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Integrability and the AdS_3/CFT_2 correspondence

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Abstract: We investigate the AdS_3/CFT$_2$ correspondence for theories with 16 supercharges using the integrability approach. We construct Green-Schwarz actions for Type IIB strings on $AdS_3 \times S^3 \times M_4$ where $M_4 = T^4$ or $S^3 \times S^1$ using the coset approach. These actions are based on a $\mathbb{Z}_4$ automorphism of the super-coset $D(2,1;\alpha) \times D(2,1;\alpha)/\mathrm{SO}(1,2) \times \mathrm{SO}(3) \times \mathrm{SO}(3)$. The equations of motion admit a representation in terms of a Lax connection, showing that the system is classically integrable. We present the finite gap equations for these actions. When $\alpha = 0\,,\ 1/2\,,\ 1$ we propose a set of quantum Bethe equations valid at all values of the coupling. The AdS_3/CFT$_2$ duals contain novel massless modes whose role remains to be explored.

Keywords: AdS-CFT Correspondence, Integrable Equations in Physics, Bethe Ansatz, Sigma Models

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Contents

1 Introduction

Several AdS/CFT systems possess integrable structures and are solvable non-perturbatively by Bethe ansatz techniques. Integrability tools such as Bethe ansatz, exact S-matrices, bootstrap and finite-gap integration proved useful in finding the exact spectrum of the

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AdS_5/CFT$_4$ and AdS_4/CFT$_3$ systems at the planar/free-string level. We believe that this is not the end of the story and that other integrable AdS/CFT systems should exist. In this paper we extend the integrability approach to the AdS_3/CFT$_2$ dual pairs with sixteen supercharges. The AdS_3 backgrounds typically arise from the D1-D5 system in type IIB string theory [1] and are dual to two-dimensional conformal field theories [2].

The AdS_3/CFT$_2$ duality may appear simpler than its higher-dimensional counterparts, partly because the conformal symmetry in two dimensions is larger and more constraining, partly because problems with string quantization on Ramond-Ramond backgrounds inevitable in higher dimensions can be circumvented in $d = 3$. Some $AdS_3 \times X$ backgrounds can be supported by pure NSNS flux, which leads to enormous simplification of the worldsheet CFT. The worldsheet sigma-model then admits an NSR description, has extended chiral symmetry, and as a consequence is solvable by representation theory of chiral algebras [3–7]. On the contrary, the RR AdS_3 sigma-model [8] does not have useful holomorphicity properties and the usual CFT methods do not work. In this respect, the AdS_3 backgrounds with the RR flux are as complicated as their higher-dimensional counterparts such as $AdS_5 \times S^5$ or $AdS_4 \times CP^3$, which however are exactly solvable due to their integrability. We would like to argue that the appropriate method to attack the problem of string quantization on the RR AdS_3 backgrounds is also the Bethe ansatz. Clear evidence for integrability in AdS_3/CFT$_2$ comes from the geometric construction of the Green-Schwarz string action on $AdS_3 \times S^3$ [9–13], and from the symmetries of the giant magnons on this background [14].

Within the context of the AdS/CFT correspondence, integrability manifests itself in a number of ways. In this paper we will be mostly concerned with the string (AdS) side of the duality, where the construction is particularly simple. The classical integrability of the string sigma-model follows from the Lax representation of the equations of motion [15]. The monodromy matrix of the Lax connection then generates an infinite set of local or non-local commuting conserved charges. Given the monodromy matrix, one can use the finite-gap methods to solve the equations of motion in terms of a much simpler set of integral equations [16, 17]. These same equations arise in the semiclassical limit of the quantum Bethe ansatz [18] that diagonalizes the exact worldsheet S-matrix [19, 20]. The experience with AdS_5/CFT$_4$ [18] and AdS_4/CFT$_3$ [21, 22] systems shows that the quantum Bethe equations are very constrained by symmetries [18, 23] and can be almost uniquely reconstructed from their semiclassical counterparts [24]. It is not clear to us if the Y-system or the TBA equations [25–28] can be reconstructed from the classical data. The semiclassical limit is encoded in the Y-system, albeit in a rather non-trivial way [29].

The key property of the $AdS_5 \times S^5$ and $AdS_4 \times CP^3$ backgrounds that guarantees their integrability is the geometric construction of the Green-Schwarz action in terms of the coset superspace [30–33].[1] If the coset admits a $\mathbb{Z}_4$ grading [34], integrability follows automatically, since the equations of motion and Maurer-Cartan equations of any $\mathbb{Z}_4$ coset can be written as a flatness conditions for a Lax connection [15]. It has been known for a long time that the Green-Schwarz sigma-model on the six-dimensional $AdS_3 \times S^3$

[1] In the $AdS_4 \times CP^3$ case the coset arises after partially fixing the kappa-symmetry [31, 33].

background is the $\mathrm{PSU}(1,1|2)\times\mathrm{PSU}(1,1|2)/\mathrm{SU}(1,1)\times\mathrm{SU}(2)$ supercoset [9–11][2] which, as one can check, possesses a $\mathbb{Z}_4$ grading and is therefore integrable [12].[3]

We will consider Type IIB strings on $AdS_3\times S^3\times M_4$ where $M_4=T^4$ or[4] $S^3\times S^1$. The case of $S^3\times S^1$ in a certain sense is more general although definitely more complicated. The $AdS_3\times S^3\times S^3\times S^1$ geometry is a supergravity solution with sixteen supercharges [36–40] and can be supported by either NSNS or RR three-form flux. In the NSNS case the duality is relatively well understood, because one can use the NSR formalism and more or less standard CFT methods to quantize string in this background [4, 39–41]. On the contrary, the AdS_3/CFT_2 duality for $AdS_3\times S^3\times S^3\times S^1$ supported by the RR flux perhaps is the most obscure case among all known AdS/CFT pairs. The AdS/CFT correspondence for $AdS_3\times S^3\times S^3\times S^1$ was discussed in [39] and at length in [40]. The string in this RR background cannot be quantized by any known method. As far as the dual CFT is concerned, very little is known about it, apart from its rather intricate symmetries. The CFT is probably a resolution of a permutation orbifold [39], and displays a number of unusual features, in particular the non-linear BPS bound [39]. All this makes identification of the spectra and the moduli spaces on the two sides of the duality problematic even at the supergravity level [40].

The radii of the two three-spheres ($R_\pm$) and the AdS radius (l) are not independent in $AdS_3\times S^3\times S^3\times S^1$. The supergravity equations of motion require them to satisfy the triangle equality:

$$\frac{1}{R_+^2}+\frac{1}{R_-^2}=\frac{1}{l^2}\,. \tag{1.1}$$

The same triangle equality arises in the invariant bilinear form of the exceptional Lie superalgebra $\mathfrak{d}(2,1;\alpha)$ [42, 43], eq. (A.3), and not just by chance — the symmetry of the corresponding AdS/CFT pair is the large $N=4$ superconformal algebra [37] whose rigid part is $\mathfrak{d}(2,1;\alpha)$. Indeed, the super-isometries of the $AdS_3\times S^3\times S^3$ background form two copies of $\mathfrak{d}(2,1;\alpha)$ [38]. The parameter α, potentially any complex number [42, 43], is related to the relative size of the two spheres and takes values between 0 and 1 when a suitable reality condition is imposed on the super-algebra. In view of the triangle equality (1.1) we shall use the trigonometric parameterization:

$$\alpha=\frac{l^2}{R_+^2}\equiv\cos^2\phi,\qquad \frac{l^2}{R_-^2}\equiv\sin^2\phi. \tag{1.2}$$

There are two simplifying limits worth mentioning. One is $\phi=0$. The radii of AdS and of one of the spheres then become equal, while the other sphere blows up to an infinite size. By re-compactifying on T^3, we get the $AdS_3\times S^3\times T^4$ background. This limiting case corresponds to the $\alpha\to 1$ degeneration of the $\mathfrak{d}(2,1;\alpha)$ algebra, which up to some abelian factors contracts to $\mathfrak{psu}(1,1|2)$. The symmetry algebra of $AdS_3\times S^3$ is indeed $\mathrm{PSU}(1,1|2)\times\mathrm{PSU}(1,1|2)$. The other special point is $\phi=\pi/4$, when the two spheres have equal sizes, $\sqrt{2}$ times smaller than the radius of AdS. The exceptional superalgebra $\mathfrak{d}(2,1;\alpha)$ (with $\alpha=1/2$) then coincides with the classical $\mathfrak{osp}(4|2)$ superalgebra from the $\mathfrak{d}(n,m)$ series.

[2] An alternative construction of the GS action on $AdS_3\times S^3$ is given in [35].

[3] The pure spinor superstring in this background is also integrable [13].

[4] When $M_4=K3$ the theory also preserves 16 supersymmetries. In orbifold limits of $K3$ our results generalise in a straightforward way.

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The plan of the paper is the following. In section 2 we recall the standard classical integrable structure of a general supercoset with $\mathbb{Z}_4$ grading. We then specify the general construction to the $AdS_3 \times S^3 \times S^3 \times S^1$ case. In section 3 we show that the supercoset action in the flat space limit reproduces the flat space GS action in a particular κ-gauge. In section 4, we demonstrate that, up to terms quadratic in fermions, our supercoset sigma model is indeed a realization of worldsheet theory of GS string on the $AdS_3 \times S^3 \times S^3 \times S^1$ background with completely fixed κ-symmetry identical to the one used in section 3. In section 5 we investigate the BMN limit of the coset sigma model in light cone gauge and compare it to the supergravity analysis in [40, 44]. We make some preliminary steps towards going beyond the strict BMN limit. In section 6 we point out that the type IIB GS string on the purely RR $AdS_3 \times S^3 \times T_4$ background can be treated as a limiting case of the $AdS_3 \times S^3 \times S^3 \times S^1$ super coset action. In section 7 we discuss the general classical integrability scheme for $\mathbb{Z}_4$ symmetric (super)cosets, and derive the finite gap equations entirely in terms of the group-theory data. We then derive classical Bethe equations for $AdS_3 \times S^3 \times S^3 \times S^1$ and discuss their BMN limit. We propose quantum Bethe equations for two special points, $\phi = \pi/4$ and $\phi = 0$, the latter case corresponding to $AdS_3 \times S^3 \times T^4$. In section 8 we conclude with a preliminary discussion of the massless modes, which is a novel feature of the AdS_3/CFT_2 duality compared to AdS_5/CFT_4 or AdS_4/CFT_3. In the appendices we collect the commutation relations of the superalgebra $\mathfrak{d}(2,1;\alpha)$, the gamma matrix conventions, the background-field expansion of the general $\mathbb{Z}_4$ coset action, and some higher order terms of the near-BMN expansion in $AdS_3 \times S^3 \times S^3 \times S^1$.

2 $\mathbb{Z}_4$ cosets

The space-time supersymmetric action of the superstring in flat space [45] can be interpreted as a coset sigma-model [46, 47]. The coset construction readily generalizes to curved space, and in particular allows one to build the Green-Schwarz action for a number of AdS-type backgrounds. The basic example is the Metsaev-Tseytlin action in $AdS_5 \times S^5$ [30] (see [48] for a recent review). As realized in [34], an important feature of the AdS-type cosets is the $\mathbb{Z}_4$ symmetry,[5] that in particular allows one to construct the Wess-Zumino term necessary for the consistency of the Green-Schwarz action. Integrability arises as a bonus symmetry in all $\mathbb{Z}_4$ cosets. Indeed, the derivation of the Lax pair for the $AdS_5 \times S^5$ sigma-model [15] does not really depend on the specifics of the background and relies solely on the existence of the $\mathbb{Z}_4$ structure.

2.1 The action and equations of motion

We start by reviewing the general construction of the sigma-model action for $\mathbb{Z}_4$ cosets. A coset G/H_0 possesses a $\mathbb{Z}_4$ symmetry, if the superalgebra $\mathfrak{g}$ admits a $\mathbb{Z}_4$ decomposition:

$$\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3, \tag{2.1}$$

consistent with the (anti-)commutation relations: $[\mathfrak{h}_n, \mathfrak{h}_m\} \subset \mathfrak{h}_{(n+m) \bmod 4}$. Equivalently the $\mathbb{Z}_4$ symmetry is associated with an order-four automorphism of the Lie superalgebra $\mathfrak{g}$,

[5]The manifestly $\mathbb{Z}_4$ invariant form of the Metsaev-Tseytlin action is given in [49].

which is a linear map $\Omega : \mathfrak{g} \to \mathfrak{g}$ that satisfies $[\Omega(X), \Omega(Y)\} = \Omega([X,Y\})$ and $\Omega^4 = \mathrm{id}$. The subspace $\mathfrak{h}_n$ then is defined as a subset of generators whose $\mathbb{Z}_4$ charge is n, in the basis in which Ω is diagonal:

$$\Omega(\mathfrak{h}_n) = \mathrm{e}^{\pi i n/2}\mathfrak{h}_n. \tag{2.2}$$

The denominator of the coset is the Lie group of the invariant subalgebra $\mathfrak{h}_0$ in the $\mathbb{Z}_4$ decomposition. The fermion number F is the $\mathbb{Z}_4$ charge $\mod 2$, so that $\mathfrak{h}_0 \oplus \mathfrak{h}_2$ is the bosonic subalgebra of $\mathfrak{g}$, and $\mathfrak{h}_1$, $\mathfrak{h}_3$ consist of Grassmann-odd generators.

The worldsheet embedding in G/H is parameterized by a coset representative $g(x) \in G$, subject to gauge transformations $g(x) \to g(x)h(x)$ with $h(x) \in H_0$. The global G-valued transformations act on $g(x)$ from the left: $g(x) \to g'g(x)$. The action and the equation of motion can be written in terms of the left-invariant current

$$J_{\mathbf{a}} = g^{-1}\partial_{\mathbf{a}} g = J_{\mathbf{a}\,0} + J_{\mathbf{a}\,1} + J_{\mathbf{a}\,2} + J_{\mathbf{a}\,3}. \tag{2.3}$$

The $\mathfrak{h}_0$ component of the current transforms as a connection under the gauge transformations: $J_{\mathbf{a}\,0} \to h^{-1}J_{\mathbf{a}\,0}h + h^{-1}\partial_{\mathbf{a}} h$. The other three components transform as matter fields in the adjoint: $J_{\mathbf{a}\,1,2,3} \to h^{-1}J_{\mathbf{a}\,1,2,3}h$.

The action of the sigma model is[6]

$$S = \int d^2x \;\mathrm{Str}\left(\sqrt{h}h^{\mathbf{ab}}J_{\mathbf{a}\,2}J_{\mathbf{b}\,2} + \varepsilon^{\mathbf{ab}}J_{\mathbf{a}\,1}J_{\mathbf{b}\,3}\right). \tag{2.4}$$

Here $\mathrm{Str}(\cdot\,\cdot)$ denotes the G and $\mathbb{Z}_4$ invariant bilinear form on $\mathfrak{g}$. This action is obviously gauge invariant and $\mathbb{Z}_4$-symmetric.

The equations of motion for this action and the Bianchi identities for the currents (the Maurer-Cartan equations) read:

$$\begin{aligned}
2D_{\mathbf{a}}\left(\sqrt{-h}h^{\mathbf{ab}}J_{\mathbf{b}\,2}\right) - \varepsilon^{\mathbf{ab}}[J_{\mathbf{a}\,1}, J_{\mathbf{b}\,1}] + \varepsilon^{\mathbf{ab}}[J_{\mathbf{a}\,3}, J_{\mathbf{b}\,3}] &= 0\\
\left(\sqrt{-h}h^{\mathbf{ab}} + \varepsilon^{\mathbf{ab}}\right)[J_{\mathbf{a}\,2}, J_{\mathbf{b}\,1}] &= 0\\
\left(\sqrt{-h}h^{\mathbf{ab}} - \varepsilon^{\mathbf{ab}}\right)[J_{\mathbf{a}\,2}, J_{\mathbf{b}\,3}] &= 0\\
\varepsilon^{\mathbf{ab}}\left(2D_{\mathbf{a}}J_{\mathbf{b}\,2} + [J_{\mathbf{a}\,1}, J_{\mathbf{b}\,1}] + [J_{\mathbf{a}\,3}, J_{\mathbf{b}\,3}]\right) &= 0\\
\varepsilon^{\mathbf{ab}}\left(D_{\mathbf{a}}J_{\mathbf{b}\,1} + [J_{\mathbf{a}\,2}, J_{\mathbf{b}\,3}]\right) &= 0\\
\varepsilon^{\mathbf{ab}}\left(D_{\mathbf{a}}J_{\mathbf{b}\,3} + [J_{\mathbf{a}\,2}, J_{\mathbf{b}\,1}]\right) &= 0\\
F_{\mathbf{ab}} + [J_{\mathbf{a}\,2}, J_{\mathbf{b}\,2}] + [J_{\mathbf{a}\,1}, J_{\mathbf{b}\,3}] + [J_{\mathbf{a}\,3}, J_{\mathbf{b}\,1}] &= 0,
\end{aligned} \tag{2.5}$$

where $D_{\mathbf{a}} = \partial_{\mathbf{a}} + [J_{\mathbf{a}\,0}, \cdot]$ and $F_{\mathbf{ab}} = \partial_{\mathbf{a}}J_{\mathbf{b}\,0} - \partial_{\mathbf{b}}J_{\mathbf{a}\,0} + [J_{\mathbf{a}\,0}, J_{\mathbf{b}\,0}]$. These equations admit a Lax representation, they are equivalent to the flatness condition for the connection [15]

$$L_{\mathbf{a}} = J_{\mathbf{a}\,0} + \frac{\mathrm{x}^2+1}{\mathrm{x}^2-1}J_{\mathbf{a}\,2} - \frac{2\mathrm{x}}{\mathrm{x}^2-1}\,\frac{1}{\sqrt{-h}}h_{\mathbf{ab}}\varepsilon^{\mathbf{bc}}J_{\mathbf{c}\,2} + \sqrt{\frac{\mathrm{x}+1}{\mathrm{x}-1}}\,J_{\mathbf{a}\,1} + \sqrt{\frac{\mathrm{x}-1}{\mathrm{x}+1}}J_{\mathbf{a}\,3}. \tag{2.6}$$

[6] We use $(+-)$ conventions for the worldsheet metric, but mostly-plus conventions for the target-space. The ε-tensor is defined such that $\varepsilon^{01} = 1$.

