Instantons and intermittency

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(Received 26 March 1996)

We describe the method for finding the non-Gaussian tails of the probability distribution function (PDF) for solutions of a stochastic differential equation, such as the convection equation for a passive scalar, the random driven Navier-Stokes equation, etc. The existence of such tails is generally regarded as a manifestation of the intermittency phenomenon. Our formalism is based on the WKB approximation in the functional integral for the conditional probability of large fluctuation. We argue that the main contribution to the functional integral is given by a coupled field-force configuration—the *instanton*. As an example, we examine the correlation functions of the passive scalar u advected by a large-scale velocity field δ correlated in time. We find the instanton determining the tails of the generating functional, and show that it is different from the instanton that determines the probability distribution function of high powers of u. We discuss the simplest instantons for the Navier-Stokes equation. [S1063-651X(96)05010-6]

PACS number(s): 47.10.+g, 47.27.−i, 05.40.+j

I. INTRODUCTION

The intermittency phenomenon [reflected in non-Gaussian, scaling-violating tails of the probability distribution function (PDF)] is believed to be the hardest part of the yet-to-be-built theory of turbulence. Neither the physical mechanism nor the mathematical properties of rare fluctuations responsible for the intermittency are known.

Now, what is the most likely force which can lead to the given rare fluctuation of the field? The main idea of this paper is that such a force is not random at all. It satisfies a well defined equation, which follows from the WKB approximation in the functional integral. Asymptotically, the fluctuations of the force around this are most likely negligible. In this respect, the method is similar to the ''optimal fluctuation'' method used in treating properties of a solid with quenched disorder (see e.g., Ref. [1]). A similar approach has been used to analyze high-order terms of the perturbation series in quantum field theory [2], and to calculate phonon attenuation due to multiple quasiparticle production [3]. The idea that PDF tails in Navier-Stokes turbulence may be obtained by minimizing the action was discussed earlier by Giles [4], who tried to find the minimum perturbatively with respect to nonlinearity. We shall see below that the extremal trajectories are nonperturbative objects, as are instantons in quantum mechanics and field theory.

The problem under consideration is quite general, and can be formulated for any field governed by a nonlinear dynamic equation and driven by a random ''force.'' Generally, the PDF of the field depends both on the statistics of the driven force and on the form of the dynamical equation. Here we are interested in the second dependence, so that we assume the force to be Gaussian. Because of nonlinearity, the PDF of the field is non-Gaussian even for a Gaussian random force. Note that a strong intermittency also appears for linear problems with ''multiplicative noise,'' for instance, for a passive scalar advected by a random velocity field.

We start with the dynamical equation

$$\partial_t u+\mathcal{L}(u)=\phi, \tag{1.1}$$

that controls the evolution of a field $u(t,\mathbf{r})$ under the action of a random ''force'' $\phi(t,\mathbf{r})$. Here $\mathcal{L}(u)$ is a nonlinear expression, it can be thought of as being local in space. Generally, both the field u and the force ϕ have a number of components. The Gaussian statistics of the force ϕ is completely characterized by the pair correlation function

$$\langle \phi(t_1,\mathbf{r}_1)\phi(t_2,\mathbf{r}_2)\rangle=\Xi(t_1-t_2,\mathbf{r}_1-\mathbf{r}_2). \tag{1.2}$$

In principle, relations (1.1) and (1.2) contain all the information about the statistics of u.

Equation (1.1) describes e.g., thermal fluctuations in hydrodynamics where it is reduced to the well known Langevin equation [5]. Then ϕ is short correlated in time and in space such that it can be treated as a white noise. For some systems, this thermal noise produces remarkable dynamical effects. Some examples are collected in Ref. [6]. Here we are interested in turbulence, where ϕ is an external ''force'' correlated on large scales in space. Turbulence was first treated in terms of Eq. (1.1) by Wyld [9], who formulated the diagram technique as a perturbation series with respect to the nonlinear term in the Navier-Stokes equation. The diagram technique cannot be applied to our problem since we are interested in nonperturbative effects. Nevertheless we can use the functional that generates the technique since it is a nonperturbative object. Such a generating functional was introduced in [10,11] for Eq. (1.1); it has the form

$$\mathcal{Z}(\lambda)\equiv\left\langle \exp\left(i\int dt\, d\mathbf{r}\, \lambda u\right)\right\rangle$$
$$=\int \mathcal{D}u\mathcal{D}p \exp\left(i\mathcal{I}+i\int dt\, d\mathbf{r}\,\lambda u\right), \tag{1.3}$$

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where p is an auxiliary field, and the effective action is

$$\mathcal{I}=\int dt\,d\mathbf{r}\,p[\partial_t u+\mathcal{L}(u)] + \frac{i}{2}\int dt\,dt'\,d\mathbf{r}\,d\mathbf{r}'\,\Xi(t-t',\mathbf{r}-\mathbf{r}')pp'. \quad (1.4)$$

The coefficients of the expansion of $\mathcal{Z}$ in λ are the correlation functions of u. The auxiliary field p determines the response functions of the system, for instance, the linear response function (Green function) is $G=\langle up\rangle$. Note the remarkable property [12]

$$\int \mathcal{D}u\mathcal{D}p\,\exp(i\mathcal{I})=1,$$

related to the causality. That is the reason why the normalization constant is unity in Eq. (1.3). This makes it possible to average any additional random field directly over $\mathcal{Z}$ if necessary.

The asymptotics of $\mathcal{Z}(\lambda)$ at large λ is determined by the saddle-point configuration (usually called the classical trajectory or instanton) which should satisfy the following equations obtained by varying the argument of the exponent in Eq. (1.3) with respect to u and p:

$$\partial_t u+\mathcal{L}(u)=-i\int dt'\,d\mathbf{r}'\,\Xi(t-t',\mathbf{r}-\mathbf{r}')p(t',\mathbf{r}'), \quad (1.5)$$

$$\partial_t p-\frac{\delta\mathcal{L}}{\delta u}p=\lambda. \quad (1.6)$$

Solutions of Eqs. (1.5) and (1.6) are generally smooth functions of t and $\mathbf{r}$. Comparing Eqs. (1.1) and (1.5) we conclude that the right-hand side of Eq. (1.5) just describes a special force configuration necessary to produce the instanton. If u_{inst} is a solution of Eq. (1.5) and (1.6), then asymptotically, at large λ,

$$\delta\ln\mathcal{Z}(\lambda)/\delta\lambda=iu_{\text{inst}}. \quad (1.7)$$

Let us discuss the boundary conditions for the saddle-point equations. Equation (1.5) implies that we should fix the value u_{in} for the field u at the initial time t_{in}. Conversely, a boundary condition for field p is implied at the remote future since, as follows from Eq. (1.6), it propagates in the negative direction in time. Minimization of the action generally requires $p\to 0$ at $t\to\infty$. For the instantons discussed below, the finiteness of the action will also require $u\to 0$ at $t\to-\infty$.

If one is interested in the simultaneous statistics of u, then the function λ can be chosen as

$$\lambda(t,\mathbf{r})=y\,\delta(t)\lambda_0(\mathbf{r}), \quad (1.8)$$

where y is a number, and λ_0 is an appropriate function of $\mathbf{r}$ depending on what spatial correlation functions we are going to study. In this case, we should find the solution for p satisfying the rule $p=0$ at $t>0$. The system (1.5) and (1.6) is thus to be treated for $t<0$ only. This corresponds to the causality principle, since only processes occurring in the past could influence the value of the simultaneous correlation functions at $t=0$. The formal ground for the rule follows from the consideration of the problem in the restricted time interval $t<t_0$, which is possible if $\lambda=0$ at $t>t_0$. Then the minimization of $\mathcal{I}+\int dt\,d\mathbf{r}\,\lambda u$ over the final value $u(t_0)$ gives $p(t_0)=0$, because of the boundary term originating from $\int dt\,d\mathbf{r}\,p\partial_t u$.

One may also be interested in the probability distribution function $\mathcal{P}(u)$ for the field u. It can be expressed via the generating functional $\mathcal{Z}(\lambda)$ by the functional Fourier transform

$$\mathcal{P}(u)=\int \mathcal{D}\lambda\,\mathcal{Z}(\lambda)\exp\left(-i\int dt\,d\mathbf{r}\,\lambda u\right). \quad (1.9)$$

We expect that the behavior of $\mathcal{P}(u)$ for large u as well as the behavior of $\mathcal{Z}(\lambda)$ for large λ is associated with some saddle-point configurations. Generally, the configurations are not always the same for both Eqs. (1.3) and (1.9). Indeed, we see from Eq. (1.9) that the tail of $\mathcal{P}(u)$ at large u corresponds to a large value of $\delta\ln\mathcal{Z}(\lambda)/\delta\lambda$ which is related to large λ only if the tails of both the PDF and the generating functional decay faster than exponent—see the example in Sec. III. Otherwise, those tails are determined by different configurations as is demonstrated in Sec. II.

