Integer-valued branching processes with immigration

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Abstract.

The notion of self-decomposability for $\mathbb{N}_0$-valued rv's as introduced by Steutel and van Harn [8] and its generalization by van Harn, Steutel and Vervaat [4], are used to study the limiting behaviour of continuous-time branching processes with immigration. This behaviour provides analogues to the behaviour of sequences of rv's obeying a certain difference equation as studied by Vervaat [10] and their continuous-time counterpart considered by Wolfe [11]. Furthermore, discrete-state analogues are given for results on stability in the processes studied by Wolfe, and for results on self-decomposability in supercritical branching processes by Yamazato [12].

Key words: branching process with immigration, stochastic difference equation, stochastic differential equation, self-decomposable (class L), stable.

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1. Introduction.

Recently, Vervaat [10] considered the following stochastic difference equation:

(1.1) $$X_n - A_n X_{n-1} + B_n \qquad (n \in \mathbb{N}),$$

where the $(A_n, B_n) \stackrel{d}{=} (A,B)$ are independent and independent of X_0. Iteration of the special case

(1.2) $$X_n = \rho X_{n-1} + B_n ,$$

with $\rho \in [0,1)$ a constant, yields

(1.3) $$X_n = \rho^k X_{n-k} + B_n^{(k)} \qquad (k = 1,2,\ldots,n),$$

with $B_n^{(k)} := \Sigma_{j=0}^{k-1} \rho^j B_{n-j} \stackrel{d}{=} \Sigma_{j=1}^{k-1} \rho^{j-1} B_j$ independent of X_{n-k}. So equation (1.2) is solved by

(1.4) $$X_n = \rho^n X_0 + \sum_{k=1}^{n} \rho^{n-k} B_k \stackrel{d}{=} \rho^n X_0 + \sum_{k=1}^{n} \rho^{k-1} B_k .$$

Under the condition (cf. [10]) that $E \log (1 + |B|) < \infty$ there is a limit X_∞ satisfying

(1.5) $$X_\infty = \sum_{k=1}^{\infty} \rho^{k-1} B_k \stackrel{d}{=} \rho X_\infty + B ,$$

or, more generally from (1.3):

(1.6) $$X_\infty \stackrel{d}{=} \rho^k X_\infty + B^{(k)}$$

with X and $B^{(k)}$ independent, or

(1.7) $$X_\infty \stackrel{d}{=} c X_\infty + X_c \qquad (c = \rho^k,\ k \in \mathbb{N}),$$

with X_∞ and X_c independent, i.e. X_∞ is "incompletely self-decomposable" (see e.g. Urbanik [9]; X_∞ is called (completely) self-decomposable if (1.7) holds for all $c \in (0,1)$).

Wolfe [11] considers the continuous-time analogue of (1.2), formally described by the stochastic differential equation

(1.8) $$dX(t) = -\delta X(t)dt + dB(t),$$

with δ a positive constant and B(t) a Lévy process. In analogy to (1.3) and (1.4) one has (all integrals exist in the sense of convergence in probability, and pathwise in the sense of formal integration by parts (cf. Jurek and Vervaat [6])).

(1.9) $$X(t) = e^{-\delta s} X(t-s) + B^{(s)}(t) \qquad (s \in (0,t]),$$

with $B^{(s)}(t) = {}_{t-s}\int^{t} \exp\{-\delta(t-u)\}dB(u) \stackrel{d}{=} {}_0\int^{s} \exp(-\delta u)dB(u)$, and specially

(1.10) $$X(t) = e^{-\delta t} X(0) + \int_0^t e^{-\delta(t-u)}\, dB(u) \stackrel{d}{=} e^{-\delta t} X(0) + \int_0^t e^{-\delta u}\, dB(u).$$

If X(t) has a limit in distribution $X(\infty)$, then analogous to (1.6) we have

(1.11) $$X(\infty) \stackrel{d}{=} e^{-\delta s} X(\infty) + B^{(s)}$$

with $X(\infty)$ and $B^{(s)}$ independent, i.e., contrary to the discrete-time case, $X(\infty)$ is (completely) self-decomposable. This is one of the results in the following theorem of Wolfe [11].

Theorem 1.1. Let X(t) be as in (1.10). Then

(i) There is a random variable $X(\infty)$ such that $X(t) \xrightarrow{d} X(\infty)$ if and only if $E \log (1 + |B(1)|) < \infty$.