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The spectral parameter $\mathtt{x}$ is an arbitrary complex number $\mathtt{x} \neq \pm 1$. Provided that the currents obey the equations of motion, the Lax connection satisfies

$$\partial_{\mathbf{a}} L_{\mathbf{b}} - \partial_{\mathbf{b}} L_{\mathbf{a}} + [L_{\mathbf{a}}, L_{\mathbf{b}}] = 0. \tag{2.7}$$

And conversely, if the connection $L_{\mathbf{a}}$ is flat for any $\mathtt{x}$, the currents satisfy the equations of motion.

The Wilson loop of the Lax connection defines an infinite set of conserved charges, which include the global Noether charges of the left group multiplication. These Noether charges are expressed in term of the gauge-invariant right current that can be obtained from the left currents $J_{\mathbf{a}\,n}$ by conjugation with the coset representative g:

$$k^{\mathbf{a}} = g\left(\sqrt{-h}h^{\mathbf{ab}} J_{\mathbf{b}\,2} - \frac{1}{2}\varepsilon^{\mathbf{ab}} J_{\mathbf{b}\,1} + \frac{1}{2}\varepsilon^{\mathbf{ab}} J_{\mathbf{b}\,3}\right) g^{-1}. \tag{2.8}$$

This current is conserved:

$$\partial_{\mathbf{a}} k^{\mathbf{a}} = 0, \tag{2.9}$$

as a consequence of the equations of motion.

2.2 $AdS_3 \times S^3 \times S^3 \times S^1$ supercoset

The conformal algebra in two dimensions is a two-fold tensor product, with two factors acting independently on the left and right movers. The cosets appropriate for the AdS_3/CFT$_2$ correspondence are thus of the form $H \times H/H_0$. If H is a superalgebra, such a coset will naturally have a $\mathbb{Z}_4$ structure. Indeed, one can define a $\mathbb{Z}_4$ automorphism on $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$ by combining the fermion parity with the permutation of the two factors:

$$\Omega = \begin{pmatrix} 0 & \mathrm{id} \\ (-1)^F & 0 \end{pmatrix}. \tag{2.10}$$

This map satisfies all necessary requirements: it preserves the (anti)-commutation relations of $\mathfrak{h} \oplus \mathfrak{h}$ and squares to $(-1)^F$ such that its forth power is the identity: $\Omega^4 = \mathrm{id}$. The $\mathbb{Z}_4$ grading associated with the automorphism (2.10) is

$$\begin{aligned} \mathfrak{h}_0 &= \{(X, X) | X \in \mathfrak{h}_{\mathrm{bos}}\} \\ \mathfrak{h}_1 &= \{(X, iX) | X \in \mathfrak{h}_{\mathrm{ferm}}\} \\ \mathfrak{h}_2 &= \{(X, -X) | X \in \mathfrak{h}_{\mathrm{bos}}\} \\ \mathfrak{h}_3 &= \{(X, -iX) | X \in \mathfrak{h}_{\mathrm{ferm}}\} \end{aligned} \tag{2.11}$$

In particular, the invariant subspace is the diagonal bosonic subalgebra.

Thus for any superalgebra H one can construct a $\mathbb{Z}_4$ invariant coset sigma-model with the global $H \times H$ symmetry. The denominator of the coset is the diagonal bosonic subgroup. The bosonic part of the action is the sigma-model with the target space $H_{\mathrm{bos}} \times H_{\mathrm{bos}}/H_{\mathrm{diag}}$ isomorphic to[7] H_{bos}. Thus constructed sigma-model will be automatically integrable.

[7] In the gauge orbit $\{(g_L h, g_R h) | h \in H_{\mathrm{bos}}\}$, we can pick a representative by taking $h = g_L^{-1}$, or in other words impose $g_L(x) = 1$ as a gauge condition. What remains is the group manifold $\{(1, g)\}$.

The construction is completely general and works for any supergroup. By taking $H = \mathrm{PSU}(1,1|2)$ we recover the action of [9–11] for the Green-Schwarz string on the $AdS_3 \times S^3$ background [12]. We can also pick $H = D(2,1;\alpha)$ whose bosonic subgroup for $0 < \alpha < 1$ is $\mathrm{SU}(1,1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$. When restricted to the bosonic fields, the action reduces to a sigma-model whose target space is the group manifold of $\mathrm{SU}(1,1)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$, namely $AdS_3 \times S^3 \times S^3$. The $D(2,1;\alpha) \times D(2,1;\alpha)/\mathrm{SU}(1,1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ coset thus describes a Green-Schwarz-type sigma-model on $AdS_3 \times S^3 \times S^3$. The equations of motion following from the coset action admit the Lax representation and consequently the model is completely integrable.

It is not immediately clear if this model is capable of describing the Green-Schwarz superstring on $AdS_3 \times S^3 \times S^3 \times S^1$, because of the missing S^1 factor. This factor has to be added by hand. Similar situation occurs in the hybrid sigma-model on $AdS_3 \times S^3 \times T^4$ [8], where the T^4 factor is completely orthogonal to the non-linear part of the action. The hybrid formalism assumes the conformal gauge from the very beginning [8], and adding an independent CFT is not a problem provided the total central charge vanishes. On the contrary, in the Green-Schwarz action *all* bosons are coupled to *all* fermions through the kinetic term $\bar{\theta}^I \partial_{\mathbf{a}} X^M \Gamma_M \partial_{\mathbf{b}} \theta^J$. The desired decoupling of a bosonic direction is essentially equivalent to setting one of the Dirac matrices to zero. This does not sound right at all. In addition, there are 32 fermions in the Green-Schwarz action compared to sixteen ($2 \times$ the number of supercharges in $\mathfrak{d}(2,1;\alpha)$) fermion degrees of freedom in the $D(2,1;\alpha) \times D(2,1;\alpha)/\mathrm{SU}(1,1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ coset. However, half of the 32 Green-Schwarz fermions are unphysical because of kappa-symmetry. In the next two sections we will demonstrate that by taking a special gauge choice of the kappa-symmetry it is possible to decouple the S^1 factor in the $AdS_3 \times S^3 \times S^3 \times S^1$ supergeomerty, and that the resulting kappa-fixed GS action coincides with the coset model plus one free boson.

The fact that the S^1 appears as an extra factor in the action without any couplings to the other fields can be anticipated from the structure of the Killing spinors [38]. In particular, the momentum Killing vector for S^1 does not appear on the right-hand-side of the anti-commutator of the Killing spinors for this background and also commutes with all the supercharges [38].[8]

3 Flat space limit

In this section we find the flat space limit of the action (2.4) for the $D(2,1;\alpha) \times D(2,1;\alpha)/\mathrm{SU}(1,1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ coset, and show that it coincides with the usual flat space Green-Schwarz action [45] in a particular kappa-gauge.

To obtain the flat space limit of the $\mathfrak{d}(2,1;\alpha)^2$ algebra we should rescale the generators as follows

$$\mathfrak{h}_0 \to \mathfrak{h}_0\,, \qquad \mathfrak{h}_2 \to R\mathfrak{h}_2 \tag{3.1}$$

$$\mathfrak{h}_1 \to \sqrt{R}\mathfrak{h}_1\,, \qquad \mathfrak{h}_3 \to \sqrt{R}\mathfrak{h}_3\,, \tag{3.2}$$

[8]We are grateful to Jerome Gauntlett for a detailed explanation of these and related results of [38].

and take the $R \to \infty$ limit. In this limit the generators of $\mathfrak{h}_0$ become the angular momenta of $\mathrm{SO}(1,2) \times \mathrm{SO}(3) \times \mathrm{SO}(3) \subset \mathrm{SO}(1,9)$, while the elements of $\mathfrak{h}_2$ become flat space-time momenta; it is easy to check that, in the $R \to \infty$ limit, the elements of $\mathfrak{h}_2$ commute with themselves and the supercharges - as is the case for flat-space momenta. In the flat space limit the anti-commutators (A.2) become

$$\left\{Q^I_{a\alpha\dot{\alpha}}\,,\,Q^J_{b\beta\dot{\beta}}\right\} = \delta^{IJ}\Big[i(\varepsilon\gamma^\mu)_{ab}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}P_\mu + \cos^2\phi\,\varepsilon_{ab}(\varepsilon\gamma^n)_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}}P_n + \sin^2\phi\,\varepsilon_{ab}\varepsilon_{\alpha\beta}(\varepsilon\gamma^{\dot{n}})_{\dot{\alpha}\dot{\beta}}P_{\dot{n}}\Big]\,, \tag{3.3}$$

where $I = 1,2$, $Q^I = Q^L - i(-1)^I Q^R \in \mathfrak{h}_{2I-1}$ and $P_\mu = S^L_\mu - S^R_\mu$, $P_n = L^L_n - L^R_n$, $P_{\dot{n}} = R^L_{\dot{n}} - R^R_{\dot{n}}$, and the indices L and R distinguish the two copies of $\mathfrak{d}(2,1;\alpha)$.

Recall that the IIB flat space supersymmetry algebra is

$$\left\{q^I_{\hat{a}}\,,\,q^J_{\hat{b}}\right\} = \delta^{IJ}\left[C\Gamma^M(1+\Gamma)\right]_{\hat{a}\hat{b}}P_M\,, \tag{3.4}$$

where $I\,,\,J = 1\,,\,2$ counts the amount of 10d supersymmetry, $\hat{a}\,,\,\hat{b} = 1,\ldots,32$ are spinor indices of $\mathrm{SO}(1,9)$, $M = 0,\ldots,9$ is the 10d vector index, C is the charge conjugation matrix, Γ^M are 32×32 Dirac matrices of $\mathrm{SO}(1,9)$ and Γ is the 10d chirality matrix.[9]

We would like to identify a sub-algebra of this flat space-time supersymmetry algebra which has the same form as the flat space limit of $\mathfrak{d}(2,1;\alpha)^2$. In particular, we want to find a subset of 16 fermionic generators which satisfy (3.3). To this end, we define the projection operators

$$K^\pm(\phi) \equiv \frac{1}{2}(1 \pm \cos\phi\,\Gamma^{012345} \pm \sin\phi\,\Gamma^{012678})\,. \tag{3.5}$$

Below, we will show that the sixteen supercharges $K^+q^I_\alpha$ satisfy (3.3). This will allow us to show that the flat space limit of the action (2.4) for $\mathfrak{g} = \mathfrak{d}(2,1;\alpha)^2$ coincides with the usual flat space Green-Schwarz action [45] in a particular kappa-gauge.

First, let us note that

$$K^\pm K^\pm = K^\pm\,, \qquad K^\pm K^\mp = 0, \tag{3.6}$$

as required for a projector. Further, $K^\pm$ commute with the 10d chirality matrix $\Gamma \equiv \Gamma^{0123456789}$ and satisfy

$$K^{\pm t}C = CK^\mp \tag{3.7}$$

where ^{t} indicates the transpose. As a result, projecting with respect to $K^\pm$ is compatible with both the 10d Majorana and Weyl conditions. With our choice of gamma matrices, the anticommutator of the supercharges q^I projected by K^+ is

$$\left\{K^+(\phi)q^I\,,\,K^+(\phi)q^J\right\} = \delta^{IJ}m(\phi)\otimes\left(i\varepsilon\gamma^\mu\otimes\varepsilon\otimes\varepsilon\,P_\mu + \cos\phi\,\varepsilon\otimes\varepsilon\gamma^n\otimes\varepsilon\,P_n + \sin\phi\,\varepsilon\otimes\varepsilon\otimes\varepsilon\gamma^{\dot{n}}P_{\dot{n}}\right)\,, \tag{3.8}$$

[9]In appendix C we define a basis for the gamma matrices that will be particularly useful to the symmetries of the problem.

where $m(\phi)$ is a degenerate 4×4 matrix

$$m(\phi)=\begin{pmatrix}0&1\\0&0\end{pmatrix}\otimes\begin{pmatrix}1+\cos\phi&-\sin\phi\\-\sin\phi&1-\cos\phi\end{pmatrix}. \tag{3.9}$$

The projected supercharges $K^+(\phi)q^I$ have the same commutation relations as the flat space limit of the $\mathfrak{d}(2,1;\alpha)^2$ supercharges (the equations (3.8) and (3.3) differ only by normalization of momenta). As a result the flat-space limit of the $\mathbb{Z}_4$ coset sigma-model action will match with the flat space Green-Schwarz action in the (fully fixed) kappa gauge

$$K^-(\phi)\theta^I=\theta^I. \tag{3.10}$$

One can explicitly check that the form of the actions is indeed equivalent; in doing this it is important to start with the 3d form of the Wess-Zumino term - just as one does for $AdS_5\times S^5$ [30].

4 Coset model vs. Green-Schwarz string

In this section we compare the $D(2,1;\alpha)\times D(2,1;\alpha)/\mathrm{SU}(1,1)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$ supercoset with the Green-Schwarz action on $AdS_3\times S^3\times S^3\times S^1$, expanded to the second order in fermions.[10] As discussed before, the agreement of the two is not at all obvious. For successful comparison one has to completely fix the kappa-symmetry in the Green-Schwarz action in such a way that the S^1 factor decouples from fermions. In this respect, the background is similar to $AdS_4\times CP^3$, for which the coset description requires partially fixed kappa-symmetry [31, 33]. A novel feature is the complete decoupling of a bosonic direction. It is known from the experience with $AdS_4\times CP^3$ that the coset kappa-symmetry gauge may become singular on certain string configurations [33, 50]. It would thus be interesting to find the full Green-Schwarz action on $AdS_3\times S^3\times S^3\times S^1$, perhaps along the lines of the type IIA $AdS_4\times CP^3$ case [33]. For technical reasons we will restrict ourselves to the quadratic part of the action, which is known in a closed form for any type IIB supergravity background [51].

The Green-Schwarz fermions couple to the metric and to the three-form RR flux as [51]:

$$\mathcal{L}_{\rm GS}=\left(\sqrt{-h}h^{\mathbf{ab}}\delta^{IJ}-\varepsilon^{\mathbf{ab}}\sigma_3^{IJ}\right)\bar\theta^I\not{E}_{\mathbf{a}}\left(D_{\mathbf{b}}\delta^{JK}+\frac{1}{48}\not{F}\not{E}_{\mathbf{b}}\sigma_1^{JK}\right)\theta^K. \tag{4.1}$$

Here $E_{\mathbf{a}}^A$ is the worldsheet projection of the vierbein:

$$E_{\mathbf{a}}^A=\partial_{\mathbf{a}}X^ME_M^A. \tag{4.2}$$

For any tangent-space tensor we define

$$\not{\mathcal{F}}=\mathcal{F}_{A_1\ldots A_n}\Gamma^{A_1\ldots A_n}.$$

The metric coupling in the covariant derivative is the standard spin connection:

$$D_{\mathbf{b}}=\partial_{\mathbf{b}}+\frac{1}{4}\not{\Omega}_{\mathbf{b}},\qquad \Omega_{\mathbf{b}}^{AB}=\partial_{\mathbf{b}}X^M\Omega_M^{AB} \tag{4.3}$$

[10] We would like to thank A. Tseytlin for suggesting this calculation to us.

The fermions are Majorana-Weyl spinors of the same chirality:

$$\Gamma\theta^I = \theta^I, \qquad \bar{\theta}^I = \theta^{It}C, \qquad I = 1,2. \tag{4.4}$$

The explicit form of the Dirac matrices, as given in the appendix C, will be important in our calculation.

