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Instantons on Noncommutative R^4 and Projection Operators

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The noncommutative version of ADHM construction of instantons, which was proposed by Nekrasov and Schwarz, is carefully studied. Noncommutative $\boldsymbol{R}^4$ is described by an algebra of operators acting in a Fock space. In the ADHM construction of instantons, one looks for zero-modes of the Dirac-like operator. The feature peculiar to the noncommutative case is that these zero-modes project out some states in the Fock space. The mechanism of these projections is clarified in the case that the gauge group is $U(1)$. In $U(N)$ cases, it is shown in some explicit examples that projections similar to those in the $U(1)$ cases also appear. A physical interpretation of these projections in the IIB matrix model is also discussed.

§1. Introduction

One of the most important reasons to consider physics in noncommutative spacetime is that the behavior of the theory at short distances is expected to become manageable, due to the noncommutativity of the spacetime coordinates. It has become clear recently that a noncommutative geometry appears [1] in a definite limit of string theory, BFSS matrix theory [2] and IIB matrix theory. [3] In these cases, noncommutativity should be relevant to the short-scale physics of D-branes.

Among D-brane systems, Dp-brane-D$(p+4)$-brane bound states are of interest because this system has two different descriptions: one in terms of the worldvolume theory of Dp-branes and another in terms of the worldvolume theory of D$(p+4)$-branes. The D-flat condition of the worldvolume theory of Dp-brane coincides with the ADHM equations, [5],[6] and Dp-branes are described as instantons in the D$(p+4)$-brane worldvolume theory. These descriptions should be equivalent, because they describe the same system, and indeed the moduli space of the worldvolume theory of Dp-branes is identical to the instanton moduli space. In the constant NS-NS B-field background in the worldvolume of D$(p+4)$-branes, the coordinates on the D$(p+4)$-branes become noncommutative, and the worldvolume theory of Dp-branes acquires a Fayet-Iliopoulos D-term. [16],[11] The equivalence of the two descriptions follows from the pioneering work of Nekrasov and Schwarz. [12] In order to construct instantons on noncommutative $\boldsymbol{R}^4$, one adds a constant (corresponding to the Fayet-Iliopoulos term) to the ADHM equations.**) The modified ADHM equations describe

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**) The case of equivariant instantons is studied in Ref. 14).

the resolutions of singularities in the moduli space of instantons on $\boldsymbol{R}^4$.[8)] This moduli space has provided an important clues to the nonperturbative aspects of string theory[17)-22)] and matrix theory.[11),23)-26)] Further studies from the viewpoints of both string theory and noncommutative geometry were recently given in Ref. 15). More recently, Braden and Nekrasov constructed instantons on blowups of $\boldsymbol{C}^2$, which are conjectured to be related to instantons on noncommutative $\boldsymbol{R}^4$.[13),*)]

In Ref. 12) Nekrasov and Schwarz explicitly constructed some instanton solutions and showed that they are non-singular. An interesting point is that they are non-singular even if their commutative counterparts in the original ADHM construction are singular, the case of so-called small instantons. In these cases, the noncommutativity of the coordinates actually eliminates the singular behavior of the field configurations. What is special to the noncommutative case is the appearance of projection operators which project out potentially dangerous states in Fock space, where Fock space is introduced to describe the noncommutative $\boldsymbol{R}^4$.[12)] The purpose of this paper is to investigate this mechanism. It is shown that this mechanism has rich structures, and it gives insight into the short-scale structures near the core of instantons on noncommutative space. An important point is that the existence of the projection forces us to express gauge fields in reduced Fock space, where some of the states have been projected out. It is shown that this modification of the Fock space corresponds to the modification of the spacetime topology.

The outline of this paper is as follows. In §2, gauge theory on noncommutative space and the ADHM construction on commutative $\boldsymbol{R}^4$ are briefly reviewed. In §3, the ADHM construction on noncommutative $\boldsymbol{R}^4$ is studied. The reason we must consider the projections is explained. In §4, the mechanism of the projections is clarified in the case that the gauge group is $U(1)$, utilizing Nakajima's beautiful results.[8),10)] In §5, it is demonstrated that similar projections also occur in the $U(N)$ case. In §6, embedding of the $U(1)$ instanton solution into the IIB matrix model is considered. The solution is understood as representing D-instantons within the D3-brane in IIB matrix model. It is shown that the role of the projection is to remove anti-D-instantons and create holes in the D3-brane worldvolume.

When the previous version of this paper was in the final stage of preparation, Ref. 13) appeared. Some issues discussed in this paper have commutative counterparts in that paper. Explanations of the modification of spacetime topology have been added to this paper after taking into due consideration the relation to Ref. 13).[**)]

[*)] In most of this paper I use the term "commutative" with the usual commutative $\boldsymbol{R}^4$ in mind, and explicitly refer to Ref. 13) when I compare our noncommutative descriptions to their commutative descriptions.

[**)] I would like to thank N. Nekrasov for explaining their work to me, and pointing out my misleading statement in the earlier version of this paper.

§2. Preliminaries

In this section we briefly review the theory of gauge fields on noncommutative $\boldsymbol{R}^4$ and ADHM construction on commutative $\boldsymbol{R}^4$, as preliminaries to the ADHM construction on noncommutative $\boldsymbol{R}^4$.

2.1. *Gauge fields on noncommutative $\boldsymbol{R}^4$*

Noncommutative $\boldsymbol{R}^4$ is described by an algebra generated by x^μ $(\mu = 1, \cdots, 4)$ obeying the commutation relations

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{2·1}$$

where $\theta^{\mu\nu}$ is real and constant. In this paper we restrict ourselves to the case in which $\theta^{\mu\nu}$ is *self-dual* and set*)

$$\theta^{12} = \theta^{34} = \frac{\zeta}{4}\ . \tag{2·2}$$

Then the algebra depends on only one constant parameter, ζ.

Next, we introduce the generators of noncommutative $\boldsymbol{C}^2 \approx \boldsymbol{R}^4$ by

$$z_1 = x_2 + ix_1, \quad z_2 = x_4 + ix_3\,. \tag{2·3}$$

Their commutation relations are

$$[z_1, \bar{z}_1] = [z_2, \bar{z}_2] = -\frac{\zeta}{2}\ . \qquad \text{(others: zero)} \tag{2·4}$$

We choose $\zeta > 0$. The commutation relations (2·1) have a group of automorphisms of the form $x^\mu \mapsto x^\mu + c^\mu$, where c^μ is a commuting real number. We denote the Lie algebra of this group by $\underline{\boldsymbol{g}}$. Following Ref. 12), we start with the algebra End $\mathcal{H}$ of operators acting in the Fock space $\mathcal{H} = \sum_{(n_1,n_2)\in \boldsymbol{Z}^2_{\geq 0}} \boldsymbol{C}\,|n_1, n_2\rangle$, where z and $\bar{z}$ are represented as creation and annihilation operators:

$$\begin{aligned}
\sqrt{\frac{2}{\zeta}} z_1 \,|n_1, n_2\rangle &= \sqrt{n_1 + 1}\,|n_1 + 1, n_2\rangle\,, & \sqrt{\frac{2}{\zeta}} \bar{z}_1 \,|n_1, n_2\rangle &= \sqrt{n_1}\,|n_1 - 1, n_2\rangle\,,\\
\sqrt{\frac{2}{\zeta}} z_2 \,|n_1, n_2\rangle &= \sqrt{n_2 + 1}\,|n_1, n_2 + 1\rangle\,, & \sqrt{\frac{2}{\zeta}} \bar{z}_2 \,|n_1, n_2\rangle &= \sqrt{n_2}\,|n_1, n_2 - 1\rangle\,.
\end{aligned} \tag{2·5}$$

The algebra End $\mathcal{H}$ has a subalgebra of operators which have finite norm. We define the norm of operators by $||a|| := \sup ||a\phi||/||\phi||$, where $a \in \mathrm{End}\,\mathcal{H}$, $|\phi\rangle \neq 0$, and $|\phi\rangle \in \mathrm{Dom}(a) \subset \mathcal{H}$. $\mathrm{Dom}(a)$ is the domain of the operator a, and $||\phi|| := \langle\phi|\phi\rangle^{1/2}$. We denote this algebra by $\mathcal{A}_\zeta$. Whenever we consider the derivative of an operator $a \in \mathcal{A}_\zeta$, we assume that it is also contained in $\mathcal{A}_\zeta$, i.e. $\partial_\mu a \in \mathcal{A}_\zeta$. Here ∂_μ is understood as the action of $\underline{\boldsymbol{g}} = \boldsymbol{R}^4$ on $\mathcal{A}_\zeta$ by translation. The $U(N)$ gauge field

*) See Ref. 15) for the meaning of this choice of parameters in string theory.

on noncommutative $\boldsymbol{R}^4$ is defined as follows. First we consider the N-dimensional vector space $\mathcal{E} := (\mathcal{A}_\zeta)^{\oplus N}$, which carries the right representation of $\mathcal{A}_\zeta$

$$\mathcal{E} \times \mathcal{A}_\zeta \ni (e,a) \mapsto ea \in \mathcal{E}, \quad \begin{aligned} &e(ab) = (ea)b, \\ &e(a+b) = ea + eb, \\ &(e+e')a = ea + e'a, \end{aligned} \tag{2·6}$$

for any $e, e' \in \mathcal{E}$ and $a, b \in \mathcal{A}_\zeta$.*) The elements of $\mathcal{E}$ can be thought of as an N-dimensional vector with entries in $\mathcal{A}_\zeta$. Let us consider the unitary action of U on an element of $\mathcal{E}$:

$$e \to Ue, \tag{2·7}$$

where U is an $N \times N$ matrix, where components are in $\mathcal{A}_\zeta$, and satisfying $UU^\dagger = U^\dagger U = \mathrm{Id}_\mathcal{H} \otimes \mathrm{Id}_N$. Here $\mathrm{Id}_\mathcal{H}$ is the identity operator in $\mathcal{A}_\zeta$, and Id_N is the $N \times N$ identity matrix. Under this unitary transformation, De, the covariant derivative of $e \in \mathcal{E}$, is required to transform covariantly:

$$De \to UDe. \tag{2·8}$$

The covariant derivative D is written as

$$D = d + A. \tag{2·9}$$

Here the $U(N)$ gauge field A is introduced to ensure covariance, as explained below. A is a matrix-valued one-form: $A = A_\mu dx^\mu$ with A_μ being an anti-hermitian $N \times N$ matrix. The action of the exterior derivative d is defined as

$$da := (\partial_\mu a)\, dx^\mu, \quad a \in \mathcal{A}_\zeta. \tag{2·10}$$

Here, the dx^μ commute with x^μ and anti-commute among themselves, and hence $d^2 a = 0$ for $a \in \mathcal{A}_\zeta$. From (2·7) and (2·8), the covariant derivative transforms as

$$D \to UDU^\dagger. \tag{2·11}$$

Hence the gauge field A transforms as

$$A \to UdU^\dagger + UAU^\dagger. \tag{2·12}$$

The field strength is defined by

$$F := D^2 = dA + A^2. \tag{2·13}$$

We can construct a gauge invariant action S by**)

$$S = -\frac{1}{4g^2} \mathrm{Tr}_{\mathcal{H},U(N)}\, F_{\mu\nu} F^{\mu\nu}. \tag{2·14}$$

*) $\mathcal{E}$ is a right module over $\mathcal{A}_\zeta$. (See, for example, Refs. 31) and 32).)