The best starting point to develop the instanton formalism is the problem of white-noise-advected passive scalar θ since it allows for a detailed analytical treatment [14–16]. It will be shown in Sec. II that both $\mathcal{P}(\theta)$ and $\mathcal{Z}(\lambda)$ have exponential tails, as was established before by Shraiman and Siggia [15] (see also [16]). By using this example, we shall explicitly demonstrate that different instanton configurations are responsible for the tails of the generating functional $\mathcal{Z}(\lambda)$ at large λ and of PDF at large θ, respectively. It is instructive to recognize the difference between the instantons: We shall show that the instanton that is responsible for large θ corresponds to a small strain and suppressed stretching. Conversely, the instanton that determines the tails of $\mathcal{Z}$ corresponds to a large value of strain.

Section III presents the first step in studying instantons of the Navier-Stokes equation. Only instantons for the two-point generating functional $\langle\exp(i\lambda(u_1-u_2)\rangle$ will be considered. The family of such instantons corresponds to the velocity fields with a linear spatial profile at $r\ll L$. Consideration of the instanton perturbations (giving the fluctuation contribution into the action) that correspond to spiral creation in the straining field of the instanton will be the subject of further publications.

II. PASSIVE SCALAR ADVECTED BY A LARGE-SCALE VELOCITY FIELD

Let us show how the general formalism described in Sec. I works for a particular problem: the advection of a passive scalar field $\theta(t,\mathbf{r})$ by an incompressible turbulent flow in d-dimensional space [13–16]. The advection is governed by the equation

$$(\partial_t+v_\alpha\nabla_\alpha-\kappa\triangle)\theta=\phi, \quad \nabla_\alpha v_\alpha=0, \quad (2.1)$$

where $\phi(t,\mathbf{r})$ is the external source, $\mathbf{v}$ is the advecting velocity, and $\triangle$ designates a Laplacian, κ being the diffusion coefficient. Both $\mathbf{v}(t,\mathbf{r})$ and $\phi(t,\mathbf{r})$ are random functions of

t and $\mathbf{r}$. We regard the statistics of the velocity and the source to be independent. Therefore, all correlation functions of θ are to be treated as averages over both statistics.

We assume that the source ϕ is δ correlated in time and spatially correlated on a scale L, and that is has Gaussian statistics completely determined by the pair correlation function

$$\langle \phi(t_1,\mathbf{r}_1)\phi(t_2,\mathbf{r}_2)\rangle=\delta(t_1-t_2)\chi(r_{12}). \tag{2.2}$$

Here $\chi(r_{12})$ as a function of the argument $r_{12}\equiv|\mathbf{r}_1-\mathbf{r}_2|$ decays on the scale L. We are interested in the behavior of the correlation functions on scales $r\ll L$. Thus only the constant $P_2=\chi(0)$ will enter all the answers. The constant P_2 has the physical meaning of the production rate of θ^2.

Following Kraichnan [13,14], we consider the case of a Gaussian velocity $\mathbf{v}$ δ correlated in time and containing only large-scale space harmonics. Then the velocity statistics is also completely determined by the pair correlation functions

$$\langle v_\alpha(t_1,\mathbf{r}_1)v_\beta(t_2,\mathbf{r}_2)\rangle=\delta(t_1-t_2)V_{\alpha\beta},$$

$$V_{\alpha\beta}=V_0\delta_{\alpha\beta}-\mathcal{K}_{\alpha\beta}(\mathbf{r}_{12}),\quad \mathcal{K}_{\alpha\beta}(0)=0. \tag{2.3}$$

Here the so-called eddy diffusivity is as follows:

$$\mathcal{K}_{\alpha\beta}=D(r^2\delta_{\alpha\beta}-r_\alpha r_\beta)+\frac{D(d-1)}{2}\delta_{\alpha\beta}r^2, \tag{2.4}$$

where d is the dimensionality of the space and isotropy of the velocity statistics being assumed. Representations (2.3) and (2.4) are valid for scales less than the velocity infrared cutoff L_u, which is supposed to be the largest scale of the problem. Then V_0 and $\mathcal{K}_{\alpha\beta}$ in Eq. (2.3) are the first two terms of the expansion of the velocity correlation function in r/L_u, so that $D\sim V_0/L_u^2$. We also presume the inequality $dDL^2\gg\kappa$ which guarantees the existence of a convective interval of scales $r_d\ll r\ll L$ where correlation functions of the passive scalar are formed mainly by stretching in the inhomogeneous velocity field. Here $r_d=2\sqrt{\kappa/[D(d-1)]}$ is the diffusion length. We thus consider the limit of high Peclet number $L/r_d\gg1$. Note that, to have finite moments of the scalar field, the Peclet number should be kept finite, since the variance of the scalar already turns into infinity in the limit of infinite Peclet number [13].

The statistics of the large-scale velocity field has a remarkable property: It follows from expressions (2.3) and (2.4) that the correlation function of the strain field $\sigma_{\alpha\beta}=\nabla_\beta v_\alpha$ is $\mathbf{r}$ independent,

$$\langle\sigma_{\alpha\beta}(t_1)\sigma_{\mu\nu}(t_2)\rangle =D[(d+1)\delta_{\alpha\mu}\delta_{\beta\nu}-\delta_{\alpha\nu}\delta_{\beta\mu}-\delta_{\alpha\beta}\delta_{\mu\nu}]\delta(t_1-t_2). \tag{2.5}$$

This means that the strain field $\sigma_{\alpha\beta}$ can be treated as a random function of time t only. That property enables one to find in detail statistical properties of the field θ [15,16]. To exploit this property, it is convenient to pass into the comoving reference frame—that is, to the frame moving with the velocity of a Lagrangian particle of the fluid. This means that we pass to the space variable $\mathbf{r}-\varrho(t)$, where $\varrho(t)$ is the Lagrangian trajectory of the particle [7,8]. We will take the particle positioned at the origin at time $t=0$, then

$$\varrho(t)=\int_0^t d\tau\,\mathbf{v}(\tau,\varrho(\tau)). \tag{2.6}$$

After the transformation $\mathbf{r}\to\mathbf{r}-\varrho(t)$, Eq. (2.1) acquires the form

$$\{\partial_t+[v_\alpha(t,\mathbf{r})-v_\alpha(t,0)]\nabla_\alpha-\kappa\triangle\}\theta=\phi. \tag{2.7}$$

It can be seen from Eqs. (2.3) and (2.4) that the statistics of $v_\alpha(t,\mathbf{r})-v_\alpha(t,0)$ coincides with the statistics of $\sigma_{\alpha\beta}r_\beta$. That means that the generating functional corresponding to (2.7) can be written as

$$\mathcal{Z}(\lambda)=\int\mathcal{D}\theta\,\mathcal{D}p\,\mathcal{D}\sigma\exp\left(-\mathcal{F}(\sigma)+i\mathcal{I}+i\int dt\,d\mathbf{r}\,\lambda\,\theta\right), \tag{2.8}$$

where $\sigma_{\alpha\beta}$ is a function of time satisfying $\sigma_{\alpha\alpha}=0$, and its PDF is $\exp(-\mathcal{F})$. The effective action I and the functional $\mathcal{F}$ in Eq. (2.8) are

$$i\mathcal{I}=i\int dt\,d\mathbf{r}(p\partial_t\theta+p\sigma_{\alpha\beta}r_\beta\nabla_\alpha\theta+\kappa\nabla p\nabla\theta)-\tfrac{1}{2}\int dt\,d\mathbf{r}_1\,d\mathbf{r}_2\,p_1\chi(r_{12})p_2, \tag{2.9}$$

$$\mathcal{F}=\frac{1}{2d(d+2)D}\int dt[(d+1)\sigma_{\alpha\beta}\sigma_{\alpha\beta}+\sigma_{\alpha\beta}\sigma_{\beta\alpha}]. \tag{2.10}$$

Note that there is a difference between Eqs. (1.3) and (2.8) which is in the presence of an additional random field $\sigma_{\alpha\beta}$.