In the units where the radius of AdS_3 is set to one, the metric and the RR three-form are

$$ds^2 = ds^2(AdS_3) + \frac{1}{\cos^2\phi}\, ds^2(S^3_+) + \frac{1}{\sin^2\phi}\, ds^2(S^3_-) + dU^2, \tag{4.5}$$

$$F = Vol(AdS_3) + \frac{1}{\cos^2\phi}\, Vol(S^3_+) + \frac{1}{\sin^2\phi}\, Vol(S^3_-), \tag{4.6}$$

where $ds^2(\mathcal{M})$ and $Vol(\mathcal{M})$ are the standard metricae and volume forms on AdS_3 and S^3, and U is the periodic coordinate on S^1.

To find an explicit form of the metric, the spin connection and the volume form, we can use the fact that $AdS_3 \times S^3 \times S^3 \times S^1$ is a group manifold of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. The metric and the volume form can then be expressed through the Maurer-Cartan forms:

$$\omega = g^{-1}dg = \omega^i t_i, \tag{4.7}$$

where t_i are the Lie-algebra generators of $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{su}_L(2)$ or $\mathfrak{su}_R(2)$, assumed to be canonically normalized: to $\eta_{ij} = \mathrm{diag}(-++)$ for $\mathfrak{sl}(2,\mathbb{R})$ and $\eta_{ij} = \delta_{ij}$ for $\mathfrak{su}(2)$. The metric and the volume form on a group manifold are given by

$$ds^2 = \eta_{ij}\omega^i \wedge \star\omega^j, \qquad Vol = f_{ijk}\omega^j \wedge \omega^j \wedge \omega^k, \tag{4.8}$$

where $f^i{}_{jk}$ are the structure constants. This suggests the following local frame on $AdS_3 \times S^3 \times S^3 \times S^1$:

$$E^A_M = \begin{cases} \omega^\mu_M, & M = 0,1,2 \\ \frac{1}{\cos\phi}\,\omega^n_M, & M = 3,4,5 \\ \frac{1}{\sin\phi}\,\omega^{\dot{n}}_M, & M = 6,7,8 \\ \delta^A_9, & M = 9. \end{cases} \tag{4.9}$$

The factors of $\cos\phi$ and $\sin\phi$ take into account that AdS_3, S^3_+ and S^3_- have different radii which satisfy the triangle equality (1.1). For the tangent-space indices we use the Lie-algebra notations from appendix A.

The spin connection on a group manifold is determined by the Maurer-Cartan equations:

$$\Omega^m_{M\,n} = -\frac{1}{2}\, f^m{}_{nl}\,\omega^l_M, \tag{4.10}$$

In our case,

$$\Omega_{M\,AB} = \begin{cases} -\frac{1}{2}\,\epsilon_{\mu\nu\lambda}\,\omega^\lambda_M, & M = 0,1,2 \\ -\frac{1}{2}\,\epsilon_{mnp}\,\omega^p_M, & M = 3,4,5 \\ -\frac{1}{2}\,\epsilon_{\dot{m}\dot{n}\dot{p}}\,\omega^{\dot{p}}_M, & M = 6,7,8 \\ 0, & M = 9. \end{cases}$$

Finally, the three-form flux (4.6) has the following tangent-space components:

$$F_{\mu\nu\lambda} = \varepsilon_{\mu\nu\lambda}, \qquad F_{mnp} = \cos\phi\,\varepsilon_{mnp}, \qquad F_{\dot{m}\dot{n}\dot{p}} = \sin\phi\,\varepsilon_{\dot{m}\dot{n}\dot{p}}. \tag{4.11}$$

Contracting the RR form with the Dirac matrices we find:

$$\not{F} = 6(\Gamma^{012} + \cos\phi\,\Gamma^{345} + \sin\phi\,\Gamma^{678}). \tag{4.12}$$

This matrix is nilpotent: $\not{F}^2 = 0$, and is proportional to the projector introduced in the previous section, eq. (3.5):

$$\not{F} = 12\Gamma^{012}K^{+}. \tag{4.13}$$

This elucidates the geometric origin of the gauge-fixing condition (3.10).

Assuming that the fermions obey the gauge-fixing condition (3.10), and taking into account the gamma-matrix identities $K^{\pm t}C = CK^{\mp}$, $K^{\pm}\Gamma^{012} = \Gamma^{012}K^{\mp}$ and $D_{\mathbf{a}}K^{\pm} = K^{\pm}D_{\mathbf{a}}$, we can bring the Green-Schwarz Lagrangian (4.1) to the following form:

$$\mathcal{L}_{\rm GS} = \left(\sqrt{-h}h^{\mathbf{ab}}\delta^{IJ} - \varepsilon^{\mathbf{ab}}\sigma_3^{IJ}\right)\bar{\theta}^I K^{+}\not{E}_{\mathbf{a}}K^{-}\left(D_{\mathbf{b}}\delta^{JK} + \frac{1}{4}\Gamma^{012}K^{+}\not{E}_{\mathbf{b}}K^{-}\sigma_1^{JK}\right)\theta^K. \tag{4.14}$$

Quite remarkably, this Lagrangian does not depend on the S^1 coordinate U. Before the gauge fixing, the fermions coupled to U via the $\partial_{\mathbf{a}}U\Gamma^9$ term in $\not{E}_{\mathbf{a}}$, but in $K^{-}\not{E}_{\mathbf{a}}K^{+}$ the $\partial_{\mathbf{a}}U\Gamma^9$ term is projected out by virtue of an easily verifiable identity $K^{+}\Gamma^9K^{-} = 0$. The S^1 factor indeed decouples from fermions after the kappa-symmetry fixing condition is imposed. It remains to show that the $AdS_3 \times S^3 \times S^3$ couplings in the Green-Schwarz action are the same as in the supercoset model.

Imposing the chiral and the kappa-symmetry conditions

$$\frac{1-\Gamma}{2}\,\theta^I = 0, \qquad K^{+}\theta^I = 0,$$

and using the explicit form of the projectors, eqs. (C.15), (C.13), we find that the worldsheet fermions acquire the following form:

$$\theta^I = |+\rangle_0 \otimes |-\rangle_\phi \otimes \theta^{I\,\alpha a\dot{\alpha}}, \tag{4.15}$$

where the spinors $|\pm\rangle_\phi$ are defined in (C.14). The remaining tri-spinor indices of $\theta^{I\,\alpha a\dot{\alpha}}$ are acted upon by the three triplets of gamma matrices γ^μ, γ^m, $\gamma^{\dot{m}}$, defined in (A.1). Plugging the gauge-fixed fermions into the Lagrangian (4.14), we can get rid of the first two spinors in the tensor product with the help of (C.16). Explicitly,

$$K^{+}\not{E}_{\mathbf{b}}K^{-} \;\to\; -i\omega^{\mu}_{\mathbf{b}}\gamma_\mu + \omega^{m}_{\mathbf{b}}\gamma_m + \omega^{\dot{m}}_{\mathbf{b}}\gamma_{\dot{m}} \equiv iV_{\mathbf{b}} \tag{4.16}$$

$$K^{-}\Gamma^{012}K^{+} \;\to\; i \tag{4.17}$$

$$\not{\Omega}_{\mathbf{b}} \;\to\; -\omega^{\mu}_{\mathbf{b}}\gamma_\mu - i\omega^{m}_{\mathbf{b}}\gamma_m - i\omega^{\dot{m}}_{\mathbf{b}}\gamma_{\dot{m}} = V_{\mathbf{b}}, \tag{4.18}$$

and the Green-Schwarz Lagrangian becomes

$$\mathcal{L}_{\rm GS} = i\left(\sqrt{-h}h^{\mathbf{ab}}\delta^{IJ} - \varepsilon^{\mathbf{ab}}\sigma_3^{IJ}\right)\bar{\theta}^I V_{\mathbf{a}}\left[\left(\partial_{\mathbf{b}} + \frac{1}{4}V_{\mathbf{b}}\right)\delta^{JK} - \frac{1}{4}V_{\mathbf{b}}\sigma_1^{JK}\right]\theta^K. \tag{4.19}$$

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According to (C.12),

$$\bar{\theta}^I_{a\alpha\dot{\alpha}} = \theta^{I\,b\beta\dot{\beta}} \epsilon_{ba}\epsilon_{\beta\alpha}\epsilon_{\dot{\beta}\dot{\alpha}}\ {}_0\langle +|\, i\sigma^2\, |-\rangle_0\ {}_\phi\langle -|\, i\sigma^2\, |+\rangle_\phi = -\theta^{I\,b\beta\dot{\beta}} \epsilon_{ba}\epsilon_{\beta\alpha}\epsilon_{\dot{\beta}\dot{\alpha}} \,. \tag{4.20}$$

This Lagrangian should be compared to the quadratic terms in the supercoset action. The expansion of the coset Lagrangian to the second order in fluctuations in an arbitrary bosonic background is given in appendix B. We can specify the general construction to the case of the $D(2,1;\alpha) \times D(2,1;\alpha)/\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ coset. The background bosonic field in (B.1) in this case is parameterized by an element of $H_{\rm bos} \times H_{\rm bos}/H_{\rm diag}$: $g_B = (g_L, g_R) \sim (g_L h, g_R h)$. Fixing the coset gauge by setting $g_L = 1$ we find:

$$g_B^{-1}\partial_{\mathbf{a}} g_B = \omega^i_{\mathbf{a}} T^R_i = \frac{1}{2}\,\omega^i_{\mathbf{a}}\left(T^L_i + T^R_i\right) - \frac{1}{2}\,\omega^i_{\mathbf{a}}\left(T^L_i - T^R_i\right), \tag{4.21}$$

and thus for the background currents (B.2) we get:

$$\begin{aligned} K_{\mathbf{a}} &= -\frac{1}{2}\,\omega^i_{\mathbf{a}}\left(T^L_i - T^R_i\right) \\ A_{\mathbf{a}} &= \frac{1}{2}\,\omega^i_{\mathbf{a}}\left(T^L_i + T^R_i\right). \end{aligned} \tag{4.22}$$

The fermion part of the fluctuation field in (B.1) can be parameterized as

$$\mathbb{X}_{1,3} = 2\theta^{2,1\,\alpha a\dot{a}}\left(Q^L_{\alpha a\dot{a}} \pm iQ^R_{\alpha a\dot{a}}\right). \tag{4.23}$$

Using the commutation relations of the $\mathfrak{d}(2,1;\alpha)$ algebra listed in appendix A, we obtain:

$$[A_{\mathbf{a}}, \theta^I] = \frac{1}{4}\,V_{\mathbf{a}}\theta^I, \qquad [K_{\mathbf{a}}, \theta^I] = -\frac{1}{4}\,V_{\mathbf{a}}\sigma_1^{IJ}\theta^J, \tag{4.24}$$

where $V_{\mathbf{a}}$ is the same combination of the background currents and Dirac matrices as in (4.16), (4.18). Plugging $K_{\mathbf{a}}$ and $A_{\mathbf{a}}$, as given in (4.24), into the supercoset Lagrangian (B.9), we find that it exactly agrees with the kappa-fixed Green-Schwarz action (4.19).

5 The BMN limit

In this section we will quantize the coset sigma-model in the light-cone gauge perturbatively in the sigma-model coupling by expanding the Lagrangian near a light-cone geodesic. Technically, this is the same background-field expansion from appendix B as we used in the previous section to compare the coset sigma-model with the Green-Schwarz string action. The quadratic action describes the BMN limit [52] of a point-like string moving along the light-cone geodesic [53] $t = x^0 = \varphi$, where t is the global AdS time, x^0 is the worldsheet time direction, and φ is an angle on $S^3 \times S^3 \times S^1$. The quadratic terms can be read off the general formula (B.9), with the background taken in the form

$$g_B = \mathrm{e}^{\,\Xi}\,, \qquad \Xi = tD + \varphi J, \tag{5.1}$$

where D is the dilatation generator of $\mathfrak{d}(2,1;\alpha)^2$ and J is the angular momentum. Higher orders in fluctuations generate the near-BMN expansion [54–56].

The light-cone frame in $AdS_3 \times S^3 \times S^3 \times S^1$ is obtained by combining the global AdS time with an angle on $S^3 \times S^3 \times S^1$. The time is conjugate to the dilatation generator[11]

$$D = S_0^L - S_0^R \tag{5.2}$$

For the angular momentum we have more freedom. Potentially one can pick an arbitrary element of $\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$:

$$J = C(L_5^L - L_5^R) + C'(R_8^L - R_8^R) + C''P, \tag{5.3}$$

where P is the generator of the extra U(1) factor. However, there are two conditions to satisfy: J has to be appropriately normalized in order for $t \pm \varphi$ to be light-cone directions, and the light-cone gauge should preserve some supersymmetry. We shall see that these two conditions uniquely determine all three coefficients in (5.3). If we relax the supersymmetry condition, there are more solutions which we discuss later in section 8.1.

In the light-cone gauge $D+J$ is fixed, and determines the internal length of the string, and $D-J$ becomes the light-cone Hamiltonian:

$$H = D - J \tag{5.4}$$

Both should be null elements of the underlying Lie algebra. This guarantees that the string moves along the light-cone as is required by the Virasoro constraints for the background (5.1). From the explicit form of the metric on $\mathfrak{d}(2,1;\alpha)$ in appendix A, we find that C, C', C'' must satisfy

$$-\frac{1}{2} + \frac{C^2}{2\cos^2\phi} + \frac{C'^2}{2\sin^2\phi} + C''^2 = 0. \tag{5.5}$$

If we also want to preserve supersymmetry, the Hamiltonian $H = D - J$ should commute with some of the supercharges. The eigenvalues of $\operatorname{ad} H$ on the odd generators of the superalgebra are proportional to $\pm 1 \mp C \mp C'$, where the signs take all eight possible values. We thus require that

$$C' = 1 - C. \tag{5.6}$$

With this choice, the zero norm condition (5.5) becomes

$$\frac{(C - \cos^2\phi)^2}{2\cos^2\phi\sin^2\phi} + C''^2 = 0. \tag{5.7}$$

The unique solution to this equation is $C'' = 0$, $C = \cos^2\phi$. Then $C' = \sin^2\phi$, and

$$J = \cos^2\phi\,(L_5^L - L_5^R) + \sin^2\phi\,(R_8^L - R_8^R) \tag{5.8}$$

The light-cone gauge fixing explicitly breaks part of the original $\mathfrak{d}(2,1;\alpha)^2$ symmetry. By inspecting the symmetries of the giant magnons in $AdS_3 \times S^3$ [57] and drawing some intuition from the spin chain picture in the dual CFT, David and Sahoo argued that the

[11] The generators of $\mathfrak{d}(2,1;\alpha)$ are defined in appendix A and the superscripts L and R denote the generators of the left and right $\mathfrak{d}(2,1;\alpha)$ algebras.

little group of the light-cone gauge in $AdS_3 \times S^3 \times T^4$ is PSU(1|1) × PSU(1|1) [14]. The $\mathfrak{psu}(1|1) \oplus \mathfrak{psu}(1|1)$ superalgebra admits a three-parametric central extension [58], and all three central charges appear in the symmetry algebra of the giant magnon [14].

Let us see how the residual symmetries arise in the coset construction of $AdS_3 \times S^3 \times S^3 \times S^1$. The global symmetry transformations act on the coset representative (B.1) from the left and in general will change the background. Only those transformations that commute with D and J can be pulled through $g_B = \mathrm{e}^{\Xi}$ and will act on $\mathrm{e}^{\mathbb{X}}$. Even these transformation will be non-linearly realized on $\mathbb{X}$, with the exception of the transformations from $\mathfrak{h}_0$, for which one can apply a compensating right gauge multiplication such that $\mathbb{X}$ will transform in the adjoint. Thus the elements of $\mathfrak{h}_0$ which commute with D and J act on the transverse field $\mathbb{X}$ by conjugations and leave the action invariant. There are three U(1) charges that satisfy this condition:

$$q_1 = S_0^L + S_0^R, \qquad q_2 = L_5^L + L_5^R, \qquad q_3 = R_8^L + R_8^R. \tag{5.9}$$

However, this is not the end of the story. By analogy with $AdS_5 \times S^5$ and $AdS_4 \times CP^3$, we may expect that the supercharges that commute with $D - J$ will be also preserved, although their algebra, as well as their action on the transverse fields, can be deformed by the gauge fixing. The supercharges that commute with $D - J$ are $Q^{L,R}_{\pm\pm\pm}$. They form two copies of the $\mathfrak{psu}(1|1)$ algebra (which is just the 2d Clifford algebra). We expect that the algebra gets centrally extended by the mechanism described in [20, 59].