**) In this paper we only consider the case in which the metric on $\boldsymbol{R}^4$ is flat: $g_{\mu\nu} = \delta_{\mu\nu}$.

For later use, let us consider the projection operator $P \in \boldsymbol{M}_N(\mathcal{A}_\zeta)$, $P^\dagger = P$, $P^2 = P$, where $\boldsymbol{M}_N(\mathcal{A}_\zeta)$ denotes the algebra of $N \times N$ matrices whose entries are in $\mathcal{A}_\zeta$. For every projection operator P, we can consider the vector space $P\mathcal{E}$:*)

$$e \in P\mathcal{E} \Longleftrightarrow e \in \mathcal{E}, \quad e = Pe. \tag{2·15}$$

We can consider the unitary action on $P\mathcal{E}$:

$$\begin{aligned} e \to U_P e, \quad & U_P = PU_P = U_P P, \\ & U_P^\dagger U_P = U_P U_P^\dagger = P. \end{aligned} \tag{2·16}$$

Then we can construct a covariant derivative D_P for $P\mathcal{E}$ by

$$D_P = Pd + A, \quad A = PA = AP. \tag{2·17}$$

Note that $D_P = PD_P$. We require $D_P e$ to transform as

$$D_P e \to U_P D_P e. \tag{2·18}$$

Then the covariant derivative D_P must transform as

$$D_P \to U_P D_P U_P^\dagger. \tag{2·19}$$

For any $e \in P\mathcal{E}$, one can show

$$\begin{aligned} U_P D_P U_P^\dagger e &= U_P(Pd + A)U_P^\dagger e = U_P d(P(U_P^\dagger e)) + U_P A U_P^\dagger e \\ &= U_P P d U_P^\dagger e + PU_P(U_P^\dagger de) + U_P A U_P^\dagger e \quad (U_P P = PU_P) \\ &= Pde + (U_P dU_P^\dagger + U_P A U_P^\dagger)e. \end{aligned} \tag{2·20}$$

Hence the gauge transformation rule of the gauge field A is given by

$$A \to U_P dU_P^\dagger + U_P A U_P^\dagger. \tag{2·21}$$

The field strength becomes

$$\begin{aligned} F &:= D_P^2 \\ &= PdA + A^2 + PdPdP. \end{aligned} \tag{2·22}$$

Indeed, for $e \in P\mathcal{E}$, one can show

$$\begin{aligned} Fe &= (Pd + A)(Pde + Ae) \\ &= Pd(Pde) + Pd(Ae) + APde + A^2 e \\ &= Pd(Pde) + PdAe + A^2 e, \end{aligned} \tag{2·23}$$

*) $P\mathcal{E}$ is a right projective module over $\mathcal{A}_\zeta$.

and since $e = Pe$ and $P^2 = P$, we can rewrite (2·23) using the following equations:

$$\begin{aligned} Pd(Pde) &= Pd(Pd(Pe)) \\ &= Pd(PdPe + Pde) \\ &= PdPdPe - PdPde + PdPde \\ &= PdPdPe. \end{aligned} \tag{2·24}$$

Hence we obtain (2·22). We can construct a gauge invariant action S_P by

$$S_P = -\frac{1}{4g^2}\mathrm{Tr}_{\mathcal{H},U(N)}\, PF_{\mu\nu}F^{\mu\nu}P. \tag{2·25}$$

The gauge field A is called "anti-self-dual", or "instanton", if its field strength satisfies the conditions

$$F^+ := \frac{1}{2}(F + *F) = 0, \tag{2·26}$$

where $*$ is the Hodge star.

2.2. *Review of ADHM construction on commutative* $\boldsymbol{R}^4$

ADHM construction [7)] is a method to obtain an anti-self-dual gauge field on $\boldsymbol{R}^4$ from solutions of some quadratic matrix equations. More specifically, in order to construct the anti-self-dual $U(N)$ gauge field with instanton number k, one starts from the following data (ADHM data):

1. A pair of complex hermitian vector spaces $V = \boldsymbol{C}^k$ and $W = \boldsymbol{C}^N$.
2. The operators $B_1, B_2 \in \mathrm{Hom}(V,V)$, $I \in \mathrm{Hom}(W,V)$, $J = \mathrm{Hom}(V,W)$ satisfying the equations

$$\begin{aligned} \mu_{\boldsymbol{R}} &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0, \\ \mu_{\boldsymbol{C}} &= [B_1, B_2] + IJ = 0. \end{aligned} \tag{2·27}$$

Next we define the Dirac-like operator $\mathcal{D}_z : V \oplus V \oplus W \to V \oplus V$ by

$$\begin{aligned} &\mathcal{D}_z = \begin{pmatrix} \tau_z \\ \sigma_z^\dagger \end{pmatrix}, \\ &\tau_z = (\, B_2 - z_2,\, B_1 - z_1,\, I \,), \\ &\sigma_z^\dagger = (\, -(B_1^\dagger - \bar{z}_1),\, B_2^\dagger - \bar{z}_2,\, J^\dagger \,). \end{aligned} \tag{2·28}$$

Equation (2·27) is equivalent to the set of equations

$$\tau_z\tau_z^\dagger = \sigma_z^\dagger\sigma_z, \quad \tau_z\sigma_z = 0, \tag{2·29}$$

which are important conditions in ADHM construction. There are N zero-modes of $\mathcal{D}_z$:

$$\mathcal{D}_z\psi^{(a)} = 0, \quad a = 1,\cdots,N. \tag{2·30}$$

We can choose an orthonormal basis in the space of these zero-modes:

$$\psi^{(a)\dagger}\psi^{(b)} = \delta^{ab}. \tag{2·31}$$

The change of basis in the space of orthonormal zero-modes $\psi^{(a)}$ becomes $U(N)$ gauge symmetry. The anti-self-dual $U(N)$ gauge field is constructed with the formula

$$A^{ab} = \psi^{(a)\dagger} d\psi^{(b)}. \tag{2·32}$$

There is an action of $U(k)$ that does not change (2·32):

$$(B_1, B_2, I, J) \longmapsto (gB_1g^{-1}, gB_2g^{-1}, gI, Jg^{-1}), \qquad g \in U(k). \tag{2·33}$$

The moduli space of the anti-self-dual $U(N)$ gauge field with instanton number k is given by

$$\mathcal{M}(k, N) = \mu_{\boldsymbol{R}}^{-1}(0) \cap \mu_{\boldsymbol{C}}^{-1}(0)/U(k), \tag{2·34}$$

where the action of $U(k)$ is the one given in (2·33). When (B_1, B_2, I, J) is a fixed point of the $U(k)$ action, $\mathcal{M}(k, N)$ is singular. Such a singularity corresponds to an instanton shrinking to zero size.

§3. ADHM construction on noncommutative $\boldsymbol{R}^4$ and the appearance of the projection operator

The singularities in (2·34) have a natural resolution.[8)] First, we modify (2·27) as

$$\begin{aligned} \mu_{\boldsymbol{R}} &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = \zeta \,\mathrm{Id}_V, \\ \mu_{\boldsymbol{C}} &= [B_1, B_2] + IJ = 0, \end{aligned} \tag{3·1}$$

and then consider the space

$$\mathcal{M}_\zeta(k, N) = \mu_{\boldsymbol{R}}^{-1}(\zeta \,\mathrm{Id}_V) \cap \mu_{\boldsymbol{C}}^{-1}(0)\,/U(k). \tag{3·2}$$

Then $\mathcal{M}_\zeta(k, N)$ is a smooth $4kN$-dimensional hyper-Kähler manifold. Although the absence of singularities is interesting from the physical point of view, construction of instantons from (3·1) cannot be done straightforwardly. The main problem in this regard is that the key equations in (2·29) are not satisfied on the usual commutative $\boldsymbol{R}^4$. However, Nekrasov and Schwarz realized that τ_z and σ_z do satisfy (2·29) if the coordinates are noncommutative, as in (2·4).[12)] Once (2·29) is satisfied, we can expect that the construction of instantons is similar to therein the usual commutative case. But there are some features peculiar to the noncommutative case. In particular, since the ADHM construction on noncommutative $\boldsymbol{R}^4$ starts from the space (3·2), in which the small instanton singularities are absent, one expects that a crucial difference will appear when the size of the instanton is small. It is interesting to study such situations and see how the effects of the noncommutativity appear.