(ii) The distribution of $X(\infty)$ is self-decomposable (class L), and hence is infinitely divisible and unimodal.

(iii) If a random variable X has a self-decomposable distribution then X is the weak limit a of process X(t) as in (1.10).

In this paper we consider integer-valued analogues of X(t) in connection with recent results on decomposability and stability for distributions on $\mathbb{N}_0$ as given in [4] and [8]. The discrete-time analogue, i.e. the $\mathbb{N}_0$-valued analogue of X_n in (1.2) is less interesting as it lacks the complete self-decomposability (compare (1.7)).

In Section 2 we give a brief review of results on discrete self-decomposable distributions; these are then used to prove analogues of Theorem 1.1 in Section 3. Section 4 contains an application of Theorem 1.1 on a special case of the stochastic difference equation (1.1). In Section 5 we give the analogues of a result by Wolfe [11] on stable distributions, and in Section 6 some extensions and analogues of limit theorems by Yamazato [11] for supercritical branching processes.

2. Self-decomposability and stability on $\mathbb{N}_0$ and branching processes.

We need some of the ideas and results from [8] and [4] for the analogues on $\mathbb{N}_0$ of (1.10) and Theorem 1.1. Here and elsewhere P_Y will denote the probability generating function (p g f) of the $\mathbb{N}_0$-valued random variable (rv) Y, and $(F_t)_{t\geq 0}$ will denote a composition semigroup of p g f's with the property $F_{s+t}(z) = F_s(F_t(z))$ $(s,t \geq 0)$, or

(2.1) $$F_{s+t} = F_s \circ F_t ,$$

and furthermore

(2.2) $$\lim_{t \downarrow 0} F_t(z) = z \quad , \quad \lim_{t \to \infty} F_t(z) = 1 \ .$$

Semigroups of p g f's of this kind are, of course, familiar in (sub-)critical branching processes (see e.g. [1] and [5]).

To stress the analogy to continuous-state versions of our results, and to shorten notations, we introduce an integer-valued analogue to scalar multiplication (see [4] and [8] for details).

Definition 1.1. Let $(F_t)_{t>0}$ be a fixed semigroup of p g f's as in (2.1), (2.2), and let X be an $\mathbb{N}_0$-valued random variable. Then for $0 < \rho \leq 1$ the $\mathbb{N}_0$-valued multiple $\rho \circledcirc X$ is defined (in distribution) by its p g f as follows

(2.3) $$P_{\rho \circledcirc X} = P_X \circ F_{-\log \rho} \ .$$

One easily verifies that, quite analogous to scalar multiplication, the operation $\circledcirc$ has the following properties:

(2.4)
$$\rho_1 \circledcirc (\rho_2 \circledcirc X) \stackrel{d}{=} \rho_1 \rho_2 \circledcirc X$$
$$\rho \circledcirc (X + Y) \stackrel{d}{=} \rho \circledcirc X + \rho \circledcirc Y \qquad \text{(X and Y independent)}$$
$$X_n \stackrel{d}{\to} X \Rightarrow \rho \circledcirc X_n \stackrel{d}{\to} \rho \circledcirc X$$
$$\rho \circledcirc X \stackrel{d}{\to} 0 \quad \text{as} \quad \rho \downarrow 0 \ .$$

For other properties we refer to [4], where it is shown that (2.3) provides all possible multiplications that satisfy (2.4) plus a linearity condition for the p.g.f's.

We now define self-decomposability and stability with respect to $\odot$. As the operation $\odot$ depends on the specific semigroup $F = (F_t)_{t \geq 0}$ under consideration we use the terms F-self-decomposable and F-stable.

Definition 2.2. An $\mathbb{N}_0$-valued rv X is called F-self-decomposable if

$$(2.5) \qquad X \overset{d}{=} \rho \odot X + X_\rho \qquad (X \text{ and } X_\rho \text{ independent; all } \rho \in (0,1));$$

X is called F-stable with exponent $\alpha \in (0,1]$ if more specially

$$(2.6) \qquad X \overset{d}{=} \rho \odot X + (1-\rho^\alpha)^{1/\alpha} \odot X' \quad (X \text{ and } X' \overset{d}{=} X \text{ independent; } \rho \in (0,1)).$$

Remark. Equivalently, (2.5) and (2.6) can be written in terms of (F_t) as follows $(t = -\log\rho,\ P = P_X)$

$$(2.5') \qquad P = (P \circ F_t) P_t \qquad (P_t \text{ a pgf; } t > 0)$$

$$(2.6') \qquad P = (P \circ F_t)(P \circ F_s) \qquad (s,t > 0;\ e^{-\alpha s} + e^{-\alpha t} = 1)$$

We shall need a number of results from [4].