The quadratic action for fluctuations around the BMN geodesic can be read off from equation (B.9). In particular, the mass-squared operator for the BMN modes is

$$\mathcal{M}^2 = \mathrm{ad}^2(D + J). \tag{5.10}$$

Given the tensor product structure of $D(2, 1; \alpha) \times D(2, 1; \alpha)$, the mass spectrum can be expressed in terms of the U(1) charges (5.9):

$$M = |q_1 + \cos^2\phi\, q_2 + \sin^2\phi\, q_3|. \tag{5.11}$$

The mass takes four possible values 0, 1, $\cos^2\phi$ or $\sin^2\phi$, in agreement with the bosonic spectrum found in [40] and the full spectrum at the special value of $\phi = \pi/4$ [44]. Each mass level contains four states: two bosons and two fermions. This is the dimension of the bifundamental multiplet of $\mathfrak{psu}(1|1) \times \mathfrak{psu}(1|1)$. The spectrum thus nicely fits into four bifundamental multiplets of the unbroken $\mathfrak{psu}(1|1) \times \mathfrak{psu}(1|1)$.

To write down the Lagrangian for the fluctuation modes, we can choose the following parameterization of the coset element in (B.1):

$$\begin{aligned}
\mathbb{X}_1 &= \chi^{a\alpha\dot\alpha}\left(\tilde{Q}^L_{a\alpha\dot\alpha} - i\tilde{Q}^R_{a\alpha\dot\alpha}\right)\\
\mathbb{X}_2 &= \frac{1}{\sqrt{2}}\,X\left(S_1^L + iS_2^L - S_1^R - iS_2^R\right) + \frac{1}{\sqrt{2}}\,\bar{X}\left(S_1^L - iS_2^L - S_1^R + iS_2^R\right)\\
&\quad+\frac{\cos\phi}{\sqrt{2}}\,Y\left(L_3^L + iL_4^L - L_3^R - iL_4^R\right) + \frac{\cos\phi}{\sqrt{2}}\,\bar{Y}\left(L_3^L - iL_4^L - L_3^R + iL_4^R\right)\\
&\quad+\frac{\sin\phi}{\sqrt{2}}\,Z\left(R_6^L + iR_7^L - R_6^R - iR_7^R\right) + \frac{\sin\phi}{\sqrt{2}}\,\bar{Z}\left(R_6^L - iR_7^L - R_6^R + iR_7^R\right)
\end{aligned}$$

$$\times \sin\phi \cos\phi\, V \left(L_5^L - R_8^L - L_5^R + R_8^R\right)$$

$$\mathbb{X}_3 = \theta^{a\alpha\dot{\alpha}} \left(\tilde{Q}^L_{a\alpha\dot{\alpha}} + i\tilde{Q}^R_{a\alpha\dot{\alpha}}\right), \tag{5.12}$$

where $\tilde{Q}_{a\alpha\dot{\alpha}}$ are defined in (A.4). Plugging (5.1), (5.2), (5.8), and (5.12) into (B.9), and adding the S^1 mode U, we get:

$$\begin{aligned}
\mathcal{L} = \sqrt{-h}h^{\mathbf{ab}} \Bigg[& -\frac{1}{2}\left(1+2X\bar{X}\right)\partial_{\mathbf{a}} t\, \partial_{\mathbf{b}} t + \frac{1}{2}\left(1 - 2\cos^4\phi\, Y\bar{Y} - 2\sin^4\phi\, Z\bar{Z}\right)\partial_{\mathbf{a}}\varphi\, \partial_{\mathbf{b}}\varphi \\
& + \partial_{\mathbf{a}}\bar{X}\,\partial_{\mathbf{b}}X + \partial_{\mathbf{a}}\bar{Y}\,\partial_{\mathbf{b}}Y + \partial_{\mathbf{a}}\bar{Z}\,\partial_{\mathbf{b}}Z + \frac{1}{2}\partial_{\mathbf{a}}V\,\partial_{\mathbf{b}}V + \frac{1}{2}\partial_{\mathbf{a}}U\,\partial_{\mathbf{b}}U\Bigg] \\
+ \sum_{a\,\alpha\,\dot{\alpha}\,b\,\beta\,\dot{\beta}=\pm} \varepsilon_{ab}\varepsilon_{\alpha\beta}\varepsilon_{\dot{\alpha}\dot{\beta}} \Bigg[& i\left(\sqrt{-h}h^{\mathbf{ab}} + \varepsilon^{\mathbf{ab}}\right)\chi^{a\alpha\dot{\alpha}} M_{\mathbf{a}}^{(b,\beta,\dot{\beta})}\partial_{\mathbf{b}}\chi^{b\beta\dot{\beta}} \\
& + i\left(\sqrt{-h}h^{\mathbf{ab}} - \varepsilon^{\mathbf{ab}}\right)\theta^{a\alpha\dot{\alpha}} M_{\mathbf{a}}^{(b,\beta,\dot{\beta})}\partial_{\mathbf{b}}\theta^{b\beta\dot{\beta}} \\
& - \left(\sqrt{-h}h^{\mathbf{ab}} + \varepsilon^{\mathbf{ab}}\right) M_{\mathbf{a}}^{(a,\alpha,\dot{\alpha})} M_{\mathbf{b}}^{(b,\beta,\dot{\beta})}\chi^{a\alpha\dot{\alpha}}\theta^{b\beta\dot{\beta}}\Bigg],
\end{aligned} \tag{5.13}$$

where

$$M_{\mathbf{a}}^{(a,\alpha,\dot{\alpha})} = a\partial_{\mathbf{a}} t + \left(\alpha\cos^2\phi + \dot{\alpha}\sin^2\phi\right)\partial_{\mathbf{a}}\varphi. \tag{5.14}$$

At the quadratic level the light-cone gauge fixing amounts in replacing the world-sheet metric by $\eta_{\mathbf{ab}} = \mathrm{diag}(+-)$, and the light-cone coordinates by their background values: $\partial_{\mathbf{a}} t = \delta^0_{\mathbf{a}} = \partial_{\mathbf{a}}\varphi$. The action can be brought to a nice 2d form by introducing the notations

$$X^1 = \frac{V+iU}{\sqrt{2}}, \qquad X^2 = Z, \qquad X^3 = Y, \qquad X^4 = X, \tag{5.15}$$

and

$$\psi_l^I = -4\alpha\dot{\alpha}m_I\theta^{-\alpha\dot{\alpha}}, \qquad \psi_r^I = -4\alpha\dot{\alpha}m_I\chi^{-\alpha\dot{\alpha}}, \qquad \bar{\psi}_l^I = \chi^{+\alpha\dot{\alpha}}, \qquad \bar{\psi}_r^I = \theta^{+\alpha\dot{\alpha}}, \tag{5.16}$$

where $I = (2\alpha + \dot{\alpha} + 5)/2$ and

$$m_I = (0, \sin^2\phi, \cos^2\phi, 1). \tag{5.17}$$

In these notations, the light-cone action becomes:

$$\mathcal{L}_{\text{l.c.}} = \sum_{I=1}^{4}\left(\partial_{\mathbf{a}}\bar{X}^I\,\partial^{\mathbf{a}}X^I - m_I^2\bar{X}^I X^I + i\bar{\psi}^I\rho^{\mathbf{a}}\partial_{\mathbf{a}}\psi^I - m_I\bar{\psi}^I\psi^I\right), \tag{5.18}$$

where we have introduced the 2d Dirac spinors

$$\psi^I = \begin{pmatrix}\psi_l^I \\ \psi_r^I\end{pmatrix}, \qquad \bar{\psi}^I = \left(\bar{\psi}_l^I \;\; \bar{\psi}_r^I\right). \tag{5.19}$$

For the 2d gamma matrices we take:

$$\rho^{\mathbf{a}} = (\sigma^1, i\sigma^2). \tag{5.20}$$

One can readily expand the action beyond the leading BMN order. For instance, the cubic interaction terms for the bosonic fields are

$$\mathcal{L}^{(3,b)} = 2\sin\phi\,\cos\phi\,\left(\sin^2\phi\,\bar{Z}Z - \cos^2\phi\,\bar{Y}Y\right)\sqrt{-h}h^{\mathbf{ab}}\left(\partial_{\mathbf{a}}V\,\partial_{\mathbf{b}}\varphi + \frac{1}{3}V\,\partial_{\mathbf{a}}\partial_{\mathbf{b}}\varphi\right), \quad (5.21)$$

which gives after the gauge fixing:

$$\mathcal{L}^{(3,b)}_{\text{l.c.}} = 2\sin\phi\,\cos\phi\,\left(\sin^2\phi\,\bar{Z}Z - \cos^2\phi\,\bar{Y}Y\right)\partial_0 V. \quad (5.22)$$

The interactions involving fermions are more complicated and although it is straightforward to include them in the near-BMN expansion, we will not do it here. In appendix D we present the quartic interaction terms for bosons.

The heaviest mode ($X^4 = X$) lies on the threshold of the decay $4 \to 2+3$, since the masses satisfy the sum rule $m_2 + m_3 = \sin^2\phi + \cos^2\phi = 1 = m_4$. A similar sum rule holds in $AdS_4 \times CP^3$. There, the massive mode disappears from the spectrum, once the quantum corrections are taken into account, by mixing with the continuum of the two-particle states [60, 61]. We expect that the same mechanism is at work here and that the heaviest mode should not be regarded as an elementary excitation.

The appearance of the massless BMN modes is quite unusual in the AdS/CFT context. In the $AdS_5 \times S^5$ and $AdS_4 \times CP^3$ backgrounds, which in other respects are very similar to $AdS_3 \times S^3 \times S^3 \times S^1$, the BMN modes are all massive. As we shall see, the massless modes hinder straightforward application of integrability methods to $AdS_3 \times S^3 \times S^3 \times S^1$. We will return later to the discussion of their origin and possible implications for the AdS_3/CFT$_2$ duality.

6 The $AdS_3 \times S^3 \times T^4$ limit

The coset action we have constructed in section 2.2 for Type IIB string theory on $AdS_3 \times S^3 \times S^3 \times S^1$ has the relative radii of the two S^3's ($R_\pm$) and the radius of S^1 as free parameters (see equation (1.1)). We can take, say, $R_- \to \infty$ and decompactify the S^3. The resulting theory should be Type IIB string theory on $AdS_3 \times S^3 \times T^4$, after periodic identifications in the resulting $\mathbb{R}^3$. In this section we show explicitly how this happens.

A coset action for Type IIB string theory on $AdS_3 \times S^3$ had been written down in [9–11, 35] and basically follows the Metsaev-Tseytlin construction for $AdS_5 \times S^5$ [30]. In the notation of our paper, this action is based on the $\mathbb{Z}_4$ automorphism (2.10) of the $\mathfrak{g} = \mathfrak{psu}(1,1|2) \oplus \mathfrak{psu}(1,1|2)$ superalgebra, which puts the six-dimensional Type IIB GS action on $AdS_3 \times S^3$ into the general framework of $\mathbb{Z}_4$ cosets. The coset has 16 fermions, half of which can be removed by fixing kappa-symmetry, leaving 8 physical degrees of freedom: the correct number for a six-dimensional GS action.

We propose that the coset action on $AdS_3 \times S^3$, when supplemented with four free bosons, in fact describes ten-dimensional Type IIB GS strings on $AdS_3 \times S^3 \times T^4$ in a suitable (fully fixed) kappa-symmetry gauge. This may appear puzzling at first, since, as we have just mentioned, the six-dimensional coset action [9–11] has only 8 physical fermions: a factor of two short of the 16 fermions required in ten dimensions. We are going to argue that the extra T^4 factor in the action changes the number of physical degrees of freedom in the coset sector. At first sight that seems impossible, since one can always go to the conformal gauge, where the T^4 completely decouples. Were this to be true, the coset would lack half of the fermion degrees of freedom. However, the decoupling of the T^4 is not complete, since the four bosons of T^4 do interact with the coset fermions through the 2d metric coupling or, in the conformal gauge, through the Virasoro constraints. The key point here is that the metric transforms non-trivially under kappa symmetry. The metric couplings of the extra bosons violate the kappa symmetry of the action and, in effect, keep all 16 fermions physical. Put differently, the addition of the extra free bosons modifies the Virasoro constraints for the model; a consequence of this modification is that kappa-symmetry of the six-dimensional action is *not* a symmetry of the ten-dimensional action. As a result the coset + T^4 model has more fermions that just the coset.[12]

We first discuss the $R_- \to \infty$ limit for the $\mathfrak{d}(2,1;\alpha)$ super-algebra. To reintroduce the dependence on R_- we re-scale $R_{\dot m} \to R_- R_{\dot m}$. We should also take $\phi \to 0$ (c.f. equations (1.2)). The relevant (anti)-commutators then reduce to

$$\begin{aligned} [R_{\dot m}, R_{\dot n}] &= 0 \\ \{Q_{a\alpha\dot\alpha}, Q_{b\beta\dot\beta}\} &= i(\varepsilon\gamma^\mu)_{ab}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta}\,S_\mu - \varepsilon_{ab}\,(\varepsilon\gamma^m)_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta}\,L_m\,. \end{aligned} \tag{6.1}$$

In the limit we also find that the AdS_3 and S^3 radii are equal $R_+ = l$ (c.f. equation (1.1)). In fact, it is easy to convince oneself that the algebra is now $\mathfrak{psu}(1,1|2)$ together with three commuting generators $R_{\dot m}$. This is a well known property of $\mathfrak{d}(2,1;\alpha)$ for $\alpha = 1$ (see for example [43]).

Our coset action is based on a $\mathbb{Z}_4$ automorphism of $\mathfrak{d}(2,1;\alpha)\oplus\mathfrak{d}(2,1;\alpha)$ constructed in section 2.2. In the $R_- \to \infty$ limit this automorphism reduces to the $\mathbb{Z}_4$ automorphism for $\mathfrak{psu}(1,1|2)\oplus\mathfrak{psu}(1,1|2)$ which is used to construct the coset action on $AdS_3\times S^3$. Taking into account the extra commuting generators $R_{\dot m}$, we then find that in the $R_- \to \infty$ limit the Type IIB coset action on $AdS_3\times S^3\times S^3\times S^1$ reduces to the coset action for $AdS_3\times S^3$ together with four free bosons for T^4. Since the original action for $AdS_3\times S^3\times S^3\times S^1$ has the interpretation of a Green-Schwarz action in the kappa-gauge (3.10), we conclude that the coset action for $AdS_3\times S^3$ together with four free bosons for T^4 is a Green-Schwarz action for $AdS_3\times S^3\times T^4$ in the kappa gauge

$$\Gamma^{012345}\theta^I = \theta^I\,. \tag{6.2}$$

7 Integrability

As we have shown in the previous sections, string theory on the $AdS_3\times S^3\times S^3\times S^1$ background is described by an integrable sigma-model. Its classical equations of motion can be solved in a quite general form by the finite-gap integration method [62]. The finite-gap method basically performs the separation of variables, always possible in an integrable system. It thus replaces the oscillator expansion in the flat space and serves as a first

[12]This is perhaps a known mechanism, although we could not find it anywhere in the literature.

step towards string quantization via Bethe ansatz. For the $AdS_5 \times S^5$ [16, 17, 63–68] and $AdS_4 \times CP^3$ [21] backgrounds the finite-gap method yields a set of coupled integral equations, which on the one hand parameterize possible classical solutions of the sigma-model and on the other hand can be regarded as the classical limit of the Bethe equations for the quantum spectrum of the string. We first describe the general scheme of finite-gap integration, as applied to the $\mathbb{Z}_4$ cosets, and then specify the general construction to the case at hand, the sigma model on $AdS_3 \times S^3 \times S^3 \times S^1$.

7.1 The general scheme

The general construction mostly follows the derivation of the finite-gap equations for $AdS_5 \times S^5$ [17], but we will use more invariant group-theoretic language, and unlike in all previous studies will not rely on an explicit supermatrix representation of the Lax connection.[13]

As usual in integrable systems, instead of solving the non-linear equations of motion one can study the linear problem for the Lax operator (2.6). The fundamental solution of the linear problem is the monodromy matrix of the Lax connection [69]:

$$\mathcal{M}(\mathtt{x}) = \mathrm{P}\exp \oint_{C_{x_* x_*}} dx^{\mathbf{a}} L_{\mathbf{a}}(x;\mathtt{x}). \tag{7.1}$$

The contour of integration $C_{x_* x_*}$ is a closed curve that links the worldsheet,[14] but is otherwise arbitrary. The canonical choice is the equal time section $x^0 = x^0_*$. Because of the flatness condition (2.7) the monodromy matrix does not change under continuous deformations of the contour.

The monodromy matrix is gauge-dependent and also depends on the base point x_*. Under the gauge transformations the monodromy matrix gets conjugated by an element of H_0: $\mathcal{M} \to h^{-1}(x_*)\mathcal{M}h(x_*)$, $h(x_*) \in H_0$. If the base point is shifted to x'_*, the monodromy matrix is conjugated by the monodromy along the curve connecting x_* and x'_*: $\mathcal{M} \to U^{-1}\mathcal{M}U$, $U \equiv U(\Gamma_{x_* x'_*}) \in G$. Both transformation leave the monodromy matrix in the same conjugacy class. Therefore, the conjugacy class of the monodromy matrix is gauge invariant and independent of the base point (time-independent).