The ADHM construction on noncommutative $\boldsymbol{R}^4$ is as follows.[12] We define the operator $\mathcal{D}_z : (V \oplus V \oplus W) \otimes \mathcal{A}_\zeta \to (V \oplus V) \otimes \mathcal{A}_\zeta$ by the same formula (2·28):

$$\begin{aligned} \mathcal{D}_z &= \begin{pmatrix} \tau_z \\ \sigma_z^\dagger \end{pmatrix}, \\ \tau_z &= (\, B_2 - z_2,\, B_1 - z_1,\, I\,), \\ \sigma_z^\dagger &= (\, -(B_1^\dagger - \bar{z}_1),\, B_2^\dagger - \bar{z}_2,\, J^\dagger\,). \end{aligned} \tag{3·3}$$

The operator $\mathcal{D}_z \mathcal{D}_z^\dagger : (V \oplus V) \otimes \mathcal{A}_\zeta \to (V \oplus V) \otimes \mathcal{A}_\zeta$ has the block diagonal form

$$\mathcal{D}_z \mathcal{D}_z^\dagger = \begin{pmatrix} \Box_z & 0 \\ 0 & \Box_z \end{pmatrix}, \qquad \Box_z \equiv \tau_z \tau_z^\dagger = \sigma_z^\dagger \sigma_z, \tag{3·4}$$

which is a consequence of (2·29) and important for ADHM construction. Next, we look for solutions to the equation

$$\mathcal{D}_z \Psi^{(a)} = 0, \quad (a = 1, \cdots, N) \tag{3·5}$$

where the components of $\Psi^{(a)}$ are *operators*: $\Psi^{(a)} : \mathcal{A}_\zeta \to (V \oplus V \oplus W) \otimes \mathcal{A}_\zeta$. If we can normalize the $\Psi^{(a)}$ as

$$\Psi^{\dagger(a)} \Psi^{(b)} = \delta^{ab}\, \mathrm{Id}_{\mathcal{H}}, \tag{3·6}$$

we can construct an anti-self-dual $U(N)$ gauge field by the same formula (2·32):

$$A^{ab} = \Psi^{(a)\dagger} d\Psi^{(b)}, \tag{3·7}$$

where a and b are $U(N)$ indices. Then the field strength becomes

$$\begin{aligned} F = F^-_{\mathrm{ADHM}} &\equiv \Psi^\dagger \left(d\tau_z^\dagger \frac{1}{\Box_z} d\tau_z + d\sigma_z \frac{1}{\Box_z} d\sigma_z^\dagger \right) \Psi \\ &= \begin{pmatrix} \psi_1^\dagger & \psi_2^\dagger & \xi^\dagger \end{pmatrix} \begin{pmatrix} dz_1 \frac{1}{\Box_z} d\bar{z}_1 + d\bar{z}_2 \frac{1}{\Box_z} dz_2 & -dz_1 \frac{1}{\Box_z} d\bar{z}_2 + d\bar{z}_2 \frac{1}{\Box_z} dz_1 & 0 \\ -dz_2 \frac{1}{\Box_z} d\bar{z}_1 + d\bar{z}_1 \frac{1}{\Box_z} dz_2 & dz_2 \frac{1}{\Box_z} d\bar{z}_2 + d\bar{z}_1 \frac{1}{\Box_z} dz_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\quad \times \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix}, \end{aligned} \tag{3·8}$$

where we have written

$$\Psi \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix} \equiv \begin{pmatrix} \Psi^{(1)} & \cdots & \Psi^{(N)} \end{pmatrix}, \qquad \begin{aligned} \psi_1 &: \boldsymbol{C}^N \otimes \mathcal{A}_\zeta \to V \otimes \mathcal{A}_\zeta, \\ \psi_2 &: \boldsymbol{C}^N \otimes \mathcal{A}_\zeta \to V \otimes \mathcal{A}_\zeta, \\ \xi &: \boldsymbol{C}^N \otimes \mathcal{A}_\zeta \to W \otimes \mathcal{A}_\zeta. \end{aligned}$$

The derivation here is similar to that in the commutative case. The field strength in (3·8) is anti-self-dual.

However, as we will see shortly, there are some states in $\mathcal{H}$ which are annihilated by $\Psi^{(a)}$ for some a. More precisely, all the components of $\Psi^{(a)}$ annihilate these states. This is not a special phenomenon, and its study is the purpose of this paper. Let us consider the case in which there is one such zero-mode $\Psi^{(1)}$. In this case, we cannot normalize $\Psi^{(1)}$ as in (3·6). We may normalize $\Psi^{(1)}$ as

$$\Psi^{(1)\dagger}\Psi^{(1)} = P, \tag{3·9}$$

where $P \in \mathcal{A}_\zeta$ is a projection operator that projects out the states annihilated by $\Psi^{(1)}$. However, the projection operator gives an additional contribution to the field strength, because the projection operator depends on z and $\bar{z}$.*) The derivative of the projection operator gives an additional contribution to the field strength, which is not anti-self-dual.

The appearance of the projection operator P indicates that we should consider the restricted vector space $P\mathcal{E}$ rather than $\mathcal{E}$. Indeed, as we will see shortly, ADHM construction works perfectly in this setting.

Let us concentrate on the simplest $U(1)$ case. Here, the covariant derivative is given by the formula (2·17):

$$D_P = Pd + A, \tag{3·10}$$

with $A = PAP$. The field strength is given by (2·22):

$$F = PdA + A^2 + PdPdP. \tag{3·11}$$

We can construct an anti-self-dual gauge field by setting

$$A = \Psi^\dagger d\Psi P, \tag{3·12}$$

where Ψ is a zero-mode of $\mathcal{D}_z$ and normalized as $\Psi^\dagger\Psi = P$. Note that $\Psi^\dagger = P\Psi^\dagger$. Let us check that (3·12) is really anti-self-dual. The first term in (3·11) becomes

$$PdA = Pd\Psi^\dagger d\Psi P - P\Psi^\dagger d\Psi dP, \tag{3·13}$$

and the last term above can be rewritten as

$$\begin{aligned} P\Psi^\dagger d\Psi dP &= P(d(\Psi^\dagger\Psi) - d\Psi^\dagger\Psi)dP \\ &= PdPdP - Pd\Psi^\dagger\Psi dP. \end{aligned} \tag{3·14}$$

The first term in (3·14) cancels $PdPdP$ in (3·11). The last term in (3·14) vanishes when acting on $e = Pe \in P\mathcal{E}$, since $\Psi dPP = -\Psi Pd(1-P)P = 0$. The second term

*) For example, $|0,0\rangle\langle 0,0| =: e^{-\frac{2}{\zeta}(z_1\bar{z}_1 + z_2\bar{z}_2)} :$, where $: \mathcal{O} :$ indicates normal ordering of the operator $\mathcal{O}$.

in (3·11) then becomes

$$
\begin{aligned}
A^2 &= P\Psi^\dagger d\Psi\Psi^\dagger d\Psi P \\
&= P(d(\Psi^\dagger\Psi)) - d\Psi^\dagger\Psi)\Psi^\dagger d\Psi P \\
&= PdP\Psi^\dagger d\Psi - Pd\Psi^\dagger\Psi\Psi^\dagger d\Psi P.
\end{aligned}
\tag{3·15}
$$

The first term in (3·15) vanishes because $PdP\Psi^\dagger = -Pd(1-P)P\Psi^\dagger = 0$. Then the field strength becomes

$$
\begin{aligned}
F &= Pd\Psi^\dagger(1-\Psi\Psi^\dagger)d\Psi P \\
&= PF^-_{\mathrm{ADHM}}P = F^-_{\mathrm{ADHM}},
\end{aligned}
\tag{3·16}
$$

where F^-_{ADHM} is defined in (3·8) and is anti-self-dual. Generalization to the $U(N)$ case is straightforward.

The absence of singular behavior in the field configuration follows rather straightforwardly from the explicit formula (3·8). Since we have normalized the zero-modes in the subspace where zero-modes do not vanish, these normalized zero-modes are well defined. Moreover, as shown in Appendix A, the operator $\Box_z$ has no zero-mode, and hence its inverse does not cause divergences. Therefore, from the explicit formula (3·8), we can see no source of divergence in either (3·8) or (3·16).

§4. $U(1)$ instantons and projection operators

4.1. *Projection operators in $U(1)$ instanton solutions and their relation to the ideal*

In the previous section it was shown how to construct anti-self-dual gauge fields in the case that the zero-mode annihilates some states. Then the natural question is, how should the states annihilated by the zero-modes be determined? In this section the answer to this question is given in the case that the gauge group is $U(1)$.

Let us consider the solution to the equation

$$
\mathcal{D}_z\,|\mathcal{U}\rangle = 0, \tag{4·1}
$$

where $|\mathcal{U}\rangle \in \mathcal{H}^{\oplus k}\oplus\mathcal{H}^{\oplus k}\oplus\mathcal{H}$; i.e. the components of $|\mathcal{U}\rangle$ are *vectors* in the Fock space $\mathcal{H}$. We call $|\mathcal{U}\rangle$ a "vector zero-mode", and call Ψ in (3·5) an "operator zero-mode". We can construct an operator zero-mode if we know all the vector zero-modes. The advantage of considering vector zero-modes is that we can relate them to the ideal discussed in Refs. 9) and 10). The point is that we can regard vector zero-modes as holomorphic vector bundles described in purely commutative terms. Noncommutativity appears when we construct an operator zero-mode treating all the vector zero-modes as a whole.

Let us write

$$
|\mathcal{U}\rangle = \begin{pmatrix} |u_1\rangle\,, \\ |u_2\rangle\,, \\ |\,f\,\rangle\,, \end{pmatrix}, \qquad
\begin{aligned}
|u_1\rangle &\equiv u_1(z_1,z_2)\,|\,0,0\,\rangle\,, \\
|u_2\rangle &\equiv u_2(z_1,z_2)\,|\,0,0\,\rangle\,, \\
|\,f\,\rangle &\equiv f(z_1,z_2)\,|\,0,0\,\rangle\,,
\end{aligned}
\tag{4·2}
$$

where $|u_1\rangle, |u_2\rangle \in \mathcal{H}^{\oplus k}$; i.e. they are vectors in $V = \boldsymbol{C}^k$ and vectors in $\mathcal{H}$, and $|f\rangle \in \mathcal{H}$. The space of solutions of (4·1) (i.e. $\ker \mathcal{D}_z = \ker \tau_z \cap \ker \sigma_z^\dagger \simeq \ker \tau_z / \mathrm{Im}\, \sigma_z$ *)) is isomorphic to the ideal $\mathcal{I}$ defined by

$$\mathcal{I} = \left\{ f(z_1, z_2) \;\middle|\; f(B_1, B_2) = 0 \right\}, \tag{4·3}$$

where B_1 and B_2 together with I and J give a solution to (3·1). In the $U(1)$ case, one can show $J = 0$, and the isomorphism is given by the inclusion of the third factor in (4·2):[9),10)]

$$\ker \tau_z / \mathrm{Im}\, \sigma_z \hookrightarrow \mathcal{O}_{\boldsymbol{C}^2} : \quad |\mathcal{U}\rangle = \begin{pmatrix} |u_1\rangle \\ |u_2\rangle \\ |f\rangle \end{pmatrix} \hookrightarrow f(z_1, z_2). \tag{4·4}$$