Theorem 2.3. An $\mathbb{N}_0$-valued rv X is F-self-decomposable if and only if its pgf P satisfies

$$(2.7) \qquad P(z) = \exp\left[-\lambda \int_z^1 \frac{1-Q(x)}{U(x)}\, dx \right],$$

where $\lambda > 0$ and Q is any pgf with $Q(0) = 0$; X is F-stable with exponent α if and only if

$$(2.8) \qquad P(z) = \exp[-\lambda\{A(z)\}^\alpha].$$

Here U and A satisfy: $U(z) = \lim_{t\downarrow 0} (F_t(z) - z)/t$ and

$$\frac{\partial}{\partial t} F_t(z) = U(F_t(z)) = U(z)\frac{\partial}{\partial z} F_t(z) , \tag{2.9}$$

$$1/U(z) = -A'(z)/A(z) . \tag{2.10}$$

The next theorem is a slight modification of Theorem 8.4 in [4]; we take $F_1'(1) = e^{-\delta}$ with an arbitrary $\delta > 0$ rather than $\delta = 1$.

<u>Theorem 2.4</u>. Let (F_t) be a semigroup of p g f's as in (2.1) with $F_1'(1) = e^{-\delta}$, and let

$$V(x) := 1 - F_{\log x}(0) \qquad (x \geq 1) . \tag{2.11}$$

For any nonnegative rv Y with Laplace transform $\psi_Y(\tau) = E\exp(-\tau Y)$ define the map $\pi = \pi^F$ (from the Laplace transforms into the p g f's) by

$$(\pi\psi_Y)(z) = \psi_Y(A^\delta(z)) , \tag{2.12}$$

with A as in (2.8). Further let $(X_n)_{n\in\mathbb{N}}$ be a sequence of $\mathbb{N}_0$-valued rv's. Then there exist $c_n \to \infty$ and a rv $\tilde{X}$ such that

$$c_n^{-1} \circledast X_n \xrightarrow{d} \tilde{X} \qquad (n \to \infty)$$

if and only if there exist $a_n \to \infty$ and a rv X such that $a_n^{-1}X_n \xrightarrow{d} X \quad (n \to \infty)$. In this case

$$a_n V((c_n)^{1/\delta}) \to \theta \quad (n \to \infty) \tag{2.13}$$

for some $\theta > 0$, and

$$P_{\tilde{X}}(z) = (\pi\psi_X)(z) . \tag{2.14}$$

Finally, we need

Theorem 2.5. If ψ_X is a self-decomposable Laplace transform, then $\pi\psi_X$ is an F-self-decomposable pgf.

We shall use the notation (cf. Example 6.6 in [4]):

(2.15) $$\Pi = \{P : P = \pi\psi_X \text{ with } \psi_X \text{ self-decomposable}\} .$$

3. A limit theorem for branching processes with immigration.

Of the four (discrete/continuous time/space) possible variants of (1.2) the discrete-time, discrete-space variant:

(3.1) $$X_n = \rho \circledast X_{n-1} + B_n ,$$

with $\mathbb{N}_0$-valued B has properties similar to (1.2), and is not very interesting from our point of view. We shall concentrate on the $\mathbb{N}_0$-valued analogues of (1.8) and (1.10), and we write (taking X(0) = 0 without essential restriction)

(3.2) $$X(t) \overset{d}{=} \int_0^t e^{-\delta(t-u)} \circledast dB(u) \overset{d}{=} \int_0^t e^{-\delta u} \circledast dB(u) ,$$

where B(u) now is a compound Poisson process:

(3.3) $$B(u) = \sum_{0<T_k\le u} C_k ,$$

with C_k iid and $\mathbb{N}_0$-valued and independent of the Poisson process generated by (T_k). Now X(t) can be written explicitly as

(3.4) $$X(t) \overset{d}{=} \sum_{0<T_k\le t} e^{-\delta(t-T_k)} \circledast C_k ,$$

where an expression of the form $A \circledcirc X$ with A a rv is interpreted as (see also (2.3))

$$P_{A\circledcirc X}(z) = \int_0^1 P_{a\circledcirc X}(z)\, dG_A(a) ,$$

where G_A is the distribution function of A.