Since the conjugacy class of $\mathcal{M}(\mathtt{x})$ is time-independent for arbitrary spectral parameter $\mathtt{x}$, it generates an infinite number of integrals of motion. The set of conjugacy classes is isomorphic to the maximal torus of G modulo Weyl group, and by choosing a Cartan basis H_l, the monodromy matrix can be locally represented as

$$\mathcal{M}(\mathtt{x}) = U^{-1}(\mathtt{x}) \exp\left(p_l(\mathtt{x})H_l\right) U(\mathtt{x}).$$

The quasi-momenta $p_l(\mathtt{x})$ are the gauge-invariant generating functions for the integrals of motion. They are defined up to transformations from the Weyl group and shifts by integer multiples of 2π.

[13] Different matrix representations give the same algebraic curve and the same set of integral equations [64]. The group-theoretic construction makes this equivalence manifest.

[14] We discuss only the tree level of string theory, and assume that the worldsheet has the topology of a cylinder.

The monodromy matrix $\mathcal{M}(\mathtt{x})$ is a meromorphic function of the spectral parameter $\mathtt{x}$ whose only possible singularities are located at $\mathtt{x} = \pm 1$. The nature of these singularities will be discussed later. On the contrary, the quasi-momenta $p(\mathtt{x})$ in general are multi-valued functions of $\mathtt{x}$, defined up to Weyl transformations, and can have branch points with the monodromy in the Weyl group. For simplicity, we only consider the case when the monodromies are elementary Weyl reflections (including generalized Weyl reflections specific to supergroups [70, 71]). An arbitrary element of the Weyl group is a product of Weyl reflections. In the Bethe-ansatz language branch points with composite monodromies correspond to stacks of Bethe roots [72, 73] describing composite quantum states of several elementary excitations.

The Weyl reflection with respect to the lth root acts on the lth quasi-momentum as $p_l(\mathtt{x}) \to p_l(\mathtt{x}) - A_{lm}p_m(\mathtt{x})$, where A_{lm} is the Cartan matrix of $\mathfrak{g}$. We denote by $\{\mathtt{a}_{l,i}\}$ the set of branch points of the quasi-momentum $p_l(\mathtt{x})$ and by $\{C_{l,i}\}$ the set of cuts that connect these branch points pairwise. The precise nature of the singularity of $p(\mathtt{x})$ at $\mathtt{a}_{l,i}$ is determined by the monodromy

$$p_l(\mathtt{x}) \to p_l(\mathtt{x}) - A_{lm}p_m(\mathtt{x}) + 2\pi n_{l,i}, \tag{7.2}$$

and depends on whether the lth root of the superalgebra is bosonic or fermionic. For the fermionic root, $A_{ll} = 0$, and the quasi-momentum shifts by a (known) function which is analytic at $\mathtt{x} = \mathtt{a}_{l,i}$: $p_l \to p_l + \cdots$. Consequently, the quasi-momentum $p_l(\mathtt{x})$ has a logarithmic singularity. For the bosonic root $A_{ll} = 2$ and, in addition to the shift by an analytic function, the quasi-momentum changes sign: $p_l \to -p_l + \cdots$. This means that $p_l(\mathtt{x})$ has a square root branch point at $\mathtt{x} = \mathtt{a}_{l,i}$.

In conclusion, the quasi-momenta are meromorphic functions on the complex plane with cuts $C_{l,i}$, and can have logarithmic (fermionic) or square root (bosonic) branch points. For the bosonic, square-root cuts, the monodromy condition (7.2) is equivalent to an equation for the continuous part of the quasi-momentum across the cut:

$$A_{lm}\not{p}_m(\mathtt{x}) = 2\pi n_{l,i}, \qquad \mathtt{x} \in C_{l,i}, \tag{7.3}$$

where we define:

$$\not{p}_l(\mathtt{x}) = \frac{1}{2}\left(p_l(\mathtt{x} + i0) + p_l(\mathtt{x} - i0)\right). \tag{7.4}$$

The same equation holds at the end-points of the fermionic cuts, in which case $p_l(\mathtt{x})$ drops out of the equation.

In addition to the branch cuts, p_l has simple poles at $\mathtt{x} = \pm 1$, where the Lax connection (2.6) itself has a singularity:

$$L_{\mathbf{a}} = \frac{1}{2\sqrt{-h}}\left(\sqrt{-h}h_{\mathbf{ab}} \pm \varepsilon_{\mathbf{ab}}\right) J_2^{\mathbf{b}} \frac{1}{\mathtt{x} \pm 1} + \cdots \qquad (\mathtt{x} \to \mp 1). \tag{7.5}$$

Hence,

$$p_l(\mathtt{x}) = \frac{1}{2}\,\frac{\kappa_l \mp 2\pi m_l}{\mathtt{x} \pm 1} + \cdots \qquad (\mathtt{x} \to \mp 1). \tag{7.6}$$

Parameterization of the residues at $\mathtt{x} = 1$ and $\mathtt{x} = -1$ by their sum and difference is a matter of convenience.

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The quasi-momenta carry all the information about the conserved quantities in the sigma-model. In particular, their asymptotic behavior at infinity determines the global symmetry charges. Indeed,

$$L_{\mathbf{a}} = g^{-1}\left(\partial_{\mathbf{a}} + \frac{1}{\mathtt{x}}\,\frac{2}{h}\,\varepsilon_{\mathbf{ab}}k^{\mathbf{b}}\right)g + \cdots \qquad (\mathtt{x}\to\infty), \tag{7.7}$$

where $k^{\mathbf{a}}$ is the global symmetry current (2.8). Thus,

$$\mathcal{M}(\mathtt{x}) = 1 + \frac{2}{\mathtt{x}}\oint dx^{\mathbf{a}}\,\frac{1}{h}\,\varepsilon_{\mathbf{ab}}k^{\mathbf{b}} + \cdots \qquad (\mathtt{x}\to\infty), \tag{7.8}$$

and

$$p_l(\mathtt{x}) = -\frac{2}{\mathtt{x}}\,Q_l \qquad (\mathtt{x}\to\infty). \tag{7.9}$$

Further coefficients of the Taylor expansion constitute an infinite set of conserved charges responsible for integrability of the model. Using quasi-momenta, one can also build the canonical set of action variables [67, 74].

The information contained in $p_l(\mathtt{x})$ is actually redundant because of the $\mathbb{Z}_4$ symmetry. The symmetry acts on the flat connection according to (2.2), and it is not hard to see that

$$\Omega(L_{\mathbf{a}}(\mathtt{x})) = L_{\mathbf{a}}(1/\mathtt{x}). \tag{7.10}$$

The action of the Ω on the Lie algebra elements, such as the Lax connection $L_{\mathbf{a}}(\mathtt{x})$, can be lifted to the group action with the help of the exponential map. Thus,

$$\Omega(\mathcal{M}(\mathtt{x})) = \mathcal{M}(1/\mathtt{x}). \tag{7.11}$$

Likewise, Ω acts on the the maximal torus, albeit the $\mathbb{Z}_4$ action is then defined up to the Weyl reflections. Given the $\mathbb{Z}_4$ action on the Cartan generators:

$$\Omega(H_l) = H_m S_{ml}, \tag{7.12}$$

one can infer the transformation properties of the quasi-momenta under the inversion in the spectral-parameter plane:

$$p_l(1/\mathtt{x}) = S_{lm}p_m(\mathtt{x}). \tag{7.13}$$

In consequence, the knowledge of the quasi-momenta in the physical region $|\mathtt{x}| > 1$ is sufficient to reconstruct them everywhere in the complex plane. In particular, all the branch points of the quasi-momenta are invariant under inversion. If $\mathtt{a}_{l,i}$ is a branch point of $p_l(\mathtt{x})$, (possibly) other quasi-momentum will have a branch point at $1/\mathtt{a}_{l,i}$.

A meromorphic function with the properties listed above can be reconstructed from the discontinuities at its cuts, which for the bosonic quasi-momenta[15] we denote by $2\pi i\rho_l(\mathtt{x})$.

[15] Calling the quasi-momenta bosonic or fermionic is a slight abuse of terminology, because all $p_l(\mathtt{x})$ are even elements of the Grassmann algebra. We call the quasi-momentum p_l fermionic if the lth node of the Dynkin diagram is fermionic.

For fermionic cuts, the monodromy condition (7.2) completely determines the discontinuity and there is no further freedom. The quasi-momenta thus admit the spectral representation:

$$p_l(\mathtt{x}) = -\frac{\kappa_l \mathtt{x} + 2\pi m_l}{\mathtt{x}^2 - 1} + \int_{C_l} d\mathtt{y}\, \frac{\rho_l(\mathtt{y})}{\mathtt{x} - \mathtt{y}} + \int_{1/C_l} d\mathtt{y}\, \frac{\tilde{\rho}_l(\mathtt{y})}{\mathtt{x} - \mathtt{y}}\,, \tag{7.14}$$

where C_l denotes the collection of cuts in the physical domain $|\mathtt{x}| > 1$ on which $p_l(\mathtt{x})$ has a discontinuity. Since the quasi-momenta inside the unit circle can be reconstructed by inversion, we chose to treat separately the cuts at $|\mathtt{x}| > 1$ and their images under $\mathtt{x} \to 1/\mathtt{x}$.

The inversion symmetry (7.13) determines the densities $\tilde{\rho}_l(\mathtt{x})$ in terms of $\rho_k(1/\mathtt{x})$, and imposes certain constraints on κ_l and m_l. We find that m_l's must be integers[16] and that κ_l and m_l satisfy[17]

$$S_{lk}\kappa_k = -\kappa_l, \qquad S_{lk}m_k = -m_l, \tag{7.15}$$

Finally,

$$p_l(\mathtt{x}) = -\frac{\kappa_l \mathtt{x} + 2\pi m_l}{\mathtt{x}^2 - 1} + \int d\mathtt{y}\, \frac{\rho_l(\mathtt{y})}{\mathtt{x} - \mathtt{y}} - S_{lm} \int \frac{d\mathtt{y}}{\mathtt{y}^2}\, \frac{\rho_m(\mathtt{y})}{\mathtt{x} - \frac{1}{\mathtt{y}}}\,. \tag{7.16}$$

The integration contours lie entirely outside the unit circle. The condition (7.3) becomes a set of integral equations for the densities:

$$A_{lm} \fint d\mathtt{y}\, \frac{\rho_m(\mathtt{y})}{\mathtt{x} - \mathtt{y}} - A_{lk}S_{km} \int \frac{d\mathtt{y}}{\mathtt{y}^2}\, \frac{\rho_m(\mathtt{y})}{\mathtt{x} - \frac{1}{\mathtt{y}}} = A_{lk} \frac{\kappa_k \mathtt{x} + 2\pi m_k}{\mathtt{x}^2 - 1} + 2\pi n_{l,i}, \qquad \mathtt{x} \in C_{l,i}. \tag{7.17}$$

For bosonic nodes of the Dynkin diagram, these equations hold on the cuts of the quasi-momenta. For the fermionic nodes, they hold at the positions of singularities. Solutions of these equations describe quasi-periodic solutions of the equations of motion in the sigma-model. The conserved charges for a given solution can be computed by expanding the quasi-momenta at infinity. The classical finite-gap equations have a direct quantum counterpart, the Bethe equations for the quantum spectrum of the sigma-model.

7.2 Classical Bethe equations for $D(2,1;\alpha) \times D(2,1;\alpha)$

The symmetrized Cartan matrix of[18] $D_+(2,1;\alpha) \times D_-(2,1;\alpha)$ is

$$A = \begin{pmatrix} 4\sin^2\phi & -2\sin^2\phi & 0 \\ -2\sin^2\phi & 0 & -2\cos^2\phi \\ 0 & -2\cos^2\phi & 4\cos^2\phi \end{pmatrix} \otimes \mathbb{1}. \tag{7.18}$$

The second factor in the tensor product acts on the $\pm$ indices. The $\mathbb{Z}_4$ symmetry operator acts on the Cartan generators simply by permutation of the two $D(2,1;\alpha)$ factors:

$$S = \mathbb{1} \otimes \sigma^1. \tag{7.19}$$

[16] Under $\mathtt{x} \to 1/\mathtt{x}$, the pole part of p_l changes sign and shifts by $-2\pi m_l$, provided that the equations below are satisfied. If m_l's are integers, the shift has no physical significance, since the quasi-momentum is defined up to an integer multiple of 2π. Typically, the integers m_l have the meaning of the winding numbers [63].

[17] Since $\Omega^2 = (-1)^F$ and H_l are Grassmann-even, the matrix S_{lm} squares to one and has eigenvalues one or minus one.

[18] In this section, we denote quantities related to the left (right) $\mathfrak{d}(2,1;\alpha)$ by $+(-)$.

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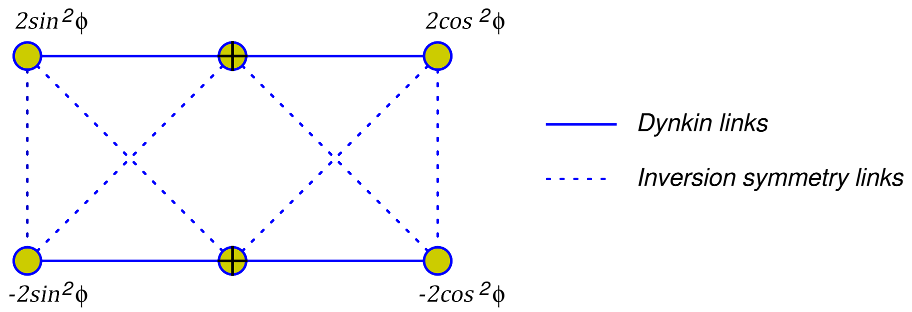

Figure 1. The Dynkin diagram of the classical Bethe equations.

The vector κ_l can be found by evaluating the monodromy matrix on the vacuum solution (5.1) and calculating the residue of the quasi-momenta at $\mathrm{x} = \pm 1$:

$$\kappa = 2\pi\mathcal{E}\begin{pmatrix}0\\1\\0\end{pmatrix}\otimes\begin{pmatrix}-1\\1\end{pmatrix}. \tag{7.20}$$

Here, $\mathcal{E} = D/\sqrt{\lambda}$ is the ratio of the energy of the string to its tension which we denote by[19] $\sqrt{\lambda}/(2\pi)$.

These data leads to the following set of classical Bethe equations:

$$\begin{aligned}\pm 4\pi\sin^2\phi\,\frac{\mathcal{E}\mathrm{x}+m}{\mathrm{x}^2-1}+2\pi n^{\pm}_{1,i} &= 4\sin^2\phi\,⨍ d\mathrm{y}\,\frac{\rho_1^{\pm}}{\mathrm{x}-\mathrm{y}}-2\sin^2\phi\int d\mathrm{y}\,\frac{\rho_2^{\pm}}{\mathrm{x}-\mathrm{y}}\\ &\quad -4\sin^2\phi\int\frac{d\mathrm{y}}{\mathrm{y}^2}\,\frac{\rho_1^{\mp}}{\mathrm{x}-\frac{1}{\mathrm{y}}}+2\sin^2\phi\int\frac{d\mathrm{y}}{\mathrm{y}^2}\,\frac{\rho_2^{\mp}}{\mathrm{x}-\frac{1}{\mathrm{y}}} \end{aligned}\tag{7.21}$$

$$\begin{aligned}2\pi n^{\pm}_{2,i} &= -2\sin^2\phi\int d\mathrm{y}\,\frac{\rho_1^{\pm}}{\mathrm{x}-\mathrm{y}}-2\cos^2\phi\int d\mathrm{y}\,\frac{\rho_3^{\pm}}{\mathrm{x}-\mathrm{y}}\\ &\quad +2\sin^2\phi\int\frac{d\mathrm{y}}{\mathrm{y}^2}\,\frac{\rho_1^{\mp}}{\mathrm{x}-\frac{1}{\mathrm{y}}}+2\cos^2\phi\int\frac{d\mathrm{y}}{\mathrm{y}^2}\,\frac{\rho_3^{\mp}}{\mathrm{x}-\frac{1}{\mathrm{y}}} \end{aligned}\tag{7.22}$$

$$\begin{aligned}\pm 4\pi\cos^2\phi\,\frac{\mathcal{E}\mathrm{x}+m}{\mathrm{x}^2-1}+2\pi n^{\pm}_{3,i} &= 4\cos^2\phi\,⨍ d\mathrm{y}\,\frac{\rho_3^{\pm}}{\mathrm{x}-\mathrm{y}}-2\cos^2\phi\int d\mathrm{y}\,\frac{\rho_2^{\pm}}{\mathrm{x}-\mathrm{y}}\\ &\quad -4\cos^2\phi\int\frac{d\mathrm{y}}{\mathrm{y}^2}\,\frac{\rho_3^{\mp}}{\mathrm{x}-\frac{1}{\mathrm{y}}}+2\cos^2\phi\int\frac{d\mathrm{y}}{\mathrm{y}^2}\,\frac{\rho_2^{\mp}}{\mathrm{x}-\frac{1}{\mathrm{y}}}\,. \end{aligned}\tag{7.23}$$

The integer m is the winding number of the string. The equations can be summarized in a diagram shown in figure 1. The nodes of the diagram correspond to the densities $\rho_n^{\pm}$. The left hand side of the equations is determined by the Dynkin labels, and the right hand side by the links. The original Dynkin links from the Cartan matrix (7.18) determine coefficients in the first term in (7.17). The second term, which is associated with the inversion symmetry, produces additional links on the Dynkin diagram, shown in figure 1 by broken lines.