A. Uniaxial instanton

Here we examine the saddle-point contribution to the generating functional $\mathcal{Z}(\lambda)$. The equations describing the saddle points are extremum conditions for $i\mathcal{I}+i\int dt\,d\mathbf{r}\,\lambda\,\theta-\mathcal{F}$. Starting from expressions (2.9) and (2.10), we find

$$\partial_t\theta+\sigma_{\alpha\beta}r_\beta\nabla_\alpha\theta-\kappa\nabla^2\theta=-i\int d\mathbf{r}'\,\chi(|\mathbf{r}-\mathbf{r}'|)p(t,\mathbf{r}'), \tag{2.11}$$

$$\partial_t p+\sigma_{\alpha\beta}r_\beta\nabla_\alpha p+\kappa\nabla^2 p=\lambda, \tag{2.12}$$

$$\sigma_{\alpha\beta}(t)=iD\int d\mathbf{r}[(d+1)r_\beta\nabla_\alpha\theta-r_\alpha\nabla_\beta\theta-\delta_{\alpha\beta}r_\gamma\nabla_\gamma\theta]p, \tag{2.13}$$

where $p=p(t,\mathbf{r})$, and $\theta=\theta(t,\mathbf{r})$. If to take into account only the saddle-point contribution described by Eqs. (2.11), (2.12), and (2.13), then

$$\mathcal{Z}(\lambda)=\left\langle\exp\left(i\int dt\,d\mathbf{r}\,\lambda\,\theta\right)\right\rangle\propto\exp(-\mathcal{F}_{\rm extr}), \tag{2.14}$$

where $\mathcal{F}_{\rm extr}$ is the saddle-point value of $\mathcal{F}-i\mathcal{I}-i\int dt\,d\mathbf{r}\,\lambda\,\theta$. From (2.9), (2.10), and (2.12), one obtains

$$\mathcal{F}_{\rm extr}=\tfrac{1}{2}\int dt\, d\mathbf{r}_1\, d\mathbf{r}_2\, p_1\chi(r_{12})p_2+\frac{1}{2d(d+2)D}\times\int dt[(d+1)\sigma_{\alpha\beta}\sigma_{\alpha\beta}+\sigma_{\alpha\beta}\sigma_{\beta\alpha}]. \quad (2.15)$$

In the following, we will be interested in simultaneous correlation functions of θ, so that we take the field λ in the form (1.8) and solve the equations for negative time $t<0$. Let us stress that for function (1.8) the term $\lambda\theta$ is not influenced by the transformation $\mathbf{r}\to\mathbf{r}-\varrho(t)$ because of $\varrho(0)=0$. Note that the system of equations (2.11), (2.12), and (2.13) with function (1.8) is invariant under the transformation

$$\sigma\to X\sigma,\quad p\to Xp,\quad t\to X^{-1}t,\quad y\to Xy,$$
$$\kappa\to X\kappa,\quad \mathcal{F}_{\rm extr}\to X\mathcal{F}_{\rm extr}, \quad (2.16)$$

where X is an arbitrary factor. This leads to the conclusion that

$$\mathcal{F}_{\rm extr}=yf(y/\kappa), \quad (2.17)$$

with the function f to be determined.

We will treat nearly single-point statistics. This means that the space support of the function λ_0 in Eq. (1.8) is taken to be much smaller than the pumping length L. From the other hand, we would like to avoid bulky formulas related to the account of diffusion. Therefore, the size of the support is believed to be much larger than the diffusion length r_d. We thus come to

$$\lambda(t,\mathbf{r})=y\,\delta(t)\,\delta_\Lambda(\mathbf{r}), \quad (2.18)$$

where $\delta_\Lambda(\mathbf{r})$ is a function with the characteristic size Λ^{-1} satisfying $L\gg\Lambda^{-1}\gg r_d$, and normalized $\int d\mathbf{r}\,\delta_\Lambda(\mathbf{r})=1$. The effective Peclet number $L\Lambda$ is thus assumed to be large. For example, we can take

$$\delta_\Lambda(\mathbf{r})=\frac{\Lambda^d}{\pi^{d/2}}\exp(-\Lambda^2r^2). \quad (2.19)$$

We thus examine the object

$$\mathcal{Z}_\Lambda=\langle\exp(iy\theta_\Lambda)\rangle, \quad (2.20)$$

where

$$\theta_\Lambda=\int d\mathbf{r}\,\delta_\Lambda(\mathbf{r})\,\theta(t=0,\mathbf{r}). \quad (2.21)$$

Keeping in mind the inequality $\Lambda^{-1}\gg r_d$, in the following we omit the diffusive terms in the equations. The extremum conditions (2.11) and (2.12) are then as follows:

$$\partial_t\theta+\sigma_{\alpha\beta}r_\beta\nabla_\alpha\theta=-i\int d\mathbf{r}'\chi p', \quad (2.22)$$

$$\partial_tp+\sigma_{\alpha\beta}r_\beta\nabla_\alpha p=y\,\delta(t)\,\delta_\Lambda(\mathbf{r}), \quad (2.23)$$

According to Eq. (2.13) the structure of the tensor $\sigma_{\alpha\beta}$ reflects the spatial symmetry of the fields $\theta(\mathbf{r})$ and $p(\mathbf{r})$. The source term in the right-hand side of Eq. (2.23) has spherical symmetry. Therefore, Eqs. (2.22) and (2.23) have a spherically symmetrical solution. However, incompressibility condition $\sigma_{\alpha\alpha}=0$ requires that $\sigma_{\alpha\beta}=0$ on that solution which makes the respective action to be infinite due to time integration. The probability of a spherical solution is thus zero. Let us show that the mostly symmetric solution with a finite action has a uniaxial form, which means that $\sigma_{\alpha\beta}$ is a diagonal matrix with the components

$$\mathrm{diag}\,\sigma=(-s,s/(d-1),\ldots). \quad (2.24)$$

As was suggested in [16], it is useful to pass to the fields

$$\tilde\theta(t,\mathbf{r})=\theta(t,e_\parallel x,e_\perp\mathbf{r}_\perp),\quad \tilde p(t,\mathbf{r})=p(t,e_\parallel x,e_\perp\mathbf{r}_\perp), \quad (2.25)$$

where x is the coordinate along the marked direction, $\mathbf{r}_\perp$ is the component of the radius vector $\mathbf{r}$ perpendicular to the direction, and

$$e_\parallel(t')=\exp\left[\int_{t'}^0 dt\,s(t)\right],\quad e_\perp^{d-1}=e_\parallel^{-1}. \quad (2.26)$$

Now Eqs. (2.22) and (2.23) can be rewritten

$$\partial_t\tilde\theta=-i\int d\mathbf{r}'\,\chi(R(t))\tilde p(t,\mathbf{r}'), \quad (2.27)$$

$$\partial_t\tilde p=y\,\delta(t)\,\delta_\Lambda(\mathbf{r})\to\tilde p=-y\,\delta_\Lambda(\mathbf{r}),\quad t<0, \quad (2.28)$$

where we presented an obvious solution for $\tilde p$ satisfying $\tilde p=0$ at $t>0$. The quantity R in Eq. (2.27) is

$$R^2=e_\parallel^2(x-x')^2+e_\perp^2(\mathbf{r}_\perp-\mathbf{r}'_\perp)^2. \quad (2.29)$$

Note that

$$\partial_tR=-s(x\partial_x+x'\partial'_x)R+\frac{s}{d-1}(\mathbf{r}_\perp\cdot\nabla_\perp+\mathbf{r}'_\perp\cdot\nabla'_\perp)R. \quad (2.30)$$

For the considered uniaxial geometry, relation (2.13) gives

$$s=-iD\int d\mathbf{r}\,\tilde p[(d-1)x\partial_x-\mathbf{r}_\perp\cdot\nabla_\perp]\tilde\theta. \quad (2.31)$$

Now, using Eqs. (2.27) and (2.28), we find

$$\partial_ts=-Dy^2\int d\mathbf{r}\,d\mathbf{r}'\,\delta_\Lambda(\mathbf{r})\,\delta_\Lambda(\mathbf{r}')\times[(d-1)x\partial_x-\mathbf{r}_\perp\cdot\nabla_\perp]\chi(R). \quad (2.32)$$

By virtue of Eq. (2.30) and the symmetry properties of the integrand in Eq. (2.32), we obtain

$$s\partial_ts=\frac{(d-1)Dy^2}{2}\partial_t\int d\mathbf{r}\,d\mathbf{r}'\,\delta_\Lambda(\mathbf{r})\,\delta_\Lambda(\mathbf{r}')\chi(R). \quad (2.33)$$

The equation has an obvious first integral, which can be established if we take into account that $s\to0$ if $t\to-\infty$ [otherwise Eq. (2.15) is infinite]:

$$s^2=(d-1)Dy^2\int d\mathbf{r}\,d\mathbf{r}'\,\delta_\Lambda(\mathbf{r})\delta_\Lambda(\mathbf{r}')\chi(R). \quad (2.34)$$

One can demonstrate that the main contribution to $\mathcal{Z}_\Lambda$ is determined by the saddle point with $s>0$. Our instanton thus describes a stretching in the x direction and a contraction in other directions while time moves backwards: $e_\parallel$ increases with increasing $|t|$ in accordance with (2.26). This means that the characteristic value of R in Eq. (2.34) can be estimated as $R\sim\Lambda^{-1}e_\parallel$. At small $|t|$, where $e_\parallel$ is not very large, $\chi(R)$ in Eq. (2.34) can be substituted for by $P_2=\chi(0)$, and we find that $s\simeq s_1$, where

$$s_1=y\sqrt{(d-1)P_2D}. \quad (2.35)$$

That leads to $e_\parallel\approx\exp(s_1|t|)$, which is correct if $R<L$, which means $|t|\lesssim s_1^{-1}\ln(L\Lambda)$. In the opposite limit $|t|\gg s_1^{-1}\ln(L\Lambda)$, the value of s tends to zero.