We shall need the following generalized analogue of a theorem of Lukacs [7].

Lemma 3.1. Let $B(u)$ be a compound Poisson process as in (3.3) with intensity λ, and let h be a continuous function on $[a,b] \subset [0,\infty)$ with $0 < h(u) \le 1$. Let X be defined by (cf. (3.2) and (3.4))

$$X = \int_a^b h(u) \circledcirc dB(u) = \sum_{a<T_k\le b} h(T_k) \circledcirc C_k .$$

Then the p g f of X equals

$$\begin{aligned} P_X(z) &= \exp\left\{ \int_a^b \log P_{B(1)}\, (F_{-\log h(u)}(z))du \right\} \\ &= \exp\left\{ -\lambda \int_a^b (1 - P_C(F_{-\log h(u)}(z))du \right\}. \end{aligned} \tag{3.5}$$

Proof. Equality of the last two expressions is obvious. To prove that $P_X(z)$ is equal to the latter of these, we proceed as indicated on p. 118 in [5]. Conditioning on the number of T_k with $a < T_k \le b$ we obtain using (2.3)

$$P_X(z) = \sum_{n=0}^{\infty} e^{-\lambda(b-a)} \frac{\lambda^n (b-a)^n}{n!}\, E \prod_{k=1}^{n} P_C(F_{-\log h(U_k)}(z)) ,$$

where the U_k are well known to be distributed as the order statistics of n independent uniform random variables on (a,b). It follows that

$$E \prod_{k=1}^{n} P_C(F_{-\log h(U_k)}(z)) = \left\{\frac{1}{b-a}\int_a^b P_C(F_{-\log h(u)}(z))\,du\right\}^n,$$

from which (3.5) is immediate. □

We now apply (3.5) to X(t) as defined by (3.2) and (3.4), i.e. with (a,b] = (0,t] and h(u) = exp(-δu). We obtain

$$(3.6)\qquad P_{X(t)}(z) = \exp\left[-\lambda\int_0^t \{1 - P_C(F_{\delta u}(z))\}du\right],$$

and comparing (3.6) with (16.3) in [5] one recognizes $P_{X(t)}$, as the p g f of the number of individuals present at time t in (sub-)critical continuous-time branching process with batch immigration, and batch size p g f P_C.

We now formulate the analogue of Theorem 1.1.

Theorem 3.2. Let X(t) be a (sub-)critical branching process with immigration as given by (3.2) and (3.4). Then

(i) There is a rv X(∞) such that $X(t) \xrightarrow{d} X(\infty)$ if and only if

$$(3.7)\qquad \int_0^1 (1 - P_C(x))/U(x)\,dx < \infty,$$

with U defined by (2.9).

(ii) The distribution of X(∞) is F-self-decomposable and hence infinitely divisible.

(iii) If a rv X has an F-self-decomposable distribution, then X is the weak limit of a branching process with immigration as given by (3.4).

Proof. From (3.6) and (2.9) we deduce, using (2.2),

$$P_X(t) = \exp\left[-\lambda/\delta \int_z^{F_{\delta t}(z)} (1 - P_C(x))/U(x)\,dx \right] \to$$

$$\to \exp\left[\lambda/\delta \int_z^{1} (1 - P_C(x))/U(x)\,dx \right] \qquad (t \to \infty) ,$$

and (i) and (ii) follow from Theorem 2.3. The converse (iii) is obtained by taking the p g f of C_k in (3.4) equal to Q in (2.7). □

Remark 1. The closest analogue to Theorem 1.1 is obtained by taking

$$F_t(z) = 1 - e^{-\delta t} + e^{-\delta t} z ,$$

the special case discussed in [8]. Equation (3.1) can now be written as

$$X_n = I_1 + \dots + I_{X_{n-1}} + B_n ,$$

where the I_j are independent with $P(I_j = 1) = 1 - P(I_j = 0) = e^{-\delta}$. This representation provides a discrete state-space analogue to (1.8).

