, but there is one which is the lightest, and the corresponding instanton corrections dominate over the classical action and the remaining corrections. It turns out that the effect of such corrections is to boost the speed at which the lightest string becomes tensionless, by bending the trajectory and making it move along the 4d dilaton direction. Other 4d strings also tend to a tensionless limit along the trajectory, but at a slower, universal rate which makes them irrelevant for the EFT breakdown. Finally, quantum corrections are such that from a certain point on the trajectory the scale of the lightest string coincides with the scale of a KK tower of states, similarly to the equi-dimensional limits analysed in [19]. In this way the lightest D-brane 4d string along the trajectory, dual to a fundamental IIB string, is able to realise the Emergent String Conjecture.

We apply a similar analysis to the vector multiplet moduli space of type I string theory on $K3 \times T^2$. This compactification yields a 4d $\mathcal{N} = 2$ EFT, but the vector multiplet sector contains both Kähler and complex structure moduli, and it is subject to several quantum corrections. Therefore its infinite distance limits are not obviously captured by the classifications made in [6, 19]. We consider several limits in this setup, taking into account the quantum corrections to the Kähler potential computed in [34]. In all such limits we observe plain decompactification limits. When compared to the analysis performed in dual setups this seems to raise a discrepancy for one of them.

The rest of the paper is structured as follows. In section 2, we describe the type IIB hypermultiplet moduli space and we introduce the different classical infinite distance trajectories that we later analyse in section 3. In there, we elaborate on the correspondence between 4d string tensions and quantum corrections and use it as an organising principle to see how at each of the trajectories endpoints a weakly-coupled fundamental type II string emerges. In section 4 we analyse infinite distance trajectories in the vector multiplet sector of type I strings on $K3 \times T^2$, obtaining always decompactification limits. For one of them, we point out a mismatch with results obtained in a dual setup. We draw our conclusions in section 5.

Several technical details have been relegated to the appendices. Appendix A keeps track of the relations between the 4d Planck mass and the string scale in the different frames used in the main text. Appendix B collects all the necessary expressions for the quantum corrected metric in hypermultiplet moduli space. Appendix C reviews how to define a contact potential for mutually local D-brane instantons including D3-branes.

2 Limits in type IIB hypermultiplet moduli space

Let us consider type IIB string theory compactified on a Calabi-Yau threefold X. The low energy four-dimensional EFT exhibits $\mathcal{N} = 2$ supersymmetry with a moduli space $\mathcal{M}$ that factorises in hyper- and vector multiplet sectors:

$$\mathcal{M} = \mathcal{M}_{\text{VM}} \times \mathcal{M}_{\text{HM}}. \tag{2.1}$$

On the one hand, the vector multiplet moduli sector $\mathcal{M}_{\text{VM}}$ is a $h^{2,1}$-dimensional special Kähler manifold. On the other hand, each hypermultiplet transforms in the fundamental representation of the SU(2) R-symmetry group, endowing $\mathcal{M}_{\text{HM}}$ with a quaternionic-Kähler structure. While the possible infinite distance limits in $\mathcal{M}_{\text{VM}}$ can be classified via mixed Hodge structure methods [6, 11, 35], the hypermultiplet sector has been comparatively largely unexplored in the context of the Swampland Distance Conjecture, except for [15] (see also [17] for a study of the type IIA mirror dual). In this section and the next, this part of the moduli space will be the main focus of our study.

In a type IIB CY compactification, hypermultiplets descend from two different sources. The first one is the universal hypermultiplet, whose bosonic content is comprised of four real scalar fields: the dilaton ϕ and the C_0-form, which combine into the axio-dilaton $\tau = C_0 + ie^{-\phi}$, and the two axions b^0 and c^0, dual to the four-dimensional components of the NS-NS and R-R two-forms B and C_2, respectively. The remaining hypermultiplets are then parameterised by $h^{1,1}$ different complexified Kähler moduli, defined as

$$z^a = M_s^2 \int_{\gamma^a} B + iJ \,, \tag{2.2}$$

where J stands for the CY Kähler form, $\text{Span}(\gamma^a) = H_2(X, \mathbb{Z})$ is a basis of integral two-cycles, and $M_s = (2\pi\sqrt{\alpha'})^{-1}$ the string scale. To define *bona fide* hypermultiplets these moduli partner with two axions, c^a and d^a, related to the R-R potentials C_2 and C_4 via

$$c^a = M_s^2 \int_{\gamma^a} C_2 \,, \qquad d_2^a = M_s^2 \int_{\gamma^a} C_4 \,, \tag{2.3}$$

with d^a being the four-dimensional dual of d_2^a. The vacuum expectation values of all these $4(h^{1,1} + 1)$ real scalar fields parameterise the hypermultiplet moduli space $\mathcal{M}_{\text{HM}}$, that can be summarised as:

$$
\begin{array}{llll}
\text{universal hypermultiplet} & \tau = C_0 + ie^{-\phi}, & b^0, & c^0 \,, \\
h^{1,1} \text{ hypermultiplets} & z^a = b^a + it^a, & c^a, & d^a \,.
\end{array}
$$

In the following we choose to work in the so-called string frame, for which the ten-dimensional supergravity action takes the form

$$S_{10D} = 2\pi M_s^8 \int_{\mathbb{R}^{1,3} \times X} d^{10}x \sqrt{-G} e^{-2\phi} R_{(10)} + \dots \tag{2.4}$$

As we are interested in paths in the moduli space leading to classical infinite distances, we focus on the non-periodic coordinates, the saxions:

$$\tau_2 = \text{Im} \, \tau = e^{-\phi} \,, \qquad t^a \,, \qquad a = 1, \dots h^{1,1}(X) \,, \tag{2.5}$$

and for simplicity we will only consider trajectories for which the vacuum expectation values of all axions vanish.

At the classical level, the hypermultiplet moduli space metric reads [36]:

$$ds^2_{\mathcal{M}_{\text{HM}}} = \frac{1}{2} \left(d\varphi_4\right)^2 + g_{a\bar{b}} dz^a d\bar{z}^b + (\text{axions}) \,, \tag{2.6}$$

where φ_4 stands for the 4d dilaton defined as

$$e^{-2\varphi_4} = \frac{1}{2}\tau_2^2 \, \mathcal{V}(t^a) \,, \tag{2.7}$$

with $\mathcal{V}$ the volume of X. This designates φ_4 as the four-dimensional physical field, rather than its ten-dimensional cousin τ_2. It moreover defines a preferred coordinate for the classical metric of the hypermultiplet moduli space, as it factorises from the Kähler moduli z^a. The classical metric $g_{a\bar{b}}$ for the latter can be obtained from a prepotential, and can be identified with that of the vector multiplet sector $\mathcal{M}_{\mathrm{VM}}^{\mathrm{IIA}}$ in type IIA compactified on the same Calabi-Yau.

Having summarised the classical structure of $\mathcal{M}_{\mathrm{HM}}$, we turn our attention to regions at its boundaries. More precisely, we consider paths of infinite distance in the saxionic directions φ_4 and t^a, aiming to understand the metric structure around its endpoints. Particularly interesting are those paths in which the classical 4d Planck mass

$$M_{\mathrm{Pl}}^2 = 4\pi e^{-2\varphi_4} M_s^2 = 2\pi\tau_2^2 \, \mathcal{V}(t^a) M_s^2 \,, \tag{2.8}$$

is kept fixed with respect to the string scale M_s. Indeed, such a constraint amounts to keep the 4d dilaton fixed, so that classically one moves in a copy of the vector multiplet $\mathcal{M}_{\mathrm{VM}}^{\mathrm{IIA}}$ obtained from type IIA compactified on X. Moreover, the D-instanton actions take the form

$$S_{\mathbf{k}} = 2\pi e^{-\varphi_4} e^{\mathcal{K}/2}|Z_{\mathbf{k}}| + 2\pi i(\text{axions}) \,, \tag{2.9}$$

where $e^{\mathcal{K}/2}|Z_{\mathbf{k}}|$ is a normalised central charge function that only depends on the Kähler moduli z^a and that, when we compactify type IIA on X, corresponds to the mass of the D-particles in 4d Planck mass units, see e.g. [13]. Therefore, as stressed in [15, 17], one may use the classification of infinite distance points performed in [6, 11, 13] to see that at endpoint regions of infinite distance geodesics with φ_4 constant, towers of instantons will develop an exponentially fast vanishing action. As such, they will significantly modify the classical metric, to the point that they could even remove the infinite distance at the quantum level [15].

Recently, an alternative classification of infinite distance geodesics in $\mathcal{M}_{\mathrm{VM}}^{\mathrm{IIA}}$ has been carried out in [19]. A key aspect of this classification is to distinguish those infinite distance paths along which the spectrum of four-dimensional particles signals a decompactification limit, against those where this does not happen. The latter, dubbed equi-dimensional limits, were conjectured to be dual to a weakly-coupled string theory, described by either a type II or heterotic string asymptotically tensionless along the path [19].

One of the main purposes of this paper is to test the Emergent String Conjecture of [19] beyond the vector multiplet moduli space of a Calabi-Yau. In particular, this section and the next will confront it with different infinite distance paths in the hypermultiplet moduli space $\mathcal{M}_{\mathrm{HM}}$ of type IIB on a Calabi-Yau X, and study which insight this brings into the analysis performed in [15].

From the discussion above, it is clear that a path in which the conjecture is trivially satisfied is $\varphi_4 \to -\infty$. There, keeping all the Kähler moduli constant, we are led to the

weak-coupling region $\tau_2 \to \infty$ in which the perturbative, D-brane and NS5-brane instanton corrections are irrelevant. Finally, from (2.8) it is trivial to see that $\frac{M_s}{M_{\text{Pl}}} \to 0$, and so the weakly-coupled string realising the Emergent String Conjecture is nothing but the type IIB fundamental string. The opposite limit can be seen as an S-dual analogue.

As mentioned above a more interesting set of paths are those for which φ_4 is constant, and so quantum corrections induced by D-instantons will significantly modify the metric. Contrary to paths in vector multiplet moduli space, now the only obvious candidates to yield towers of massless particles are Kaluza-Klein modes and string winding modes. As a result one can build many more examples of equi-dimensional limits, providing further non-trivial tests to the Emergent String Conjecture.

The D1-brane limit

One particular path that has been investigated in detail in [15] corresponds to the large-volume $\mathcal{V} \to \infty$, strong-coupling, Im $\tau \to 0$, limit. This trajectory can be parameterised by a variable $\sigma \in \mathbb{R}$, upon imposing the following scalings to the saxionic coordinates:

$$\text{D1:} \qquad \tau_2(\sigma) = \tau_2(0)\, e^{-\frac{3}{2}\sigma}\,, \qquad t^a(\sigma) = t^a(0)\, e^{\sigma}\,, \qquad \sigma \to \infty\,, \tag{2.10}$$

where $t^a(0)$, $\tau_2(0)$ denote the starting point of the path in the interior of the moduli space. At the classical level, these scalings correspond to a constant 4d dilaton, thus ensuring that the ratio between the Planck mass and the string scale does not change:

$$\frac{M_s}{M_{\text{Pl}}} \sim \text{const}\,. \tag{2.11}$$

While the Planck mass remains constant along the path, other mass scales might of course vary. One of them is the Kaluza-Klein scale

$$\frac{M_{\text{KK}}^2}{M_{\text{Pl}}^2} \sim e^{-\sigma}\,, \tag{2.12}$$

which one might think leads to a decompactification limit. However, this is not so if one takes into account the tension of D1-strings extended along four dimensions, which also vanishes asymptotically. In Planck mass units, we find:

$$\frac{T_{\text{D1}}}{M_{\text{Pl}}^2} = \tau_2 \frac{M_s^2}{M_{\text{Pl}}^2} \sim e^{-\frac{3}{2}\sigma} \longrightarrow 0\,, \tag{2.13}$$

and it therefore decreases faster than the Kaluza-Klein scale. Following the reasoning of [19], this limit qualifies as equi-dimensional, and the obvious candidate for the emergent string is the above D1-brane. Indeed, one can check that any other 4d string obtained by dimensional reduction will be heavier. In the following, as D1-strings are the lightest strings becoming tensionless in this limit, we will dub the path (2.10) the **D1 limit**.

While the presence of these tensionless strings were already noted in [15], its significance for the low energy physics was not investigated. The associated effective theory was then obtained by truncating the spectrum of the fundamental type IIB string to its zero modes

and integrating out all massive states. The appearance of a tensionless string that is not the fundamental string therefore signals a breakdown of the theory, as a tower of states with masses below the cutoff of the original EFT, given by M_s, arises. To capture the four-dimensional physics, one should therefore look at the effective theory that can be obtained from the D1-string instead of the effective theory of the fundamental string we started with. This philosophy is in line with the idea of emergent strings at points of infinite distance in moduli space [9, 16, 19].

To get a handle on the effective theory coming from the D1-string, we can make use of S-duality, which in type IIB exchanges D1- and fundamental strings. Thus the effective theory that governs the D1 limit (2.10) is just a variant of a compactification of the usual type IIB supergravity. The original limit is however subject to a large number of non-perturbative instanton corrections which significantly modify the original moduli space metric. As we will see, a similar statement will hold after S-duality. Hence, any classical statements made in this section will have to be reconsidered when taking quantum corrections into account.

S-duality and fundamental strings

To understand the low energy physics in the D1 limit, one can use an S-duality transformation to obtain a limit in which fundamental strings become tensionless. Here, the four-dimensional EFT inherits S-duality from the $\mathrm{SL}(2,\mathbb{Z})$ symmetry of type IIB string theory, which amounts to

$$\tau \longrightarrow -\frac{1}{\tau}\,, \qquad t^a \longrightarrow |\tau| t^a\,, \qquad \begin{pmatrix} b^a \\ c^a \end{pmatrix} \longrightarrow \begin{pmatrix} -c^a \\ b^a \end{pmatrix}\,. \tag{2.14}$$

Importantly, the 4d dilaton (2.7) is not invariant under this transformation, $e^{-2\varphi_4} \to \tau_2^{-1} e^{-2\varphi_4}$, and so the quotient M_s/M_{Pl} will vary from one frame to the other. To keep track of the changes, let us denote the string scale after S-duality by M_s' which — as reviewed in appendix A—is related to the original string scale by $M_s'^2 = \tau_2 M_s^2$. Unsurprisingly, this resembles the tension of D1-strings in the D1 limit as upon S-duality those are exchanged with fundamental strings. Applying (2.14) to the path (2.10), one finds the following path in moduli space

$$\text{F1:} \qquad \tau_2'(\sigma) = \tau_2'(0)\, e^{+\frac{3}{2}\sigma}\,, \qquad t'^a(\sigma) = t'^a(0)\, e^{-\frac{\sigma}{2}}\,, \qquad \sigma \to \infty\,, \tag{2.15}$$

along which the ratio M_s'/M_{Pl} no longer remains constant but instead decreases. Indeed, in the transformed limit (2.15) we thus get a tensionless fundamental string

$$\frac{T_{\mathrm{F1}}'}{M_{\mathrm{Pl}}^2} = \frac{M_s'^2}{M_{\mathrm{Pl}}^2} = e^{2\varphi_4'} \sim e^{-\frac{3}{2}\sigma} \longrightarrow 0\,. \tag{2.16}$$

The fundamental string becoming tensionless fastest, we refer to (2.15) as the **F1 limit**. Since this path in moduli space drives us to a region of weak coupling and small volume, we again expect this limit to be heavily quantum corrected. In particular, the small-volume point might not belong to the quantum Kähler moduli space. We come back to this issue and its implications in the next section.

T-dualities and D3-branes

It was shown in [19] that to reach an infinite distance point while keeping a constant volume, the Calabi-Yau manifold must exhibit either an elliptic, K3- or a T^4-fibration, such that the fibre shrinks to zero size in that limit. Before delving into quantum corrections, we would like to consider a limit of this sort, which moreover is related to the previous two by some duality.

On the one hand, for K3- and T^4-fibrations a possible duality to the D1 limit holding at the quantum level is rather difficult to establish in general, if existent at all. On the other hand, the case of an elliptic fibration has a well-defined duality transform where the geometric limit of vanishing fibre can be related to the D1 limit — and by extension to the F1 limit. As such, one may use it as a starting point to investigate other (quantum-corrected) infinite distance limits in hypermultiplet moduli space in the spirit of [19].