Let us define the "ideal state" by

$$|\varphi\rangle \in \text{ideal states of } \mathcal{I} \iff \exists f(z_1, z_2) \in \mathcal{I}, \quad |\varphi\rangle = f(z_1, z_2)\, |0, 0\rangle \tag{4·5}$$

and denote the space of all the ideal states by $\mathcal{H}_{\mathcal{I}}$. We define $\mathcal{H}_{/\mathcal{I}}$ **) as a subspace in $\mathcal{H}$ orthogonal to $\mathcal{H}_{\mathcal{I}}$:

$$|g\rangle \in \mathcal{H}_{/\mathcal{I}} \iff \forall f(z_1, z_2) \in \mathcal{I}, \quad \langle 0|\, f^\dagger(\bar{z}_1, \bar{z}_2)\, |g\rangle = 0. \tag{4·6}$$

Here $\mathcal{H}_{/\mathcal{I}}$ is a k-dimensional space.[10)] Let us denote the complete basis of $\mathcal{H}_{/\mathcal{I}}$ by $|g_\alpha\rangle, \alpha = 1, 2, \cdots, k$, and the orthonormalized complete basis of $\mathcal{H}_{\mathcal{I}}$ by $|f_i\rangle, i = k+1, k+2, \cdots$. Altogether, they span the complete basis of $\mathcal{H}$. We can label them by the positive integer n:

$$\{\, |h_n\rangle,\ n \in \boldsymbol{Z}_+ \} = \{\, |g_\alpha\rangle, |f_i\rangle,\ \alpha = 1, 2, \cdots, k,\ i = k+1, k+2, \cdots \}. \tag{4·7}$$

As we can see from (4·4), the zero-modes (4·1) are completely determined by the ideal $f_i(z_1, z_2)$:[10)]

$$|\mathcal{U}(f_i)\rangle = \begin{pmatrix} |u_1(f_i)\rangle \\ |u_2(f_i)\rangle \\ |f_i\rangle \end{pmatrix}. \tag{4·8}$$

We can construct the operator zero-mode (3·5) with the formula

$$\Psi = \sum_i \sum_n (\Psi)_{in}\, |\mathcal{U}(f_i)\rangle\, \langle h_n|, \tag{4·9}$$

*) Note that since $\tau_z \sigma_z = 0$ [see (2·29)], $\ker \tau_z / \mathrm{Im}\, \sigma_z$ is well defined. The relation $\ker \tau_z \cap \ker \sigma_z^\dagger \simeq \ker \tau_z / \mathrm{Im}\, \sigma_z$ is understood as follows: The condition $\ker \sigma_z^\dagger\, |\mathcal{U}\rangle = 0$ fixes the "gauge freedom" mod $\mathrm{Im}\, \sigma_z$ in $\ker \tau_z / \mathrm{Im}\, \sigma_z$.

**) The meaning of this notation is as follows: $\mathcal{H}_{/\mathcal{I}}$ corresponds to $\boldsymbol{C}[z_1, z_2]/\mathcal{I}$, where $\boldsymbol{C}[z_1, z_2]$ is the ring of polynomials of z_1 and z_2.

where $(\Psi)_{in}$ is a commuting number. From (4·9), one can see that there are infinitely many operator zero-modes. Since the Fock space $\mathcal{H}$ is divided into two orthogonal subspaces, $\mathcal{H}_{\mathcal{I}}$ and $\mathcal{H}_{/\mathcal{I}}$, through the isomorphism (4·3), it is natural to restrict the action of operators to $\mathcal{H}_{\mathcal{I}}$. We call Ψ_0 the "minimal operator zero-mode" if it has the form

$$\Psi_0 = \sum_{i,j} (\Psi_0)_{ij} \left|\mathcal{U}(f_i)\right\rangle \left\langle f_j\right|, \qquad |f_j\rangle \in \mathcal{H}_{\mathcal{I}},$$
$$\forall j,\ \exists i \quad \text{such that} \quad (\Psi_0)_{ij} \neq 0; \tag{4·10}$$

i.e. $(\Psi_0)_{in} = 0$ for $n = \alpha = 1, 2, \cdots, k$. We call it the "normalized" minimal operator zero-mode if it is normalized in $\mathcal{H}_{\mathcal{I}}$:

$$\Psi_0^\dagger \Psi_0 = P_{\mathcal{I}}, \tag{4·11}$$

where $P_{\mathcal{I}}$ is a projection operator which represents the projection on $\mathcal{H}_{\mathcal{I}}$, the space of ideal states. The uniqueness of the normalized minimal operator zero-mode up to the gauge transformation (2·16) is demonstrated in Appendix B. This implies that the normalized minimal operator zero-mode contains minimal information regarding the ideal (4·3). From the above definition, the minimal operator zero-mode annihilates states in $\mathcal{H}_{/\mathcal{I}}$; i.e. $\Psi_0 \left|\varphi\right\rangle = 0$ for $|\varphi\rangle \in \mathcal{H}_{/\mathcal{I}}$. Note that if we write

$$\Psi_0 = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \xi \end{pmatrix}, \tag{4·12}$$

then $\xi \left|\varphi\right\rangle = 0 \Rightarrow \psi_1 \left|\varphi\right\rangle = \psi_2 \left|\varphi\right\rangle = 0$. Hence the states annihilated by the minimal operator zero-mode Ψ_0 are completely determined by the third factor ξ in (4·12).

An interesting point is that the noncommutative operators appear from the ideal described in purely commutative terms by treating an infinite number of elements of the ideal simultaneously.

As an illustration, let us construct a $U(1)$ one-instanton solution from the ideal. First, let us recall the $U(1)$ one-instanton solution constructed in Ref. 12). The solution to the modified ADHM equations (3·1) is given by

$$B_1 = B_2 = 0, \qquad I = \sqrt{\zeta}\,, \qquad J = 0. \tag{4·13}$$

The following is a solution to the equation $\mathcal{D}_z \tilde{\Psi}_0 = 0$:

$$\tilde{\Psi}_0 = \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\xi} \end{pmatrix} = \begin{pmatrix} \sqrt{\zeta}\bar{z}_2 \\ \sqrt{\zeta}\bar{z}_1 \\ (z_1\bar{z}_1 + z_2\bar{z}_2) \end{pmatrix}. \tag{4·14}$$

Note that all the components of $\tilde{\Psi}_0$ annihilate $|0,0\rangle$. As a consequence, $\tilde{\Psi}_0^\dagger \tilde{\Psi}_0 = (z_1\bar{z}_1 + z_2\bar{z}_2)(z_1\bar{z}_1 + z_2\bar{z}_2 + \zeta)$ annihilates $|0,0\rangle$. Therefore, the inverse of $(z_1\bar{z}_1 + z_2\bar{z}_2)$ is only defined in the subspace of the Fock space where $|0,0\rangle$ is projected out:

$$(z_1\bar{z}_1 + z_2\bar{z}_2)^{-1} := P(z_1\bar{z}_1 + z_2\bar{z}_2)^{-1}P, \tag{4·15}$$

where P is the projection operator that projects out $|0,0\rangle$:

$$P = \mathrm{Id}_{\mathcal{H}} - |0,0\rangle\langle 0,0| \,. \tag{4·16}$$

Therefore

$$\Psi_0 = \tilde{\Psi}_0(\tilde{\Psi}_0^\dagger \tilde{\Psi}_0)^{-1/2} \tag{4·17}$$

is normalized as $\Psi_0^\dagger \Psi_0 = P$.

Let us reconstruct this zero-mode from the ideal. The ideal that corresponds to (4·13) is generated by z_1 and z_2, which we represent as $\mathcal{I} = (z_1, z_2)$. The basis vector of $\mathcal{H}_{/\mathcal{I}}$ is $|0,0\rangle$, which is orthogonal to all the ideal states. We can use $|n_1, n_2\rangle, (n_1, n_2) \neq (0,0)$, as basis vectors of $\mathcal{H}_{\mathcal{I}}$, the space of ideal states. The solutions of $\mathcal{D}_z |\mathcal{U}\rangle = 0$ are given by

$$|\mathcal{U}_{n_1 n_2}\rangle = \begin{pmatrix} |u_{1 n_1 n_2}\rangle \\ |u_{2 n_1 n_2}\rangle \\ |\, f_{n_1 n_2}\,\rangle \end{pmatrix} = \begin{pmatrix} \sqrt{n_2}\,|n_1, n_2 - 1\rangle \\ \sqrt{n_1}\,|n_1 - 1, n_2\rangle \\ \frac{1}{\sqrt{2}}(n_1 + n_2)\,|n_1, n_2\rangle \end{pmatrix}, \quad (n_1, n_2) \neq (0,0). \tag{4·18}$$

From (4·18), we obtain the operator zero-mode $\mathcal{D}_z \Psi = 0$:

$$\Psi = \sum_{(m_1,m_2)\neq(0,0)} \sum_{(n_1,n_2)} (\Psi)_{(m_1,m_2)(n_1,n_2)} \,|\mathcal{U}_{m_1 m_2}\rangle\langle n_1, n_2| \,. \tag{4·19}$$

The normalized minimal operator zero-mode Ψ_0 is required to satisfy

$$\begin{aligned} \Psi_0 &= \sum_{(m_1,m_2)\neq(0,0)} \sum_{(n_1,n_2)\neq(0,0)} (\Psi)_{(m_1,m_2)(n_1,n_2)} \,|\mathcal{U}_{m_1 m_2}\rangle\langle n_1, n_2| \,, \\ \Psi_0^\dagger \Psi_0 &= \mathrm{Id}_{\mathcal{H}} - |0,0\rangle\langle 0,0| \,. \end{aligned} \tag{4·20}$$

From the normalization condition in (4·20), we obtain

$$\begin{aligned} \sum_{(m_1,m_2)\neq(0,0)} & \frac{1}{2}(m_1 + m_2)(m_1 + m_2 + 2)\,(\Psi^\dagger)_{(l_1,l_2)(m_1,m_2)} (\Psi)_{(m_1,m_2)(n_1,n_2)} \\ &= \delta_{(l_1,l_2)(n_1,n_2)}. \end{aligned} \tag{4·21}$$

The solution of (4·21) is

$$(\Psi_0)_{(m_1,m_2)(n_1,n_2)} = \sqrt{\frac{2}{(n_1 + n_2)(n_1 + n_2 + 2)}}\,\delta_{(m_1,m_2)(n_1,n_2)} \,. \tag{4·22}$$

Equations (4·20) and (4·22) are equivalent to (4·14) and (4·17).