The above analysis shows that the main contribution to $\mathcal{F}_{\rm extr}$ [Eq. (2.15)] is associated with the region of integration $|t|\lesssim s_1^{-1}\ln(L\Lambda)$, when stretching from the distance Λ^{-1} to distance L takes place. The first term in Eq. (2.15) can be written as $\frac{1}{2}y^2P_2s_1^{-1}\ln(L\Lambda)$. Substituting Eqs. (2.24) and (2.35) into the second term of Eq. (2.15), we find

$$\mathcal{F}_{\rm extr}=\left(\frac{P_2y^2}{(d-1)D}\right)^{1/2}\ln(L\Lambda). \quad (2.36)$$

Note that the expression is in agreement with Eq. (2.17), since we considered the case $r_d\Lambda\ll1$, where the answer should be κ independent. It is also possible to restore $\mathcal{Z}_\Lambda(y)$ in the limit $\Lambda\to\infty$; that is, for the single-point object. For this we should recognize that generally $\mathcal{F}_{\rm extr}$ is a function of the dimensionless parameter Λr_d, and use property Eq. (2.17). Then in the limit $r_d\Lambda\gg1$, where the Λ dependence should disappear, we obtain

$$\mathcal{F}_{\rm extr}=\left(\frac{P_2y^2}{(d-1)D}\right)^{1/2}\left\{\ln(L/r_d)+\frac{1}{4}\ln\left(\frac{P_2}{D}y^2(d-1)\right)\right\}. \quad (2.37)$$

Note the nontrivial dependence of this single-point object on y. This is a consequence of the time dependence of the effective diffusion cutoff, which can be seen at the direct solution with an explicit account of the diffusion.

Above, we considered the simplest case of the uniaxial strain matrix $\sigma_{\alpha\beta}$. It is not very difficult to generalize the scheme to the case where principal axes of $\sigma_{\alpha\beta}$ are fixed (that is, they do not depend on time). The answer shows that it is the uniaxial solution that gives the minimum value of $\mathcal{F}_{\rm extr}$, and therefore only this contribution should be taken into account. The fact that the symmetry of the solution is lower than the symmetry of the source means spontaneous symmetry breaking. Note that symmetry breaking in the random $N\times N$ matrix process with SU(N)-symmetric statistics was noticed previously by Dorokhov [17], who showed that the mean Lyapunov exponents are nonzero and that a largest exponent exists. In our case, this means the existence of a mean stretching and spectral cascade of the passive scalar. It is interesting to note that the irreversibility of scalar turbulence and related symmetry breaking naturally appears in our field-theoretical formalism, precisely in the way it appeared in the classical cases of ferromagnetism, superconductivity, etc. The probability of a less symmetric state is larger than that of a symmetric one.

As long as we are interested in the tail of the generating function $\mathcal{Z}_\Lambda(y)$ at large y, the instanton contributions (2.36) or (2.37) give the correct answer. However, it is not enough to consider that contribution to obtain the tails of the PDF, because the respective tail of $\mathcal{Z}(y)$ is exponential. Indeed, we shall see below that the tails of $\mathcal{P}(\theta)$ are determined by the contributions at moderate y. We thus face the problem of finding $\mathcal{Z}(y)$ at arbitrary y. Fortunately, the tails of $\mathcal{P}(\theta)$ are also determined by the instanton contribution; however, this instanton is different from the above uniaxial solution, which represents the situation where stretching occurs along one marked direction. It is obvious that, if the direction slowly varies in time, the value of the effective action will not be essentially influenced. The role of such soft fluctuations is expected to be negligible if the characteristic time $s_1^{-1}\ln(L\Lambda)$ of the stretching is small enough. We thus conclude, taking into account Eq. (2.35), that expression (2.36) is correct at large y. At moderate y, the fluctuations of the stretching direction should be taken into account; it is the topic of Sec. II B. There we shall explicitly integrate over the soft mode, and obtain different equations for the instanton.

B. Isotropic instanton

Here we take into account the fluctuations of the stretching direction which were neglected in Sec. III A. For that purpose, it is useful to introduce the variable measuring the stretching rate along the current stretching direction (the direction of the maximal Lyapunov exponent) determined by the strain field $\sigma_{\alpha\beta}=\nabla_\beta v_\alpha$. For this aim, it is useful to perform the transformation of the fields θ and p, generalizing Eq. (2.25) for an arbitrary $\sigma_{\alpha\beta}$ [16]. That is, let us pass to the fields

$$\tilde\theta(t,\mathbf{r})=\theta(t,M_{\alpha\beta}r_\beta),\quad \tilde p(t,\mathbf{r})=p(t,M_{\alpha\beta}r_\beta), \quad (2.38)$$

with the $d\times d$ matrix $M_{\alpha\beta}$ controlled by the equation

$$\partial_t\hat M=\hat\sigma\hat M,\quad \hat M(t=0)=\hat 1, \quad (2.39)$$

with a formal solution

$$\hat M=T\exp\left(\int_0^t dt'\,\hat\sigma(t')\right). \quad (2.40)$$

The symbol T designates the antichronological ordering for negative t. Note that $\det\hat M=1$ due to the incompressibility condition $\mathrm{tr}\hat\sigma=\nabla_\alpha v_\alpha=0$. Performing the substitution in Eq. (2.9) and passing to the space variable $\hat M\mathbf{r}$ (the Jacobian of the transformation is equal to unity due to $\det\hat M=1$), one obtains

$$i\mathcal{I}=i\int dt\,d\mathbf{r}\,\tilde p\,\partial_t\tilde\theta-\frac{1}{2}\int dt\,d\mathbf{r}_1d\mathbf{r}_2\tilde p_1\chi(R)\tilde p_2, \quad (2.41)$$

where

$$R_\alpha = M_{\alpha\beta}(r_{1\beta} - r_{2\beta}). \tag{2.42}$$

We see that only $\mathbf{R}$ is σ dependent in Eq. (2.41) and, moreover, only its absolute value R enters the effective action. Just that value is a measure of the stretching irrespective of the directions of the current main axes of the matrix $\hat{\sigma}$. The statistics of R can be established starting from the PDF $\exp(-\mathcal{F})$; see, e.g., [18]. The answer is that, for negative times, R can be written as

$$R(t) = \exp\left(\int_t^0 dt'\, \zeta(t') \right) |\mathbf{r}_1 - \mathbf{r}_2|, \tag{2.43}$$

with the random variable ζ having PDF $\exp(-\mathcal{F}_\zeta)$ with

$$\mathcal{F}_\zeta = \int dt \frac{1}{2D(d-1)} \left(\zeta - \frac{d(d-1)}{2} D \right)^2. \tag{2.44}$$

The generating functional (2.20) is thus rewritten as

$$\mathcal{Z}_\Lambda = \int \mathcal{D}\tilde{\theta}\, \mathcal{D}\tilde{p}\, \mathcal{D}\zeta \exp(iy\,\theta_\Lambda + i\mathcal{I} - \mathcal{F}_\zeta), \tag{2.45}$$

where θ_Λ is defined by Eq. (2.21).

We have performed the exact transformation of the statistical weight introducing the variable ζ which measures the stretching rate. Gaussian integration over $\tilde{p}$ and $\tilde{\theta}$, and subsequent minimization of the action with respect to ζ, is equivalent to the following system:

$$\partial_t \tilde{p} = y\,\delta(t)\,\delta_\lambda(\mathbf{r}), \tag{2.46}$$

$$\partial_t \tilde{\theta}(t,\mathbf{r}_1) = -i \int d\mathbf{r}_2\, \chi[R(t)] \tilde{p}(t,\mathbf{r}_2), \tag{2.47}$$

$$\zeta(t') = \frac{d(d-1)}{2} D - \frac{d-1}{2} D \int_{-\infty}^{t'} dt\, d\mathbf{r}_1\, d\mathbf{r}_2\, \tilde{p}_1 \tilde{p}_2 \frac{\partial \chi}{\partial R} R. \tag{2.48}$$

Equation (2.46) has the same form as Eq. (2.28), and consequently has the same solution $\tilde{p} = -y\,\delta_\Lambda(\mathbf{r})$. It follows from Eqs. (2.45) and (2.46) that, in the saddle-point approximation, $\mathcal{Z}_\Lambda \propto \exp(-\mathcal{F}_{\rm extr})$, where

$$\mathcal{F}_{\rm extr} = \frac{1}{2} \int dt\, d\mathbf{r}_1\, d\mathbf{r}_2\, \tilde{p}_1\, \tilde{p}_2 \chi(R) + \frac{1}{2D(d-1)} \times \int dt \left(\zeta - \frac{(d-1)d}{2} D \right)^2. \tag{2.49}$$

It follows from Eq. (2.43) that $\partial_t R = -\zeta R$. Using that, we can find the first integral of Eq. (2.48),

$$\zeta^2 = \frac{d^2(d-1)^2}{4} D^2 + (d-1) D y^2 \times \int d\mathbf{r}_1 d\mathbf{r}_2\, \delta_\Lambda(\mathbf{r}_1)\, \delta_\Lambda(\mathbf{r}_2) \chi(R). \tag{2.50}$$