To establish such a duality, we will assume that the Calabi-Yau X admits an elliptic fibration over a two complex-dimensional Kähler base, B_2:

$$\pi : \ X \longrightarrow B_2 \,, \tag{2.17}$$

with fibres denoted by $\mathcal{E}$. As we are interested in well-behaved compactifications, we require the elliptic fibration to be smooth so that the base admits at most divisors with singularities of type I_1 in the Kodaira-Néron classification.

We can then divide the set of divisors generating the Kähler cone into vertical divisors

$$D_\alpha = \pi^* \tilde{D}_\alpha \,, \qquad \alpha = 2, \dots h^{1,1}(B_2) + 1 \,, \tag{2.18}$$

obtained by pulling back the Kähler cone generators $\tilde{D}_\alpha$ of the base to the full fibration, and the divisor,

$$D_1 = S_0 + \pi^* c_1(B_2) \,, \tag{2.19}$$

related to the zero section S_0 and the first Chern class of the base. The Poincaré dual of these divisors, J_α and J_1 respectively, are used to expand the Kähler form of the elliptic fibration in terms of $h^{1,1}(B_2) + 1$ moduli:

$$J = t^1 J_1 + \sum_{\alpha=2}^{h^{1,1}(B_2)+1} t^\alpha J_\alpha \,, \tag{2.20}$$

where the first modulus t^1 parametrises the volume of the fibre.

The limit we want to consider is then the regime where the volume of the elliptic fibre vanishes while keeping the ratio between the Planck mass and the string scale fixed. In the original D1 limit (2.10) all moduli are blown up, and we can therefore obtain a shrinking fibre by performing a double T-duality:

$$t^1 \to \frac{1}{t^1} \,. \tag{2.21}$$

However, for non-trivial fibrations we have to account for possible I_1-singularities and curvature of the base. This simple double T-duality should then be replaced by a certain

Fourier-Mukai transform[1] [37], which has been shown to corresponds to a monodromy action on the Kähler moduli space [38].

In terms of the complexified Kähler moduli, the Fourier-Mukai transform acts as [39]:

$$z^1 \longrightarrow -\frac{1}{z^1}\,, \qquad z^\alpha \longrightarrow z^\alpha + \frac{1}{2}K^\alpha z^1 + \frac{K^\alpha}{2z^1}\,, \tag{2.22}$$

where K^α denotes the coefficients of the first Chern class of the base expanded in the Kähler cone basis, $c_1(B_2) = K^\alpha J_\alpha$, and account for the non-trivial fibration structure. To keep a constant string-to-Planck-scale ratio the dilaton further needs to be adjusted. As shown in appendix A the correct transformation is given to leading order by:

$$\tau_2 \longrightarrow |z^1|\tau_2 + \dots \tag{2.23}$$

The ellipses correspond to terms involving K^α which will not play any rôle in the limit we consider. In this new frame the dual path indeed corresponds to shrinking the elliptic fibre and blowing up the volume of the base while going to strong coupling, as can be seen from the new scalings:

$$\text{D3:} \qquad \tau_2'' = e^{-\frac{1}{2}\sigma}\tau_2''(0)\,, \qquad t''^1(\sigma) = e^{-\sigma}t''^1(0)\,, \qquad t''^\alpha = e^\sigma t''^\alpha(0)\,. \tag{2.24}$$

Again, to facilitate the comparison quantities in this frame will be double primed. Note that the limit we obtain corresponds to a so-called limit with co-scaling in the nomenclature of [19]: while the string-to-Planck-scale ratio remains constant, the volume blows up and is compensated by a scaling to strong coupling.

Moreover, similarly to what happened with S-duality the lightest string is not a D1-brane in that limit. Under the double T-duality, D1-branes are mapped to D3-branes wrapping the elliptic fibre, and these are the lightest ones becoming asymptotically tensionless. Indeed, in the limit (2.24) one obtains:

$$\frac{T''_{\text{D3}|\mathcal{E}}}{M_{\text{Pl}}^2} = \tau_2''t''^1\left(\frac{M_s''}{M_{\text{Pl}}}\right)^2 \sim e^{-\frac{3}{2}\sigma} \longrightarrow 0\,. \tag{2.25}$$

We use the notation $T''_{\text{D3}|\mathcal{E}}$ to emphasise that in the dual frame, tensionless strings at the endpoint come from D3-branes wrapping the fibre. As any other string can be shown to remain heavier, we again use the nomenclature denoting a frame by its asymptotically lightest string and refer to (2.24) as the **D3 limit**.

To summarise, we have related in this section three different classical limits where different types of 4d strings become tensionless by using part of the type IIB U-duality group. The scalings of the moduli in all three regimes are summarised in table 1, and the dualities relating them illustrated in figure 1. As discussed in the next section, the classical approximation breaks down along the path in each frame, and one has to take quantum corrections into account. As we will see, the dominant quantum effects are different in each frame and they will play a key rôle in understanding the four-dimensional EFT physics. In particular they will allow us to refine previous arguments in [15] and to show how the emergent strings proposal of [19] is realised in our setup.

---

[1]Such a transform has already been used in the SDC context in [13] to relate asymptotically massless D-brane particle states at large fibre volume to the corresponding states at small volume.

Limit	τ_2	t^1	t^α
D1	$e^{-3/2\sigma}$	e^{σ}	e^{σ}
F1	$e^{3/2\sigma}$	$e^{-\sigma/2}$	$e^{-\sigma/2}$
D3	$e^{-1/2\sigma}$	$e^{-\sigma}$	e^{σ}

Table 1. Scaling of the saxionic fields in the different duality frames.

D1 limit
$t^a \to \infty \,, \tau_2 \to 0$

F1 limit
$t'^a \to 0 \,, \tau'_2 \to \infty$

S-duality

Fourier–Mukai

$\mathcal{M}_{\mathrm{HM}}$

D3 limit
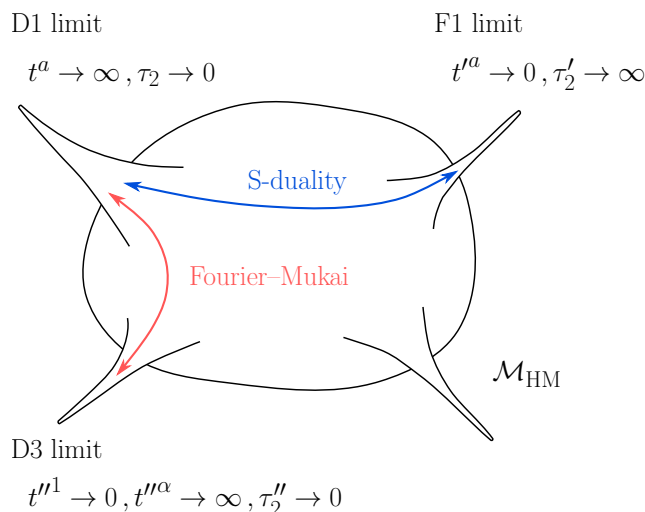$t''^1 \to 0 \,, t''^\alpha \to \infty \,, \tau''_2 \to 0$

Figure 1. Different classical infinite distance limits discussed in section 2, taken along the hypermultiplet sector of type IIB Calabi-Yau compactifications. The name of the frame refers to the lightest string in each of the limits.

3 Quantum corrections to moduli space geometry

Having established the classical infinite distance limits that we want to investigate, we now address how quantum corrections affect the hypermultiplet moduli space along them. As we will see, in each of the limits a different set of quantum corrections will significantly change the classical moduli space geometry, along the lines of [15]. As we are particularly interested in testing the Emergent String Conjecture in this setup, parallel to this analysis we will keep track of how the different 4d string tensions behave along the limits. In this sense, a useful organising principle to compare non-perturbative quantum corrections with string tensions will be the following: whenever a non-perturbative correction becomes significant, the tension of a certain 4d string becomes lighter than M_s^2, and the other way around.[2] This will allow us to sharpen the arguments of [15], and to see how each of the different limits — which remain of infinite distance — leads to a weakly-coupled type II string theory emerging at the trajectory's endpoint. Notice that as a consequence of this emergence, at the far end of the trajectory the hypermultiplet metric will have to be of the classical form (2.6). This simplification will also be crucial in interpreting our results.

---

[2]This reflects into different regions of the moduli space as shown in figure 2 below.

3.1 Quantum corrections in the D1 limit

Quantum corrections along the D1 limit (2.10) were considered in detail in [15]. Particular attention was given to the corrections to the hypermultiplet metric, which in general can be determined by studying the quaternionic-Kähler structure of $\mathcal{M}_{\text{HM}}$ in twistor space (see the reviews [40, 41] and references therein). In the case of the D1 limit this structure simplifies, because corrections due to D3-, D5-and NS5-brane instantons can be safely neglected, and one can encode the corrections to the metric in a contact potential [42] first computed in [43]. The structure of this contact potential χ reads

$$\chi = \chi_{\text{cl}} + \chi_{\text{corr}} \,, \tag{3.1}$$

where

$$\chi_{\text{cl}} = \frac{1}{12} \tau_2^2 \mathcal{K}_{abc} t^a t^b t^c \tag{3.2}$$

is the classical contribution to the potential, with $\mathcal{K}_{abc}$ the triple-intersection numbers of X and

$$\chi_{\text{corr}} = \frac{\tau_2^2}{8(2\pi)^3} \sum_{\mathbf{k} \geq \mathbf{0}} n_{\mathbf{k}}^{(0)} \sum_{(m,n) \in \mathbb{Z}^2 \backslash 0} \frac{1 + 2\pi |m\tau + n| k_a t^a}{|m\tau + n|^3} e^{-S_{m,n}^{\mathbf{k}}} \,, \tag{3.3}$$

the relevant perturbative and non-perturbative corrections. Here $\mathbf{k}$ is a vector of $h^{1,1}$ entries $k_a \in \mathbb{N}$, such that $k_a \gamma^a$ spans homology classes in $H_2^+(X, \mathbb{Z})$. Terms with $\mathbf{k} \neq \mathbf{0}$ represent corrections coming from Euclidean (m, n)-strings wrapping two-cycle classes with non-vanishing genus-zero Gopakumar-Vafa invariant $n_{\mathbf{k}}^{(0)}$, with action

$$S_{m,n}^{\mathbf{k}} = 2\pi k_a \left(|m\tau + n| t^a - imc^a - inb^a \right) \,. \tag{3.4}$$

The contribution from the term $\mathbf{k} = \mathbf{0}$ corresponds to the D(-1)-brane corrections, together with the one-loop corrections on g_s and α', if one sets $n_{\mathbf{k}=\mathbf{0}}^{(0)} = -\chi_E(X)$, that is to (minus) the Euler characteristic of X.

It was found in [15] that:

i) $\chi \sim \chi_{\text{corr}} \gg \chi_{\text{cl}}$ as one proceeds along the D1 limit. Since χ_{cl} is nothing but the classical 4d dilaton (2.7), one can see χ as its quantum-corrected version [44], and in particular the quantum-corrected string-to-Planck-scale ratio. Therefore, even if the D1 limit was designed to keep such a ratio constant at the classical level, D-instanton corrections make it blow up. One can moreover identify this growth with the number of instanton species $N_{sp}(\sigma)$ that acquire a small action at a given point σ along the trajectory (2.10), so that

$$M_s \sim \frac{M_{\text{Pl}}}{\sqrt{N_{sp}(\sigma)}} \,, \tag{3.5}$$

and the string scale can be interpreted as some sort of species scale for D-instantons.

ii) Even if in general the presence of χ_{corr} distorts the hypermultiplet metric away from its classical form (2.6), as one proceeds along the D1 limit a simplification occurs, in

the sense that the scalings of the different metric factors coincide with those of

$$\mathrm{d}s^2_{\mathcal{M}_{\mathrm{HM}}} = \frac{1}{2}\left(\mathrm{d}\log\chi\right)^2 + g_{a\bar{b}}\mathrm{d}z^a\mathrm{d}\bar{z}^b + (\text{axions})\,, \tag{3.6}$$

$$g_{a\bar{b}} = \partial_{z^a}\partial_{\bar{z}^b}K \qquad \text{with} \qquad K = -\log\chi\,, \tag{3.7}$$

where χ is to be evaluated for large values of σ in (2.10).

To determine such scalings one needs to evaluate the dependence of χ and its derivatives with the trajectory coordinate σ. The strategy followed in [15] made use of the standard split of χ_{corr} into three distinct sources [45],

$$\chi_{\mathrm{corr}} = \chi_{\mathrm{pert}} + \chi_{\mathrm{WS}} + \chi_{\mathrm{D}}\,. \tag{3.8}$$

Here the first term is the perturbative piece, interpreted as the one-loop corrections in α' and g_s,

$$\chi_{\mathrm{pert}} = -\frac{\chi_E(X)}{4(2\pi)^3}\left(\zeta(3)\tau_2^2 + \frac{\pi^2}{3}\right)\,, \tag{3.9}$$

and controlled by the Euler density of the compactification manifold $\chi_E(X)$. The second contribution encodes worldsheet instantons associated to any effective two-cycle, and can be written as a sum weighted by genus-zero Gopakumar-Vafa invariants $n_{\boldsymbol{k}}^{(0)}$:

$$\chi_{\mathrm{WS}} = \frac{\tau_2^2}{4(2\pi)^3}\sum_{\boldsymbol{k}>0} n_{\boldsymbol{k}}^{(0)}\mathrm{Re}\left(\mathrm{Li}_3(e^{2\pi i k_a z^a}) + 2\pi k_a t^a \mathrm{Li}_2(e^{2\pi i k_a z^a})\right)\,, \tag{3.10}$$

with $\mathrm{Li}_s(x) = \sum_{r>0} r^{-s}x^r$ the polylogarithm function. Finally, the last term corresponds to D1- and D(-1)-instanton corrections:

$$\chi_{\mathrm{D}} = \frac{\tau_2}{8\pi^2}\sum_{k_\Lambda\neq 0} n_{\boldsymbol{k}}^{(0)}\sum_{m>0}\frac{|k_\Lambda z^\Lambda|}{m}\cos(2\pi m k_\Lambda\zeta^\Lambda)K_1(2\pi m|k_\Lambda z^\Lambda|\tau_2)\,, \tag{3.11}$$

$$z^\Lambda = (1, z^a)\,, \qquad \zeta^\Lambda = (\tau_1, \tau_1 b^a - c^a)\,,$$

with K_1 the modified Bessel function. The sum is performed over $k_\Lambda = (k_0, k_a) \neq 0$, such that $k_0 \in \mathbb{Z}$ and $k_a \in \mathbb{N}$, interpreted as the D(-1) and D1-instanton charges, respectively.

As one proceeds along the trajectory (2.10) the three terms in (3.8) will have very different behaviours. This is already to be expected from the scaling of the different instanton actions involved, namely

$$S_{k_0}^{\mathrm{D}(-1)} = \tau_2 k_0 \stackrel{(2.10)}{\longrightarrow} e^{-\frac{3}{2}\sigma}\,, \tag{3.12a}$$

$$S_{\boldsymbol{k}}^{\mathrm{D1}} = \tau_2 k_a z^a \stackrel{(2.10)}{\longrightarrow} e^{-\frac{1}{2}\sigma}\,, \tag{3.12b}$$

$$S_{\boldsymbol{k}}^{\mathrm{WS}} = k_a z^a \stackrel{(2.10)}{\longrightarrow} e^{+\sigma}\,, \tag{3.12c}$$

and so the corrections to the moduli space metric mainly arise due to D1- and D(-1)-instantons. Indeed, one can check that along the D1 limit the contributions of χ_{pert} and

χ_{WS} are negligible compared to that of χ_{D}, and that the growth of the latter can be described by using that for small arguments, the modified Bessel function behaves as

$$K_1(x) \sim \frac{1}{x} \qquad \text{as } x \to 0\,.$$ (3.13)

At the end of the day, one obtains that the significant contributions to the contact potential take the form

$$\chi \sim \frac{1}{12}\tau_2^2 \mathcal{K}_{abc}t^a t^b t^c + \frac{1}{96\pi}\left[-\chi_E(X) + \sum_{\mathbf{k}>0}^{\Omega} n_{\mathbf{k}}^{(0)}\right]\sum_{k_0}^{k_0^{\max}} 1\,,$$ (3.14)

where the sum over $k_\Lambda = (k_0, k_a)$ is cut off by $k_0^{\max}$ and Ω, which contain those instanton charges for which the condition

$$\text{Re}(S_{k^\Lambda}^{\text{D}}) = 2\pi|k_0 e^{-\frac{3}{2}\sigma} + k_a e^{-\frac{1}{2}\sigma}| \ll 1$$ (3.15)

is satisfied. This cutoff is introduced to parametrize the approximation of the Bessel function in (3.14) and can physically be interpreted as distinguishing instantons that contribute significantly to χ from those that are exponentially suppressed. As mentioned previously, as we move along the trajectory (2.10), a growing number of instantons acquire a small action and hence the second term in (3.14) eventually dominates over the classical part. Finally, in [15] the condition (3.15) was approximated as

$$\text{Re}(S_{k^\Lambda}^{\text{D}}) = |k_0 e^{-\frac{3}{2}\sigma} + k_a e^{-\frac{1}{2}\sigma}| \leq e^{-\epsilon\sigma}\,,$$ (3.16)

for $\epsilon \in \mathbb{R}$ some fixed parameter. This imposes the following cutoff on the instanton sum

$$k_0 \leq e^{(3/2-\epsilon)\sigma}\,, \qquad k_a \leq e^{(1/2-\epsilon)\sigma}\,.$$ (3.17)

Then, whenever $\epsilon > 0$ the hard cutoff on the action of the instantons in the sum decreases gradually. This parallels the behaviour of wrapped D-brane particles contributing to loop corrections of the moduli space metric of type II vector multiplet moduli spaces, as considered in the emergence proposal [6]. In that case, the gradually decreasing cutoff is identified with the species scale, leading to a break down of the associated effective theory. It was then found in [15] that for $\epsilon > 0$ the quantum corrections to the metric remove the infinite distance, while the path (2.10) remains of infinite length for $\epsilon = 0$.