4.2. *Some $U(1)$ instanton solutions*

Construction of the operator zero-mode from the vector zero-modes is useful for the purpose of understanding the notion of the minimal operator zero-mode. But in some simple cases, it is easier to directly look for the operator zero-modes. It is interesting to observe that the operator zero-modes that are most naturally obtained actually annihilate states in $\mathcal{H}_{/\mathcal{I}}$.

***U*(1) two-instanton solution**[*)]

Let us study the two-instanton solutions that are degenerateing at the origin. The corresponding solution to the matrix equations (3·1) is given by

$$B_1 = \begin{pmatrix} 0 & \sqrt{\zeta}\lambda_1 \\ 0 & 0 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & \sqrt{\zeta}\lambda_2 \\ 0 & 0 \end{pmatrix}, \; I = \begin{pmatrix} 0 \\ \sqrt{2\zeta} \end{pmatrix}, \; J = 0, \qquad (4·23)$$

where λ_1 and λ_2 are complex numbers satisfying $|\lambda_1|^2 + |\lambda_2|^2 = 1$. Note that B_1 and B_2 are upper-half triangular matrices. λ_1 and λ_2 are parameters that represents the direction between two instantons before they collide and degenerate.[10) 20)] The corresponding ideal is $\mathcal{I} = \left(z_1^2,\, -\lambda_2 z_1 + \lambda_1 z_2\right)$. Hence the states orthogonal to all the ideal states are annihilated by $\bar{z}_1^2,\ -\lambda_2^*\bar{z}_1 + \lambda_1^*\bar{z}_2$. The states annihilated by $\bar{z}_1^2$ is $|0, n_2\rangle$ and $|1, n_2\rangle$ for all non-negative integers n_2. In order to describe the states annihilated by $-\lambda_2^*\bar{z}_1 + \lambda_1^*\bar{z}_2$, it is simpler to use the basis constructed by the rotated creation and annihilation operators z' and $\bar{z}'$ defined as follows:

$$\begin{aligned}
&z'_1 \equiv \lambda_1^* z_1 + \lambda_2^* z_2, \qquad z'_2 \equiv -\lambda_2 z_1 + \lambda_1 z_2,\\
&|0,0\rangle = |0,0\rangle_\lambda\,,\\
&\sqrt{\frac{2}{\zeta}} z'_1\, |n'_1, n'_2\rangle_\lambda = \sqrt{n'_1 + 1}\, |n'_1 + 1, n'_2\rangle_\lambda\,, \quad \sqrt{\frac{2}{\zeta}} \bar{z}'_1\, |n'_1, n'_2\rangle_\lambda = \sqrt{n'_1}\, |n'_1 - 1, n'_2\rangle_\lambda\,,\\
&\sqrt{\frac{2}{\zeta}} z'_2\, |n'_1, n'_2\rangle_\lambda = \sqrt{n'_2 + 1}\, |n'_1, n'_2 + 1\rangle_\lambda\,, \quad \sqrt{\frac{2}{\zeta}} \bar{z}'_2\, |n'_1, n'_2\rangle_\lambda = \sqrt{n'_2}\, |n'_1, n'_2 - 1\rangle_\lambda\,.
\end{aligned} \qquad (4·24)$$

Then the states annihilated by $\bar{z}'_2 = -\lambda_2^*\bar{z}_1 + \lambda_1^*\bar{z}_2$ are $|n'_1, 0\rangle_\lambda$ for all non-negative n'_1. Therefore the basis vectors of the states orthogonal to all the ideal states are $|0, 0\rangle$ and $|1, 0\rangle_\lambda$. Now let us study operator zero-mode. The (unnormalized) minimal operator zero-mode can be directly obtained from (4·23):

$$\tilde{\Psi}_0 = \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\xi} \end{pmatrix}, \qquad \tilde{\psi}_1 = \begin{pmatrix} \sqrt{\zeta}\bar{z}_2\bar{z}'_1 \\ \bar{z}_2\frac{\zeta}{2}(\hat{N} - 1) + \zeta\lambda_1\bar{z}'_2 \end{pmatrix},$$

$$\tilde{\psi}_2 = \begin{pmatrix} \sqrt{\zeta}\bar{z}_1\bar{z}'_1 \\ \bar{z}_1\frac{\zeta}{2}(\hat{N} - 1) - \zeta\lambda_2\bar{z}'_2 \end{pmatrix},$$

$$\tilde{\xi} = \frac{1}{\sqrt{2\zeta}}\left(\frac{\zeta}{2}\right)^2 \left(\hat{N}(\hat{N} - 1) + 2n'_2\right), \qquad (4·25)$$

---

*) Although we only consider solutions of the matrix equation (3·1) and do not construct a gauge field explicitly, we call the solutions "instanton solutions", because in principle we can construct instantons from the matrix data. We regard k as the number of instantons.

where $\frac{\zeta}{2}\hat{N} \equiv z_1\bar{z}_1 + z_2\bar{z}_2$, $\frac{\zeta}{2}\hat{n}'_2 \equiv z'_2\bar{z}'_2$. The zero-mode represented by (4·25) is truly minimal: $|0,0\rangle$ and $|1,0\rangle_\lambda$ are annihilated by all the components of $\tilde{\Psi}_0$.

$\boldsymbol{U(1)}$ three-instanton solutions

Let us consider the $k=3$ solution corresponding to the following simple ideal:*)

$$\mathcal{I} = \left\{ f(z_1,z_2) = \sum_{n_1,n_2} a_{n_1n_2} z_1^{n_1} z_2^{n_2} \;\middle|\; \begin{array}{l} a_{n_1n_2} = 0 \text{ when } (n_1,n_2) \text{ belongs} \\ \text{to the Young tableau (Y1).} \end{array} \right\}, \tag{4·26}$$

$$\begin{array}{c} \begin{array}{|c|c} \hline (1,0) & \\ \hline (0,0) & \multicolumn{1}{|c|}{(0,1)} \\ \hline \end{array} \\ \text{(Y1)} \end{array} \quad . \tag{4·27}$$

The solution to (3·1) is given by

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{\zeta} \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & \sqrt{\zeta} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 0 \\ 0 \\ \sqrt{3\zeta} \end{pmatrix}, \qquad J = 0. \tag{4·28}$$

We can find the (unnormalized) minimal operator zero-mode:

$$\tilde{\Psi}_0 = \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\xi} \end{pmatrix}, \quad \tilde{\psi}_1 = \begin{pmatrix} \sqrt{\zeta}\bar{z}_2^2 \\ \sqrt{\zeta}\bar{z}_1\bar{z}_2 \\ \frac{\zeta}{2}\hat{N}\bar{z}_2 \end{pmatrix}, \quad \tilde{\psi}_2 = \begin{pmatrix} \sqrt{\zeta}\bar{z}_1\bar{z}_2 \\ \sqrt{\zeta}\bar{z}_1^2 \\ \frac{\zeta}{2}\hat{N}\bar{z}_1 \end{pmatrix},$$
$$\tilde{\xi} = \frac{1}{\sqrt{3\zeta}}\left(\frac{\zeta}{2}\right)^2 \hat{N}(\hat{N}-1). \tag{4·29}$$

The zero-mode represented by (4·29) actually annihilates $|0,0\rangle$, $|1,0\rangle$, $|0,1\rangle$ and hence is minimal.

Next consider the ideal corresponding to the following Young tableau (Y2):

$$\begin{array}{c} \begin{array}{|c|} \hline (2,0) \\ \hline (1,0) \\ \hline (0,0) \\ \hline \end{array} \\ \text{(Y2)} \end{array} \quad . \tag{4·30}$$

The solution to (3·1) is given by

$$B_1 = \begin{pmatrix} 0 & \sqrt{\zeta} & 0 \\ 0 & 0 & \sqrt{2\zeta} \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = 0, \quad I = \begin{pmatrix} 0 \\ 0 \\ \sqrt{3\zeta} \end{pmatrix}, \quad J = 0. \tag{4·31}$$

*) This kind of ideal corresponds to fixed points of the T^2 action in Refs. 9) and 10).

The (unnormalized) minimal operator zero-mode is given as

$$\tilde{\Psi}_0 = \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\xi} \end{pmatrix}, \quad \tilde{\psi}_1 = \begin{pmatrix} 2\zeta \bar{z}_1^2 \bar{z}_2 \\ \sqrt{2\zeta}\frac{\zeta}{2}\hat{N}\bar{z}_1\bar{z}_2 \\ \left(\frac{\zeta}{2}\right)^2 \left\{(\hat{N}+1)(\hat{N}+4) - 2(\hat{n}_1 - 1)\right\}\bar{z}_2 \end{pmatrix},$$

$$\tilde{\psi}_2 = \begin{pmatrix} 2\zeta \bar{z}_1^3 \\ \sqrt{2\zeta}\frac{\zeta}{2}\hat{N}\bar{z}_1^2 \\ \left(\frac{\zeta}{2}\right)^2 \left\{(\hat{N}+1)\hat{N} - 2\hat{n}_1\right\}\bar{z}_1 \end{pmatrix},$$

$$\tilde{\xi} = \frac{1}{\sqrt{3\zeta}}\left(\frac{\zeta}{2}\right)^3 \hat{N}\left\{\hat{N}(\hat{N}+3) - 2(3\hat{n}_1 - 1)\right\}. \qquad (4\cdot32)$$

We can check that the zero-mode represented by (4·32) annihilates $|0,0\rangle, |1,0\rangle, |2,0\rangle$.

§5. $U(N)$ instantons and projection operators

In the previous section we clarified the notion of the minimal operator zero-mode for the $U(1)$ case. In this section we study the $U(2)$ instanton solutions and observe that the projection of states by zero-modes also occurs. Since the $U(N)$ instanton solutions are essentially embeddings of $U(2)$ instanton solutions in $U(N)$, this implies that the projection of states is a general phenomenon in the ADHM construction of instantons on noncommutative $\boldsymbol{R}^4$. In the following, we make two observations:

1. The minimal operator zero-mode appears in the $U(1)$ subgroup of the $U(2)$ gauge group. It annihilates some states even when the size of the instanton is not small.
2. When the size of the instanton becomes small, only the contribution from the $U(1)$ subgroup described by the minimal operator zero-mode remains.