[19] In the AdS_5/CFT_4 case, λ is also the 't Hooft coupling of the dual super-Yang-Mills theory. The precise nature of the parameter λ in the CFT_2 dual of the $AdS_3\times S^3\times S^3\times S^1$ background is not clear to us.

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The light-cone energy ($\mathcal{E} - \mathcal{J} = (D - J)/\sqrt{\lambda}$) and the worldsheet momentum of the solution are given by

$$\mathcal{P} = -2 \sum_{s=\pm} s \left(\sin^2 \phi \int \frac{d\mathrm{x}}{\mathrm{x}} \, \rho_1^s(\mathrm{x}) + \cos^2 \phi \int \frac{d\mathrm{x}}{\mathrm{x}} \, \rho_3^s(\mathrm{x}) \right) \tag{7.24}$$

$$\mathcal{E} - \mathcal{J} = 2 \sum_{s=\pm} \left(\sin^2 \phi \int \frac{d\mathrm{x}}{\mathrm{x}^2} \, \rho_1^s(\mathrm{x}) + \cos^2 \phi \int \frac{d\mathrm{x}}{\mathrm{x}^2} \, \rho_3^s(\mathrm{x}) \right). \tag{7.25}$$

Physical states should in addition satisfy the level-matching condition $\mathcal{P} \in 2\pi\mathbb{Z}$.

7.3 BMN limit

The trivial solution of the finite-gap equations, with zero densities, describes the BMN vacuum. The BMN modes correspond to the vanishingly small cuts whose position is determined by the no-force condition, the vanishing of the left-hand side the Bethe equations. For instance, the left hand side of (7.21) vanishes at

$$\frac{1}{\mathrm{x}_n} = \frac{\mathcal{J}}{n} \left(\sin^2 \phi - \sqrt{\sin^4 \phi + \frac{n^2}{\mathcal{J}^2}} \right), \tag{7.26}$$

where we have set the winding number m to zero and also neglected the difference between $\mathcal{E}$, which enters the classical Bethe equations, and $\mathcal{J}$, which plays the role of the length of the string in the light-cone gauge. This is justified for small deviations from the BMN vacuum. The set of points $\{\mathrm{x}_n\}$ determines the locus at which short cuts with infinitesimal filling fractions can emerge. The solution with infinitesimal cuts corresponds to exciting a number of BMN modes. Their occupation numbers N_n are proportional to the filling fractions of the cuts:

$$S_n = \int_{C_{1,n}} d\mathrm{x} \, \rho(\mathrm{x}). \tag{7.27}$$

The precise relationship between the occupation numbers are the filling fractions is derived in [16]:

$$S_n = \frac{\pi N_n}{\sqrt{\lambda}} \left(1 + \sqrt{1 + \frac{n^2}{\mathcal{J}^2 \sin^4 \phi}} \right).$$

The energy then is

$$\mathcal{E} - \mathcal{J} = \sum_n \frac{2 S_n \sin^2 \phi}{\mathrm{x}_n^2} = \frac{2\pi}{\sqrt{\lambda}} \sum_n N_n \left(\sqrt{\sin^4 \phi + \frac{n^2}{\mathcal{J}^2}} - \sin^2 \phi \right),$$

Because the length of the string in the light-cone gauge is $2\pi\mathcal{J}$, the combination $n/\mathcal{J}$ plays the role of the worldsheet momentum. The spectrum of small fluctuations thus describes particles with the dispersion relation

$$\varepsilon(p) = \sqrt{p^2 + \sin^4 \phi}. \tag{7.28}$$

Similarly, the densities $\rho_3^{\pm}(\mathtt{x})$ describe particles with mass $\cos^2\phi$. The heavy modes are more tricky. They correspond to stacks [72, 73] that cross from node 1 to node 3 through node 2. The stack, roughly speaking, is a set of overlapping densities on different nodes. In this particular case it is a simultaneous solution of a pair of equations

$$\frac{2\sin^2\phi\,\mathcal{J}\mathtt{x}}{\mathtt{x}^2-1}=2\pi n,\qquad \frac{2\cos^2\phi\,\mathcal{J}\mathtt{x}}{\mathtt{x}^2-1}=2\pi m. \tag{7.29}$$

The solution is only possible in the thermodynamic limit $n\sim m\sim\mathcal{J}\to\infty$, since n and m must satisfy $n/m=\tan^2\phi$, in which case the stack corresponds to a particle of mass 1. The stacks are also responsible for the correct four-fold degeneracy at each mass level. For instance, the bosonic members of the $\mathrm{PSU}(1|1)\times\mathrm{PSU}(1|1)$ multiplet of mass $\sin^2\phi$ are the single-node solutions for the densities ρ_1^+ and ρ_1^-. The fermions in the same multiplet are the 1^+-2^+ and 1^--2^- stacks.

We have correctly reproduced the massive part of the BMN spectrum. However, the massless modes, that we also found in section 5, are completely missing. As we explain below in section 8, the finite-gap equations do not capture massless modes and describe only those solutions of the sigma-model in which the massless modes are not excited. Although this makes our analysis incomplete, we will proceed with quantization of the classical Bethe equations obtained above.

7.4 Quantum Bethe equations

In the previously studied cases of $AdS_5\times S^5$ and $AdS_4\times CP^3$ the relationship between classical and quantum Bethe equations follows a regular pattern dictated by the structure of the Dynkin diagram. At the quantum level, the densities $\rho_l(\mathtt{x})$ describe macroscopic distributions of the Bethe roots, the solutions of the quantum Bethe equations. When the number of roots is very large and the sums over the roots can be replaced by the integrals over their densities, the quantum Bethe equations reduce to the finite-gap integral equations for the densities. By way of observation, one can notice that the quantum Bethe equations for $AdS_5\times S^5$ and $AdS_4\times CP^3$ can be inverse engineered from the finite gap equations by applying a simple set of regular rules for each element in the Dynkin diagram. The key observation is that only three distinct structures appear in the finite-gap equations, namely:

(i) the normal Dynkin links;

(ii) the inversion symmetry links between a momentum-carrying node and "wrong" fermionic nodes (all possible links of type appear in the equations);

(iii) the inversions symmetry links that connect the momentum-carrying nodes pairwise.

It is quite remarkable that the classical Bethe equations in our case also contain only these three types of structural elements. We can thus apply the same set of rules as in AdS_5/CFT_4 and AdS_4/CFT_3 to discretize the classical Bethe equations derived above. There is one subtlety though. The discretization is straightforward only when the elements of the Cartan matrix are integers, which is not the case for the $\mathfrak{d}(2,1;\alpha)$ algebra in general.

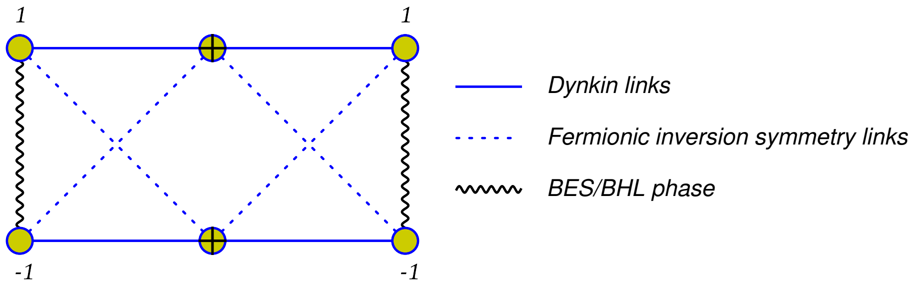

Figure 2. The Dynkin diagram of the asymptotic Bethe ansatz.

The two exceptions are $\phi = \pi/4$ (considered here) and $\phi = 0$ (discussed below), and we will restrict our attention to these two special cases.

When $\phi = \pi/4$, the $\mathfrak{d}(2,1;\alpha)$ superalgebra coincides with $\mathfrak{osp}(4|2)$ whose Cartan matrix has integer entries. In quantum theory, the cuts of the classical spectral curve get discretized and become the arrays of Bethe roots. The asymptotic Bethe ansatz determines the positions of the roots in the spectral plane, $\mathrm{x}_{l,i}$, through a system of discrete functional equations (the Bethe equations). The coupling constant of the sigma-model (playing the role of $\hbar$) $2\pi/\sqrt{\lambda}$ does not appear in the equations explicitly and only enters through the quantum parameters $\mathrm{x}^{\pm}$ defined by the Jukovsky map:

$$\mathrm{x}^{\pm} + \frac{1}{\mathrm{x}^{\pm}} = \mathrm{x} + \frac{1}{\mathrm{x}} \pm \frac{i}{2h(\lambda)} \,. \tag{7.30}$$

The function $h(\lambda)$ cannot determined by integrability alone, but at strong coupling should behave as

$$h(\lambda) \approx \frac{\sqrt{\lambda}}{2\pi} \qquad (\lambda \to \infty)\,, \tag{7.31}$$

in order to reproduce the correct dispersion relation $\varepsilon(p) = \sqrt{p^2 + 1/4}$.

The Bethe equations are constructed according to the Dynkin diagram in figure 2, which is obtained from the classical Dynkin diagram 1 by

(i) assigning $\mathrm{x} - \mathrm{x}^{\pm}$ type interactions to the normal links;

(ii) assigning the $\mathrm{x} - 1/\mathrm{x}^{\pm}$ type interactions to the anomalous fermionic links;

(iii) and finally associating the BES/BHL phase with the anomalous bosonic links.

The conjectured set of Bethe equations thus reads[20]

$$\left(\frac{\mathrm{x}^{+}_{1,j}}{\mathrm{x}^{-}_{1,j}}\right)^{L} = \prod_{k\neq j} \frac{\mathrm{x}^{+}_{1,j} - \mathrm{x}^{-}_{1,k}}{\mathrm{x}^{-}_{1,j} - \mathrm{x}^{+}_{1,k}} \frac{1 - \frac{1}{\mathrm{x}^{+}_{1,j}\mathrm{x}^{-}_{1,k}}}{1 - \frac{1}{\mathrm{x}^{-}_{1,j}\mathrm{x}^{+}_{1,k}}} \sigma^2(\mathrm{x}_{1,j}, \mathrm{x}_{1,k})$$
$$\times \prod_{k} \frac{\mathrm{x}^{-}_{1,j} - \mathrm{x}_{2,k}}{\mathrm{x}^{+}_{1,j} - \mathrm{x}_{2,k}} \prod_{k} \frac{1 - \frac{1}{\mathrm{x}^{-}_{1,j}\mathrm{x}_{\bar{2},k}}}{1 - \frac{1}{\mathrm{x}^{+}_{1,j}\mathrm{x}_{\bar{2},k}}} \prod_{k} \sigma^{-2}(\mathrm{x}_{1,j}, \mathrm{x}_{\bar{1},k})$$

[20]We denote by 1, 2, and 3 the upper nodes of $OSp_{+}(4|2)$ and by $\bar{1}$, $\bar{2}$, and $\bar{3}$ the lower nodes of $OSp_{-}(4|2)$. The nodes 1 and 3 ($\bar{1}$ and $\bar{3}$) carry momentum 1 (-1).

$$
\begin{aligned}
1 &= \prod_k \frac{\mathtt{x}_{2,j}-\mathtt{x}^+_{1,k}}{\mathtt{x}_{2,j}-\mathtt{x}^-_{1,k}} \prod_k \frac{\mathtt{x}_{2,j}-\mathtt{x}^+_{3,k}}{\mathtt{x}_{2,j}-\mathtt{x}^-_{3,k}} \prod_k \frac{1-\frac{1}{\mathtt{x}_{2,j}\mathtt{x}^+_{\bar{1},k}}}{1-\frac{1}{\mathtt{x}_{2,j}\mathtt{x}^-_{\bar{1},k}}} \prod_k \frac{1-\frac{1}{\mathtt{x}_{2,j}\mathtt{x}^+_{\bar{3},k}}}{1-\frac{1}{\mathtt{x}_{2,j}\mathtt{x}^-_{\bar{3},k}}} \\
\left(\frac{\mathtt{x}^+_{3,j}}{\mathtt{x}^-_{3,j}}\right)^L &= \prod_{k\neq j} \frac{\mathtt{x}^+_{3,j}-\mathtt{x}^-_{3,k}}{\mathtt{x}^-_{3,j}-\mathtt{x}^+_{3,k}} \, \frac{1-\frac{1}{\mathtt{x}^+_{3,j}\mathtt{x}^-_{3,k}}}{1-\frac{1}{\mathtt{x}^-_{3,j}\mathtt{x}^+_{3,k}}} \, \sigma^2(\mathtt{x}_{3,j},\mathtt{x}_{3,k}) \\
&\quad \times \prod_k \frac{\mathtt{x}^-_{3,j}-\mathtt{x}_{2,k}}{\mathtt{x}^+_{3,j}-\mathtt{x}_{2,k}} \prod_k \frac{1-\frac{1}{\mathtt{x}^-_{3,j}\mathtt{x}_{\bar{2},k}}}{1-\frac{1}{\mathtt{x}^+_{3,j}\mathtt{x}_{\bar{2},k}}} \prod_k \sigma^{-2}(\mathtt{x}_{3,j},\mathtt{x}_{\bar{3},k}) \\
\left(\frac{\mathtt{x}^-_{\bar{1},j}}{\mathtt{x}^+_{\bar{1},j}}\right)^L &= \prod_{k\neq j} \frac{\mathtt{x}^+_{\bar{1},j}-\mathtt{x}^-_{\bar{1},k}}{\mathtt{x}^-_{\bar{1},j}-\mathtt{x}^+_{\bar{1},k}} \, \frac{1-\frac{1}{\mathtt{x}^+_{\bar{1},j}\mathtt{x}^-_{\bar{1},k}}}{1-\frac{1}{\mathtt{x}^-_{\bar{1},j}\mathtt{x}^+_{\bar{1},k}}} \, \sigma^2(\mathtt{x}_{\bar{1},j},\mathtt{x}_{\bar{1},k}) \\
&\quad \times \prod_k \frac{\mathtt{x}^-_{\bar{1},j}-\mathtt{x}_{\bar{2},k}}{\mathtt{x}^+_{\bar{1},j}-\mathtt{x}_{\bar{2},k}} \prod_k \frac{1-\frac{1}{\mathtt{x}^-_{\bar{1},j}\mathtt{x}_{2,k}}}{1-\frac{1}{\mathtt{x}^+_{\bar{1},j}\mathtt{x}_{2,k}}} \prod_k \sigma^{-2}(\mathtt{x}_{\bar{1},j},\mathtt{x}_{1,k}) \\
1 &= \prod_k \frac{\mathtt{x}_{\bar{2},j}-\mathtt{x}^+_{\bar{1},k}}{\mathtt{x}_{\bar{2},j}-\mathtt{x}^-_{\bar{1},k}} \prod_k \frac{\mathtt{x}_{\bar{2},j}-\mathtt{x}^+_{\bar{3},k}}{\mathtt{x}_{\bar{2},j}-\mathtt{x}^-_{\bar{3},k}} \prod_k \frac{1-\frac{1}{\mathtt{x}_{\bar{2},j}\mathtt{x}^+_{1,k}}}{1-\frac{1}{\mathtt{x}_{\bar{2},j}\mathtt{x}^-_{1,k}}} \prod_k \frac{1-\frac{1}{\mathtt{x}_{\bar{2},j}\mathtt{x}^+_{3,k}}}{1-\frac{1}{\mathtt{x}_{\bar{2},j}\mathtt{x}^-_{3,k}}} \\
\left(\frac{\mathtt{x}^-_{\bar{3},j}}{\mathtt{x}^+_{\bar{3},j}}\right)^L &= \prod_{k\neq j} \frac{\mathtt{x}^+_{\bar{3},j}-\mathtt{x}^-_{\bar{3},k}}{\mathtt{x}^-_{\bar{3},j}-\mathtt{x}^+_{\bar{3},k}} \, \frac{1-\frac{1}{\mathtt{x}^+_{\bar{3},j}\mathtt{x}^-_{\bar{3},k}}}{1-\frac{1}{\mathtt{x}^-_{\bar{3},j}\mathtt{x}^+_{\bar{3},k}}} \, \sigma^2(\mathtt{x}_{\bar{3},j},\mathtt{x}_{\bar{3},k}) \\
&\quad \times \prod_k \frac{\mathtt{x}^-_{\bar{3},j}-\mathtt{x}_{\bar{2},k}}{\mathtt{x}^+_{\bar{3},j}-\mathtt{x}_{\bar{2},k}} \prod_k \frac{1-\frac{1}{\mathtt{x}^-_{\bar{3},j}\mathtt{x}_{2,k}}}{1-\frac{1}{\mathtt{x}^+_{\bar{3},j}\mathtt{x}_{2,k}}} \prod_k \sigma^{-2}(\mathtt{x}_{\bar{3},j},\mathtt{x}_{3,k})
\end{aligned}
\tag{7.32}
$$

The energy and momentum are given by

$$
\begin{aligned}
E &= ih(\lambda) \sum_{l=1,3,\bar{1},\bar{3}} \sum_j \left(\frac{1}{\mathtt{x}^+_{l,j}} - \frac{1}{\mathtt{x}^-_{l,j}}\right) \\
\mathrm{e}^{\frac{2i\pi P}{\sqrt{\lambda}}} &= \frac{\prod_{l=1,3}}{\prod_{l=\bar{1},\bar{3}}} \prod_j \frac{\mathtt{x}^+_{l,j}}{\mathtt{x}^-_{l,j}} \equiv 1.
\end{aligned}
\tag{7.33}
$$

The elementary excitation associated to single Bethe roots then have the dispersion relation

$$
\varepsilon(p) = \sqrt{4h^2(\lambda)\sin^2\frac{\pi p}{\sqrt{\lambda}} + \frac{1}{4}},
$$

which reproduces the dispersion relation of the BMN modes provided that the function $h(\lambda)$ behaves at strong coupling as (7.31).