The constant here is established using the property $\zeta \to (d-1)dD/2$ at $t \to -\infty$ following from $\theta \to 0$ at $t \to -\infty$ [Eq. (2.49) is infinite otherwise]. The only dynamical variable of the instanton is thus the stretching rate given by a scalar ζ, the direction of the stretching does not enter the expressions. That is why we call this instanton isotropic (not to be confused with a symmetric one, where stretching is absent). The characteristic value of R on the right-hand side of Eq. (2.50) can be estimated as

$$R(t') \sim \Lambda^{-1} \exp\left(\int_{t'}^0 dt\, \zeta(t) \right). \tag{2.51}$$

If $R \ll L$, then the integral on the right-hand side of Eq. (2.50) is approximately equal to P_2; if $R \gg L$, then the integral is negligible. This means that there are two different time intervals. At large $|t|$, it is $\zeta \simeq (d-1)dD/2$, and at small $|t|$ it is $\zeta \simeq \zeta_1$, where

$$\zeta_1^2 = \frac{(d-1)^2 d^2}{4} D^2 + (d-1) D y^2 P_2. \tag{2.52}$$

The boundary between the regions is at $|t| \sim \zeta_1^{-1} \ln(L\Lambda)$. The main contribution to $\mathcal{F}_{\rm extr}$, (2.49), is associated with the region $|t| < t_1 = \zeta_1^{-1} \ln(L\Lambda)$. Again, this has a simple physical meaning: the action (i.e., probability) is determined by the time interval when the given piece of the scalar is stretched until the pumping correlation scale L. The first term in Eq. (2.49) can be substituted for by $y^2 P_2 t_1/2$, and the second one can be substituted for by $(2D(d-1))^{-1}(\zeta_1 - (d-1)dD/2)^2 t_1$. Using Eq. (2.52), we find

$$\mathcal{F}_{\rm extr} = \left[\left(\frac{d^2}{4} + \frac{P_2 y^2}{(d-1)D} \right)^{1/2} - \frac{d}{2} \right] \ln(L\Lambda). \tag{2.53}$$

Comparing expressions (2.53) and (2.36), we conclude that fluctuations of the stretching direction can be neglected if $y^2 \gg D d^3 P_2^{-1}$. Let us stress that at $y^2 \sim D d^3 P_2^{-1}$, the value of Eq. (2.53) is much larger than unity. That means that violation of Eq. (2.36) is not associated with a destructing saddle-point regime; it is rather related to an incorrect calculation of soft fluctuations in the saddle-point regime. Note also that the role of fluctuations increases with increasing space dimensionality d.

C. Probability distribution functions

The scheme proposed in the preceding subsections can also be applied to calculating PDF $\mathcal{P}_\Lambda(\vartheta)$ of the quantity θ_Λ [Eq. (2.21)]. Let us start from the average

$$\langle \theta_\Lambda^{2n} \rangle = \int \mathcal{D}\theta\, \mathcal{D}p\, \mathcal{D}\sigma \exp(i\mathcal{I} - \mathcal{F}_\sigma + 2n \ln \theta_\Lambda). \tag{2.54}$$

The saddle-point contribution to $\langle \theta_\Lambda^{2n} \rangle$ is determined by the extrema of $i\mathcal{I} - \mathcal{F}_\sigma + 2n \ln \theta_\Lambda$ which coincide with Eqs. (2.11), (2.12), and (2.13) if we substitute

$$y \to -\frac{2ni}{\theta_\Lambda}. \tag{2.55}$$

Then an attempt to find the analog of the uniaxial instanton fails. The formal reason for this is in additional i in Eq. (2.55). The physical reason is that the uniaxial instanton is an adequate object for the statistics of fast processes, whereas $\langle\theta_\Lambda^{2n}\rangle$ is determined by slow processes with suppressed stretching.

To find a solution, we should pass to the isotropic instanton. That means that we should perform the same transformation of the fields as in Sec. II B, which leads to the saddle-point equations (2.46), (2.47), and (2.48), with Eq. (2.55). The equations have a solution of the same type as considered above, with

$$\zeta_1^2=\frac{d^2(d-1)^2}{4}D^2-(d-1)DP_2\frac{4n^2}{\theta_\Lambda^2}. \tag{2.56}$$

The value of θ_Λ in Eq. (2.56) is the parameter which can be found from the equation analogous to Eq. (2.47), which now reads

$$\partial_t\tilde{\theta}=\frac{2n}{\theta_\Lambda}\int d\mathbf{r}_2\chi(R)\delta_\Lambda(\mathbf{r}_2). \tag{2.57}$$

As previously, the integral on the right-hand side of Eq. (2.57) for $\mathbf{r}_1=0$ is equal to P_2 if $|t|\lesssim\ln(L\Lambda)/\zeta_1$, and is negligible otherwise. We thus come to the conclusion that

$$\theta_\Lambda^2\simeq\theta^2(t=0,\mathbf{r}=0)\simeq 2nP_2\frac{\ln(L\Lambda)}{\zeta_1}. \tag{2.58}$$

Substituting the relation into Eq. (2.56), we find the equation on ζ_1 leading to

$$\zeta_1=(d-1)D\left\{-\frac{n}{\ln(L\Lambda)}+\left(\frac{n^2}{\ln^2(L\Lambda)}+\frac{d^2}{4}\right)^{1/2}\right\}. \tag{2.59}$$

We see that ζ_1 decreases with increasing n, and consequently the characteristic time $\ln(L\Lambda)/\zeta_1$ increases with increasing n. Now substituting Eq. (2.59) into Eq. (2.58), one obtains

$$\theta_{\Lambda n}^2=\frac{8nP_2\ln(L\Lambda)}{d^2D(d-1)}\left\{\frac{n}{\ln(L\Lambda)}+\left(\frac{n^2}{\ln^2(L\Lambda)}+\frac{d^2}{4}\right)^{1/2}\right\}. \tag{2.60}$$

It is not very difficult to recognize that the main contribution to the saddle-point value of $i\mathcal{I}-\mathcal{F}_\zeta+2n\ln\theta_\Lambda$ is determined by the last term. This means that

$$\langle\theta_\Lambda^{2n}\rangle\propto\exp(-\mathcal{F}_{\rm extr})\propto\theta_{\Lambda n}^{2n}, \tag{2.61}$$

with $\theta_{\Lambda n}$ from Eq. (2.60).

The same result can be deduced by the alternative method. That is, starting from Eq. (2.53), we can calculate the tail of the PDF $\mathcal{P}_\Lambda(\vartheta)$ for the quantity θ_Λ, Eq. (2.21). The function $\mathcal{P}_\Lambda(\vartheta)$ is the Fourier transform of $\mathcal{Z}_\Lambda(y)$:

$$\mathcal{P}_\Lambda(\vartheta)=\int dy\exp(-iy\vartheta)\mathcal{Z}_\Lambda(y)$$
$$\propto\int dy\exp(-iy\vartheta-\mathcal{F}_{\rm extr}). \tag{2.62}$$

Here, substituting Eq. (2.49) and calculating the integral over y by the saddle-point method [18], we find

$$\mathcal{P}_\Lambda(\vartheta)\propto\exp\left\{\frac{d}{2}\ln(L\Lambda)\left[1-\left(1+\frac{d-1}{P_2}D\frac{\vartheta^2}{\ln^2(L\Lambda)}\right)^{1/2}\right]\right\}, \tag{2.63}$$

which is in agreement with [15,18,16]. Formally, expression (2.63) is valid at $\vartheta\to\infty$, but it really covers the whole region of ϑ because the PDF is Gaussian at small ϑ [16]. The distant tails of the PDF are exponential, as was established by Shraiman and Siggia [15]. Note that the value of the Lyapunov exponent ζ_1 corresponding to the saddle point in Eq. (2.62) is

$$\zeta_{\rm extr}=\frac{d}{2}(d-1)D\left(1+\frac{d-1}{P_2}D\frac{\vartheta^2}{\ln^2(L\Lambda)}\right)^{-1/2}. \tag{2.64}$$

We see that the value decreases with increasing ϑ whereas the value ζ_1, Eq. (2.52), increases with increasing y. The physical meaning is quite transparent here: to observe a large fluctuation of the scalar one needs suppressed stretching. Note also that the value of y corresponding to the extremum point is

$$y_{\rm extr}^2=-\frac{(d-1)d^2D}{4P_2}\frac{\vartheta^2}{\vartheta^2+P_2/((d-1)D)}. \tag{2.65}$$

This means that $|y_{\rm extr}^2|<d^3D/P_2$, and consequently the extremum point lies beyond the applicability region of the approximation (2.36). This is the reason why Eq. (2.36) does not restore $\mathcal{P}_\Lambda(\vartheta)$.

Now we can calculate $\langle\theta_\Lambda^{2n}\rangle$ starting from the definition

$$\langle\theta_\Lambda^{2n}\rangle=\int d\vartheta\,\vartheta^{2n}\mathcal{P}_\Lambda(\vartheta). \tag{2.66}$$

This integral can be calculated, again using the saddle-point method. The result coincides, of course, with Eq. (2.61). We thus conclude that Eq. (2.49) or (2.63) cover both cases of slow and fast processes. This means that an account of fluctuations of the direction of stretching (performed in Sec. II B) gives us a tool for finding the tails of both the PDF and the generating functional.