Although we have not defined a minimal operator zero-mode for the $U(N)$ case, zero-modes similar to the minimal operator zero-mode in the $U(1)$ case appear in explicit solutions. Hence in the above we have also referred to them as minimal operator zero-modes. The second observation can be understood as follows. We may define a "small instanton" on noncommutative $\boldsymbol{R}^4$ as the $J = 0$ solution of the modified ADHM equations (3·1). Then the solution is essentially the embedding of a $U(1)$ instanton in $U(N)$.

$U(2)$ one-instanton solution

The solution to the modified ADHM equations (3·1) is given by*)

$$B_1 = B_2 = 0, \quad I = \begin{pmatrix} \sqrt{\rho^2+\zeta} & 0 \end{pmatrix}, \quad J^\dagger = \begin{pmatrix} 0 & \rho \end{pmatrix}, \qquad (5\cdot1)$$

*) There is of course a family of solutions with different orientation in the gauge group $U(2)$. The conclusions in this case, however, are the same.

where ρ is a real non-negative number that parameterizes the size of the instanton. The two "orthonormal" operator zero-modes of $\mathcal{D}_z$ are given by

$$\Psi^{(1)} = \begin{pmatrix} \psi_1^{(1)} \\ \psi_2^{(1)} \\ \xi^{(1)} \end{pmatrix} = \begin{pmatrix} \sqrt{\rho^2+\zeta}\,\bar{z}_2 \\ \sqrt{\rho^2+\zeta}\,\bar{z}_1 \\ (z_1\bar{z}_1 + z_2\bar{z}_2) \\ 0 \end{pmatrix} \Big((z_1\bar{z}_1+z_2\bar{z}_2)(z_1\bar{z}_1+z_2\bar{z}_2+\zeta+\rho^2)\Big)^{-1/2}, \tag{5·2}$$

$$\Psi^{(2)} = \begin{pmatrix} \psi_1^{(2)} \\ \psi_2^{(2)} \\ \xi^{(2)} \end{pmatrix}$$
$$= \begin{pmatrix} -\rho z_1 \\ \rho z_2 \\ 0 \\ (z_1\bar{z}_1+z_2\bar{z}_2+\zeta) \end{pmatrix} \Big((z_1\bar{z}_1+z_2\bar{z}_2+\zeta)\Big(z_1\bar{z}_1+z_2\bar{z}_2+\zeta+\rho^2\Big)\Big)^{-1/2}. \tag{5·3}$$

The zero-mode $\Psi^{(1)}$ is a straightforward modification of (4·17). $\Psi^{(1)}$ annihilates $|0,0\rangle$ for any values of ρ, and is normalized in the subspace where $|0,0\rangle$ is projected out. The zero-mode $\Psi^{(2)}$ annihilates no state in $\mathcal{H}$ and is manifestly non-singular even if $\rho=0$. When $\rho=0$, we have $\psi_1^{(2)} = \psi_2^{(2)} = 0$, and from (3·8), $\Psi^{(2)}$ does not contribute to the field strength. Therefore the structure of the instanton at $\rho=0$ is completely determined by the $U(1)$ subgroup described by the minimal operator zero-mode $\Psi^{(1)}$.

$U(2)$ two-instanton solution

We can also construct a two-instanton solution and check the assertion made in the beginning of this section. Here we only construct one simple solution. The solution of the modified ADHM equations (3·1) is given by

$$B_1 = \begin{pmatrix} 0 & \sqrt{\zeta} \\ 0 & 0 \end{pmatrix},\ B_2 = 0,\ I = \begin{pmatrix} 0 & 0 \\ \sqrt{2(\rho^2+\zeta)} & 0 \end{pmatrix},\ J^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & \rho \end{pmatrix}. \tag{5·4}$$

We can obtain two (unnormalized) orthogonal zero-modes:

$$\Psi^{(1)} = \begin{pmatrix} \psi_1^{(1)} \\ \psi_2^{(1)} \\ \xi^{(1)} \end{pmatrix}, \qquad \psi_1^{(1)} = \begin{pmatrix} \sqrt{\rho^2+\zeta}\sqrt{\zeta}\bar{z}_1\bar{z}_2 \\ \sqrt{\rho^2+\zeta}\bar{z}_2(z_1\bar{z}_1+z_2\bar{z}_2+\frac{\zeta}{2}) \end{pmatrix},$$
$$\psi_2^{(1)} = \begin{pmatrix} \sqrt{\rho^2+\zeta}\sqrt{\zeta}\bar{z}_1^2 \\ \sqrt{\rho^2+\zeta}\bar{z}_1(z_1\bar{z}_1+z_2\bar{z}_2-\frac{\zeta}{2}) \end{pmatrix},$$
$$\xi^{(1)} = \begin{pmatrix} \frac{1}{\sqrt{2}}(z_1\bar{z}_1+z_2\bar{z}_2)(z_1\bar{z}_1+z_2\bar{z}_2-\frac{\zeta}{2})+\zeta z_2\bar{z}_2 \\ 0 \end{pmatrix}, \tag{5·5}$$

and

$$\Psi^{(2)} = \begin{pmatrix} \psi_1^{(2)} \\ \psi_2^{(2)} \\ \xi^{(2)} \end{pmatrix}, \qquad \psi_1^{(2)} = \begin{pmatrix} \rho\sqrt{\zeta}\bar{z}_2 z_2 \\ \rho z_1(z_1\bar{z}_1 + z_2\bar{z}_2 + \frac{\zeta}{2}) \end{pmatrix},$$

$$\psi_2^{(2)} = \begin{pmatrix} \rho\sqrt{\zeta}\bar{z}_1 z_2 \\ \rho z_2(z_1\bar{z}_1 + z_2\bar{z}_2 + \frac{\zeta}{2}) \end{pmatrix},$$

$$\xi^{(2)} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}\left((z_1\bar{z}_1 + z_2\bar{z}_2)(z_1\bar{z}_1 + z_2\bar{z}_2 + \frac{\zeta}{2}) + \zeta(z_2\bar{z}_2 + \frac{\zeta}{2})\right) \end{pmatrix}. \qquad (5{\cdot}6)$$

Here, $\Psi^{(1)}$ is a slight modification of (4·25) with $(\lambda_1, \lambda_2) = (1, 0)$. It annihilates $|0, 0\rangle$ and $|1, 0\rangle$. $\Psi^{(2)}$ is apparently non-singular, and $\psi_1^{(2)} = \psi_2^{(2)} = 0$ when $\rho = 0$. Hence, when the size of the instanton is small, only the $U(1)$ subgroup described by $\Psi^{(1)}$ contributes to the field strength.

§6. D-instanton creates a hole in the D3-brane

The existence of the projection operator forces us to consider the reduced Fock space. In this section it is shown that the projection can be interpreted as a modification of spacetime topology. Usual Yang-Mills theory cannot describe such a spacetime topology change. However, as we see below, the IIB matrix model [3), 4)] gives an appropriate framework. The action of the IIB matrix model is obtained by dimensionally reducing the ten-dimensional $U(N)$ super Yang-Mills theory down to zero dimensions:[*)]

$$S = -\frac{1}{g^2}\mathrm{Tr}\left(\frac{1}{4}[X_\mu, X_\nu][X^\mu, X^\nu] + \frac{1}{2}\bar{\Theta}\Gamma_\mu[X^\mu, \Theta]\right), \qquad (6{\cdot}1)$$

where X_μ and Θ are $N \times N$ hermitian matrices and each component of Θ is a Majorana-Weyl spinor. The action (6·1) has the following $\mathcal{N} = 2$ supersymmetry:

$$\begin{aligned} \delta^{(1)}\Theta &= \frac{i}{2}[X_\mu, X_\nu]\Gamma^{\mu\nu}\epsilon^{(1)}, \\ \delta^{(1)}X_\mu &= i\bar{\epsilon}^{(1)}\Gamma_\mu\Theta, \\ \delta^{(2)}\Theta &= \epsilon^{(2)}, \\ \delta^{(2)}X_\mu &= 0. \end{aligned} \qquad (6{\cdot}2)$$

The classical equation of motion is given by

$$[X_\mu, [X_\mu, X_\nu]] = 0. \qquad (6{\cdot}3)$$

*) We have slightly changed the notation from that in the previous sections: In this section, N denotes the rank of the gauge group of the IIB matrix model. We only consider $U(1)$ instantons in the following.

The IIB matrix model has classical D-brane solutions:

$$X_\mu = i\hat{\partial}_\mu,$$
$$[i\hat{\partial}_\mu, i\hat{\partial}_\nu] = -iB_{\mu\nu}, \tag{6·4}$$

where the components of the matrix $B_{\mu\nu}$ are real constants. Hereafter we consider the (Euclidean) D3-brane solution; i.e., the rank of $B_{\mu\nu}$ is four and $B_{\mu\nu} = 0$ when $\mu, \nu \neq 1, 2, 3, 4$. We define "coordinate matrices" $\hat{x}^\mu$ by

$$\hat{x}^\mu = -i\theta^{\mu\nu}\hat{\partial}_\nu, \tag{6·5}$$

where $\theta^{\mu\nu}$ is an inverse matrix of $B_{\mu\nu}$. Then their commutation relations are the same as those in (2·1):

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}. \tag{6·6}$$

Hence by defining $\theta^{\mu\nu}$ (or equivalently $B_{\mu\nu}$) as self-dual as in (2·2); i.e. $\theta^{12} = \theta^{34} = \frac{\zeta}{4}$, and representing operators by infinite rank matrices,[*)] we can embed the instanton solution (3·12) in the IIB matrix model:

$$X_\mu = P(i\hat{\partial}_\mu + iA_\mu)P, \tag{6·7}$$

where A_μ is the $U(1)$ instanton solution obtained through the ADHM construction:

$$A_\mu = \Psi^\dagger[\hat{\partial}_\mu, \Psi]P. \tag{6·8}$$

Here Ψ is a zero-mode (3·5), and P is the projection operator determined by the zero-mode, as described in §4. From (6·7) the solution can be represented within reduced Fock space $P\mathcal{H} := \sum_{(n_1,n_2)\in \boldsymbol{Z}^2_{\geq 0}} \boldsymbol{C}(P\,|n_1, n_2\rangle)$. Therefore, the solution is realized by $N \times N$ matrices with $N = (\dim\mathcal{H} - k)$, where k is an instanton number. Note that in (6·7), the instanton and geometry (D3-brane) are combined into a single solution. Indeed, we can rewrite (6·7) into a simpler form:

$$\begin{aligned} X_\mu &= P(i\hat{\partial}_\mu + iA_\mu)P \\ &= P(i\hat{\partial}_\mu)P + P(i\Psi^\dagger\hat{\partial}_\mu\Psi)P - P(i\Psi^\dagger\Psi P\hat{\partial}_\mu)P \\ &= iP\Psi^\dagger\hat{\partial}_\mu\Psi P = i\Psi^\dagger\hat{\partial}_\mu\Psi. \end{aligned} \tag{6·9}$$

From (6·7) we obtain

$$[X_\mu, X_\nu] = P(-iB_{\mu\nu} - F^-_{\mu\nu\,\mathrm{ADHM}})P. \tag{6·10}$$

The derivation is similar to (3·12)–(3·16), and $F^-_{\mu\nu\,\mathrm{ADHM}}$ is anti-self-dual. From (6·10) it is easy to check that X_μ in (6·9) solves the equation of motion (6·3).