In the following we would like to revisit the approximation (3.15), in light of the connection between instantons of low action and the 4d strings of low tension, and the rôle the latter play in the Emergent String Conjecture.

Instantons and light strings

The two features found in [15] related to (3.5) and (3.6) are very suggestive from the viewpoint of the Emergent String Conjecture. However, the fundamental string associated to them is not the lightest 4d string, and so it cannot realise the Emergent String Conjecture. Indeed, as already mentioned, along the trajectory (2.10) a second effect occurs,
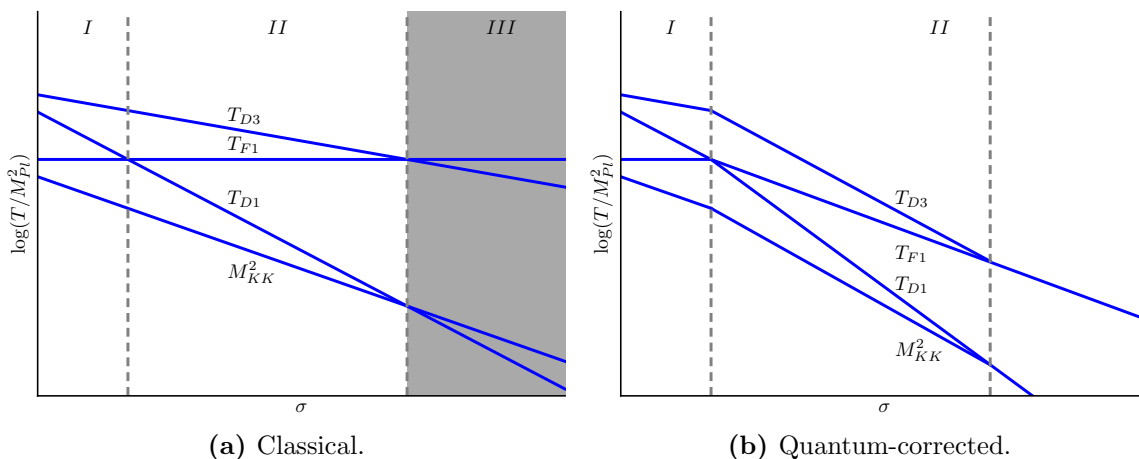
Figure 2. Tensions of the various 4d strings and the Kaluza-Klein scale along the path (2.10), and the three associated regions. Including quantum corrections Region III ceases to exist as the tension of a D3-string never goes below the string scale in that case.

namely that the D1-brane oscillation modes become lighter than those of the fundamental string. This sort of effect is not a coincidence, and goes hand in hand with instanton effects significantly modifying the classical metric. Indeed, the D1-brane tension goes below the fundamental string one in the region $\tau_2 < 1$, which is when D(-1)-brane instantons start to significantly modify the classical metric, as captured by (3.15). This is both true classically and after the corrections that change the string-to-Planck-scale ratio as in (3.5) have been taken into account.

Similarly, a D3-brane wrapped on an internal two-cycle of X will be seen as a 4d string whose tension in units of M_s^2 decreases as $t^a \tau_2 \sim e^{-\sigma/2}$, just like the action of a D1-brane instanton wrapped on the same two-cycle. One can then separate the trajectory (3.6) into three regions according to the relevance of D-brane instantons and their corresponding 4d strings:

Region I	$T_{\mathrm{F1}} < T_{\mathrm{D1}} < T_{\mathrm{D3}}$	$\chi_{\mathrm{cl}} > \chi_{\mathrm{D}}$
Region II	$T_{\mathrm{D1}} < T_{\mathrm{F1}} < T_{\mathrm{D3}}$	$\chi_{\mathrm{D}(-1)} > \chi_{\mathrm{cl}} > \chi_{\mathrm{D1}}$
Region III	$T_{\mathrm{D1}} < T_{\mathrm{D3}} < T_{\mathrm{F1}}$	$\chi_{\mathrm{D}(-1)}, \chi_{\mathrm{D1}} > \chi_{\mathrm{cl}}$

These three regions are illustrated for the classical tensions in figure 2a. We then see that, even if one cannot directly associate a scale to the action of an instanton, one can relate their magnitude to ratios of string scales. As we will see in section 3.3, in the present setup this relation can be easily understood in terms of the c-map to the vector multiplet sector of type IIA CY compactifications. It would be interesting to see if it can also be extended to compactifications with lower supersymmetry, along the lines of the analysis in [31].

As follows from the discussion of section 2, each of the strings above is dual to a critical fundamental string, and so their presence will non-trivially modify the effective field theory [16, 19]. One should therefore consider separately each of these regions when trying to extract the hypermultiplet metric from the corrected contact potential (3.1).

Indeed, Region I can be seen as a classical region of the trajectory in which the instanton corrections are not very significant, and the string scale is the standard cutoff scale below which we only have moduli and their Kaluza-Klein excitations. The contact potential, whose classical piece is derived from a supergravity action obtained from such string, can be straightforwardly trusted in such a region. In Region II the fundamental string is no longer the lightest, being replaced by the D1-string. Having all the D1-string oscillation modes below the string scale, it is a priori not clear whether the use of the contact potential is still valid. A more reasonable approach would be to perform an S-duality transform that exchanges D1- and F1-strings and extract the metric from the contact potential in this new duality frame. We implement such a strategy in the next subsection. It turns out that, due to the covariance of the contact potential, the information obtained in both frames is essentially the same, and so the approach of [15] can still be applied to analyse this region. The same conclusion does not hold, however, for Region III. In that region not only we have multiple 4d strings whose tension is below M_s^2, it also happens that the lightest of them, the D1-string, is below the Kaluza-Klein scale. As such the interpretation of the hypermultiplet metric in terms of the contact potential becomes more subtle, as we discuss below.

Given the differences between the above regions, it is unlikely that an expression like (3.14) with a simple cutoff condition of the form (3.15) will capture the physics in each of them. In the following we will seek a more suitable expression for the contact potential that will allow us to infer the corrections to the hypermultiplet metric in each region separately. To that end, one may consider the initial expression for the corrections given by (3.3) and, following [43, 45], split it in two terms:

$$\chi_{(-1)} = -\frac{\tau_2^2}{8(2\pi)^3} \chi_E(X) \sum_{(m,n)\in\mathbb{Z}^2\backslash 0} \frac{1}{|m\tau + n|^3} \,, \tag{3.18}$$

$$\chi_{(1)} = \frac{\tau_2^2}{8(2\pi)^3} \sum_{\mathbf{k}>0} n_{\mathbf{k}}^{(0)} \sum_{(m,n)\in\mathbb{Z}^2\backslash 0} \frac{1 + 2\pi|m\tau + n|k_a t^a}{|m\tau + n|^3} e^{-S_{m,n}^{\mathbf{k}}} \,. \tag{3.19}$$

Along the D1 limit (2.10) only terms with $n = 0$ contribute significantly to the sums that define both quantities. For the first one those give

$$\chi_{(-1)} \simeq -\frac{\tau_2^{-1}}{4(2\pi)^3} \chi_E(X)\zeta(3)\,, \tag{3.20}$$

while for the second one finds

$$\chi_{(1)} \simeq \frac{\tau_2^{-1}}{8(2\pi)^3} \sum_{\mathbf{k}>0} n_{\mathbf{k}}^{(0)} \sum_{m\neq 0} \frac{1 + 2\pi|m|\tau_2 k_a t^a}{|m|^3} e^{-2\pi|m|\tau_2 k_a t^a} \simeq -\frac{\tau_2^{-1}}{4(2\pi)^3}\zeta(3) \sum_{\mathbf{k}>0}^{\Omega} n_{\mathbf{k}}^{(0)}\,, \tag{3.21}$$

using that the series $\sum_{k>0} 1/k^3$ converges quickly to $\zeta(3)$ for the first few terms, and defining Ω such that $2\pi\tau_2 t^a k_a \ll 1$. We finally arrive at the expression

$$\chi_{\text{corr}} \simeq \frac{\tau_2^{-1}}{4(2\pi)^3}\zeta(3)\left[-\chi_E(X) + \sum_{\mathbf{k}>0}^{\Omega} n_{\mathbf{k}}^{(0)}\right],\tag{3.22}$$

which refines the correction term in (3.14), in the sense that it gives a precise estimate for $k_0^{\max} \sim \tau_2^{-1}$. Notice that this corresponds to the value $\epsilon = 0$ in the approximation (3.17) for the D(-1)-charge k_0, while for k_a the bound could be different. Using the results of [15] hints to a quantum-corrected trajectory of infinite length. In the next subsection we will rederive these results in an S-dual frame, and argue that the infinite distance indeed occurs but because the 4d dilaton varies along the quantum corrected trajectory. Infinite excursions along Kähler coordinates are, in contrast, obstructed by quantum corrections.

3.2 S-duality and the F1 limit

As already pointed out, as soon as $g_s > 1$ in the D1 limit we enter Region II, in which the D(-1)-instanton corrections start to be relevant and D1-strings become lighter than fundamental strings. Due to the appearance of this lighter string, the EFT that one derives from the fundamental string breaks down. The appropriate strategy to derive the hypermultiplet metric would then be to consider an S-dual frame in which the rôles of D1- and F1-strings have been interchanged, and therefore the lightest string is the fundamental one. Then, one may consider the contact potential corresponding to such fundamental string and derive the hypermultiplet metric from it. Equivalently, one may analyse the quantum corrections to the contact potential that arises along the F1 limit (2.15), as we will do in this section.

By construction, the contact potential χ transforms covariantly under any $\mathrm{SL}(2,\mathbb{Z})$ transformation. In particular under S-duality it transforms as

$$\chi \longrightarrow \chi' = \frac{\chi}{|\tau|}, \tag{3.23}$$

where as in section 2 we will use primes to denote quantities in the F1 limit. As a result, the quantum corrections contributing to χ are reshuffled and, e.g., the actions of worldsheet and D1-brane instantons are exchanged with one another:

$$S_{\mathbf{k}}^{\mathrm{WS}}(t') = S_{\mathbf{k}}^{\mathrm{D1}}(\tau, t). \tag{3.24}$$

It is easy to check that in the F1 limit the dominating corrections to the contact potential within (3.3) will come from χ_{pert} and χ_{WS}, and so it can be approximated by

$$
\begin{aligned}
\chi' \sim{}& \frac{1}{12}(\tau_2')^2 \mathcal{K}_{abc} t'^a t'^b t'^c - \frac{\chi_E(X)}{4(2\pi)^3}\zeta(3)(\tau_2')^2 \\
& + \frac{(\tau_2')^2}{4(2\pi)^3}\sum_{\mathbf{k}>0} n_{\mathbf{k}}^{(0)} \mathrm{Re}\left[\mathrm{Li}_3\left(e^{-2\pi k_a t'^a}\right) + 2\pi k_a t'^a \mathrm{Li}_2\left(e^{-2\pi k_a t'^a}\right)\right].
\end{aligned}
\tag{3.25}
$$

Then, using that at small arguments polylogarithms behave as

$$\mathrm{Li}_s(x) \sim x \quad \text{as} \quad x \to 0, \tag{3.26}$$

the dominating part of the contact potential is found to be

$$\chi' \sim \frac{1}{12}(\tau_2')^2 \mathcal{K}_{abc} t'^a t'^b t'^c + \frac{(\tau_2')^2}{32\pi^3}\zeta(3)\left[-\chi_E(X) + \sum_{\mathbf{k}>0}^{\Omega} n_{\mathbf{k}}^{(0)}\right]. \tag{3.27}$$

Taking into account the transformation (3.23), this reproduces the previous result (3.22). It follows that in the F1 limit perturbative α' corrections play the rôle that D(-1) effects had in the D1 limit while, as stated before, D1-brane instantons are replaced by worldsheet instantons. In the region in which the latter are subdominant, the corrected 4d dilaton behaves as

$$\chi' = \text{const.} (\tau_2')^2 \sim e^{3\sigma}. \tag{3.28}$$

By inserting this result in the metric (3.6), one sees that the 4d dilaton direction provides a constant contribution to the norm of the velocity vector along the trajectory (2.15). In the following we would like to argue that this behaviour still holds when we proceed along the trajectory deep into Region II and beyond, with the constant in (3.28) being a minimum quantum-corrected volume after which the Kähler coordinates cannot decrease any further. In this sense, quantum corrections would heavily deform the classical trajectory (2.15), preventing it to advance in the Kähler directions and inducing a variation of constant velocity along the 4d dilaton direction.

In order to support this claim let us consider a related setup, the quantum Kähler moduli space $\mathcal{M}_{QKM}$ describing the vector multiplet sector in type IIA string theory on a Calabi-Yau X, see [46]. Such a moduli space, recently considered in [19] in the context of infinite distance limits, is related to the one probed in the F1 limit by a standard c-map. As it is well known, mirror symmetry maps $\mathcal{M}_{QKM}$ to the classical complex structure moduli space $\mathcal{M}_{cs}$ of the mirror Calabi-Yau Y. The mirror map is defined near the large complex structure point, which can be chosen to be at $y = 0 \in \mathcal{M}_{cs}$. Close to this point the periods of the holomorphic three-form Ω_Y,

$$X^A = \int_{\gamma_A} \Omega_Y, \qquad F_B = \int_{\gamma^B} \Omega_Y, \qquad A, B = 0, \dots h^{2,1}(Y), \tag{3.29}$$

where (γ_A, γ^B) is a symplectic basis of three-cycles on Y, take the form

$$X^0 = 1 + \mathcal{O}(y^a), \qquad X^a = \frac{1}{2\pi i} \log(y^a) + \mathcal{O}(y^a), \qquad a = 1, \dots h^{2,1}(Y). \tag{3.30}$$

These can be used to define flat coordinates of $\mathcal{M}_{cs}$, that near the large complex structure point $y = 0$ take the form

$$z^a(y^a) = \frac{X^a}{X^0} = \frac{1}{2\pi i} \log(y^a) + \mathcal{O}(y^a). \tag{3.31}$$

Close to the point $y = 0$, the logarithmic term in (3.31) dominates and we can identify z^a with the classical complexified Kähler moduli of X via the classical mirror map $z^a = \frac{1}{2\pi i} \log y^a$. Away from $y = 0$ this naïve map will however start to fail, and it is the variables z^a, extended by analytic continuation, that must be considered as the coordinates of the full quantum Kähler moduli space of X. For instance, approaching a small volume point which naïvely would correspond to the limit $y^a \to 1$, one sees that the approximation $z = \frac{1}{2\pi i} \log y^a$ is no longer valid. As a consequence of the polynomial term, instead of going to the limit where $t^a = \text{Im } z^a \to 0$, the Kähler coordinates get frozen to a constant value,

and in the limit where the classical volume of the Calabi-Yau shrinks to zero, one ends up with a limit where the quantum volume is just a constant. For specific examples such as the quintic and other models with few Kähler moduli, this behaviour has been explicitly verified in the context of the SDC in [8, 14].