In much the same way we can find the PDF for the differences $\Delta\theta=\theta(\mathbf{r})-\theta(-\mathbf{r})$. Instead of Eq. (2.18), we use

$$\lambda(\mathbf{r}_1)=y[\delta_\Lambda(\mathbf{r}_1-\mathbf{r})-\delta_\Lambda(\mathbf{r}_1+\mathbf{r})]. \tag{2.67}$$

We consider the isotropic instanton and use Eqs. (2.49) and (2.50) with $\tilde{p}(t,\mathbf{r}_1)=-y[\delta_\Lambda(\mathbf{r}_1-\mathbf{r})-\delta_\Lambda(\mathbf{r}_1+\mathbf{r})]$. Then, we find for the integral in Eq. (2.49),

$$\int d\mathbf{r}_1\,d\mathbf{r}_2\,\tilde{p}_1\tilde{p}_2\chi(R)$$
$$\times\begin{cases}=-4M^2\chi''(0)r^2, & -t_1<t<0, \quad M<L/r,\\ =2P_2, & -(t_1+t_2)<t<-t_1, \quad L/r<M<L\Lambda,\\ \to 0, & t<-(t_1+t_2), \quad M>L\Lambda,\end{cases} \tag{2.68}$$

where $M=R/|\mathbf{r}_1-\mathbf{r}_2|$; that is, $\partial_tM=-\zeta M$. It follows from Eq. (2.50) that for $-t_1<t<0$, $\zeta\approx(d-1)dD/2$; for $-(t_1+t_2)<t<-t_1$, $\zeta\approx\zeta_2$, where

$$\zeta_2^2 = \frac{(d-1)^2 d^2}{4} D^2 + 2(d-1) D y^2 P_2; \qquad (2.69)$$

and for $t < -(t_1+t_2)$, again $\zeta \approx (d-1)dD/2$.

The main contribution to the extremum value (2.49) is determined by the region $-(t_1+t_2) < t < -t_1$, it is

$$\mathcal{F}_{\rm extr} = \left[\left(\frac{d^2}{4} + \frac{2P_2 y^2}{(d-1)D} \right)^{1/2} - \frac{d}{2} \right] \ln(r\Lambda), \qquad (2.70)$$

instead of Eq. (2.53). Then we can find the PDF for the difference taking the integral of Eq. (2.62) type. Instead of Eq. (2.63), one obtains

$$\mathcal{P}_\Lambda(\Delta\vartheta) \propto \exp\left\{ \frac{d}{2} \ln(r\Lambda) \left[1 - \left(1 + \frac{d-1}{2P_2} D \frac{(\Delta\vartheta)^2}{\ln^2(r\Lambda)} \right)^{1/2} \right] \right\}. \qquad (2.71)$$

The tails of the PDF are exponential while the core is Gaussian [for $(\Delta\vartheta)^2 \ll P_2 \ln(r\Lambda)/d(d-1)D$]:

$$\mathcal{P}_\Lambda(\Delta\vartheta) \propto \exp\left\{ -\frac{d(d-1)D}{8P_2} \frac{(\Delta\vartheta)^2}{\ln(r\Lambda)} \right\}. \qquad (2.72)$$

That PDF can be used for calculating moments $\langle (\Delta\vartheta)^{2n} \rangle$ with $n \ll ln(r\Lambda)$; in particular, Eq. (2.72) gives

$$\langle (\Delta\vartheta)^2 \rangle \approx \frac{4P_2 \ln(r\Lambda)}{d(d-1)D}, \quad \langle (\Delta\vartheta)^4 \rangle \approx \frac{48 P_2^2 \ln^2(r\Lambda)}{d^2(d-1)^2 D^2}, \qquad (2.73)$$

which exactly corresponds to the answers obtained at $d=2$ in [16,19].

D. Discussion

We consider the statistics of the passive scalar advected by the random velocity field in the framework of the instanton formalism. The consideration is very instructive, since it reveals some nontrivial peculiarities of the formalism. First, we see that a direct solution of the saddle-point equations gives us an answer which satisfactorily describes the tail of the generating functional $\mathcal{Z}(\lambda)$, but cannot serve to restore the tail of PDF $\mathcal{P}(\vartheta)$. The physical reason for this lies in the difference between the processes forming the tails: The tail near $\mathcal{Z}$ is related to the fast stretching process, with a characteristic time decreasing as λ increases, while the tail near $\mathcal{P}$ is related to slow stretching, with a characteristic time increasing as ϑ increases. This conclusion can be directly extracted from Eqs. (2.35) and (2.64). For slow processes, those fluctuations of the stretching direction which are relevant do not destroy the saddle-point (instanton) regime but renormalize the naive answer. For that particular problem, the fluctuations can be explicitly taken into account after the special transformations of the fields. Although the trick cannot be widely generalized, it shows the direction of improving naive answers. In a general case, we expect that a direct solution of the saddle-point equations will produce nonsymmetric instantons with a degeneracy parameter (like the direction of the marked axis in the case considered). Then there exists the ''Goldstone'' mode related to slightly nonhomogeneous variations of the parameter. Such a mode is soft since those variations weakly influence the action. Therefore, the fluctuations related to the soft mode are relevant, and should be taken explicitly into account: If the extremum is not steep at some directions in the functional space, then integration over the respective degrees of freedom should be performed explicitly. The contribution of that integration depends on the lifetime of the instanton, as was discussed above. The well-known analogy from the theory of phase transitions is that Goldstone modes appearing due to spontaneous symmetry breaking in the low-temperature phase may destroy the long-range order (i.e., restore the symmetry).

III. SIMPLEST INSTANTON OF AN INCOMPRESSIBLE VELOCITY FIELD

Here we describe the first step in considering a much more complicated problem of finding the tails of the PDF for a velocity field in three-dimensional incompressible turbulence. We consider two-point statistics, and show that an instanton with a linear spatial profile naturally appears as a basic flow.

The effective action (1.4) for the Navier-Stokes equation can be written as follows:

$$\begin{aligned}\mathcal{I} = \int dt\, d\mathbf{r} (&p_\alpha \partial_t v_\alpha + p_\alpha v_\beta \nabla_\beta v_\alpha - \nu p_\alpha \nabla^2 v_\alpha \\ &+ p_\alpha \nabla_\alpha P + Q \nabla_\alpha v_\alpha) \\ &+ \frac{i}{2} \int dt\, dt'\, d\mathbf{r}\, d\mathbf{r}'\, \Xi(t-t', \mathbf{r}-\mathbf{r}') p_\alpha p'_\alpha. \end{aligned} \qquad (3.1)$$

The additional independent fields P and Q play the role of Lagrange multipliers enforcing the incompressibility conditions $\nabla_\alpha v_\alpha = 0$ and the analogous condition $\nabla_\alpha p_\alpha = 0$ for the response field p_α. The field P has the meaning of pressure (divided by the mass density ρ). The origin of the terms with the fields P and Q in the effective action is related to the continuity equation $\partial_t \rho + \nabla_\alpha(\rho v_\alpha) = 0$, which should be incorporated into the effective action; Q is just the auxiliary (response) field corresponding to the equation. At the condition that all velocities are much smaller than the sound velocity, it is possible to neglect the time derivative in $\partial_t \rho + \nabla_\alpha(\rho v_\alpha) = 0$ and variations of the mass density, which leads to the term $Q \nabla_\alpha v_\alpha$ in Eq. (3.1). While variations of the mass density can be neglected, variations in the pressure are relevant. Therefore, it is natural to pass from the integration over the mass density to the integration over the pressure as it is implied in Eq. (3.1).