[*)] (6·4) is not satisfied in the $U(N)$ IIB matrix model with finite N.

Let us consider the supersymmetry transformation in this background:

$$\begin{aligned}\delta^{(1)}\Theta &= \frac{i}{2}[X_\mu, X_\nu]\Gamma^{\mu\nu}\epsilon^{(1)} \\ &= \frac{i}{2}P\left(-iB_{\mu\nu} - F^{-}_{\mu\nu\mathrm{ADHM}}\frac{1+\Gamma_5}{2}\right)P\Gamma^{\mu\nu}\epsilon^{(1)}, \\ \delta^{(2)}\Theta &= \epsilon^{(2)}. \end{aligned} \qquad (6\cdot11)$$

From (6·11) we can see that the solution (6·7) preserves one fourth of the supersymmetry: [4)]

$$\begin{aligned}\Gamma_5\epsilon^{(1)} &= -\epsilon^{(1)}, \\ \epsilon^{(2)} &= -\frac{1}{2}PB_{\mu\nu}P\Gamma^{\mu\nu}\epsilon^{(1)}. \end{aligned} \qquad (6\cdot12)$$

Note that the projection operator is an identity operator in the reduced Fock space $P\mathcal{H}$. Hence, the second supersymmetry transformation is proportional to the identity matrix in the $U(N)$ IIB matrix model, with $N = \dim\mathcal{H} - k$.

The physical interpretation of the projection in this setting is as follows. The matrix $B_{\mu\nu}$ in (6·4) is interpreted as a NS-NS B-field in the D3-brane worldvolume. [4)] We have set $B_{\mu\nu}$ *self-dual*. Since the self-dual B-field in the D3-brane induces negative D-instanton charge,[*)] we can regard that the D3-brane is made of infinitely many constituent anti-D-instantons. Now let us consider D-instantons within this infinite number of anti-D-instantons. In order for this configuration to become BPS, it is necessary to change the configurations of constituent anti-D-instantons. The projection removes anti-D-instantons at the positions of D-instantons and creates holes in the D3-brane worldvolume.

We can express the holes created by the projection by rewriting the above operator formulas using ordinary functions and the star-product. More precisely, we map operators to normal symbols (see Appendix C). [**)] For example, consider the projection corresponding to the ideal $\mathcal{I}$ generated by $(z_1 - w_1^i, z_2 - w_2^i)\,(i = 1, \cdots, k)$. Then, all normal symbols corresponding to operators acting in the reduced Fock space $\mathrm{End}\,P\mathcal{H}$ vanish at $(z_1, z_2) = (w_1^i, w_2^i)\,(i = 1, \cdots, k)$. This is equivalent to the assertion that the points $(z_1, z_2) = (w_1^i, w_2^i)\,(i = 1, \cdots, k)$ do not exist, or appear as holes.

Using operator symbols, one can show that the projection removes k units of the anti-D-instanton charge. Let us calculate the (anti-)instanton number in the case

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*) Our convention is: D-instanton $\sim$ instanton $\sim$ *anti*-self-dual.

**) Here we use normal symbols only to give concrete expressions of holes in $\boldsymbol{R}^4 \approx \boldsymbol{C}^2$. It may be interesting to formulate field theory on noncommutative $\boldsymbol{R}^4$ using normal symbols. It may also be interesting to investigate the relation to superstring theory. These are, however, beyond the scope of this paper.

that there are no D-instantons. Using (C·4), we have

$$\frac{1}{16\pi^2}\int d^4x B_{\mu\nu}\tilde{B}^{\mu\nu} = \frac{1}{16\pi^2}\left(2\pi\frac{\zeta}{4}\right)^2 \mathrm{Tr}_{\mathcal{H}}\, 4\left(\frac{4}{\zeta}\right)^2 = \mathrm{Tr}_{\mathcal{H}},$$
$$\tilde{B}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}B_{\rho\sigma}. \tag{6·13}$$

Since the projection reduces the dimension of the Fock space by k, it removes k units of anti-D-instanton charge. (Of course there are also contributions from the anti-self-dual part. Here we only mention the role of the projection.) This result also supports the idea that the projection removes anti-D-instantons.

§7. Conclusions and speculations

In this paper we have found that the appearance of projection operators is a general phenomenon in the ADHM construction on noncommutative $\boldsymbol{R}^4$. We have determined how to treat these projections. The existence of the projection operator forces us to consider gauge fields on reduced Fock space. Since noncommutative $\boldsymbol{R}^4$ is defined by an algebra over the entire Fock space, this projection implies the change of the spacetime topology from (noncommutative) $\boldsymbol{R}^4$. In order to describe such a change of the spacetime topology, it seems appropriate to consider a theory which can describe both the gauge theory and the geometry. Therefore we have embedded the instanton solution in the IIB matrix model. In the IIB matrix model, the instanton and the geometry are combined into a single classical solution.

In Ref. 13) it is conjectured that the $U(1)$ instanton on noncommutative $\boldsymbol{R}^4 \approx \boldsymbol{C}^2$ can be transformed into a $U(1)$ instanton on the commutative Kähler manifold which is a blowup of $\boldsymbol{C}^2$, via the field redefinition described in Ref. 15). The ideal used to describe the projection in this paper is essentially the same as that used to describe the blowup in Ref. 13). Since both instantons are constructed from the same ADHM data, the correspondence is of course one-to-one. It is interesting to understood the correspondence as a field redefinition along the lines of Ref. 15).

In §6 we have embedded instanton solutions into the IIB matrix model. Instantons on noncommutative $\boldsymbol{R}^4$ represent D-instantons within the D3-brane worldvolume. We interpret the D3-brane as bound states of infinitely many anti-D-instantons. Then the bound states of D-instantons and the D3-brane can be interpreted as bound states of D-instantons and anti-D-instantons. As shown in (6·11) and (6·12), this co-existence of positive and negative D-instanton charges preserves one fourth of the supersymmetry. However, anti-D-instantons are removed at the positions of D-instantons. This fact strongly suggests a relation to brane-anti-brane pair annihilation. [27] The IIB matrix model describes the above D-instanton-D3-brane bound states simply as its classical solution. This fact indicates the power of the IIB matrix model in the description of the fate of brane-anti-brane unstable systems. It is also straightforward to embed the noncommutative instanton solution into the

BFSS matrix model. It is interesting to study the instanton solution in the IIB matrix model or the BFSS matrix model from the point of view of brane-anti-brane pair annihilation.[28] In order to classify the topological charges which will be preserved during pair annihilations, investigations from K-theoretical viewpoints may be important.[29),30]

From the above considerations, the D3-brane may be regarded as a kind of "Dirac sea" for D-instantons. This gives new viewpoints for the second quantization of branes.[20),21]

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Appendix A
—— *The Absence of a Zero-Mode of* $\Box_z$ ——

In this appendix we show that $\Box_z$ in (3·4) has no zero-mode. Suppose

$$\Box_z |v\rangle = 0 \tag{A·1}$$

for some $|v\rangle$, where $|v\rangle \in \mathcal{H}^{\oplus k}$ (i.e. $|v\rangle$ is a vector in $V = \boldsymbol{C}^k$ and a vector in $\mathcal{H}$). Then,

$$\begin{aligned}
&\langle v| \Box_z |v\rangle = 0 \\
\Rightarrow\ &\langle v| \tau_z \tau_z^\dagger |v\rangle \\
&= \langle v| (B_1 - z_1)(B_1^\dagger - \bar{z}_1) |v\rangle + \langle v| (B_2 - z_2)(B_2^\dagger - \bar{z}_2) |v\rangle + \langle v| II^\dagger |v\rangle = 0, \\
&\langle v| \sigma_z^\dagger \sigma_z |v\rangle \\
&= \langle v| (B_1^\dagger - \bar{z}_1)(B_1 - z_1) |v\rangle + \langle v| (B_2^\dagger - \bar{z}_2)(B_2 - z_2) |v\rangle + \langle v| J^\dagger J |v\rangle = 0.
\end{aligned} \tag{A·2}$$

Since the norm of vectors in V is non-negative, we have

$$\begin{aligned}
&(B_1^\dagger - \bar{z}_1) |v\rangle = 0, \quad (B_2^\dagger - \bar{z}_2) |v\rangle = 0, \quad I^\dagger |v\rangle = 0, \\
&(B_1 - z_1) |v\rangle = 0, \quad (B_2 - z_2) |v\rangle = 0, \quad J |v\rangle = 0.
\end{aligned} \tag{A·3}$$

From (A·3), we obtain

$$\begin{aligned}
\langle v| \zeta |v\rangle &= \langle v| [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J |v\rangle \\
&= \langle v| [z_1, \bar{z}_1] + [z_2, \bar{z}_2] |v\rangle \\
&= - \langle v| \zeta |v\rangle .
\end{aligned} \tag{A·4}$$

This implies that $|v\rangle = 0$.