Via the c-map, one may embed the above IIA vector moduli space as a totally geodesic submanifold of $\mathcal{M}_{HM}$ of type IIB on the same Calabi-Yau, and then apply the above observations to the trajectory along the F1 limit. One sees that it is precisely when the worldsheet instantons should be as relevant as the perturbative α' corrections, or equivalently when the string scale goes below the Kaluza-Klein scale, that the Kähler moduli used to define the trajectory are no longer flat coordinates of the quantum moduli space. Using the appropriate coordinates one sees that the classical ones stop at a constant value:

$$t'^a \xrightarrow{(2.15)} t'^a_{QM} = \text{const} . \tag{3.32}$$

Nevertheless, the trajectory in moduli space does not stop, as it continues to progress at a constant rate along the 4d dilaton coordinate towards weaker and weaker coupling.

Going back to the original trajectory describing the D1 limit, this implies that quantum corrections shield the classical trajectory (2.10) from entering Region III in figure 2b, while instead inducing a geodesic variation along the 4d dilaton coordinate. In particular, this picture allows one to evaluate the sum over the D1-instantons charges in the D1 limit. For this, notice that since the quantum Kähler moduli in the F1 limit are constant, the worldsheet instanton actions are as well:

$$\text{Re}(S_{WS}) \xrightarrow{(2.15)} \text{const.} \tag{3.33}$$

Now, the D1-instanton actions in the dual D1 limit (2.10) are interchanged with those of worldsheet instanton actions via S-duality

$$\text{Re}(S_{D1}) = \tau_2 k_a t^a = k_a (t')^a = \text{const.} \tag{3.34}$$

Thus, as one moves along the trajectory (2.10), only the tower of D(−1)-instantons acquire an asymptotically vanishing action, while the D1-instanton actions asymptote to constant values. Notice that the action of the D(−1)-instantons itself is just given by the 10d dilaton. As argued in appendix B the 10d dilaton does not receive quantum corrections itself, allowing the D(−1)-instanton action to asymptotically vanish.

3.3 Quantum emergent strings

The emerging picture from the analysis of this section so far can be summarised as follows, see also figure 2. As we cross from Region I to Region II in the D1 limit, D(−1)-instantons start getting dominant and we should change the duality frame to consider the S-dual F1 limit, in which the fundamental string is the lightest. There the quantum corrections due to the D(−1)-instantons are mapped to α' perturbative corrections, which become stronger as we proceed towards smaller volumes in the F1 limit. Eventually perturbative and non-perturbative α' corrections prevent the quantum volume of the Calabi-Yau to

vanish, obstructing the trajectory to enter Region III. Instead, they induce a constant velocity variation along the 4d dilaton coordinate, sourcing an infinite distance geodesic towards weak coupling.

According to the Emergent String Conjecture, as we proceed along this trajectory there should be a unique string that becomes tensionless, which is dual to either a type II or a heterotic fundamental string. To see how this happens, let us consider the lightest string in the D1 limit. The tension of such a D1-string, including the effects of quantum corrections is given by

$$\frac{T_{D1}}{M_{\text{Pl}}^2} = \frac{T_{D1}}{M_s^2}\frac{M_s^2}{M_{\text{Pl}}^2} = \frac{T_{D1}}{M_s^2}\frac{1}{\chi}\,, \tag{3.35}$$

with χ the quantum-corrected 4d dilaton (3.3). Through (3.22), one can infer that it grows as $\chi \sim e^{3\sigma/2}$ in the far end of the trajectory. What remains to be determined is the scaling of the string tension in string units. Classically, it is given by

$$\frac{T_{D1}}{M_s^2} = \tau_2 \sim e^{-\frac{3}{2}\sigma} \tag{3.36}$$

that is by the variation of the 10d dilaton. By S-duality, the same relation should hold after quantum corrections. Finally, as argued in appendix B the 10d dilaton, seen as a coordinate in $\mathcal{M}_{\text{HM}}$, is not affected by quantum corrections. As a result (3.36) should hold along the whole trajectory, and we obtain a tensionless D1-string in the limit (2.10) with tension

$$\frac{T_{D1}}{M_{\text{Pl}}^2} \sim e^{-3\sigma}\,. \tag{3.37}$$

One may apply the same reasoning to see how the tension of other strings scale in the quantum-corrected regime. For instance, the tension of a D3-string wrapped on some curve $\gamma_a \subset X$ behaves as:

$$\frac{T_{D3}}{M_{\text{Pl}}^2} \sim \begin{cases} e^{-\sigma/2}\,, & \text{classically}\,, \\ \chi^{-1}\,, & \text{after quantum corrections}\,, \end{cases} \tag{3.38}$$

reflecting the fact that the tension of the D3-string in string units scales like the action of the D1-instanton wrapped on the same curve. As a consequence the quantum corrections increase the speed at which the D3-string becomes tensionless to $\chi^{-1} \sim e^{-3\sigma/2}$, but not as much as they do for the D1-string. Similar results hold for the other strings that can be obtained by wrapping branes on suitable cycles in X. Thus, all tension of 4d strings, including the tension of the F1-string which scales like $M_s^2/M_{\text{Pl}}^2 \sim e^{-3\sigma/2}$, are parametrically higher than the tension of the D1-string, which thus remains to be the lightest string also in the heavily quantum-corrected regime.

Via S-duality, the D1-string maps to the fundamental string probing an internal manifold of constant volume due to the quantum corrections. Thus, we see that in the D1 limit we find a tensionless string that is dual to a weakly-coupled fundamental type IIB string

with tension

$$\frac{T'_{F1}}{M_{\rm Pl}^2} \sim e^{-3\sigma}\,, \tag{3.39}$$

realising the Emergent String Conjecture in this infinite distance limit. In some sense this shows that the classical limit (2.10) that one initially wanted to reach does not exist as part of the quantum-corrected moduli space, and that is diverted to a limit in which a weakly-coupled, tensionless fundamental string is obtained. Notice that the endpoint of the trajectory is at finite distance in Kähler moduli space, since the quantum volume asymptotes to a constant. However, since the trajectory proceeds with constant velocity towards a weak 4d coupling limit, $\chi \to \infty$, the length of the trajectory remains infinite. As we comment in appendix B, this behaviour seems to be reflected by the metric components of the Kähler moduli, for which the infinite distance behaviour is removed when taking into account quantum corrections. On the other hand, the quantum-corrected metric component corresponding to τ_2 reproduces the classical scaling and therefore yields the infinite part to the distance in moduli space.

To complete the above picture, one has to ensure that the D1 limit does not yield a decompactification towards a higher-dimensional theory before the lightest string becomes tensionless. Approximating the Kaluza-Klein scale as

$$\frac{M_{\rm KK}^2}{M_{\rm Pl}^2} \simeq \frac{1}{\mathcal{V}(X)^{1/3}} \frac{M_s^2}{M_{\rm Pl}^2}\,, \tag{3.40}$$

and deducing the quantum-corrected scaling of the Kähler moduli via S-duality

$$t^a = \tau'_2 t'^a_{\rm QM} \sim e^{3/2\sigma}\,, \tag{3.41}$$

one obtains that the mass of KK modes scales as

$$\frac{M_{\rm KK}^2}{M_{\rm Pl}^2} \simeq e^{-3\sigma}\,. \tag{3.42}$$

Notice that this matches the scaling of the tension of the D1-string, and so both scales are comparable in the quantum-corrected regime.[3] However, since the tower of string states is denser than the KK tower, this limit is equi-dimensional [19]. Similar results can be obtained from a direct analysis of the F1 limit.

The c-map and vector multiplets

It is quite instructive to compare the above trajectories in hypermultiplet moduli space to analogous ones that one could follow in vector multiplet moduli space. The simplest way to do so is to apply the c-map construction [36, 47], which in our case involves compactifying type IIB on $\mathbb{R}^{1,2} \times S^1 \times X$ and performing a T-duality on the S^1. Under this operation, the type IIB hypermultiplet moduli space $\mathcal{M}_{\rm HM}^{\rm IIB}$ is mapped to the dimensional reduction of type IIA vector multiplet $\mathcal{M}_{\rm VM}^{\rm IIA}$ on a circle, which is the twisted hypermultiplet moduli

---

[3]That is, the trajectory is obstructed to cross to Region III in figure 2 exactly when $T_{\rm D1}^{1/2} \sim M_{\rm KK}$.

space $\mathcal{M}_{\mathrm{THM}}^{\mathrm{IIA}}$. In the following we will apply such construction along the trajectory of the D1 limit (2.10), since an analogous path in $\mathcal{M}_{\mathrm{VM}}^{\mathrm{IIA}}$ has been analysed in the context of the Swampland Distance [13] and Emergent String [19] conjectures.

One may trivially embed the classical trajectory (2.10) in the moduli space of $S^1 \times X$, with S^1 a circle whose radius in string units $L_B = R_B M_s$ remains constant along the trajectory. Upon T-duality on this circle we obtain type IIA on $S^1 \times X$, such that [48]

$$L_A \equiv R_A M_s = \frac{1}{L_B}\,, \tag{3.43a}$$

$$g_{4,A}^{-1} = g_{4,B}^{-1} L_B = R_B M_{\mathrm{Pl},B}\,, \tag{3.43b}$$

$$R_A M_{\mathrm{Pl},A} = g_{4,A}^{-1} L_A = g_{4,B}^{-1}\,, \tag{3.43c}$$

where R_A is the S^1 radius in the type IIA side and g_4, M_{Pl} stand for the 4d dilaton and 4d Planck mass in either the type IIA or IIB side of the duality. After the T-duality the Kähler moduli of X and the 10d dilaton follow the same trajectory (2.10) as they did in the type IIB side, but this now forms part of the twisted hypermultiplet moduli space $\mathcal{M}_{\mathrm{THM}}^{\mathrm{IIA}}$. Notice that classically all the quantities in (3.43) are constant. As such one can also interpret this trajectory as a classical path of infinite distance in $\mathcal{M}_{\mathrm{VM}}^{\mathrm{IIA}}$.

One interesting feature of this duality is that it maps pairs of instantons and strings in $\mathbb{R}^{1,2} \times S^1$ to particles in $\mathbb{R}^{1,2} \times S^1$. Indeed, type IIB D(−1)-instantons and D1-strings wrapped on the S^1 are mapped to D0-branes on the type IIA side, respectively wrapping and not wrapping the S^1. The same statement applies to D1-instantons and D3-branes wrapping a two-cycle γ^a of X, which map to type IIA D2-branes wrapping the same cycle. Because both elements of a pair go to the same object in type IIA, namely a particle, it follows the tension of the string T and the instanton action S must be related as $T = M_s^2 \mathrm{Re}S$. This gives a rationale as to why certain non-perturbative effects become important as soon as the tension of a string is comparable with that of a fundamental string.

On the type IIB side and from a four-dimensional viewpoint, proceeding along the trajectory (2.10) results in several strings lowering their tension with respect to M_{Pl}^2. From a three-dimensional perspective, these strings will be seen as particles when they wrap the S^1. The mass of such particles with respect to the 3d Planck mass reads

$$M_{\mathrm{F1}} = M_s^2 R_B = \frac{M_{\mathrm{Pl},3}}{\chi}\,, \qquad M_{\mathrm{D1}} = T_{\mathrm{D1}} R_B = \frac{M_{\mathrm{Pl},3}}{\tau_2 \chi}\,, \tag{3.44}$$

for the lightest winding modes of the fundamental and D1-brane, respectively. Notice that these relations are a direct consequence of (3.5), which was interpreted in terms of a species scale. In the present setup, the rôle of species scale is played by the winding mode scales of the F1- and D1-strings, respectively, and the fact that these scales decrease exponentially along the infinite distance trajectory signals that one approaches a decompactification limit, which tensionless but codimension one strings cannot prevent. This is even more manifest after T-duality to type IIA, where $M_{\mathrm{F1}} = 1/R_A$ and $M_{\mathrm{D1}} \to M_{\mathrm{D0}} = 1/R_{11}$ are related to the Kaluza-Klein scales of the c-map and M-theory radii, respectively. While the M-theory radius grows even at the classical level, quantum corrections imply that $R_A M_{\mathrm{Pl},3}$

also grows exponentially, although at a slower rate. Above these two scales one finds M-theory compactified on $T^2 \times X$, and the quantum corrections leading to χ can be computed along the lines of [49]. Finally, it is easy to see that by reducing M-theory to type IIA along R_A and performing a c-map along R_{11} one again recovers type IIB on $S^1 \times X$ but now with a trajectory that corresponds to the F1 limit (2.15). We thus observe a tight relation between limits of tensionless strings and decompactification limits.

In fact, our results are very suggestive when compared to the observations made in [19] on the type IIA vector multiplet moduli space. Indeed, since the D(-1)-instantons are c-mapped to D0-branes, taking their corrections into account in $\mathcal{M}_{\mathrm{HM}}^{\mathrm{IIB}}$ can be interpreted as integrating-in D0 branes in $\mathcal{M}_{\mathrm{VM}}^{\mathrm{IIA}}$. As discussed above, the analogous of the D1 limit in $\mathcal{M}_{\mathrm{VM}}^{\mathrm{IIA}}$ is the strong coupling/M-theory limit in which D0-branes become massless. Following [19], to appropriately analyse this setup one should consider M-theory in 5d, which from a four-dimensional type IIA perspective is at infinite distance since one has to blow up the radius of the M-theory circle. Then, having a constant 5d Planck mass forces the M-theory Kähler moduli to be constant. Therefore, in this analogous setup of the D1 limit the infinite distance along the Kähler directions is also removed due to the inclusion of quantum corrections, while the infinite distance coming from scaling the dilaton is still present. The difference is that while we see a tensionless fundamental string in $\mathcal{M}_{\mathrm{HM}}^{\mathrm{IIB}}$, the analogous limit in $\mathcal{M}_{\mathrm{VM}}^{\mathrm{IIA}}$ yields a decompactification.

3.4 Vanishing fibre limit

We would now like to extend the above discussion to the third limit established in the classical hypermultiplet moduli space, the D3 limit (2.24). Since we know that the quantum corrected D1 limit is dual to the weak-coupling limit of type IIB string theory, we only have to establish the duality between the D3 limit and the D1 limit at quantum level to show that there is a weakly-coupled, tensionless fundamental string theory that describes the asymptotics of the D3 limit. As explained in section 2, the D3 limit (2.24) can be reached starting from the D1 limit in cases where the compactification manifold is an elliptically fibred Calabi-Yau threefold by performing two T-dualities along the fibre or, more precisely, a particular Fourier-Mukai transform.