For this special F, the F-self-decomposable distributions are *unimodal*; this can be proved in close analogy to Wolfe's proof for distributions on [0,∞) (see [8] for details). The function U(x) now simplifies to $\delta(1 - x)$, and X(t) is a pure death process with immigration, which can be interpreted as the number of customers in an M/M/∞ queue with batch arrivals of size C. It follows that the stationary distribution of this number is unimodal.

Remark 2. If $X(t)$ is subcritical, then the condition (3.7) is equivalent to $E \log (1 + C) < \infty$.

Remark 3. Theorem 1.1 together with the concepts of self-decomposability and stability for non-lattice rv's could be generalized in a similar way; this would require detailed results on continuous-time branching processes with continuous state space.

4. Embedded discrete-time processes.

In this section we use Theorem 1.1 to give a probabilistic proof of a theorem by Vervaat [10], which he proved analytically. We then give the corresponding result for $\mathbb{N}_0$-valued variables.

Throughout this section U, U_n are uniformly distributed on $(0,1)$ and C, C_n are nonnegative with $E \log (1 + C) < \infty$; all these rv's are independent.

Theorem 4.1 [10]. Let $\delta > 0$ and let the rv X satisfy

$$X \stackrel{d}{=} U^{\delta}(X + C) , \tag{4.1}$$

where in the right-hand side U, X and C are independent, and U and C are as above. Then X is self-decomposable.

Proof. By Theorem 1.6 of [10] equation (4.1) has a unique solution. Now consider the special case of (1.10) where $B(u)$ is a compound Poisson process (T_n, C_n) as in (3.3). Then $X(t) \stackrel{d}{\to} X(\infty)$ as $t \to \infty$, and $X(\infty)$ satisfies (compare (3.4))

$$X(\infty) \stackrel{d}{=} \sum_{1}^{\infty} e^{-\delta T_n} C_n = \sum_{1}^{\infty} U_1^{\delta} \cdots U_n^{\delta} C_n \stackrel{d}{=} U_1^{\delta} (X(\infty) + C_1) ,$$

so $X \stackrel{d}{=} X(\infty)$ and therefore X is self-decomposable by Theorem 1.1. □

Completely analogously we have

Theorem 4.2. Let $\delta > 0$ and let the $\mathbb{N}_0$-valued rv X satisfy

$$X \overset{d}{=} U^{\delta} \circledcirc (X + C) , \tag{4.2}$$

with U, X and C in the right-hand side independent and U and C as above with C $\mathbb{N}_0$-valued. Further $\circledcirc$ is defined as in (2.3) and (3.4). Then X is seld-decomposable (cf. (2.5)).

Proof. As for (4.1) it can be shown that (4.2) has a unique solution X with $X \overset{d}{=} X(\infty)$ and $X(\infty)$ satisfying (cf. (3.4))

$$X(\infty) \overset{d}{=} \sum_{1}^{\infty} e^{-\delta T_n} \circledcirc C_n \overset{d}{=} U_1^{\delta} \circledcirc (X(\infty) + C_1) .$$

It now follows from Theorem 3.2 that X is F-self-decomposable. □

Remark. Another way of looking at $X(\infty)$ in Theorem 4.1 is to regard it as the limit of the embedded discrete-time process $(Y_n)^{\infty}$, with $Y_n = X(T_n)$ and $X(t)$ as in (1.9). Now take $s = T_1$ and put $C_n = B^{(T_1)}(T_n) \overset{d}{=} B^{(T_1)}$. Then Y_n satisfies

$$Y_n = U_n^{\delta} Y_{n-1} + C_n , \tag{4.3}$$

with U_n, Y_n and C_n independent. Equation (4.3) is a special case of (1.1) and we have (cf. [10]) $Y_n \to Y$ with $Y = X + C$ and $X = U^{\delta} Y$ as before. A similar remark goes for Theorem 4.2.

5. Stable distributions.

In this section we obtain the analogue for $\mathbb{N}_0$-valued processes of the following result of Wolfe [11].

Theorem 5.1. Let X(t) be as in (1.10) and let $t_0 > 0$. Then

$$X(\infty) \stackrel{d}{=} B(t_0) \tag{5.1}$$

if and only if $X(\infty)$ is strictly stable with exponent $(\delta t_0)^{-1}$.