The BES/BHL phase [75, 76] admits the following integral representation [77]:

$$
\sigma(\mathtt{x},\mathtt{y}) = \exp\left[\sum_{r,s=\pm} \frac{rs}{4\pi^2} \oint \frac{dz\,dw}{(\mathtt{x}^r - z)(\mathtt{y}^s - w)} \, \ln\frac{\Gamma\left(1 + ih(\lambda)\left(z + \frac{1}{z} - w - \frac{1}{w}\right)\right)}{\Gamma\left(1 - ih(\lambda)\left(z + \frac{1}{z} - w - \frac{1}{w}\right)\right)}\right], \tag{7.34}
$$

where the integration contour is the unit circle: $|z| = 1$, $|w| = 1$. From this representation one can readily infer the analytic structure and the symmetry properties of $\sigma(\mathtt{x}, \mathtt{y})$ [78, 79]. At the lowest order in the strong-coupling expansion, $h \to \infty$, the dressing factor reduces to the AFS phase [24]:

$$\sigma(\mathtt{x}, \mathtt{y}) \approx \frac{1 - \frac{1}{\mathtt{x}^-\mathtt{y}^+}}{1 - \frac{1}{\mathtt{x}^+\mathtt{y}^-}} \left[\frac{\left(1 - \frac{1}{\mathtt{x}^+\mathtt{y}^-}\right)\left(1 - \frac{1}{\mathtt{x}^-\mathtt{y}^+}\right)}{\left(1 - \frac{1}{\mathtt{x}^+\mathtt{y}^+}\right)\left(1 - \frac{1}{\mathtt{x}^-\mathtt{y}^-}\right)} \right]^{ih\left(\mathtt{x}+\frac{1}{\mathtt{x}}-\mathtt{y}-\frac{1}{\mathtt{y}}\right)} \qquad (h \to \infty), \tag{7.35}$$

and this is all one needs to reproduce the classical Bethe equations, which indeed follow from the quantum equations above in the $h \to \infty$ limit, upon the identification

$$\rho_l^+(\mathtt{x}) = \frac{1}{h} \sum_j \frac{\mathtt{x}_{l,j}^2}{\mathtt{x}_{l,j}^2 - 1}\, \delta(\mathtt{x} - \mathtt{x}_{l,j}), \qquad \rho_l^-(\mathtt{x}) = \frac{1}{h} \sum_j \frac{\mathtt{x}_{\bar{l},j}^2}{\mathtt{x}_{\bar{l},j}^2 - 1}\, \delta(\mathtt{x} - \mathtt{x}_{\bar{l},j}). \tag{7.36}$$

As we discussed above, the classical finite-gap equations describe only massive modes of the string. The sole reason to quantize this incomplete set of equations is their striking similarity to the finite-gap equations in $AdS_5 \times S^5$ and $AdS_4 \times CP^3$. In those cases, the resulting quantum Bethe equations describe the full asymptotic spectrum at any coupling. For sure, this cannot be true here, because the massless modes are missing already at the classical level. Our best hope is that the equations above represent a truncation of the hypothetical complete Bethe ansatz system that also takes into account the massless modes of the string. In an integrable system, in which action-angle variables separate, such truncations are in many cases possible. For instance, the Bethe equations for the $\mathfrak{su}(2)$ sector of AdS_5/CFT_4 [24] can be reconstructed from the finite-gap equations on $S^3 \times R^1$ [16], which ignores most part of the string modes in $AdS_5 \times S^5$. At the moment we cannot justify that ignoring the massless modes we get a consistent truncation of the Bethe equations in AdS_3/CFT_2. It is also possible that the inclusion of massless modes modifies the equations in a more essential way than just adding extra nodes into the Dynkin diagram.

7.5 $AdS_3 \times S^3 \times T^4$

We can apply the same general formalism to the coset part of the $AdS_3 \times S^3 \times T^4$ background, which is the sigma-model on $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)/\mathrm{SU}(1,1) \times \mathrm{SU}(2)$. The Cartan matrix of $\mathrm{PSU}(1,1|2) \times \mathrm{PSU}(1,1|2)$ is

$$A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{pmatrix} \otimes \mathbb{1}. \tag{7.37}$$

The $\mathbb{Z}_4$ symmetry operator again is the permutation:

$$S = \mathbb{1} \otimes \sigma^1. \tag{7.38}$$

From these data we infer the classical Bethe equations (figure 3):

$$2\pi n_{1,i}^{\pm} = -\int d\mathtt{y}\, \frac{\rho_2^{\pm}}{\mathtt{x} - \mathtt{y}} + \int \frac{d\mathtt{y}}{\mathtt{y}^2}\, \frac{\rho_2^{\mp}}{\mathtt{x} - \frac{1}{\mathtt{y}}} \tag{7.39}$$

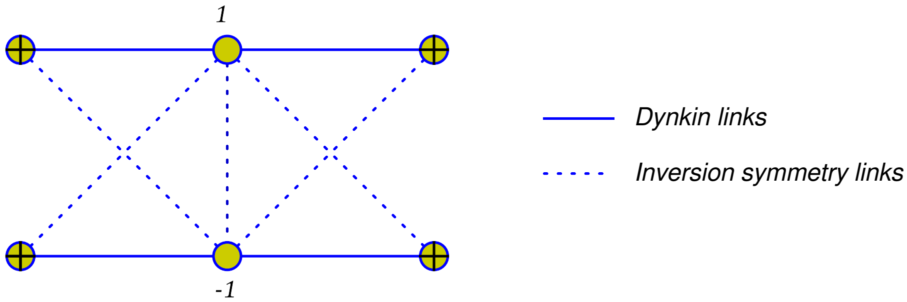

Figure 3. The Dynkin diagram for the PSU(1, 1|2) × PSU(1, 1|2)/SU(1, 1) × SU(2) coset.

$$\begin{aligned}\pm 4\pi \frac{\mathcal{E}\mathrm{x}+m}{\mathrm{x}^2-1}+2\pi n^{\pm}_{2,i} &= -\int dy\, \frac{\rho_1^{\pm}}{\mathrm{x}-\mathrm{y}}+2\fint dy\, \frac{\rho_2^{\pm}}{\mathrm{x}-\mathrm{y}}-\int dy\, \frac{\rho_3^{\pm}}{\mathrm{x}-\mathrm{y}} \\ &\quad +\int \frac{dy}{\mathrm{y}^2}\, \frac{\rho_1^{\mp}}{\mathrm{x}-\frac{1}{\mathrm{y}}}-2\int \frac{dy}{\mathrm{y}^2}\, \frac{\rho_2^{\mp}}{\mathrm{x}-\frac{1}{\mathrm{y}}}+\int \frac{dy}{\mathrm{y}^2}\, \frac{\rho_3^{\mp}}{\mathrm{x}-\frac{1}{\mathrm{y}}} \qquad (7.40)\\ 2\pi n^{\pm}_{3,i} &= -\int dy\, \frac{\rho_2^{\pm}}{\mathrm{x}-\mathrm{y}}+\int \frac{dy}{\mathrm{y}^2}\, \frac{\rho_2^{\mp}}{\mathrm{x}-\frac{1}{\mathrm{y}}} \qquad (7.41)\end{aligned}$$

The quantum Bethe equations can be reconstructed by applying the same set of rules as before (figure 4):

$$\begin{aligned}1 &= \prod_k \frac{\mathrm{x}_{1,j}-\mathrm{x}^+_{2,k}}{\mathrm{x}_{1,j}-\mathrm{x}^-_{2,k}} \prod_k \frac{1-\frac{1}{\mathrm{x}_{1,j}\mathrm{x}^+_{\bar{2},k}}}{1-\frac{1}{\mathrm{x}_{1,j}\mathrm{x}^-_{\bar{2},k}}} \\ \left(\frac{\mathrm{x}^+_{2,j}}{\mathrm{x}^-_{2,j}}\right)^L &= \prod_{k\neq j} \frac{\mathrm{x}^+_{2,j}-\mathrm{x}^-_{2,k}}{\mathrm{x}^-_{2,j}-\mathrm{x}^+_{2,k}}\, \frac{1-\frac{1}{\mathrm{x}^+_{2,j}\mathrm{x}^-_{2,k}}}{1-\frac{1}{\mathrm{x}^-_{2,j}\mathrm{x}^+_{2,k}}}\, \sigma^2(\mathrm{x}_{2,j},\mathrm{x}_{2,k}) \prod_k \frac{\mathrm{x}^-_{2,j}-\mathrm{x}_{1,k}}{\mathrm{x}^+_{2,j}-\mathrm{x}_{1,k}} \prod_k \frac{\mathrm{x}^-_{2,j}-\mathrm{x}_{3,k}}{\mathrm{x}^+_{2,j}-\mathrm{x}_{3,k}} \\ &\quad \times \prod_k \frac{1-\frac{1}{\mathrm{x}^-_{2,j}\mathrm{x}_{\bar{1},k}}}{1-\frac{1}{\mathrm{x}^+_{2,j}\mathrm{x}_{\bar{1},k}}} \prod_k \frac{1-\frac{1}{\mathrm{x}^-_{2,j}\mathrm{x}_{\bar{3},k}}}{1-\frac{1}{\mathrm{x}^+_{2,j}\mathrm{x}_{\bar{3},k}}} \prod_k \sigma^{-2}(\mathrm{x}_{2,j},\mathrm{x}_{\bar{2},k}) \\ 1 &= \prod_k \frac{\mathrm{x}_{3,j}-\mathrm{x}^+_{2,k}}{\mathrm{x}_{3,j}-\mathrm{x}^-_{2,k}} \prod_k \frac{1-\frac{1}{\mathrm{x}_{3,j}\mathrm{x}^+_{\bar{2},k}}}{1-\frac{1}{\mathrm{x}_{3,j}\mathrm{x}^-_{\bar{2},k}}} \\ 1 &= \prod_k \frac{\mathrm{x}_{\bar{1},j}-\mathrm{x}^+_{\bar{2},k}}{\mathrm{x}_{\bar{1},j}-\mathrm{x}^-_{\bar{2},k}} \prod_k \frac{1-\frac{1}{\mathrm{x}_{\bar{1},j}\mathrm{x}^+_{2,k}}}{1-\frac{1}{\mathrm{x}_{\bar{1},j}\mathrm{x}^-_{2,k}}} \\ \left(\frac{\mathrm{x}^+_{\bar{2},j}}{\mathrm{x}^-_{\bar{2},j}}\right)^L &= \prod_{k\neq j} \frac{\mathrm{x}^+_{\bar{2},j}-\mathrm{x}^-_{\bar{2},k}}{\mathrm{x}^-_{\bar{2},j}-\mathrm{x}^+_{\bar{2},k}}\, \frac{1-\frac{1}{\mathrm{x}^+_{\bar{2},j}\mathrm{x}^-_{\bar{2},k}}}{1-\frac{1}{\mathrm{x}^-_{\bar{2},j}\mathrm{x}^+_{\bar{2},k}}}\, \sigma^2(\mathrm{x}_{\bar{2},j},\mathrm{x}_{\bar{2},k}) \prod_k \frac{\mathrm{x}^-_{\bar{2},j}-\mathrm{x}_{\bar{1},k}}{\mathrm{x}^+_{\bar{2},j}-\mathrm{x}_{\bar{2},k}} \prod_k \frac{\mathrm{x}^-_{\bar{2},j}-\mathrm{x}_{\bar{3},k}}{\mathrm{x}^+_{\bar{2},j}-\mathrm{x}_{\bar{3},k}} \\ &\quad \times \prod_k \frac{1-\frac{1}{\mathrm{x}^-_{\bar{2},j}\mathrm{x}_{1,k}}}{1-\frac{1}{\mathrm{x}^+_{\bar{2},j}\mathrm{x}_{1,k}}} \prod_k \frac{1-\frac{1}{\mathrm{x}^-_{\bar{2},j}\mathrm{x}_{3,k}}}{1-\frac{1}{\mathrm{x}^+_{\bar{2},j}\mathrm{x}_{3,k}}} \prod_k \sigma^{-2}(\mathrm{x}_{\bar{2},j},\mathrm{x}_{2,k})\end{aligned}$$

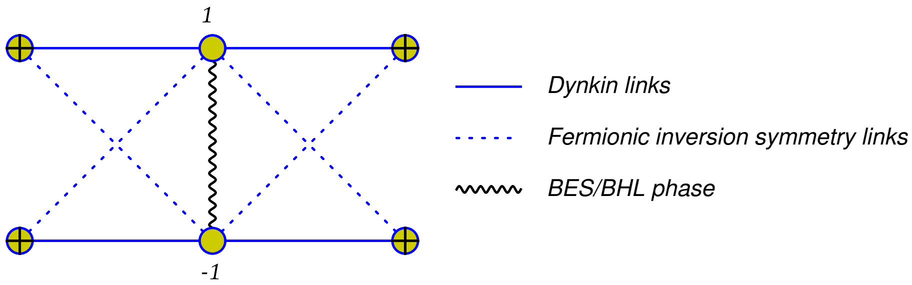

Figure 4. The Dynkin diagram for the quantum Bethe equations.

$$1 = \prod_k \frac{\mathrm{x}_{\bar{3},j} - \mathrm{x}^+_{2,k}}{\mathrm{x}_{\bar{3},j} - \mathrm{x}^-_{2,k}} \prod_k \frac{1 - \frac{1}{\mathrm{x}_{\bar{3},j}\mathrm{x}^+_{2,k}}}{1 - \frac{1}{\mathrm{x}_{\bar{3},j}\mathrm{x}^-_{2,k}}} \,. \tag{7.42}$$

Again it is not too hard to show that these equations describe $4_B + 4_F$ modes with mass 1, and do not capture $4_B + 4_F$ massless modes. So the remarks at the end of section 7.4 apply to this case as well.

8 Outlook: some comments on the massless modes

A new feature of the AdS_3/CFT_2 dual theories, as compared with the AdS_5/CFT_4 and AdS_4/CFT_3 cases, is the presence of massless modes. We have encountered these massless modes as appearing either from the free boson(s) on S^1 or T^4, or from an extra massless mode in the BMN limit of the $AdS_3 \times S^3 \times S^3 \times S^1$ theory; there are also fermionic massless modes related to the bosonic modes by supersymmetry. The integrability techniques we have used throughout this paper have little to say about the massless degrees of freedom.

It is not difficult to see why the finite-gap methods fail to capture the massless modes, just by looking at the bosonic string on the $S^3 \times S^3 \times \mathbb{R}^1$ subspace of $AdS_3 \times S^3 \times S^3 \times S^1$. The $\mathbb{R}^1$ factor is the global time direction in AdS_3. At the classical level, all three factors are independent and related only by the Virasoro constraints. It is known that the Lax connection, in effect, imposes the constraints automatically [80]. In the case at hand the Lax connection actually over-imposes the Virasoro by setting the energy momentum tensors $T^{1,2}_{\pm\pm}$ for each of the three-spheres to a constant independently, and thus kills two degrees of freedom instead of one — we want only the sum $T^1_{\pm\pm} + T^2_{\pm\pm}$ to be constrained. The massless excitation arises from the combination of the longitudinal modes on the two spheres such that their Virasoro-violating contributions mutually cancel. The finite-gap method eliminates this mode from the very beginning.