However, in the new frame the relevant quantum corrections do not come anymore solely from D1/D(-1)-instantons and the procedure used so far to incorporate quantum effects on the hypermultiplet moduli space has to be adapted. To achieve this, we must first consider which class of instantons we expect to contribute significantly to the metric. As Fourier-Mukai transforms map instantons into instantons, we are assured that we only need to consider the transformation of the relevant instanton actions.[4] In the case at hand these are found in the new frame through the transformation of the moduli (2.22):

$$\mathrm{Re}(S_{k_\Lambda}^D) = \tau_2 |k_\Lambda z^\Lambda| \longrightarrow \tau_2 \left| \left(k_1 + \frac{1}{2} k_\alpha K^\alpha \right) + k_0 z^1 + k_\alpha \left(z^1 z^\alpha + \frac{1}{2} K^\alpha (z^1)^2 \right) \right|, \quad (3.45)$$

---

[4]Via the c-map, this can be seen by formulating BPS particles in the type IIA side in a categorical language, where the Fourier-Mukai transform is an equivalence [37, 41].

where k_0 stands for the D(−1)-charge in the original duality frame, and we singled out the D1-instanton charge along the elliptic fibre, k_1. After the transformation, the first term is associated to a D(−1)-instanton with charge $q_0 = k_1 + \frac{1}{2}k_\alpha K^\alpha$, while the second is the action an instanton of charge $q_1 = k_0$ coming from a D1-brane wrapping the elliptic fibre. Finally, the last term in (3.45) has precisely the form of an instanton of charge q^α associated to D3-branes wrapping a vertical divisor:

$$\mathrm{Re}(S_{D3|_{D_\alpha}}) = q^\alpha \,\mathrm{Vol}(D_\alpha)\,, \qquad \mathrm{Vol}(D_\alpha) = \eta_{\alpha\beta}\left(t^1 t^\beta + \frac{1}{2}K^\beta(t^1)\right)\,, \tag{3.46}$$

where $\eta_{\alpha\beta} = D_\alpha \cdot D_\beta$ denotes the intersection pairing of vertical divisors. Generalising the notation used in the definition of the D1/D(−1)-instanton action, we can write the r.h.s. of (3.45) as

$$\mathrm{Re}(S^D_{k_A}) = \tau_2 \, k_A z^A\,, \tag{3.47}$$

$$k_A = (k_0, k_1, k_\alpha, 0)\,, \qquad z^A = (1, z^1, z^\alpha, \mathrm{Vol}(D_a))\,. \tag{3.48}$$

In this language, the Fourier-Mukai transformation acts, as expected, on the lattice of charges:

$$\mathrm{Re}(S^D_{k^A}) \longrightarrow \mathrm{Re}(S^D_{q^A}) = \tau_2 |q_A z^A|\,. \tag{3.49}$$

The charges after the transformation are related to the original D1/D(−1)-charges k_A by matrix multiplication (cf. [13]):

$$q_A = \begin{pmatrix} k_1 + \frac{1}{2}k_\alpha K^\alpha \\ k_1 \\ 0 \\ \eta^{\alpha\beta}k_\beta \end{pmatrix} = \begin{pmatrix} 0 & -1 & \frac{1}{2}K^\alpha & 0 \\ 1 & 0 & 0 & \frac{1}{2}K_\alpha \\ 0 & 0 & 0 & -\eta_{\alpha\beta} \\ 0 & 0 & \eta^{\alpha\beta} & 0 \end{pmatrix} k_A\,, \tag{3.50}$$

where $\eta^{\alpha\beta}$ is the inverse of the intersection pairing and $K_\alpha = \eta_{\alpha\beta}K^\alpha$. We conclude that D1/D(−1)-instantons are transformed into D3/D1/D(−1)-instantons that have a D1-charge only along the fibre. In the case of a trivial fibration, the Chern class of the base vanishes and we obtain the expected behaviour under a double T-duality, and the presence of D3-instantons thus accounts for the non-trivial curvature of the base under the transform (2.22).

Unfortunately, one cannot directly use the contact potential (3.8) to take into account quantum corrections, as it does not include contributions from D3-brane instantons. However, as reviewed in appendix C, corrections to the 4d dilaton are known for generic, mutually local D-brane instantons:

$$\chi = \frac{1}{6}\tau_2^2 \mathcal{K}_{abc} t^a t^b t^c - \frac{\chi(X)}{192\pi} + \frac{\tau_2}{8\pi^2}\sum_q \Omega(q)\sum_{m=1}^\infty \frac{\mathrm{Re}(S^D_q)}{m}K_1(2\pi m\mathrm{Re}(S^D_q))\,. \tag{3.51}$$

The sum is taken over all D-instantons of charge q, and Ω denotes the associated Donaldson-Thomas invariant. In the case of D1/D(−1)-instantons, the latter reduce to either the Euler density or Gopakumar-Vafa invariants and one recovers equation (3.11).

Limit	D(−1)	D1|$_{\mathcal{E}}$	D1|$_{\mathcal{C}}$	D3|$_{D_0}$	D3|$_{D_\alpha}$	D5|$_X$	NS5|$_X$	WS|$_{\mathcal{E}}$	WS|$_{\mathcal{C}}$
D1, eq. (2.10)	$e^{-\frac{3}{2}\sigma}$	$e^{-\frac{1}{2}\sigma}$	$e^{-\frac{1}{2}\sigma}$	$e^{\frac{1}{2}\sigma}$	$e^{\frac{1}{2}\sigma}$	$e^{\frac{3}{2}\sigma}$	e^0	e^σ	e^σ
F1, eq. (2.15)	$e^{\frac{3}{2}\sigma}$	e^σ	e^σ	$e^{\frac{1}{2}\sigma}$	$e^{\frac{1}{2}\sigma}$	e^0	$e^{\frac{3}{2}\sigma}$	$e^{-\frac{1}{2}\sigma}$	$e^{-\frac{1}{2}\sigma}$
D3, eq. (2.24)	$e^{-\frac{1}{2}\sigma}$	$e^{-\frac{3}{2}\sigma}$	$e^{\frac{1}{2}\sigma}$	$e^{-\frac{3}{2}\sigma}$	$e^{-\frac{1}{2}\sigma}$	$e^{\frac{1}{2}\sigma}$	e^0	$e^{-\sigma}$	e^σ

Table 2. Asymptotic scaling of the real part of the action for the different instantons in each limit. $\mathcal{C} \subset B_2$ denotes a curve in the base.

The D3 limit differs from the original D1 limit only by the Donaldson-Thomas invariants, as the asymptotics of the classical instanton action are the same, see equation (3.49). As already alluded to earlier, in the present case, the Fourier-Mukai transform corresponds to a monodromy action M on the Kähler moduli space [38]. We can therefore use that Donaldson-Thomas invariants do not change under monodromies, i.e. $\Omega(Mk) = \Omega(k)$ to conclude that χ in the D1 and D3 limit coincide:

$$\chi'' \sim \chi_{\text{class.}} + \frac{\tau_2^{-1}}{4(2\pi)^3}\zeta(3)\left[-\chi_E(X) + \sum_{\mathbf{k}>\mathbf{0}}^{\Omega} n_{\mathbf{k}}^{(0)}\right] \sim \chi\,, \tag{3.52}$$

with the second term behaving as the quantity in brackets in (3.22). The only difference is the interpretation: after T-dualising, k_0 counts the instanton charge of a D1-instanton wrapped along the fibre rather than a D(−1)-instanton in the original limit.

Note that one needs to ensure that other types of instantons, such as those coming from NS5- and D5-branes, are also parametrically smaller. The classical scaling of the actions for the instantons in all the limits keeping the Planck mass constant are given in table 2. As one can see, D5-branes do not acquire a vanishing action in either frames. A similar statement also holds for NS5-instantons. Furthermore, perturbative corrections are missing in table 2, since there is no obvious way how to express them in terms of instanton actions. As noted in the previous subsection, in the F1 limit they take over the rôle of the D(−1)-instantons in the D1 limit.

Again, the fact that χ'' is not constant tells us that the fundamental string scale in the new limit changes in terms of the Planck mass. We saw that classically the string coming from a D3-string is the lightest with classical tension

$$\frac{T_{D3}''}{M_{\text{Pl}}^2} = \tau_2''(t^1)''\frac{1}{\chi_{\text{cl}}''}\,. \tag{3.53}$$

One may however fear that this string no longer becomes tensionless the fastest after quantum corrections. Notice that in the present setup quantum corrections could obstruct the limit in which the area of the fibre vanishes, analogously to the obstruction of the small-volume point in the F1 limit, or the correction to the D3-brane tension in the D1 limit, see (3.38).

We can answer this question via duality to the D1 limit, by using that the D3-string tension in string units is mapped under the Fourier-Mukai transform to τ_2

$$\tau_2''(t^1)'' \sim \tau_2\,. \tag{3.54}$$

As discussed in appendix B, the scaling of τ_2 in the D1 limit is not affected by quantum corrections. Therefore, the tension of the D3-brane wrapped on the fibre should not be affected by the quantum corrections relevant in the D3 limit. In other words, the D3/D1/D(-1)-instantons above do not affect the vanishing of the fibre volume in this limit. Taking into account the corrected expression for χ in (3.52), we obtain that the tension of the D3-strings in the D3 limit behaves as

$$\frac{T''_{D3}}{M_{\text{Pl}}^2} \sim e^{-3\sigma}\,, \tag{3.55}$$

which matches the corrected tension of the D1-string in the D1 limit and thus by extension the corrected tension of the F1 string in the F1 limit.

We thus conclude that after taking into account instanton corrections, we can effectively describe the vanishing-fibre limit in moduli space by a tensionless, weakly-coupled type IIB string theory.

4 Infinite distance limits in type I string theory

We have so far investigated how quantum corrections alter the geometry around classical infinite distance points in the hypermultiplet sector of type IIB string theory. We showed that taking these corrections into account, a tensionless string arises in three infinite distance limits, where quantum effects are under control, which is dual to a tensionless and weakly-coupled fundamental string of type IIB string theory. Moreover these strings already arose classically and instantons only enhanced this behaviour.

In this section, we would like to address yet another string theory setup which has infinite distance limits and in which instanton corrections play an important rôle, namely four-dimensional $\mathcal{N} = 2$ compactifications of type I string theory.

Type I string theory is well known to be related to type IIB string theory by modding out worldsheet parity, arriving to a theory of unoriented strings where half of the supersymmetry gets broken. To obtain a four-dimensional $\mathcal{N} = 2$ EFT, we are therefore led to consider a type I string theory compactified on

$$X = K3 \times T^2\,. \tag{4.1}$$

To make contact with the setup analysed in the D3 limit we can interpret this as a compactification on a trivial elliptic fibration over a $K3$ base. While somewhat similar, there are important differences between the two. Indeed, the volume of the torus, its complex structure and the dilaton now give rise to vector multiplets. Accordingly, it is now the vector multiplet moduli space that receives quantum corrections in the form of instantons, rather than the hypermultiplet sector. Finally, as a consequence of the mixing of the complex structure and Kähler sector, it is a priori harder to classify points at infinite distance in terms of mixed Hodge structures [6].

For definiteness, the following analysis will focus on the BSGP orbifold [50, 51] which can be thought of as the $T^4/\mathbb{Z}_2 \times T^2$ orbifold limit of $K3 \times T^2$. The reason for this choice is that all relevant quantum contributions are known in this case [34].

This model has an unbroken U(16) gauge group realised by D9-branes, while the D5-gauge group gets broken due to the Green-Schwarz mechanism. In the following, we will ignore Wilson line moduli such that the part of the vector multiplet moduli space that we are interested in is spanned by three physical vector multiplet moduli:

$$S = e^{-\phi} T \mathcal{V} + i\, b\,, \qquad S' = e^{-\phi} T + i\, b'\,, \qquad U\,. \tag{4.2}$$

The complexified 4d dilaton is denoted by S; U is the complex structure of the torus; $T = \mathrm{Vol}(T^2) M_s^2$ and $\mathcal{V} = \mathrm{Vol}(K3) M_s^4$ are respectively the volumes of the torus and the K3, measured in units of the string scale M_s, which type I string theory inherits from type IIB string theory. Finally $e^{-\phi}$ is the 10d type I dilaton and the two axions, b and b' are respectively obtained by dualising the type I two-form or reducing it along the torus.

In terms of the physical fields, the Planck mass is given by:

$$M_{\mathrm{Pl}}^2 = e^{-\phi} \mathrm{Re}(S) M_s^2\,. \tag{4.3}$$

From the emergent string proposal [19] and the type IIB examples discussed previously, we would like to check whether tensionless four-dimensionless strings emerge at points of infinite distance. Obvious candidates for such strings are type I D1-branes in Minkowski spacetime, whose tension is given by

$$T_{\mathrm{D1}} = e^{-\phi} M_s^2\,. \tag{4.4}$$

From the definition of the Planck mass, we see that the real part of S controls the D1-brane tension in Planck units.

4.1 Small torus limit

Having introduced the moduli space for type I string theory compactified on $K3 \times T^2$, we now want to look for classical infinite distance limits in this moduli space. We focus on limits that keep S constant, and one possible case for the type I vector multiplet moduli (4.2) that we can consider is

$$S = \mathrm{const.}\,, \qquad \mathrm{Re}\, S' \sim e^{-\sigma}\,, \qquad U = \mathrm{const.}\,, \qquad b = b' = 0\,. \tag{4.5}$$

Since S' is related to the volume of the torus, we can think of this limit as a small torus limit and taking $K3 \times T^2$ as a trivial elliptic fibration, we can interpret it as the type I analogue of the D3 limit in type IIB hypermultiplet moduli space. Unlike the previous type IIB hypermultiplet analysis we can obtain the vector multiplet geometry directly from a Kähler potential, which classically reads

$$K_{\mathrm{cl}} = -\log(S + \bar{S}) - \log(S' + \bar{S}') - \log(U + \bar{U})\,. \tag{4.6}$$

In the limit (4.5), the only relevant component of the classical metric is given by:

$$g_{S'\bar{S}'} = \frac{1}{(S' + \bar{S}')^2} \sim e^{2\sigma}\,, \tag{4.7}$$

such that the associated line element gives indeed rise to an infinite distance.

Classically, this limit keeps the ratio $T_{\mathrm{D1}}/M_{\mathrm{Pl}}^2$ constant, and the type I string does not become tensionless in Planck units. On the other hand, D1-branes wrapping one of the torus one-cycles are becoming massless in this classical limit, giving rise to a KK-like tower

$$\frac{M_{\mathrm{D1}}^2}{M_{\mathrm{Pl}}^2} \sim n^2 \frac{\mathrm{Re}\, S'}{\mathrm{Re}\, S} \sim n^2 e^{-\sigma}\,. \tag{4.8}$$

We would thus argue that the limit (4.5) is a decompactification limit. Still, these are classical statements and the intuition we earned from the previous sections tells us that we should take into account quantum corrections as we consider a limit for which a cycle shrinks to zero size. From our previous experience, we expect either perturbative or instanton corrections to dominate.

For the BSGP orbifold, the relevant corrections to the Kähler potential have been computed in [34] and take the following form:

$$K = -\log\left(S + \bar{S} + V_{\text{1-loop}} + V_{\mathrm{D1}}\right) - \log(S' + \bar{S}') - \log(U + \bar{U})\,. \tag{4.9}$$

As one can see the only relevant corrections are to the 4d dilaton. Similarly to type IIB hypermultiplets, these are of both perturbative and non-perturbative origin, the latter corresponding to D1-instantons wrapping the torus:

$$V_{\text{1-loop}} = -\frac{4\pi}{6} \frac{E(iU, 2)}{S' + \bar{S}'}\,, \tag{4.10a}$$

$$V_{\mathrm{D1}} = -\frac{1}{2\pi} \sum_{k > j \geq 0,\, p > 0} \frac{e^{-2\pi k p S'}}{(kp)^2} \left(\frac{\hat{A}_K(\mathcal{U})}{(S' + \bar{S}')} - \frac{2ikp}{\mathcal{U} - \bar{\mathcal{U}}} \frac{E_{10}(\mathcal{U})}{\eta^{24}(\mathcal{U})}\right)\,. \tag{4.10b}$$

Let us shortly review the properties of these corrections. On the one hand, the perturbative contributions $V_{\text{1-loop}}$ are phrased in terms of non-holomorphic Eisenstein series of order k:

$$E(U, k) \equiv \frac{1}{\zeta(2k)} \sum_{(j_1, j_2) \neq (0,0)} \frac{(\mathrm{Im}\, U)^k}{|j_1 + j_2 U|^{2k}}\,. \tag{4.11}$$

On the other hand, the D1-instanton corrections V_{D1} can be expressed in terms of quasi-modular forms related to the usual Eisenstein series E_{2k}, depending on complex structure of the instanton worldsheet wrapping the torus:

$$\hat{\mathcal{A}}_K(\mathcal{U}) = \frac{1}{12\eta^{24}}\left(\hat{E}_2 E_4 E_6 + 2E_6^2 + 3E_4^3\right)\,, \qquad \mathcal{U} = \frac{j + ipU}{k}\,. \tag{4.12}$$

Note that $\hat{\mathcal{A}}_K$ is only quasi-modular due to the appearance of $\hat{E}_2(\tau) = E_2(\tau) - \frac{3}{\pi \mathrm{Im}(\tau)}$.

In the limit $\mathrm{Re}(S') \to 0$ both perturbative and instanton corrections become important, dominating over the classical $S + \bar{S}$ term of the Kähler potential. In particular:

$$V_{\text{1-loop}} + V_{\mathrm{D1}} \sim \frac{1}{S' + \bar{S}'} \sim e^{\sigma}\,. \tag{4.13}$$

By $\mathcal{N} = 2$ supersymmetry, these corrections induce at the same time a modification of the gauge kinetic function of the U(16) open string gauge group [34]:

$$f_{\mathrm{U}(16)} = S + \frac{\pi S'}{2} - 12 \log \eta(iU) - \frac{1}{2} \sum_{k > j \geq 0,\, p > 0} \frac{e^{-2\pi k p S'}}{(kp)} \mathcal{A}_f(\mathcal{U})\,, \tag{4.14}$$

where

$$\mathcal{A}_f = \frac{1}{12\eta^{24}}\left(E_2 E_4 E_6 - \frac{5}{12}E_6^2 - \frac{7}{12}E_4^3\right),\tag{4.15}$$

and η denotes the Dedekind function.