Proof. Let $\psi = \log \varphi_{B(1)}$ with φ_B the moment generating function of B. Then by Lukacs' theorem (the analogue of Lemma 3.1) (5.1) is equivalent to

$$t_0 \, \psi(s) = \int_0^\infty (e^{-\delta u} s)\, du ,$$

and differentiation yields $t_0 \, \delta \, \psi'(s) = \psi(s)/s$, and so $\psi(s) = s^{1/(\delta t_0)}$. □

Theorem 5.2. Let X(t) be as in (3.2) and let $t_0 > 0$. Then

$$X(\infty) \stackrel{d}{=} B(t_0) \tag{5.2}$$

if and only if $X(\infty)$ is F-stable with exponent $(\delta t_0)^{-1}$.

Proof. Let $R = \log P_{B(1)}$. Then by Lemma 3.1 (5.2) is equivalent to

$$t_0 \, R(z) = \int_0^\infty R(F_{\delta u}(z))\, du ,$$

and so, on account of (2.9) and (2.10) (see also (2.2))

$$\delta t_0 \, R'(z) = \delta \int_0^\infty R'(F_{\delta u}(z)) F'_{\delta u}(z)\, du = \frac{A'(z)}{A(z)} \int_0^z R'(x)\, dx ,$$

and hence

$$R'(z)/R(z) = \frac{1}{\delta t_0} A'(z)/A(z) ,$$

or $P(z) := \exp(R(z)) = \exp(-\lambda\{A(z)\}^{1/(\delta t_0)})$ for some $\lambda > 0$. The result now follows from Theorem 2.3. □

Remark. In his paper Wolfe [11] considers the relation $X(\infty) \overset{d}{=} b\,B(t_0)$ for some $b > 0$ (not necessarily $b = 1$). This leads to a differential equation for ψ of the form

(5.3) $$s^{-1}\psi(s) = b t_0 \delta \psi'(bs) ,$$

which is satisfied by $\psi(s) = c_1 s^{\alpha_1} + c_2 s^{\alpha_2}$, with real α_j satisfying $\delta t_0 \alpha b^{\alpha} = 1$. It is by no means obvious however that these are the only solutions of (5.3), and the argument in [11] seems insufficient. The same problem occurs for a generalized version of Theorem 5.2.

6. Some analogies for supercritical branching processes.

In the present section we derive a discrete-state analogue of the following result which slightly generalizes a theorem of Yamazato [12]. For a continuous-state, continuous-time analogue see Biggins and Shanbbag [2].

Theorem 6.1. Let either $T = \mathbb{N}_0$ or $T = [0,\infty)$, and let $(X_t)_{t\in T}$ be a branching process with $P(X_0 = 1) = 1$, $P(X_1 > 0) = 1$ and $E X_1 =: m \in (1,\infty)$. Then there is a positive function c on T and a random variable W such that

(6.1) $$\lim_{t\to\infty} \frac{c(t+s)}{c(t)} = m^s \qquad \text{for } s \in T ,$$

(6.2) $$P(\lim_{t\to\infty} X_t/c(t) = W) = 1 ,$$

$P(W > 0) = 1$, and the characteristic function (ch. f) φ_W of W satisfies

(6.3) $\varphi_W/\varphi_{m^{-u}W}$ is a ch. f. for all $u \in T$.

In particular, φ_W is self-decomposable if $T = [0,\infty)$.

Proof. Statements (6.1) and (6.2) are contained in Theorem I.10.3 of [1] ($q = 0$) and its continuous-time analogue as indicated on pages 112 and 113 of [1]. In fact, it can be shown that, apart from null sets, $[W = 0] = [X_t \to 0]$, which has probability zero since $P(X_1 > 0) = 1$. Hence $P(W > 0) = 1$. Since $P(X_n \geq 1) = 1$ for all $u \in T$ we have for fixed u

$$X_{t+n} \overset{d}{=} X_t'(1) + X_t''(X_u - 1) ,$$

where $X_t'(1)$ and $X_t''(X_u - 1)$ are independent branching processes with the same offspring distribution, but with 1 and $X_u - 1$ individuals in the zeroth generation. It follows that

$$\lim_{t\to\infty} \{X_t'(1) + X_t''(X_u - 1)\}/c(t + u) = W'$$

exists with probability one, with $W' \overset{d}{=} W$. Moreover, by (6.1)

$$\lim_{t\to\infty} \frac{X_t'(1)}{c(t+u)} = \lim_{t\to\infty} \frac{X_t'(1)}{c(t)} \cdot \frac{c(t)}{c(t+u)} \overset{d}{=} W/m^u$$

exists with probability one, and consequently so does

$$\lim_{t\to\infty} \frac{X_t''(X_u - 1)}{c(t+u)} \overset{d}{=} R_u \quad , \text{say} .$$