We are going to examine the generating functional for the velocity,

$$\begin{aligned}\mathcal{Z}(\boldsymbol{\lambda}) &\equiv \left\langle \exp\left(i \int dt\, d\mathbf{r}\, \boldsymbol{\lambda} \cdot \mathbf{v} \right) \right\rangle \\ &= \int \mathcal{D}p\, \mathcal{D}v\, \mathcal{D}P\, \mathcal{D}Q \exp\left(i\mathcal{I} + i \int dt\, d\mathbf{r} \cdot \boldsymbol{\lambda} \mathbf{v} \right). \end{aligned} \qquad (3.2)$$

The extremum conditions for the argument of the exponent in Eq. (3.2) determining the Navier-Stokes instanton read

$$\partial_t v_\alpha(\mathbf{r})+v_\beta(t,\mathbf{r})\nabla_\beta v_\alpha(t,\mathbf{r})-\nu\nabla^2 v_\alpha(t,\mathbf{r})+\nabla_\alpha P(t,\mathbf{r})$$
$$=-i\int dt'\int \frac{d^dk}{(2\pi)^d}\exp(i\mathbf{k}\cdot\mathbf{r})\Xi(t-t',\mathbf{k})p_\alpha(t',\mathbf{k}),\tag{3.3}$$

$$\partial_t p_\alpha(t,\mathbf{r})-p_\beta(t,\mathbf{r})\nabla_\alpha v_\beta(t,\mathbf{r})+v_\beta(t,\mathbf{r})\nabla_\beta p_\alpha(t,\mathbf{r})$$
$$+\nu\nabla^2 p_\alpha(t,\mathbf{r})+\nabla_\alpha Q(t,\mathbf{r})=\lambda_\alpha(t,\mathbf{r}),\tag{3.4}$$

where $\Xi(\mathbf{k})$ and $p_\alpha(\mathbf{k})$ are Fourier transforms of $\Xi(\mathbf{r})$ and $p_\alpha(\mathbf{r})$, respectively. In Eqs. (3.3) and (3.4) the conditions $\nabla_\alpha v_\alpha=0$ and $\nabla_\alpha p_\alpha=0$, which originate from varying over the fields P and Q, are also implied. Then the values of the fields P and Q can also be found from the conditions. This gives the relations

$$\nabla^2 P=-\nabla_\alpha(v_\beta\nabla_\beta v_\alpha),\tag{3.5}$$

$$\nabla^2 Q=\nabla_\alpha(p_\beta\nabla_\alpha v_\beta-v_\beta\nabla_\beta p_\alpha).\tag{3.6}$$

In the following we consider the simultaneous correlation functions of the velocity differences $\langle[\mathbf{v}(0,\boldsymbol{\rho}/2)-\mathbf{v}(0,-\boldsymbol{\rho}/2)]^{2n}\rangle$, where $\boldsymbol{\rho}$ is the separation between the points. The functional generating such functions is extracted from $\mathcal{Z}(\lambda)$ if one obtains

$$\lambda_\alpha=yn_\alpha\delta(t)[\delta(\mathbf{r}-\boldsymbol{\rho}/2)-\delta(\mathbf{r}+\boldsymbol{\rho}/2)],\tag{3.7}$$

where $\mathbf{n}$ is a unit vector. As was explained in Sec. I, the presence of such a term on the right-hand side of Eq. (3.4) means that we should solve the problem at negative times t with the final condition

$$p_\alpha(0,\mathbf{r})=-y(\delta_{\alpha\beta}-\nabla_\alpha\nabla_\beta\nabla^{-2})n_\beta[\delta(\mathbf{r}-\boldsymbol{\rho}/2)-\delta(\mathbf{r}+\boldsymbol{\rho}/2)].\tag{3.8}$$

We assume that the pumping correlation function Ξ is δ correlated in time: $\Xi(t,\mathbf{r})=\delta(t)\chi(\mathbf{r})$. Then the system of equations (3.3)–(3.6) is invariant under the transformation analogous to Eq. (2.17):

$$t\to X^{-1}t,\quad \mathbf{v}\to X\mathbf{v},\quad P\to X^2P,\quad Q\to X^3Q,$$
$$\nu\to X\nu,\quad \lambda\to X\lambda,\quad \mathbf{p}\to X^3\mathbf{p},\tag{3.9}$$

where X is an arbitrary factor. For the function (3.8) the transformation (3.9) means $y\to X^2y$. The extremum value $\mathcal{F}_{\rm extr}$ of the argument of the exponent in Eq. (3.2) transforms as $\mathcal{F}_{\rm extr}\to X^3\mathcal{F}_{\rm extr}$ at Eq. (3.9). This leads to the conclusion that

$$\mathcal{Z}(y)\propto\exp(-\mathcal{F}_{\rm extr}),\quad \mathcal{F}_{\rm extr}=y^{3/2}f(y/\nu^2),\tag{3.10}$$

with the function f to be determined. We expect that in the limit $y\to\infty$ a ν dependence in the function disappears. Then we conclude $\mathcal{F}_{\rm extr}\propto y^{3/2}$.

The characteristic wave vector k_0 in the correlation function $\chi(\mathbf{k})$ of the pumping force is of the order of the inverse pumping length L. Then examining the region $\mathbf{r}\ll L$ one can expand the exponent $\exp(i\mathbf{k}\cdot\mathbf{r})$ in Eq. (3.3) into the series over $\mathbf{k}\cdot\mathbf{r}$. The first term of the expansion produces the zero contribution to the right-hand side of Eq. (3.3) because of the structure of the field p determined by condition (3.8): The condition means that at $t=0$ $\mathbf{p}(\mathbf{r})=-\mathbf{p}(-\mathbf{r})$, the property is reproduced by the equations, so that $\mathbf{p}(\mathbf{k})=-\mathbf{p}(-\mathbf{k})$ at any time t. Thus the leading term of the expansion of the right-hand side is linear in $\mathbf{r}$. This means that Eq. (3.3) admits a linear profile as a solution in the region $|\mathbf{r}|\ll L$,

$$v_\alpha=\sigma_{\alpha\beta}(t)r_\beta,\quad \sigma_{\alpha\alpha}=0.\tag{3.11}$$

Let us emphasize that Eq. (3.3) may well have other instanton solutions with more complicated profiles; their analysis is left for future studies. For Eq. (3.11), we obtain, from Eq. (3.3),

$$\partial_t\sigma_{\alpha\gamma}+\sigma_{\alpha\beta}\sigma_{\beta\gamma}-\frac{1}{d}\delta_{\alpha\gamma}(\sigma_{\mu\nu}\sigma_{\nu\mu})=\int\frac{d^dk}{(2\pi)^d}k_\gamma p_\alpha(\mathbf{k})\chi(\mathbf{k}).\tag{3.12}$$

Here we substituted the expression for the pressure,

$$P=-\frac{1}{d}(\sigma_{\mu\nu}\sigma_{\nu\mu})r^2,\tag{3.13}$$

which provides for the condition $\nabla_\alpha v_\alpha=0$. Note that P is defined up to a harmonic function; expression (3.13) is chosen because of its isotropy.

For the linear velocity profile, Eq. (3.4) can be rewritten in Fourier representation as

$$\partial_t p_\alpha-\sigma_{\beta\alpha}p_\beta-\sigma_{\beta\gamma}k_\beta\frac{\partial}{\partial k_\gamma}p_\alpha-\nu k^2p_\alpha+ik_\alpha Q=0,\tag{3.14}$$

which should be solved with the condition following from Eq. (3.8):

$$p_\alpha(t=0,\mathbf{k})=2iy\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right)n_\beta\sin(\mathbf{k}\cdot\boldsymbol{\rho}/2).\tag{3.15}$$

The characteristic wave vector $\mathbf{k}$ in Eq. (3.12) is of the order of L^{-1}. Thus we can expand $\sin(\mathbf{k}\cdot\boldsymbol{\rho}/2)$ in $\mathbf{k}\cdot\boldsymbol{\rho}$ and keep only the first nonvanishing term of the expansion $\propto\mathbf{k}\cdot\boldsymbol{\rho}$. As was discussed in Sec. I, the response field $\mathbf{p}(\mathbf{r},t)$ propagates backwards in time, starting with the initial value (3.15) at $t=0$. We shall see that for a long time (determined by a small viscosity) the field $\mathbf{p}(t,\mathbf{k})$ at $k\sim L^{-1}$ has the same structure $\propto\mathbf{k}\cdot\boldsymbol{\rho}$.

There is a general family of the flows with linear profiles—see Sec. III C below. We start by considering the simplest case. We assume below that the point separation $\boldsymbol{\rho}$ is directed along the same vector n_α as the measured velocity components: $\rho_\alpha=n_\alpha\rho$. Then the problem possesses the axial symmetry, which allows us to look for the following uniaxial strain matrix:

$$\sigma_{\alpha\beta}=s(\delta_{\alpha\beta}-dn_\alpha n_\beta).\tag{3.16}$$

The same symmetry admits the anzatz

$$p_\alpha(t,\mathbf{k})=\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right)iyn_\beta\phi(t,z)\mathbf{k}\cdot\mathbf{n}\rho,\tag{3.17}$$

correct in the limit of small k. In Eq. (3.17), $z=\mathbf{k}\cdot\mathbf{n}/k$, and the function $\phi(t,z)$ to be found has the initial (final) value $\phi(t=0,z)=1$. Substituting Eq. (3.15) and (3.17) into Eq. (3.14) we find $Q=0$ and

$$s^{-1}(\partial_t-\nu k^2)\phi+dz(1-z^2)\partial_z\phi+2[(d-1)-dz^2]\phi=0. \tag{3.18}$$

Substituting expression (3.17) into the right-hand side of Eq. (3.12), we find

$$\int\frac{d^dk}{(2\pi)^d}k_\gamma p_\alpha(\mathbf{k})\chi(\mathbf{k})=G(dn_\alpha n_\gamma-\delta_{\alpha\gamma}), \tag{3.19}$$

where

$$G=iy\rho C\int_{-1}^{1}dz\,z^2(1-z^2)^{(d-1)/2}\phi(t,z) \tag{3.20}$$

and

$$C=\frac{S_{d-1}}{(2\pi)^d(d-1)}\int_0^\infty dk\,k^{d+1}\chi(k). \tag{3.21}$$

Here S_d is the area of the unit sphere in d-dimensional space $S_d=2\pi^{d/2}/\Gamma(d/2)$. The constant C can be estimated as $C\sim\mathcal{E}/L^2$, where $\mathcal{E}=\langle v_\alpha\partial_t v_\alpha\rangle$ is the energy dissipation rate. Now, substituting Eqs. (3.16) and (3.19) into Eq. (3.12), we find

$$\partial_t s=(d-2)s^2-G. \tag{3.22}$$

Our next problem is to find G as a functional of s, to close this set of equations. We have to solve Eq. (3.18) for ϕ. Here viscous and inviscid cases are slightly different. We start by considering an inviscid Euler equation, we then account for the viscosity.