In the regime (4.2), the gauge kinetic function also gets dominated by instanton effects, and depends on the precise value the variable U through the modular function $\mathcal{A}_f(\mathcal{U})$. For cases where the torus complex structure is large, $\mathrm{Re}(U) \gg 1$, the imaginary part of $\mathcal{U}$ will generically also be large, such that we can use the approximation

$$\mathcal{A}_f(\mathcal{U}) = -\frac{49}{2} + \mathcal{O}\left(e^{2\pi i \mathcal{U}}\right).\tag{4.16}$$

The sum over instanton numbers in (4.14) will yield a diverging contribution in the limit $S' \to 0$ where the instanton action vanishes and thus the exponential suppression is lifted. We can make the rough estimate

$$f_{\mathrm{U}(16)} \sim S + \frac{49}{4}\sum_{k>j\geq 0,\,p>0}\frac{e^{-2\pi kpS'}}{(kp)} \sim S + \frac{49}{4}\sum_{p>0}\frac{1}{2\pi p^2 S'},\tag{4.17}$$

where we used that the summand does not explicitly depend on j and imposed $k < (2\pi p S')^{-1}$ in order for the exponential to be $\mathcal{O}(1)$. We thus conclude that in the limit $S' \to 0$ the gauge kinetic function of the open string gauge group blows up, yielding a weakly-coupled gauge theory.

Along the trajectory (4.5), we can track the tensions of BPS strings by looking at the decay constants of the axions they couple to magnetically. For instance, the D1-string couples magnetically to $\mathrm{Im}S$, and its tension is given in Planck units by the decay constant $f_{\mathrm{Im}S}$:

$$\frac{T_{D1}}{M_{\mathrm{Pl}}^2} = f_{\mathrm{Im}S} = \sqrt{\partial_S \partial_{\bar{S}} K}.\tag{4.18}$$

Classically, the tension of the D1-string is constant since

$$\partial_S \partial_{\bar{S}} K_{\mathrm{cl}} = \frac{1}{(S+\bar{S})^2},\tag{4.19}$$

but as the Kähler potential receives quantum corrections, we find that the corrected D1-tension scales as

$$\left.\frac{T_{\mathrm{D1}}}{M_{\mathrm{Pl}}^2}\right|_{\mathrm{corr}} = \sqrt{\partial_S \partial_{\bar{S}} K_{\mathrm{corr}}} = (\mathrm{Re}(S_{\mathrm{eff}}))^{-1} \sim \mathrm{Re}(S') \sim e^{-\sigma},$$
$$\mathrm{Re}(S_{\mathrm{eff}}) = \frac{1}{2}\left(S + \bar{S} + V_{\text{1-loop}} + V_{\mathrm{D1}}\right),\tag{4.20}$$

where we used the approximation (4.13). Consequently the type I string in fact becomes tensionless after taking into account instanton corrections. This is in stark contrast with the behaviour of the limits we studied in type IIB. There, quantum corrections were merely

making the strings become tensionless at a faster rate. In the present case however, they are strong enough to completely overwhelm the classically finite tension and drive it to zero.

Still, we have to check whether there are no other 4d strings becoming tensionless in this limit at the same rate as the D1-string or faster. Candidates for these strings arise from D5-branes wrapped on the entire K3 or wrapped on the torus times a 2-cycle inside K3. The latter string couples magnetically to axions that are part of the hypermultiplet moduli space and their tension is hence not affected by our limit in vector multiplet moduli space. The string coming from the D5-brane wrapped on $K3$ on the other hand couples magnetically to $\operatorname{Im} S'$. Hence its tension scales like

$$\frac{T_{D5|_{K3}}}{M_{\rm Pl}^2} = f_{\operatorname{Im} S'} = \frac{1}{\sqrt{(S' + \bar{S}')}} \to \infty \,, \tag{4.21}$$

such that the string becomes infinitely heavy asymptotically. The only tensionless string is thus the D1-string as type I string theory does not have any D3- or D7-branes.

Similarly, as we are going to a regime where the D1-string becomes very light, the type I supergravity description might not be valid anymore. Using the philosophy of the previous sections, we should instead look at the effective theory that can be derived from the tensionless string: the D1-string itself. As it is well known, the type I setup studied so far is S-dual to the heterotic string on $K3 \times T^2$ and the classical limit (4.5) is translated to the heterotic limit

$$\operatorname{Re} S_H = \text{const.} \,, \qquad \operatorname{Re} T_H \sim e^{-\sigma} \,, \qquad U = \text{const.} \,, \tag{4.22}$$

where S_H is the heterotic 4d dilaton, T_H the volume of the torus in units of the heterotic string scale M_H and U, as before, the torus complex structure. The heterotic string on $K3 \times T^2$ receives quantum corrections from winding and momentum states along the internal directions. Their effect is analogous to the type I effect studied above,[5] such that the corrected trajectory in the heterotic frame is given by:

$$\operatorname{Re} S_H|_{\rm corr.} \sim e^{\sigma} \,, \qquad \operatorname{Re} T_H \sim e^{-\sigma} \,, \qquad U = \text{const.} \,, \tag{4.23}$$

and heterotic string becomes weakly coupled. As a consequence, the heterotic string scale M_H vanishes asymptotically when measured in Planck units:

$$\frac{M_H^2}{M_{\rm Pl}^2} = (\operatorname{Re} S_H)^{-1} \to e^{-\sigma} \,, \tag{4.24}$$

in agreement the D1-string becoming tensionless in the dual Type I limit.

Still, the question remains of whether the heterotic string tension is comparable with the lowest KK scale (in which case we would indeed have an emergent string limit) or if its tension is above some KK scale, yielding a decompactification limit. The lowest KK-like tower in this setup, with masses given by (4.8), corresponds here to heterotic

---

[5]In fact, type I corrections in [34] have been calculated via duality to the heterotic string.

winding modes. We see that, simply using the new scalings of the fields in the vector multiplet (4.23), we obtain:

$$\frac{M^2_{\text{winding}}}{M^2_{\text{Pl}}} \sim \frac{\operatorname{Re} T_H}{\operatorname{Re} S_H} \sim e^{-2\sigma} \,. \tag{4.25}$$

As this is parametrically below the heterotic string scale, we are still in a decompactification regime even after taking into account quantum corrections. The effect of the quantum corrections would then just be to obstruct the small torus limit $T_H \to 0$ at finite coupling by pushing us towards a weak-coupling limit. This is in agreement with instanton corrections breaking the $\mathrm{SL}(2,\mathbb{Z})_{T_H}$ part of the modular group. However, it seems to contradict the results obtained in a dual setup, as we now discuss.

Comparison to dual setup

The vector multiplet moduli space of $\mathcal{N} = 2$ supergravity theories has already been investigated in other setups, namely that of Calabi-Yau three-folds in type IIA [13, 19]. Heterotic string theory on $K3 \times T^2$ is dual to type IIA on a K3-fibred Calabi-Yau threefold $Z : K3 \to \mathbb{P}^1$ which at the same time admits an elliptic fibration over a Hirzebruch surface $Z : T^2 \to F_n$ [52]. A concrete realisation is $W\mathbb{P}_{1,1,2,8,12}(24)$, which has $h^{(1,1)} = 3$ and therefore gives rise to three vector multiplets that can be identified with S, T_H and U [52]. The three complexified Kähler moduli z_a, $a = 1, 2, 3$ describe respectively the volume of the elliptic fibre, the volume of the $\mathbb{P}^1$-fibre of F_n, and the volume of the base $\mathbb{P}^1$. The explicit identification with S, T_H, U is given by:

$$z_1 = U\,, \qquad z_2 = T_H - U\,, \qquad z_3 = S_H - \left(1 - \frac{n}{2}\right) U - \frac{n}{2} T_H\,. \tag{4.26}$$

Moreover, the volume of the K3 fibre, which can be thought of as the restriction of the elliptic fibration to the fibre $\mathbb{P}^1$ is

$$\mathcal{V}(K3)_{\text{IIA}} = U T_H\,. \tag{4.27}$$

We thus see that the heterotic limit $T_H \to 0$ corresponds to the limit where the K3 fibre volume vanishes. Our analysis shows that the type I/heterotic limit is heavily corrected by quantum effects, just as it happens in [19].

However, the result of [19] implies that the quantum obstructions to the limit of vanishing K3 volume give rise to an emergent tensionless heterotic string. In fact a mirror symmetry calculation reveals that the volume of the K3-fibre $\mathcal{V}(K3)_{\text{IIA}}$ cannot shrink to values smaller than one in type IIA string units. The point that is reached by this trajectory then corresponds to the conifold point. Compared to our heterotic/type I setup this indicates that T_H should also have a minimal value and therefore should be constant in the quantum-corrected limit. This result seems in direct contradiction with the quantum-corrected limit (4.23).

The analogue of the conifold point of type IIA in our heterotic/type I setup is the point of self-dual radius of the torus where one U(1) gauge group enhances to SU(2) due to winding modes that become massless. One could therefore wonder whether in the above

discussion we missed effects due to the 1-loop singularity at the gauge enhancement point or due to other sources, such that (4.23) is not the actual limit after taking all quantum corrections into account. Otherwise, our result would mean that one can proceed through this conifold point in the type IIA setup, towards a decompactification limit.

4.2 Large complex structure

As a second possible limit, we consider the limit of large complex structure of the T^2, i.e.

$$\operatorname{Re} U \sim e^{\sigma} \to \infty \,, \tag{4.28}$$

with $\operatorname{Re} S$ and $\operatorname{Re} S'$ kept fixed. Note first that our conventions in this section follow [34] which can differ from other definitions of the complex structure by a factor of i; second, the large and small complex structure limits are equivalent due to the presence of an unbroken $\mathrm{SL}(2, \mathbb{Z})_U$ modular symmetry.

Applying the same reasoning as in the previous section, the D1-string and the D5-brane wrapped on $K3$ have tension

$$\frac{T_{D1}}{M_{\mathrm{Pl}}^2} = (\operatorname{Re} S)^{-1} = \text{const.} \,, \qquad \frac{T_{D5}|_{K3}}{M_{\mathrm{Pl}}^2} = (\operatorname{Re} S')^{-1} = \text{const.} \tag{4.29}$$

and thus no tensionless strings appear, as the tension of D5-branes wrapped on T^2 times a 2-cycle in K3 is again controlled by a modulus in the hypermultiplet sector. On the other hand, one can notice that D1-strings wrapped along one of the one-cycles of the torus become massless in this limit, as classically their mass scales as

$$\frac{M_{D1}}{M_{\mathrm{Pl}}} \sim \frac{q_1 + U q_2}{\sqrt{U + \bar{U}}} \,, \tag{4.30}$$

where q_1 and q_2 denote charges in a suitable basis of the two one-cycles of the torus. We thus see that as long as $q_2 = 0$, the D1-particles become massless:

$$\frac{M_{D1}|_{q_2=0}}{M_{\mathrm{Pl}}} \sim \frac{q_1}{\sqrt{U + \bar{U}}} \sim e^{-\sigma/2} \,. \tag{4.31}$$

To get a better intuition of the situation, let us perform a T-duality along one of the cycles. Doing so, the type I theory gets mapped to type I' theory, D1-branes getting mapped to D0-branes localised on D8-branes, and the complex structure of the torus exchanged with the Kähler volume. The precise duality map is not relevant for the following discussion, the important point being that the Buscher rules tell us that the type I' string coupling has to blow up:

$$e^{-\phi^{I'}} \to e^{-\sigma/2} \,. \tag{4.32}$$

Accordingly, the D0-branes then become massless:

$$\frac{M_{D0}}{M_{\mathrm{Pl}}} = e^{-\phi^{I'}} \to e^{-\sigma/2} \,, \tag{4.33}$$

as expected from T-duality. The D0-branes are, however, nothing but the Kaluza-Klein modes of Hořava-Witten theory, i.e. the strong coupling limit of type I' theory compactified on $K3 \times T^2$ times an extra circle.

We might ask whether the limit (4.28) is also heavily affected by quantum corrections. At the level of the Kähler potential we see that the corrections (4.10) due to the scaling of $\mathrm{Re}\,U \to \infty$ are either suppressed, or at most constants exponentially suppressed by $\mathrm{Re}\,S'$. We thus do not expect the effect of quantum corrections on the Kähler potential — and by extension the string tensions — to be as significant as in the limit analysed in the previous section. The gauge kinetic function, however, is again corrected and potentially blows up. A similar effect should be observable in the T-dual setup. The effects of quantum corrections to the gauge kinetic functions in type I'/Hořava-Witten theory have already been studied in great detail in [28] and will not be repeated here.

Comparison to type IIA

Again we can compare the result for the large/small complex structure limit with the IIA dual introduced in the previous subsection. The limits of large/small complex structure U on the type I side map, via the duality transform outlined above, to $z_1 \to 0/\infty$ in the type IIA geometry. Due to the identification of z_1 with the volume of the elliptic fibre, this corresponds to a T^2 limit in the language of [19]. The fact that the $\mathrm{SL}(2, \mathbb{Z})_U$ modular symmetry is unbroken reflects that in type IIA the points $z_1 \to 0$ and $z_1 \to \infty$ are the same and related by a monodromy, the Fourier-Mukai transform. Due to this monodromy the type IIA prepotential is a modular function of z^1 [39] consistent with quantum corrections on the type I side arranging in modular functions of U.

Finally, we have seen that the $U \to \infty$ limit is dual to a decompactification limit of type I' string theory to Hořava-Witten theory in one dimension higher. This result is in agreement with the T^2-type limits in type IIA Calabi-Yau compactifications, which were shown to yield a decompactification to M-theory in one dimension higher [19].

5 Conclusions

In this work we have tested the Emergent String Conjecture [19] in regimes where quantum corrections significantly modify the moduli space metric. We have done so in two different classes of 4d $\mathcal{N} = 2$ compactifications. We have first considered infinite distance limits along the hypermultiplet moduli space of type IIB Calabi-Yau compactifications, a setup already analysed in [15]. Second, we have considered trajectories in the vector multiplet moduli space of type I string theory on $K3 \times T^2$, whose infinite distance limits do not fall into the classifications made in [6, 19].

As pointed out in [15], classically the hypermultiplet moduli space of type IIB on a CY has two types of infinite distance trajectories. The first one corresponds to moving along the 4d dilaton towards a weak-coupling limit, and the second type consists of moving in Kähler moduli space keeping the (classical) 4d dilaton constant. Via the c-map, this second type of trajectories can be related to the ones in vector multiplet moduli space of type IIA on the same CY, analysed in [13, 19]. The main difference is that, while in type IIA a

tower of particles asymptotes to zero mass, in type IIB a tower of instantons and 4d strings tend to zero action and tension, respectively. This behaviour for the strings was noticed in [15], but its significance was not investigated. Here, in light of the recent Emergent String Conjecture, we have taken a direct interest on identifying these light strings and in unveiling what their presence implies. In particular, we have labelled each of the limits of section 2 in terms of its lightest string, see figure 1. At the end of the day, we find that this lightest string tension signals the breakdown of the 4d effective field theory, and that it corresponds to the emerging string of the proposal made in [19].

The emerging picture from our analysis can be described as follows. As we proceed along an infinite distance trajectory of the second type, D-brane instanton corrections start to dominate over the classical contribution to the moduli space metric. The relevance of each instanton tower can be estimated by comparing the tension of the D-brane 4d strings with that of the fundamental string. This relation between string tensions has been depicted in figure 2 for the D1 limit, which was also analysed in [15]. There, as soon as the D1-string tension goes below the F1-string tension, namely when crossing to Region II, D(−1)-instanton corrections dominate over the classical moduli space metric. Their effect implies that the 4d dilaton is no longer constant, but instead is forced to flow towards a weak-coupling limit. Quantum effects become even more dramatic when the 4d tension of D3-brane wrapping 2-cycles of the CY become comparable to the F1-string tension, i.e. when crossing to Region III. In there, D1-brane instanton effects should start to dominate. However, via S-duality to the F1 limit and mirror symmetry, we find an obstruction for their action to vanish. In terms of the analysis in [15] and in appendix B, this can be understood as a quantum obstruction for the Kähler coordinates to contribute to an infinite distance trajectory. However, due to quantum corrections the trajectory also proceeds along the 4d dilaton coordinate. A refinement of the analysis in [15] shows that the metric in this direction remains essentially unaffected by quantum corrections. As a result the trajectory ends up being of infinite distance and, while classically it started as a trajectory with constant 4d dilaton, it ends as a trajectory approaching weak coupling exponentially fast. This weak-coupling limit is maintained when we turn to a dual frame in which the lightest string is nothing but the fundamental type IIB string, realising the Emergent String Conjecture.