We conclude that a rv R_u exists such that

$$W \overset{d}{=} m^{-u} W + R_u \qquad \text{(W and } R_u \text{ independent)},$$

which is equivalent to (6.3). □

Remark. If $E X_1 \log X_1 < \infty$, then we may take $c(t) = m^t$ in Theorem 6.1 (cf. [1], Theorem I.10.1).

We now formulate our discrete-state analogue.

Theorem 6.2. Let $(X_t)_{t \in T}$ be a branching process as in Theorem 4.1, and let $\odot$ be defined as in (2.3). Then there is a positive function $\tilde{c}$ on T and a random variable $\tilde{W}$ such that

$$\text{(6.4)} \qquad \lim_{t\to\infty} \frac{\tilde{c}(t+s)}{\tilde{c}(t)} = m^s \qquad \text{for } s \in T,$$

$$(1/\tilde{c}(t)) \odot X_t \overset{d}{\to} \tilde{W} \qquad (t \to \infty),$$

$P(\tilde{W} > 0) = 1$, and the p g f $P_{\tilde{W}}$ of $\tilde{W}$ satisfies

$$P_{\tilde{W}} / P_{m^{-s} \odot \tilde{W}} \quad \text{is a p g f for all } s \in T.$$

In particular, if $T = [0,\infty)$ then P_W is F-self-decomposable, and even $P_{\tilde{W}} \in \Pi$ (cf. (2.15)).

Proof. Let the function c be as in Theorem 6.1. Then $X_t/c(t) \to W$ as $t \to \infty$. We now apply Theorem 2.4 (to an arbitrary sequence $t_n \to \infty$, $t_n \in T$), and it follows that we can choose $\tilde{c}$ such that (cf. (2.11) and (2.13))

$$\text{(6.5)} \qquad \lim_{t\to\infty} c(t)\, V\left((\tilde{c}(t))^{1/\delta}\right) = 1,$$

and it follows that

$$\lim_{t\to\infty} (1/\tilde{c}(t)) \circledast X(t) \overset{d}{\to} \tilde{W}$$

with $P_{\tilde{W}} = \pi\,\psi_W$ (cf. (2.12) and (2.14)). Moreover by properties of the map π (see [3], Lemma 5.1) and by (6.3)

$$\frac{P_{\tilde{W}}(z)}{P_{\tilde{W}}(m^{-s}z)} = \frac{\pi\,\psi_W(z)}{\pi\,\psi_W(m^{-s}z)} = \pi\left(\frac{\psi_W}{\psi_W(m^{-s}\,.)}\right)(z)$$

is a pgf for all $s \in T$, in particular $P_{\tilde{W}} \in \Pi$ (cf. (2.15)) if $T = [0,\infty)$. As $P(W > 0) = 1$ by Theorem 6.1, $\psi_W(A^{\delta}(z)) \to 0$ as $z \downarrow 0$, i.e. $P(\tilde{W} > 0) = 1$. Finally, V varies regularly at ∞ with exponent $-\delta$ (cf. [4], (3.16)), so the inverse $\tilde{V}$ of V varies regularly at 0 with exponent $-\delta^{-1}$ (cf. de Haan [3], Gorollary 1.2.1.5, p.24). Consequently, by (6.1), (6.5) and [3], Corollary 1.2.1.2

$$\frac{\tilde{c}(t+s)}{\tilde{c}(t)} = \left[\frac{\tilde{V}(1/c(t+s))}{\tilde{V}(1/c(t))}\right]^{\delta} \to m^{s}$$

as $t \to \infty$, $t \in T$ for all $s \in T$, which proves (6.4). □

Remark. If $E X_1 \log X_1 < \infty$ and also $E Y_1 \log(Y_1 + 1) < \infty$, where Y_t is the (sub-critical) branching process corresponding to (F_t), we may choose $\tilde{c}(t) = m^{t}$ (cf. remark following theorem 6.1 and Remark 8.6 in [4]).

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