A. Instanton of the Euler equation

Neglecting viscosity in Eq. (3.18), we obtain a general solution

$$\phi=h^2z^{-2}F\left(\frac{z^2h^{-2d}}{1-z^2}\right),\quad h(t)=\exp\left(\int_0^t s(t')dt'\right). \tag{3.23}$$

The initial condition $\phi(0,z)=1$ fixes the function F:

$$\phi(z)=\frac{h^{2-2d}}{1-z^2+z^2h^{-2d}}. \tag{3.24}$$

We obtain the system of equations

$$\dot{s}=(d-2)s^2-G(h),\quad \dot{h}=sh. \tag{3.25}$$

This system for the variable $q=h^{2-d}$ becomes the usual potential problem $\ddot{q}=-U'(q)$ with the potential

$$U(q)=-(d-2)\int dq\,qG(q^{1/(2-d)}).$$

For $d=3$,

$$G(h)=\frac{iy\rho Ch^2}{h^{2d}-1}\left[\frac{2}{1-h^{-2d}}-\frac{1}{3}-\frac{\ln(2h^{2d}-1)}{h^{2d}-1}\right].$$

The relevant solution, which vanishes at $t=-\infty$, corresponds to the zero energy in this potential $[s(t)\propto-1/t$ as $t\to-\infty]$. Therefore, $\dot{q}^2=2U(0)-2U(q)$ and

$$\dot{q}_{t=0}=\sqrt{2[U(0)-U(1)]}=C_1\sqrt{\mathcal{E}y\rho}/L. \tag{3.26}$$

Then the strain at the moment $t=0$ becomes $\sigma_{\alpha\beta}=\dot{q}(\delta_{\alpha\beta}-dn_\alpha n_\beta)/q(2-d)$. In accordance with Eq. (1.7), the logarithmic derivative of the $\mathcal{Z}$ functional is related to the average initial value of the velocity difference $\mathcal{Z}'(y)/\mathcal{Z}(y)=n_\alpha\langle v_\alpha(\rho,0)-v_\alpha(-\rho,0)\rangle$. In the leading WKB approximation at large y, this average can be replaced by the contribution from the instanton solution:

$$(\ln\mathcal{Z})'(y)=2\rho n_\alpha n_\beta\sigma_{\alpha\beta}=\frac{2\dot{q}(d-1)}{q(d-2)}=C_2\sqrt{\mathcal{E}y\rho^3L^{-2}}. \tag{3.27}$$

Finally, we obtain the surprisingly simple result

$$\mathcal{Z}(y)\propto\exp[C_3\sqrt{\mathcal{E}(y\rho)^3L^{-2}}]. \tag{3.28}$$

with the dimensionless constants C_1, C_2, and C_3 to be calculated. This result is in agreement with the general form Eq. (3.10), it also contains ρ dependence.

B. Account of viscosity

When the viscous terms are kept, the solution is modified as follows. With the same assumption $k_0\rho\simeq\rho/L\ll1$, we can still look for the uniform strain solution. The viscosity will drop from the velocity equation, but not from the response field equation (3.14). The extra term νk^2p can be compensated for by the extra time dependent exponential

$$p_\alpha(t,\mathbf{k})=\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right)iyn_\beta\phi(t,z)\mathbf{k}\cdot\mathbf{n}\rho\exp[\nu R(t,z)k^2]. \tag{3.29}$$

The balance of k^0 and k^1 terms in the equation is the same as before. The balance of k^2 terms gives the equation

$$\dot{R}=1+s\hat{L}R=1-ds\,z(1-z^2)\frac{\partial R}{\partial z}+2s(1-dz^2)R, \tag{3.30}$$

with the boundary condition $R(0)=0$. The substitution $R(z,t)=A(t)+B(t)z^2$ reduces the partial differential equation to two ordinary differential equations,

$$\dot{A}=1+2As,$$

$$\dot{B}=2s(B-Bd-Ad).$$

The solution is expressed via $s(t)$; at $t\to-\infty$, it grows linearly: $R\approx t$. The influence of the viscosity on our solution is weak, it smears the peaks at p and manifests itself when $\nu R\simeq L^2$, i.e., at $t\simeq L^2/\nu$. That time should be much larger than the time of instanton formation $\sqrt{L^2/\mathcal{E}y\rho}$. Our asymptotic expression (3.28) is insensitive to viscosity if $\nu \ll L\sqrt{\mathcal{E} y \rho}$; i.e., the Reynolds number on the instanton is large.

C. Instanton family

Considering a more general strain does not change basic conclusions of this section. Let us describe, for instance, a general three-dimensional symmetric flow of the type considered in [21]. In the cylindrical coordinates with the z axis along $\boldsymbol{\rho}$, the velocity vector field at $r \ll L$ is given by

$$\mathbf{u} = (u_r, u_\theta, u_z) = (-\sigma r/2, \omega r/2, \sigma z). \tag{3.31}$$

Here the vorticity has only a z component $\omega(t)$ which is a function of time as well as strain $\sigma(t)$. The pressure is now of the form

$$P = -g r^2 - e[r^2 \sin\theta \cos\theta + rz(\sin\theta + \cos\theta)].$$

Particular details of the solution depend on the relation between g and e. The diagonal elements (proportional to g) are determined locally from the Poisson equation $\Delta P = -\mathrm{div}(\mathbf{u}\cdot\nabla\mathbf{u})$. Note that the off-diagonal pressure elements are generally determined by the global structure of the flow. In our case, the value of e is determined by the distant asymptotics $u \to 0, P \to \mathrm{const}$ at $r \to \infty$, and matching conditions at $r \simeq L$ which depend on the particular choice of the pumping χ. The global description of the flows for the whole instanton family is still ahead of us. As far as the functional dependence of the respective $\mathcal{Z}(y, \boldsymbol{\rho})$ is concerned, it is the same for the whole family and does not depend on the large-scale behavior of the pumping. Considering, for instance, the case $g = 0$ [opposite to the diagonal case (3.13) considered above], we obtain $\omega = \sqrt{3} s$, and a system of equations similar to Eq. (3.25),

$$\dot{s} = s^2 - G'(h), \quad \dot{h} = sh, \tag{3.32}$$

with another yet qualitatively similar function G'. For the variable $q(t)$, related to $s(t)$ by $\dot{q} = -sq$, the Newton equation appears with a potential energy U that allow for a single solution (zero energy separatrix) vanishing as $s(t) \propto 1/t$ at $t \to -\infty$. The basic result $\ln \mathcal{Z}(y, \rho) \propto (y\rho)^{3/2}\sqrt{\mathcal{E}}/L$ is valid for the whole family in agreement with Eq. (3.10).

D. Discussion

The particular instanton found has the scaling

$$\delta u(\rho) = u(\rho/2) - u(-\rho/2) \propto \rho, \tag{3.33}$$

which would give the asymptotics of the right tail of the PDF $\mathcal{P}(\delta u, \rho) \propto \exp[-(\delta u/\rho)^3]$ obtained by the Fourier transform of $\mathcal{Z}(y)$. It is unclear at the moment if there are flows where such asymptotics take place; most probably, this simplest instanton does not realize the main extremum of the action. Note that the similar instanton with the linear profile is found for the Burgers problem [20], where it indeed determines the right tail ($\delta u > 0$) of the velocity PDF due to sawtooth waves. The general analysis of the whole family of instanton solutions for the two-point velocity statistics at the framework of the Navier-Stokes equation will be published elsewhere. Also, the crucial problem of the contribution to the action from the fluctuations against the instanton background will be considered. It is clear that, in the straining flow of the instanton, any vorticity perturbation produces a spiral with the accumulation point at the velocity null. The scaling of the perturbation contribution is different from Eq. (3.33); for instance, it will give Kolmogorov's $\frac{5}{3}$ law for the pair correlation function as in the Lundgren example [22]. The analysis of the instanton fluctuations will be the subject of further publications. Note that the instanton formalism provides a natural (and long-expected) tool for incorporating numerous results on particular solutions of the Navier-Stokes equations into the statistical theory of turbulence.

ACKNOWLEDGMENTS

We are grateful to E. Balkovsky and M. Chertkov for useful discussions. This work was partially supported by the National Science Foundation under Contract No. PHYS-90-21984 (A.M.), by the Minerva Center for Nonlinear Physics (I.K.), by the Minerva Einshtein Center (V.L.) and by the Rashi Foundation (G.F.).

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