This picture is not only realised in the D1 limit analysed in section 3.1, but also in the limits of vanishing fibre in which the lightest string is a D3-brane wrapping a 2-cycle of the CY, as discussed in section 3.4. Still, we only analysed a small subset of all possible limits in the type IIB hypermultiplet sector. It would therefore be important to classify all possible limits and to check whether the observed pattern of emerging strings holds in general. In [19] all possible infinite distance limits in the type IIA vector multiplet moduli space, subject to a constant string-to-Planck mass ratio, have been classified. Interestingly, when applied to the type IIA CY vector multiplet moduli space, each of the limits considered in this work corresponds to a decompactification limit. As the next step one should apply the known of the classification of infinite distance in the type IIA vector multiplet sector to the hypermultiplet moduli space of type IIB CY compactifications, to see if the decompactification limit in vector multiplet moduli space corresponding to the vanishing

fibre limit for a T^4-fibrations also turns into an emergent string limit when applied to the hypermultiplet sector, as we have observed for our limits. On the other hand if the Calabi-Yau is K3-fibred an emergent heterotic string appears in the IIA vector multiplet moduli space. It would then be very interesting to understand the physics of this limit in the IIB hypermultiplet analogue.

Instanton corrections also play a non-trivial rôle along infinite distance trajectories in the vector multiplet moduli space of type I string theory on $K3 \times T^2$. For trajectories along which the torus shrinks while the D9-brane gauge coupling is classically kept constant, D1-instantons wrapped on the torus together with perturbative corrections become dominant over the classical contribution to the Kähler potential. These instantons obstruct the finite coupling limit, in the sense that they bend the classical trajectory towards a weak-coupling limit in which the D1-string becomes asymptotically tensionless. In other words, to arrive to the point of vanishing T^2 area quantum corrections force us to go to vanishing coupling. This reflects that the classical $\mathrm{SL}(2, \mathbb{Z})$ symmetry, that exchanges large and small torus areas, is broken at finite coupling. On the other hand limits of large/small complex structure are less sensitive to the D1-instantons and can be reached without the D1-string becoming tensionless.

We have found that, naively, each of these limits correspond to a decompactification limit. On the one hand in the case of the complex structure limits this is to be expected, because they are dual to decompactification limits to Hořava-Witten theory. On the other hand, the small torus limit is supposed to be dual to a K3-fibre vanishing volume limit of the kind analysed in [19]. It was there obtained that this is an equi-dimensional limit in which an heterotic string emerges, due to a quantum obstruction for the K3 fibre to reach a vanishing volume. We may directly analyse this dual heterotic setup by simply applying S-duality to type I on $K3 \times T^2$. In this frame the K3 volume obstruction would translate into an obstruction for a vanishing T^2 area in string units, which would then imply an emerging string limit. As there is a priori no evident mechanism for such an obstruction, we find a naïve contradiction with the type IIA result. It would be interesting to carry a more detailed analysis that sheds more light into this puzzle, and in particular to describe the dual map between the type I and type IIA moduli spaces at the quantum level. In this context one should essentially understand whether the map itself receives quantum corrections or if the corrections calculated in [34] already capture all the quantum corrections to the type IIA quantum Kähler moduli space.

Our analysis showed the importance of quantum corrections for the Emergent String Conjecture in the type IIB hypermultiplet moduli space, and it would be interesting to extend our analysis to dual setups like for the hypermultiplet sector of type IIA Calabi-Yau compactifications, see e.g. [17]. It would be particularly interesting to study limits for which the type IIB mirror dual is not in a geometric phase, as they might give new insights into the Emergent String Conjecture and its relation to instanton corrections to the asymptotic moduli space geometry.

Finally, it would be important to investigate the effect of instanton corrections in $\mathcal{N} = 1$ theories, where they not only correct the field space metric but they also induce contributions to the superpotential. Quantum corrections to the $\mathcal{N} = 1$ superpotential

and their effect for the SDC have been investigated in [28]. Along these lines one could consider quantum corrections in $\mathcal{N} = 1$ setups — both for the superpotential and Kähler potential — and confront their effects with the Emergent String Conjecture.

Acknowledgments

A Planck mass, string scale, and duality frames

We review here the relation between the Planck mass and the string scale, how it behaves in the three frames discussed in section 2 and 3, and the relations between them. In terms of string frame variables, the Planck mass is given by

$$M_{\text{Pl}}^2 = 2\pi\tau_2^2 \, \mathcal{V}(t) \, M_s^2 \, , \qquad \mathcal{V}(t) = \frac{1}{6}\mathcal{K}_{abc}t^a t^b t^c \, , \tag{A.1}$$

where $\mathcal{V}(t)$ is the volume of the compact manifold X expressed in terms of the triple-intersection numbers $\mathcal{K}_{abc}$ and $\tau_2 = e^{-\phi}$ is the 10d dilaton. In the entirety of this work, we want the Planck-mass-to-string-scale ratio $\frac{M_{\text{Pl}}^2}{M_s^2}$ to remain constant. However, when performing a duality transform, its interpretation might change, which means that if we want to compare a given quantity, such as the rate at which a string becomes tensionless, one needs to keep track of this change. The purpose of this appendix is to provide some details that supplement the main text.

S-duality: here, the transform is a strong-weak duality and the dilaton transforms as:

$$\tau \longrightarrow \tau' = -\frac{1}{\tau} \, , \qquad \tau_2 \longrightarrow \tau_2' = \frac{\tau_2}{|\tau|^2} \, , \tag{A.2}$$

and should not have any effect on *dimensionful* Kähler moduli,

$$Z^a = \frac{z^a}{M_s^2} \longrightarrow Z'^a = Z^a \, , \qquad z^a = b^a + it^a \, . \tag{A.3}$$

To define the dimensionless moduli in the dual frame, one must then make use of both string scales,

$$z'^a = Z'^a (M_s')^2 = z^a \left(\frac{M_s'}{M_s}\right)^2 \, , \tag{A.4}$$

such that the Planck mass in terms of the transformed coordinates is given by

$$\frac{M_{\rm Pl}^2}{M_s^2} = 2\pi(\tau_2')^2 \mathcal{V}(t') \left(\frac{M_s'}{M_s}\right)^2 = 2\pi \frac{\tau_2^2}{|\tau|^4} \mathcal{V}(t) \left(\frac{M_s'}{M_s}\right)^8 . \tag{A.5}$$

In order for this ratio to remain constant, we must require $\left(M_s'^2/M_s^2\right) = |\tau|$, or equivalently in terms of the dimensionless moduli,

$$z'^a = z^a |\tau| , \tag{A.6}$$

which is the transformation used in equation (2.14). Note that when the axions have vanishing vacuum expectation values as is the case in the main text, the dilaton takes the simple transformation $\tau_2 \to 1/\tau_2$.

Fourier-Mukai transform: in order to reach the D3 limit in cases where the compact manifold is an elliptic fibration, $X \to B_2$, one needs to perform two T-dualities, corresponding to an inversion of the dimensionful fibre volume modulus.

$$T^1 = \frac{t^1}{M_s^2} \longrightarrow T''^1 = \frac{1}{M_s^4 T_1} . \tag{A.7}$$

In our case, this corresponds to a particular case of a Fourier-Mukai transform acting on the dimensionless moduli as dictated by (2.22). For simplicity, we will dispense with axion here, as their vacuum expectation values is set to zero in cases of interest. One finds:

$$t^1 \longrightarrow \frac{1}{t^1} , \qquad t^\alpha \longrightarrow t^\alpha + \frac{K^\alpha}{2}\left(t^1 - \frac{1}{t^1}\right) , \tag{A.8}$$

such that

$$(t^1)'' = \frac{1}{t^1}\left(\frac{M_s''}{M_s}\right)^2 = \frac{1}{t^1} . \tag{A.9}$$

We used that, conversely to S-duality, these two T-dualities do not change the string scale: $M_s'' = M_s$.

From the requirement that the Planck mass should be unchanged under this double T-duality, we can infer the transformation of the 10d dilaton — the Buscher's rule — for a non-trivial fibration. To that end, we need the triple-intersection numbers of the elliptically fibred threefold, which are related to the intersection pairing of base divisors, $\eta_{\alpha\beta} = D_\alpha \cdot D_\beta$, and the coefficients of the first Chern class of the base $c_1(B_2) = K^\alpha D_\alpha$ (see e.g. [13]):

$$\mathcal{K}_{111} = \eta_{\alpha\beta}K^\alpha K^\beta = K^\alpha K_\alpha , \qquad \mathcal{K}_{11\alpha} = \eta_{\alpha\beta}K^\beta = K_\alpha , \tag{A.10}$$

$$\mathcal{K}_{1\alpha\beta} = \eta_{\alpha\beta} , \qquad \mathcal{K}_{\alpha\beta\gamma} = 0 , \tag{A.11}$$

where we used the intersection pairing, $\eta_{\alpha\beta}$ to lower indices. The divisors are chosen as in equations (2.18) and (2.19). The volume is then given by

$$\mathcal{V} = \frac{1}{6}\mathcal{K}_{abc}t^a t^b t^c = \frac{1}{6}K^\alpha K_\alpha (t^1)^3 + \frac{1}{2}K_\alpha t^\alpha (t^1)^2 + \frac{1}{2}t_\alpha t^\alpha t^1 . \tag{A.12}$$

For completeness, we also give here the volume of these divisors, using $\mathrm{Vol}(D_a) = \frac{\partial \mathcal{F}}{\partial t^a}$, where $\mathcal{F}$ is the classical prepotential:

$$\mathrm{Vol}(D_1) = \frac{1}{2}K^\alpha K_\alpha (t^1)^2 + K_\alpha t^\alpha t^1 + \frac{1}{2}t^\alpha t_\alpha \,, \tag{A.13}$$

$$\mathrm{Vol}(D_\alpha) = t_\alpha t^1 + \frac{1}{2}K_\alpha (t^1)^2 \,. \tag{A.14}$$

The volume transforms under the transformation (A.8) as:

$$\mathcal{V}(t) \longrightarrow \mathcal{V}''(t) = \left(\frac{1}{(t^1)^2}\mathcal{V}(t) + \frac{\eta_{\alpha\beta}K^\alpha K^\beta}{24}((t^1)^{-3} - t^1) \right) \,, \tag{A.15}$$

such that, in a fashion analogous to S-duality, we see that maintaining a constant Planck-to-string-mass ratio can be achieved by demanding that the 10d dilaton to transforms as

$$\tau_2 \longrightarrow \tau_2'' = t^1 \tau_2 \sqrt{1 - \frac{(t^1)^3 \eta_{\alpha\beta}K^\alpha K^\beta}{24\mathcal{V}} - \frac{\eta_{\alpha\beta}K^\alpha K^\beta}{24\mathcal{V}t^1}} \,. \tag{A.16}$$

The terms proportional to $\eta_{\alpha\beta}K^\alpha K^\beta$ can be understood as corrections to the usual Buscher's rules due to the non-trivial elliptic fibration. We hence see that in the limit $t^a \to \infty$, corresponding to the D1 limit in the main text, the transformation simplifies

$$\tau_2'' \sim t^1 \tau_2 \,, \tag{A.17}$$

which is the transformation we use for the D3 limit, see equation (2.23).

B Moduli space metrics from contact potentials

We collect in this appendix some of the expressions needed to evaluate the moduli space metric subject to quantum corrections in the different limits. In order to know how the line element behaves as one scales the Kähler moduli t^a and the 10d dilaton τ_2, to reach the different limits introduced in section 2, one needs the metric of the hypermultiplet moduli space (2.6):

$$\mathrm{d}s^2_{\mathcal{M}_{\mathrm{HM}}} = \frac{1}{2}\,(\mathrm{d}\varphi_4)^2 + g_{a\bar{b}}\mathrm{d}z^a \mathrm{d}\bar{z}^b + (\mathrm{axions}) \,. \tag{B.1}$$

While for specific cases the exact hypermultiplet moduli space metric can be calculated explicitly using twistorial methods [53], in the limit that we take certain simplifications allow for a simpler approach as shown in appendices C and D of [15]. In this approach the components of $g_{a\bar{b}}$ can be obtained — restricting ourselves to trajectories in the submanifold of $\mathcal{M}_H$ corresponding to vanishing axion vevs — by taking derivatives of the contact potential χ. In the D1 limit the contact potential is dominated by D1/D($-$1)-instantons,

which alter the classical moduli space metric to:

$$
\begin{aligned}
g_{a\bar{b}} &= -\partial_{z^a}\partial_{\bar{z}^b}\log\chi \\
&= \frac{1}{\chi^2}\left[-\frac{\tau_2^2}{8}i\mathcal{K}_{ace}t^c t^e - \frac{\tau_2^2}{8\pi}\sum_{k_\Lambda} n_{\mathbf{k}}^{(0)}\sum_m k_\Lambda \bar{z}^\Lambda k_a K_0\left(2\pi m|k_\Lambda z^\Lambda|\tau_2\right) \right] \\
&\quad \times \left[\frac{\tau_2^2}{8}i\mathcal{K}_{bdf}t^d t^f - \frac{\tau_2^2}{8\pi}\sum_{k_\Lambda} n_{\mathbf{k}}^{(0)}\sum_m k_\Lambda z^\Lambda k_b K_0\left(2\pi m|k_\Lambda z^\Lambda|\tau_2\right) \right] \\
&\quad -\frac{1}{\chi}\left[\frac{\tau_2^2}{8}\mathcal{K}_{abc}t^c - \frac{\tau_2^2}{8\pi}\sum_{k_\Lambda} n_{\mathbf{k}}^{(0)}\sum_m k_a k_b K_0\left(2\pi m|k_\Lambda z^\Lambda|\tau_2\right) \right. \\
&\quad \left. +\frac{\tau_2^2}{16\pi}\sum_{k_\Lambda} n_{\mathbf{k}}^{(0)}\sum_m k_a k_b 2\pi m|k_\Lambda z^\Lambda|\tau_2 K_1\left(2\pi m|k_\Lambda z^\Lambda|\tau_2\right) \right],
\end{aligned}
\tag{B.2}
$$

where the sums over the instanton corrections are proportional to D1-instanton charges k_a. These terms can be evaluated by performing a Poisson resummation over m, resulting in a sum over a new integer n, itself dominated by the zero mode $n = 0$. Up to order one factors, one obtains the following approximate expression:

$$
g_{a\bar{b}} \supset \frac{\tau_2^2}{\chi}\sum_{k_\Lambda} n_{\mathbf{k}}^{(0)}\frac{k_a k_b}{\tau_2|k_\Lambda z^\Lambda|}\,.
\tag{B.3}
$$

In the region of the asymptotic moduli space along the D1 limit where D(-1)-instantons are the only significant contribution, (B.2) simplifies considerably as we can set the $k_a = 0$. The leading contribution to the metric along the Kähler directions is then given by

$$
g_{a\bar{b}} \sim g_{a\bar{b}}^{\text{cl}}\frac{\chi^{\text{cl}}}{\chi}\,,
\tag{B.4}
$$

where $g_{a\bar{b}}^{\text{cl}} \sim e^{-2\sigma}$ is the corresponding component of the classical metric that yields the classical infinite distance. In the presence of D(-1)-instantons, we have

$$
\chi \gg \chi^{\text{cl}}
\tag{B.5}
$$

and the metric component along the Kähler directions in the region where only D(-1)-instantons contribute is much smaller than its classical counterpart. Following the analysis in [15], using the corrected metric to calculate the line element along the Kähler directions, their contribution alone would not yield an infinite distance. This turns out also to be true for the quantum corrected metric in the S-dual F1 limit, where we can relate this effect with the fact that these classical coordinates are not the flat ones, and that taking that into account obstructs the limit of vanishing CY volume.