Appendix B

—— *The Uniqueness of the Normalized Minimal Operator Zero-Mode* ——

In this appendix we demonstrate the uniqueness of the normalized minimal operator zero-mode (up to a gauge transformation) when the gauge group is $U(1)$. Let us consider the operator zero-mode with the following form:

$$\Psi_0 = \sum_{i,j} (\Psi_0)_{ij} \left|\mathcal{U}(f_i)\right\rangle \left\langle f_j\right| . \tag{B·1}$$

Then its norm is

$$\Psi_0^\dagger \Psi_0 = \sum (\Psi_0^\dagger)_{ik} (\Psi_0)_{lj} \left|f_i\right\rangle \left\langle \mathcal{U}(f_k) | \mathcal{U}(f_l)\right\rangle \left\langle f_j\right| , \tag{B·2}$$

where

$$\left\langle \mathcal{U}(f_k) | \mathcal{U}(f_l)\right\rangle = \left\langle u_1(f_k) | u_1(f_l)\right\rangle + \left\langle u_2(f_k) | u_2(f_l)\right\rangle + \left\langle f_k | f_l\right\rangle . \tag{B·3}$$

Let us rewrite the equation $\mathcal{D}_z \left|\mathcal{U}(f_i)\right\rangle = 0$ as

$$\boldsymbol{D}\, \boldsymbol{u}(\boldsymbol{f}_i) = -\boldsymbol{f}_i, \tag{B·4}$$

where

$$\boldsymbol{D} = \begin{pmatrix} B_2 - z_2 & B_1 - z_1 \\ -(B_1^\dagger - \bar{z}_1) & B_2^\dagger - \bar{z}_2 \end{pmatrix}, \quad \boldsymbol{u}(\boldsymbol{f}_i) = \begin{pmatrix} \left|u_1(f_i)\right\rangle \\ \left|u_2(f_i)\right\rangle \end{pmatrix}, \quad \boldsymbol{f}_i = \begin{pmatrix} \left|f_i\right\rangle I \\ 0 \end{pmatrix}. \tag{B·5}$$

Since the correspondence between the elements of the ideal and vector zero-modes is one-to-one, we can consider the inverse operator of $\boldsymbol{D}$:

$$\boldsymbol{u}(\boldsymbol{f}_i) = -\frac{1}{\boldsymbol{D}} \boldsymbol{f}_i. \tag{B·6}$$

Then, (B·3) can be written as

$$\begin{aligned} &\left\langle \mathcal{U}(f_k) | \mathcal{U}(f_l)\right\rangle \\ &= \boldsymbol{u}^\dagger(\boldsymbol{f}_k)\, \boldsymbol{u}(\boldsymbol{f}_l) + \boldsymbol{f}_k^\dagger \boldsymbol{f}_l = \boldsymbol{f}_k^\dagger \left(\frac{1}{\boldsymbol{D}\boldsymbol{D}^\dagger} + 1 \right) \boldsymbol{f}_l \\ &= \left\langle f_k\right| I^\dagger \left(\left(\frac{1}{\boldsymbol{D}\boldsymbol{D}^\dagger} \right)_{1\dot{1}} + 1 \right) I \left|f_l\right\rangle , \end{aligned} \tag{B·7}$$

where we denote the components of $(\boldsymbol{D}\boldsymbol{D}^\dagger)^{-1}$ as

$$(\boldsymbol{D}\boldsymbol{D}^\dagger)^{-1} = \begin{pmatrix} (\boldsymbol{D}\boldsymbol{D}^\dagger)^{-1}_{1\dot{1}} & (\boldsymbol{D}\boldsymbol{D}^\dagger)^{-1}_{1\dot{2}} \\ (\boldsymbol{D}\boldsymbol{D}^\dagger)^{-1}_{2\dot{1}} & (\boldsymbol{D}\boldsymbol{D}^\dagger)^{-1}_{2\dot{2}} \end{pmatrix}. \tag{B·8}$$

From (B·7), the matrix $C_{kl} = \left\langle \mathcal{U}(f_k) | \mathcal{U}(f_l)\right\rangle$ has no zero-eigenvalue vector, and we can consider $(C^{-1})_{kl}$. The normalized minimal operator zero-mode is uniquely determined (up to a gauge transformation):

$$(\Psi_0)_{ij} = (C^{-1/2})_{ij}. \tag{B·9}$$

Appendix C
—— Calculations Using the Method of Operator Symbols ——

We can represent the equations over the algebra $\mathcal{A}_\zeta$ by mapping operators to ordinary c-number functions (operator symbols) and using the star product. Some calculations become simpler when we use operator symbols. The map from operators to ordinary functions depends on operator ordering procedures. In order to express holes in the D3-brane (see §6), we utilize a normal symbol that corresponds to the normal ordering. Here we review the properties of this normal symbol. (For more detailed discussion on the operator symbols, see for example[33] and references therein.) In this appendix, we use the symbol $\hat{}$ to denote operators: the $\hat{x}^\mu$ are noncommutative operators and the x^μ are c-number coordinates of $\boldsymbol{R}^4$.

Let us consider a normal-ordered operator of the form

$$\hat{f}(\hat{x}) = \int \frac{d^4k}{(2\pi)^4}\,\tilde{f}(k) : e^{ik\hat{x}} : , \tag{C·1}$$

where $k\hat{x} := k_\mu \hat{x}^\mu$. Here $:\mathcal{O}:$ denotes the normal ordering of the operator $\mathcal{O}$. For the operator valued function (C·1), the corresponding normal symbol is defined by

$$f_N(x) = \int \frac{d^4k}{(2\pi)^4}\,\tilde{f}(k)\; e^{ikx} , \tag{C·2}$$

where the x^μ are commuting coordinates of $\boldsymbol{R}^4$. We define Ω_N as a map from operators to the normal symbols:

$$\Omega_N(\hat{f}(\hat{x})) = f_N(x) := \int \frac{d^4k}{(2\pi)^4}\left(\left(2\pi\frac{\zeta}{4}\right)^2 \mathrm{Tr}_{\mathcal{H}}\left\{\hat{f}(\hat{x})\;\; : e^{-ik\hat{x}} :\right\}\right) e^{ikx}. \tag{C·3}$$

Note that from the relation $\mathrm{Tr}_{\mathcal{H}}\Big\{: \exp(ik\hat{x}) :\Big\} = \left(2\pi\frac{4}{\zeta}\right)^2 \delta^{(4)}(k)$, it follows that

$$\left(2\pi\frac{\zeta}{4}\right)^2 \mathrm{Tr}_{\mathcal{H}}\,\hat{f}(\hat{x}) = \int d^4x\, f_N(x). \tag{C·4}$$

The inverse map of Ω_N is given by

$$\Omega_N^{-1}(f(x)) = \hat{f}^N(\hat{x}) := \int \frac{d^4k}{(2\pi)^4}\left(\int d^4x f(x)\; e^{-ikx}\right) : e^{ik\hat{x}} : \,. \tag{C·5}$$

The star product of functions is defined by

$$f(x) \star_{\Omega_N} g(x) := \Omega_N(\Omega_N^{-1}(f(x))\Omega_N^{-1}(g(x)))\,. \tag{C·6}$$

Since

$$: e^{ik\hat{x}} :: e^{ik\hat{x}} := e^{\bar{w}\hat{z}} e^{w\hat{\bar{z}}} e^{\bar{w}'\hat{z}} e^{w'\hat{\bar{z}}} = e^{\frac{\zeta}{2}w\bar{w}'} e^{(\bar{w}+\bar{w}')\hat{z}} e^{(w+w')\hat{\bar{z}}}, \tag{C·7}$$

where

$$w_1 = -\frac{i}{2}(k_2 + ik_1), \quad w_2 = -\frac{i}{2}(k_4 + ik_3),$$
$$\bar{w}\hat{z} = \bar{w}_1\hat{z}_1 + \bar{w}_2\hat{z}_2, \quad \text{etc.,} \tag{C·8}$$

the explicit form of the star product is given by

$$f(z,\bar{z}) \star_{\Omega_N} g(z,\bar{z}) = e^{\frac{\zeta}{2}\frac{\partial}{\partial\bar{z}}\frac{\partial}{\partial z'}} f(z,\bar{z})g(z',\bar{z}')\Big|_{z'=z,\bar{z}'=\bar{z}}. \tag{C·9}$$

From the definition (C·6), it is seen that the star product is associative:

$$(f(x) \star_{\Omega_N} g(x)) \star_{\Omega_N} h(x) = f(x) \star_{\Omega_N} (g(x) \star_{\Omega_N} h(x)). \tag{C·10}$$

If we use coherent states, the expression of the normal symbol becomes simpler. The coherent states $|\bar{z}_1, \bar{z}_2\rangle$ are eigenstates of the annihilation operators $\hat{\bar{z}}_1, \hat{\bar{z}}_2$:

$$\hat{\bar{z}}_1 |\bar{z}_1, \bar{z}_2\rangle = \bar{z}_1 |\bar{z}_1, \bar{z}_2\rangle ,$$
$$\hat{\bar{z}}_2 |\bar{z}_1, \bar{z}_2\rangle = \bar{z}_2 |\bar{z}_1, \bar{z}_2\rangle . \tag{C·11}$$

Then the normal symbol of the operator $\hat{f}$ is given by

$$f_N(z,\bar{z}) = \langle \bar{z}_1, \bar{z}_2| \hat{f} |\bar{z}_1, \bar{z}_2\rangle . \tag{C·12}$$

(C·12) follows from (C·1), (C·2) and the relation

$$\langle \bar{z}_1, \bar{z}_2| : e^{ik\hat{x}} : |\bar{z}_1, \bar{z}_2\rangle = \langle \bar{z}_1, \bar{z}_2| e^{\bar{w}\hat{z}} e^{w\hat{\bar{z}}} |\bar{z}_1, \bar{z}_2\rangle = e^{\bar{w}z} e^{w\bar{z}}$$
$$= e^{ikx}. \tag{C·13}$$

Here we have normalized the coherent states as $\langle \bar{z}_1, \bar{z}_2 |\bar{z}_1, \bar{z}_2\rangle = 1$. From (C·12) it is easy to see that the normal symbol $f_N(z,\bar{z})$ vanishes at (z_1, z_2) when the corresponding operator $\hat{f}$ annihilates $|\bar{z}_1, \bar{z}_2\rangle$ or $\langle \bar{z}_1, \bar{z}_2|$; i.e.

$$\hat{f} |\bar{z}_1, \bar{z}_2\rangle = 0 \ \text{ or } \ \langle \bar{z}_1, \bar{z}_2| \hat{f} = 0 \quad \Longrightarrow \quad f_N(z,\bar{z}) = 0. \tag{C·14}$$

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