As we saw that the limit we want to take in the Kähler direction is obstructed by quantum corrections such that the initial scaling of the Kähler coordinates is altered, one may now ask what happens to the assumed scaling of the 10d dilaton in the D1 limit. We

can therefore look at the effective metric components in the direction corresponding to the 10d dilaton that can be obtained from the metric of the 4d dilaton:

$$g_{\tau\tau} = \frac{1}{\chi^2}\left(\frac{\partial\chi}{\partial\tau_2}\right)^2, \qquad g_{\tau t^a} = \frac{1}{\chi^2}\left(\frac{\partial\chi}{\partial\tau_2}\right)\left(\frac{\partial\chi}{\partial t^a}\right). \tag{B.6}$$

We are mainly interested in the first term for which one finds:

$$\frac{1}{\chi^2}\left(\frac{\partial\chi}{\partial\tau_2}\right)^2 = \frac{1}{\chi^2}\left[\frac{\tau_2}{12}\mathcal{K}_{abc}t^a t^b t^c - \frac{\tau_2}{8\pi}\sum_{\mathbf{k_\Lambda}}n_{\mathbf{k}}^{(0)}\sum_m |k_\Lambda z^\Lambda|^2 K_0\left(2\pi m|k_\Lambda z^\Lambda|\tau_2\right)\right]^2. \tag{B.7}$$

Note that the sum over the instantons now contains a term proportional to the D(-1)-instanton charge k_0. Thus, for the region of the D1 limit in which only D(-1)-instantons correct the geometry significantly, the instanton sum does not vanish in contrast to what happened to $g_{a\bar{b}}$ in the same region. Poisson resumming the instanton sum, we obtain the leading term for $g_{\tau\tau}$:

$$\frac{1}{\chi^2}\left(\frac{\partial\chi}{\partial\tau_2}\right)^2 \sim \frac{1}{\chi^2}\left[\frac{1}{16\pi}\sum_{\mathbf{k_\Lambda}}n_{\mathbf{k}}^{(0)}|k_\Lambda z^\Lambda|\right]^2. \tag{B.8}$$

Evaluating the sum over k_0 we obtain

$$g_{\tau\tau} \simeq \frac{1}{\chi^2}\left[\frac{\chi_E(X)}{16\pi}(k_0^{\max})^2\right]^2 \sim (k_0^{\max})^2, \tag{B.9}$$

which should be compared with the classical scaling,

$$g_{\tau\tau}^{\mathrm{cl}} \simeq e^{3\sigma}. \tag{B.10}$$

We thus see that the metric in the τ_2 direction receives strong quantum corrections but since $k_0^{\max} \sim e^{3/2\sigma}$ the scaling of the classical metric is recovered. The scaling of the 10d dilaton — interpreted as a coordinate of the moduli space — therefore does not receive any quantum corrections itself. In section 3, we use this fact to infer that the scaling of the tension of a D1-string in string units is not affected by quantum corrections.

C Contact potential for mutually local instantons

As shown in section 3, D1/D(-1)-instantons are mapped to D3/D1/D(-1)-instantons under a Fourier-Mukai transform. We show here how they can be incorporated to the quantum corrected 4d dilaton.

For generic instanton effects (excluding NS5-instantons) and with vanishing axion vacuum expectation values, these corrections are shown to be given by [40, 41]:

$$e^{-2\varphi_4} = \frac{1}{12}\tau_2^2 \mathcal{K}_{abc}t^a t^b t^c - \frac{\chi_E(X)}{192\pi}$$
$$- \frac{i\tau_2}{64\pi^2}e^{K/2}\sum_\gamma \Omega(\gamma)\int_{l_\gamma}\frac{dt}{t}\left(t^{-1}Z_\gamma - t\bar{Z}_\gamma\right)\log\left(1 - \lambda_D(\gamma)\mathcal{X}_\gamma\right). \tag{C.1}$$

The so-far-undefined quantities denote the following: the Donaldson-Thomas invariants $\Omega(\gamma)$, and the central charge Z_γ associated to the instanton charge vector γ; the integral is performed along a BPS ray l_γ; the integrand depends on a Fourier mode $\mathcal{X}_\gamma$. In cases where the instantons of interest are mutually local, which is the case here as D3/D1/D(-1)-instantons are obtained by the monodromy transformation associated to the Fourier-Mukai transform, $\mathcal{X}_\gamma$ reduces to the so-called "semi-flat" Fourier mode:

$$\mathcal{X}_\gamma^{\rm sf}(t) = \exp\left[-\pi i \tau_2 e^{-\mathcal{K}/2}\left(t^{-1}Z_\gamma - t\bar{Z}_\gamma\right)\right].\tag{C.2}$$

Using a Taylor expansion for the logarithm around $x = 1$, the integral then reduces to a Bessel function, such that the quantum corrections to the 4d dilaton associated to D3/D1/D(-1)-instantons are given by:

$$e^{-2\varphi} = \frac{1}{12}\tau_2^2 \mathcal{K}_{abc}t^a t^b t^c - \frac{\chi(X)}{192\pi} + \frac{\tau_2}{8\pi^2}\sum_\gamma \Omega(\gamma)\sum_{m=1}^\infty \frac{\mathrm{Re}(S_\gamma)}{m}K_1(2\pi m\mathrm{Re}(S_\gamma)),\tag{C.3}$$

where the instanton action S_γ has been found in equation (3.45). One can see that, as expected, it takes the form similar to the correction associated to the original D1/D(-1)-instantons, which as explained in section 3 is related to the fact that the Fourier-Mukai transform can be interpreted as a rotation in the space of charges.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY 4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

[1] C. Vafa, *The string landscape and the swampland*, `hep-th/0509212` [INSPIRE].

[2] T.D. Brennan, F. Carta and C. Vafa, *The String Landscape, the Swampland and the Missing Corner*, `PoS(TASI2017)015` [`arXiv:1711.00864`] [INSPIRE].

[3] E. Palti, *The Swampland: Introduction and Review*, *Fortsch. Phys.* **67** (2019) 1900037 [`arXiv:1903.06239`] [INSPIRE].

[4] H. Ooguri and C. Vafa, *On the Geometry of the String Landscape and the Swampland*, *Nucl. Phys.* **B 766** (2007) 21 [`hep-th/0605264`] [INSPIRE].

[5] E. Palti, *The Weak Gravity Conjecture and Scalar Fields*, *JHEP* **08** (2017) 034 [`arXiv:1705.04328`] [INSPIRE].

[6] T.W. Grimm, E. Palti and I. Valenzuela, *Infinite Distances in Field Space and Massless Towers of States*, *JHEP* **08** (2018) 143 [`arXiv:1802.08264`] [INSPIRE].

[7] B. Heidenreich, M. Reece and T. Rudelius, *Emergence of Weak Coupling at Large Distance in Quantum Gravity*, *Phys. Rev. Lett.* **121** (2018) 051601 [`arXiv:1802.08698`] [INSPIRE].

[8] R. Blumenhagen, D. Kläwer, L. Schlechter and F. Wolf, *The Refined Swampland Distance Conjecture in Calabi-Yau Moduli Spaces*, *JHEP* **06** (2018) 052 [`arXiv:1803.04989`] [INSPIRE].

[9] S.-J. Lee, W. Lerche and T. Weigand, *Tensionless Strings and the Weak Gravity Conjecture*, *JHEP* **10** (2018) 164 [arXiv:1808.05958] [INSPIRE].

[10] S.-J. Lee, W. Lerche and T. Weigand, *A Stringy Test of the Scalar Weak Gravity Conjecture*, *Nucl. Phys.* **B 938** (2019) 321 [arXiv:1810.05169] [INSPIRE].

[11] T.W. Grimm, C. Li and E. Palti, *Infinite Distance Networks in Field Space and Charge Orbits*, *JHEP* **03** (2019) 016 [arXiv:1811.02571] [INSPIRE].

[12] G. Buratti, J. Calderón and A.M. Uranga, *Transplanckian axion monodromy!?*, *JHEP* **05** (2019) 176 [arXiv:1812.05016] [INSPIRE].

[13] P. Corvilain, T.W. Grimm and I. Valenzuela, *The Swampland Distance Conjecture for Kähler moduli*, *JHEP* **08** (2019) 075 [arXiv:1812.07548] [INSPIRE].

[14] A. Joshi and A. Klemm, *Swampland Distance Conjecture for One-Parameter Calabi-Yau Threefolds*, *JHEP* **08** (2019) 086 [arXiv:1903.00596] [INSPIRE].

[15] F. Marchesano and M. Wiesner, *Instantons and infinite distances*, *JHEP* **08** (2019) 088 [arXiv:1904.04848] [INSPIRE].

[16] S.-J. Lee, W. Lerche and T. Weigand, *Emergent Strings, Duality and Weak Coupling Limits for Two-Form Fields*, arXiv:1904.06344 [INSPIRE].

[17] T.W. Grimm and D. Van De Heisteeg, *Infinite Distances and the Axion Weak Gravity Conjecture*, *JHEP* **03** (2020) 020 [arXiv:1905.00901] [INSPIRE].

[18] D. Erkinger and J. Knapp, *Refined swampland distance conjecture and exotic hybrid Calabi-Yaus*, *JHEP* **07** (2019) 029 [arXiv:1905.05225] [INSPIRE].

[19] S.-J. Lee, W. Lerche and T. Weigand, *Emergent Strings from Infinite Distance Limits*, arXiv:1910.01135 [INSPIRE].

[20] F. Baume and E. Palti, *Backreacted Axion Field Ranges in String Theory*, *JHEP* **08** (2016) 043 [arXiv:1602.06517] [INSPIRE].

[21] D. Klaewer and E. Palti, *Super-Planckian Spatial Field Variations and Quantum Gravity*, *JHEP* **01** (2017) 088 [arXiv:1610.00010] [INSPIRE].

[22] I. Valenzuela, *Backreaction Issues in Axion Monodromy and Minkowski 4-forms*, *JHEP* **06** (2017) 098 [arXiv:1611.00394] [INSPIRE].

[23] R. Blumenhagen, I. Valenzuela and F. Wolf, *The Swampland Conjecture and F-term Axion Monodromy Inflation*, *JHEP* **07** (2017) 145 [arXiv:1703.05776] [INSPIRE].

[24] A. Hebecker, P. Henkenjohann and L.T. Witkowski, *Flat Monodromies and a Moduli Space Size Conjecture*, *JHEP* **12** (2017) 033 [arXiv:1708.06761] [INSPIRE].

[25] A. Landete and G. Shiu, *Mass Hierarchies and Dynamical Field Range*, *Phys. Rev.* **D 98** (2018) 066012 [arXiv:1806.01874] [INSPIRE].

[26] H. Ooguri, E. Palti, G. Shiu and C. Vafa, *Distance and de Sitter Conjectures on the Swampland*, *Phys. Lett.* **B 788** (2019) 180 [arXiv:1810.05506] [INSPIRE].

[27] A. Hebecker, D. Junghans and A. Schachner, *Large Field Ranges from Aligned and Misaligned Winding*, *JHEP* **03** (2019) 192 [arXiv:1812.05626] [INSPIRE].

[28] E. Gonzalo, L.E. Ibáñez and A.M. Uranga, *Modular Symmetries and the Swampland Conjectures*, *JHEP* **05** (2019) 105 [arXiv:1812.06520] [INSPIRE].

[29] S.-J. Lee, W. Lerche and T. Weigand, *Modular Fluxes, Elliptic Genera and Weak Gravity Conjectures in Four Dimensions*, *JHEP* **08** (2019) 104 [arXiv:1901.08065] [InSPIRE].

[30] R. Blumenhagen, D. Kläwer and L. Schlechter, *Swampland Variations on a Theme by KKLT*, *JHEP* **05** (2019) 152 [arXiv:1902.07724] [InSPIRE].

[31] A. Font, A. Herráez and L.E. Ibáñez, *The Swampland Distance Conjecture and Towers of Tensionless Branes*, *JHEP* **08** (2019) 044 [arXiv:1904.05379] [InSPIRE].

[32] D. Harlow, *Wormholes, Emergent Gauge Fields and the Weak Gravity Conjecture*, *JHEP* **01** (2016) 122 [arXiv:1510.07911] [InSPIRE].

[33] B. Heidenreich, M. Reece and T. Rudelius, *The Weak Gravity Conjecture and Emergence from an Ultraviolet Cutoff*, *Eur. Phys. J.* **C 78** (2018) 337 [arXiv:1712.01868] [InSPIRE].

[34] P.G. Camara and E. Dudas, *Multi-instanton and string loop corrections in toroidal orbifold models*, *JHEP* **08** (2008) 069 [arXiv:0806.3102] [InSPIRE].

[35] T.W. Grimm, F. Ruehle and D. van de Heisteeg, *Classifying Calabi-Yau threefolds using infinite distance limits*, arXiv:1910.02963 [InSPIRE].

[36] S. Ferrara and S. Sabharwal, *Quaternionic Manifolds for Type II Superstring Vacua of Calabi-Yau Spaces*, *Nucl. Phys.* **B 332** (1990) 317 [InSPIRE].

[37] B. Andreas and D. Hernandez Ruiperez, *Fourier Mukai transforms and applications to string theory*, math/0412328.

[38] C.F. Cota, A. Klemm and T. Schimannek, *Topological strings on genus one fibered Calabi-Yau 3-folds and string dualities*, *JHEP* **11** (2019) 170 [arXiv:1910.01988] [InSPIRE].

[39] T. Schimannek, *Modularity from Monodromy*, *JHEP* **05** (2019) 024 [arXiv:1902.08215] [InSPIRE].

[40] S. Alexandrov, *Twistor Approach to String Compactifications: a Review*, *Phys. Rept.* **522** (2013) 1 [arXiv:1111.2892] [InSPIRE].

[41] S. Alexandrov, J. Manschot, D. Persson and B. Pioline, *Quantum hypermultiplet moduli spaces in $N = 2$ string vacua: a review*, *Proc. Symp. Pure Math.* **90** (2015) 181 [arXiv:1304.0766] [InSPIRE].

[42] B. de Wit and F. Saueressig, *Off-shell $N = 2$ tensor supermultiplets*, *JHEP* **09** (2006) 062 [hep-th/0606148] [InSPIRE].

[43] D. Robles-Llana, M. Roček, F. Saueressig, U. Theis and S. Vandoren, *Nonperturbative corrections to 4D string theory effective actions from $\mathrm{SL}(2,\mathbb{Z})$ duality and supersymmetry*, *Phys. Rev. Lett.* **98** (2007) 211602 [hep-th/0612027] [InSPIRE].

[44] S. Alexandrov, B. Pioline, F. Saueressig and S. Vandoren, *D-instantons and twistors*, *JHEP* **03** (2009) 044 [arXiv:0812.4219] [InSPIRE].

[45] D. Robles-Llana, F. Saueressig, U. Theis and S. Vandoren, *Membrane instantons from mirror symmetry*, *Commun. Num. Theor. Phys.* **1** (2007) 681 [arXiv:0707.0838] [InSPIRE].

[46] K. Hori et al., *Mirror symmetry*, vol. 1 of *Clay mathematics monographs*, AMS, Providence, U.S.A., (2003).

[47] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of Type II Superstrings and the Moduli of Superconformal Field Theories*, *Int. J. Mod. Phys.* **A 4** (1989) 2475 [InSPIRE].

[48] N. Seiberg and S.H. Shenker, *Hypermultiplet moduli space and string compactification to three-dimensions*, *Phys. Lett.* **B 388** (1996) 521 [`hep-th/9608086`] [ɪɴSPIRE].

[49] A. Collinucci, P. Soler and A.M. Uranga, *Non-perturbative effects and wall-crossing from topological strings*, *JHEP* **11** (2009) 025 [`arXiv:0904.1133`] [ɪɴSPIRE].

[50] M. Bianchi and A. Sagnotti, *Twist symmetry and open string Wilson lines*, *Nucl. Phys.* **B 361** (1991) 519 [ɪɴSPIRE].

[51] E.G. Gimon and J. Polchinski, *Consistency conditions for orientifolds and d manifolds*, *Phys. Rev.* **D 54** (1996) 1667 [`hep-th/9601038`] [ɪɴSPIRE].

[52] I. Garcia-Etxebarria, H. Hayashi, R. Savelli and G. Shiu, *On quantum corrected Kähler potentials in F-theory*, *JHEP* **03** (2013) 005 [`arXiv:1212.4831`] [ɪɴSPIRE].

[53] S. Alexandrov and S. Banerjee, *Hypermultiplet metric and D-instantons*, *JHEP* **02** (2015) 176 [`arXiv:1412.8182`] [ɪɴSPIRE].