By applying the finite-gap method to the bosonic $AdS_5 \times S^5$ or $AdS_4 \times CP^3$ backgrounds one imposes two constraints and thus eliminates just the right amount of unphysical, longitudinal modes of the string. In $AdS_3 \times S^3 \times S^3$, we eliminate three degrees of freedom, one of which is actually physical. Refs. [81–83] discuss how longitudinal modes can be incorporated in the classical/quantum Bethe ansatz. In the case of the S^3 sigma-model discussed there, the longitudinal degree of freedom corresponds to the extra rapidity variables in the

Bethe equations. Perhaps our equations from section 7 could also be augmented by on or more extra rapidity node(s) in order to describe the missing massless degrees of freedom.

In this section we collect together some first steps we have taken towards understanding the zero-modes. We hope to return to these issues in the future.

8.1 Semi-classical lightlike geodesics

Throughout this section we ignore the spacetime S^1, as it plays no role in the classical analysis - its equation of motion is decoupled from the others, apart from the Virasoro constraints, and so can be set to zero for simplicity. We use complex embedding coordinates X_i, Y_i and Z_i for AdS_3 and the two S^3's

$$-X^1X_1 + X^2X_2 = -l^2\,, \qquad Y^1Y_1 + Y^2Y_2 = R_+^2\,, \qquad Z^1Z_1 + Z^2Z_2 = R_-^2\,, \tag{8.1}$$

where $X^i \equiv X_i^*$, $Y^i \equiv Y_i^*$ and $Z^i \equiv Z_i^*$. The equations of motion are

$$-\partial^2 X_i + \Lambda_X X_i = 0\,, \qquad l^2\Lambda_X = -X^i\partial^2 X_i\,, \tag{8.2}$$

$$-\partial^2 Y_i + \Lambda_Y Y_i = 0\,, \qquad R_+^2\Lambda_Y = Y^i\partial^2 Y_i\,, \tag{8.3}$$

$$-\partial^2 Z_i + \Lambda_Z Z_i = 0\,, \qquad R_-^2\Lambda_Z = Z^i\partial^2 Z_i\,. \tag{8.4}$$

The Virasoro constraints are

$$-\dot{X}^i\dot{X}_i - X^{i\prime}X_i' + \dot{Y}^i\dot{Y}_i + Y^{i\prime}Y_i' + \dot{Z}^i\dot{Z}_i + Z^{i\prime}Z_i' = 0\,, \tag{8.5}$$

$$-\dot{X}^iX_i' + \dot{Y}^iY_i' + \dot{Z}^iZ_i' + \text{c.c.} = 0\,. \tag{8.6}$$

The Cartan generators of the Noether charges are

$$E = \frac{i}{4\pi\alpha'}\int_0^{2\pi} d\sigma\,\left(X_1\dot{X}^1 - \dot{X}_1X^1\right), \tag{8.7}$$

$$J = \frac{i}{4\pi\alpha'}\int_0^{2\pi} d\sigma\,\left(X_2\dot{X}^2 - \dot{X}_2X^2\right), \tag{8.8}$$

$$J_i^+ = \frac{i}{4\pi\alpha'}\int_0^{2\pi} d\sigma\,\left(Y_i\dot{Y}^i - \dot{Y}_iY^i\right), \tag{8.9}$$

$$J_i^- = \frac{i}{4\pi\alpha'}\int_0^{2\pi} d\sigma\,\left(Z_i\dot{Z}^i - \dot{Z}_iZ^i\right), \tag{8.10}$$

where above there is no summation on the index $i = 1, 2$. Using the global symmetry of the background a point-particle world-line can always be rotated to

$$X_1 = le^{i\kappa(t)}\,, \qquad Y_1 = R_+e^{i\omega_1(t)}\,, \qquad Z_1 = R_-e^{i\omega_2(t)}\,, \tag{8.11}$$

for some, as yet undetermined, κ, ω_i functions of the world-line parameter t. Simple solutions of the equations of motion[21] can be found for

$$\kappa(t) = \kappa t\,, \qquad \omega_i(t) = \omega_i t\,, \tag{8.12}$$

[21] A particular case of such solutions was analyzed in [84].

for constant κ, ω_i. In this case the Lagrange multipliers are constant

$$\Lambda_X = \kappa^2\,, \qquad \Lambda_Y = \omega_1^2\,, \qquad \Lambda_Z = \omega_2^2\,, \tag{8.13}$$

and the Virasoro constraints reduce to

$$\kappa^2 = \omega_1^2 + \omega_2^2\,, \tag{8.14}$$

which can be solved in terms of an angle variable θ

$$\omega_1 = \kappa\cos\theta\,, \qquad \omega_2 = \kappa\sin\theta\,. \tag{8.15}$$

The angle θ denotes the relative angle between the great circles on the two S^3 factors. In the coset language the fact that such a one-parameter family of geodesics exists corresponds to the fact that (for $C'' = 0$) equation (5.5) is solved by

$$C = \cos\phi\cos\theta\,, \qquad C' = \sin\phi\sin\theta\,. \tag{8.16}$$

The Noether charges of these solutions are

$$\begin{aligned} E &= \frac{l^2}{\alpha'}\kappa\,, \\ J_1^+ &= \frac{R_+^2}{\alpha'}\omega_1 = \kappa\frac{l^2}{\alpha'}\frac{\cos\theta}{\cos^2\phi}\,, \qquad J_1^- = \frac{R_-^2}{\alpha'}\omega_2 = \kappa\frac{l^2}{\alpha'}\frac{\sin\theta}{\sin^2\phi}\,. \end{aligned} \tag{8.17}$$

These solutions correspond to a different choice of the light-like geodesic, compared to the supersymmetric case studied in section 5. One picks a different linear combination of the Noether charges to define the angular momentum:

$$\mathcal{J} \equiv j^+\cos^2\phi J_1^+ + j^-\sin^2\phi J_1^- \tag{8.18}$$

where the ϕ dependence is chosen for later convenience. Since $\mathcal{J}$ has to be suitably normalised (in the measure given by equation (A.3)), we must take

$$j_+^2\cos^2\phi + j_-^2\sin^2\phi = 1\,, \tag{8.19}$$

in other words we should choose them as

$$j_+ = \frac{\cos\lambda}{\cos\phi}\,, \qquad j_- = \frac{\sin\lambda}{\sin\phi}\,, \tag{8.20}$$

for a free parameter λ. The choice of $j_+ = j_- = 1$, or $\lambda = \phi$ corresponds to the BPS geodesic of section 5. Evaluating $E - \mathcal{J}$ on solutions (8.11)–(8.12) we find

$$E - \mathcal{J} = \frac{l^2}{\alpha'}\left(\kappa - j^+\omega_1 - j^-\omega_2\right) = \kappa\frac{l^2}{\alpha'}\left(1 - \cos(\lambda - \theta)\right)\,. \tag{8.21}$$

The BPS groundstate corresponds to

$$\omega_1 = \kappa\cos\lambda\cos\phi\,, \qquad \omega_1 = \kappa\sin\lambda\sin\phi\,. \tag{8.22}$$

In particular, for the choice $j_\pm = 1$ made in section 5, we find $\omega_1 = \kappa \cos^2\phi$ and $\omega_2 = \kappa \sin^2\phi$. If we define $\mathcal{J}^\perp$ as the operator 'orthogonal' to $\mathcal{J}$ in the coset metric

$$\mathcal{J}^\perp = \sin\lambda \cos\phi J_1^+ - \cos\lambda \sin\phi J_1^- \,, \tag{8.23}$$

then our light-like geodesic solutions carry the following $\mathcal{J}^\perp$ charges

$$\frac{\kappa l^2}{\alpha'} (\sin\lambda \cos\theta \sec\phi - \cos\lambda \sin\theta \csc\phi) \,. \tag{8.24}$$

In particular, the BPS groundstate has $\mathcal{J}^\perp = 0$. The existence of a continuous family of light-like geodesics is a semi-classical manifestation of the massless modes we have encountered; it is a new feature of the AdS_3 background.

8.2 Large charge limit

The finite gap equations do not capture the above semi-classical solutions. One way to see this is to consider the large charge limit of the string action [85–92]. For simplicity, we restrict ourselves initially to a bosonic subsector of our action which is the analogue of the SU(2) subsector originally considered in [85]. To take the large charge limit one redefines the global coordinates as follows

$$X_1 = l e^{it(\tau\ \sigma)} \,, \qquad X_2 = 0 \,. \tag{8.25}$$

$$Y_i = R_+ e^{iv(\tau\ \sigma)} V_i \,, \tag{8.26}$$

$$Z_i = R_- e^{iw(\tau\ \sigma)} W_i \,. \tag{8.27}$$

After taking $t = \kappa\tau$ (κ is the large-spin parameter here), redefining

$$v = \cos^2\phi\ \kappa\tau + \tilde{v}(\tau\,,\,\sigma) \,, \tag{8.28}$$

$$w = \sin^2\phi\ \kappa\tau + \tilde{w}(\tau\,,\,\sigma) \,, \tag{8.29}$$

and rescaling $\tau \to \kappa\tau$, to leading order in κ, the Virasoro constraint (8.6) reduces to

$$\partial_\sigma \tilde{v} + \partial_\sigma \tilde{w} + i V^i \partial_\sigma V_i + i W^i \partial_\sigma W_i = 0 \,, \tag{8.30}$$

while the other Virasoro constraint (8.5) determines $\partial_\tau(\tilde{v} + \tilde{w})$ in terms of the other fields. The coordinates conjugate to the charges $\mathcal{J}$ and $\mathcal{J}^\perp$ are

$$k = \tilde{v} + \tilde{w} \,, \tag{8.31}$$

$$k^\perp = (1 + \tan\phi)\tilde{v} + (1 - \cot\phi)\tilde{w} \,. \tag{8.32}$$

and we can eliminate the dependence of the Lagrangian on k through the Virasoro constraints. The Lagrangian then reduces to

$$\frac{\alpha'}{l^2} \mathcal{L}_{AdS_3\times S^3\times S^3\times S^1} \to 2iV^i\partial_\tau V_i + 2iW^i\partial_\tau W_i - \frac{|D_\sigma V_i|^2}{\cos^2\phi} - \frac{|D_\sigma W_i|^2}{\sin^2\phi} - (\mathcal{D}_\sigma k^\perp)^2 \,, \tag{8.33}$$

where

$$\mathcal{D}_\sigma k^\perp \equiv \partial_\sigma k^\perp + i(1 + \tan\phi) V^i \partial_\sigma V_i + i(1 - \cot\phi) W^i \partial_\sigma W_i \,, \tag{8.34}$$

$$D_\sigma V_i \equiv \partial_\sigma V_i - V^j \partial_\sigma V_j V_i \,, \tag{8.35}$$
$$D_\sigma W_i \equiv \partial_\sigma W_i - W^j \partial_\sigma W_j W_i \,, \tag{8.36}$$

Notice that in this limit the field $k^\perp$ has no time derivative in the action. In the finite gap equations one simply ignores the $\tilde{w}$ dependent term above; the other terms reduce to standard SU(2)/U(1) Landau-Lifshitz sigma models with a normalisation of the kinetic term that reflects the relative size of the two CP^1 factors.

We can also construct the Landau-Lifshitz sigma model for the full theory. To do this we need to identify the sub-group H which preserve the vacuum of the model. From the BMN analysis in section 5 the stability sub-group in this case is $H = (\mathrm{U}(1)^3 \ltimes \mathrm{PSU}(1|1))^2$ where the bosonic generators are $S_0^{L,R}$, $L_5^{L,R}$, $R_8^{L,R}$ and the fermionic generators are $Q_{\pm\pm\pm}^{L,R}$. H can be thought of as two copies of the maximal central extension of $\mathrm{PSU}(1|1)^2$.[22] The Landau-Lifshitz sigma model can then be constructed on the coset

$$\frac{D(2,1;\alpha)^2}{\mathrm{U}(1)^6 \ltimes \mathrm{PSU}(1|1)^2}\,, \tag{8.37}$$

using the definition given in [92] for a Landau-Lifshitz sigma model on a general coset G/H. This sigma model will have $6+6$ degrees of freedom as is evident from counting the super-dimension of the coset.[23]

9 Discussion

It would be very interesting to understand how the Bethe equations we postulate arise in the dual CFT. The CFT dual for $M_4 = T^4$ is well understood (see for example [2, 93–98]). It is a deformation of the symmetric product orbifold $\mathrm{Sym}^{Q_1 Q_5}(M_4)$, where Q_1, Q_5 are the numbers of coincident D1- and D5-branes correspondingly, such that the latter are wrapping M_4 and the former are transverse to M_4. These CFTs possess "small" $\mathcal{N} = (4,4)$ super conformal symmetry with four supercurrents.[24] The BMN limit of this dual pair was investigated in [99–102]. It should be possible to construct a suitable spin-chain which could presumably match our Bethe equations at weak coupling and perhaps shed some light on the problem of massless modes. Indeed some first steps in this direction have been taken in [14]. By contrast the CFT dual for $M_4 = S^3 \times S^1$ is probably one of the most obscure amongst AdS/CFT pairs [4, 36–41]. The symmetries dictate that this CFT has a "large" $\mathcal{N} = (4,4)$ super-conformal symmetry, which appears to be more difficult to treat. As a result, to date a suitable dual CFT candidate has not been identified. In fact, a number of potential duals do not appear to satisfy the requirements of the duality [40]. In this context, we hope that unraveling the integrability structures of superstring theory on this background may be helpful for further understanding of correct CFT dual for this background.

[22] This central extension has been discussed in [14].

[23] Just as in the discussion above this sigma model will not describe the $\tilde{w}$ massless mode, nor the corresponding massless fermionic zero modes.

[24] The dependence on just the product $Q_1 Q_5$ is due to U-duality which guarantee equivalence of theories with the same product $Q_1 Q_5$.

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We have derived the classical Bethe equations for the supercoset model on the $AdS_3 \times S^3 \times S^3 \times S^1$ background. We also made a guess for the quantum Bethe equations. But both of these equations miss the massless string modes. In addition, we only see the rigid, $D(2,1;\alpha) \times D(2,1;\alpha)$ part of the target-space Virasoro symmetry. We believe that these two problems are related, and that understanding the target space symmetry enhancement may also shed light on the massless modes. The appearance of the target-space Virasoro algebra is understood at the classical supergravity level [103] and at the quantum level in the NSNS AdS_3 backgrounds [3, 104], including $AdS_3 \times S^3 \times S^3 \times S^1$ [4]. In an interesting recent development, the target space Virasoro generators were constructed for the string on $AdS_3 \times S^3 \times T^4$ with the RR flux [105]. Perhaps incorporating these results into the integrability approach can provide the missing information on the massless degrees of freedom on the worldsheet.

Note added. When we were preparing this paper for publication, ref. [106] appeared on the ArXiv, in which the spin chain for the symmetric orbifold CFT on T^4 was constructed. It would be extremely interesting to see if the $\lambda \to 0$ limit of the Bethe equations (7.42) is capable of capturing a part of the spectrum of this spin chain.

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A The $\mathfrak{d}(2,1;\alpha)$ superalgebra

The bosonic subalgebra of $\mathfrak{d}(2,1;\alpha)$ consists of three commuting $\mathfrak{sl}(2)$'s. The supercharges are in their tri-spinor representation. We are interested in the real form of $\mathfrak{d}(2,1;\alpha)$ in which one of the $sl(2)$'s is non-compact and the other two are compact, so that the bosonic subalgebra is $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. With this choice of the real form the parameter α must lie between zero and one, and it is convenient to introduce the trigonometric parameterization $\alpha = \cos^2\phi$. We denote the $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{su}(2)_+$, and $\mathfrak{su}(2)_-$ generators by S_μ $(\mu = 0,1,2)$, L_n $(n = 3,4,5)$, and $R_{\dot{n}}$ $(\dot{n} = 6,7,8)$. The supercharges are $Q_{a\alpha\dot{\alpha}}$. Their spinor indices a, α, and $\dot{\alpha}$ take values $+$ or $-$.

To describe the action of the $\mathfrak{sl}(2)$ generators on the supercharges we introduce three sets of Pauli matrices:

$$\gamma^\mu = (i\sigma^2, \sigma^1, \sigma^3), \qquad \gamma^n = (\sigma^1, \sigma^2, \sigma^3), \qquad \gamma^{\dot{n}} = (\sigma^1, \sigma^2, \sigma^3). \tag{A.1}$$

The (anti-)commutation relations of $\mathfrak{d}(2,1;\cos^2\phi)$ then read:

$$\begin{aligned}
[S_\mu, S_\nu] &= \epsilon_{\mu\nu\lambda} S^\lambda \\
[L_m, L_n] &= \epsilon_{mnp} L^p \\
[R_{\dot m}, R_{\dot n}] &= \epsilon_{\dot m\dot n\dot p} R^{\dot p} \\
[S_\mu, Q_{a\alpha\dot\alpha}] &= -\frac{1}{2}\, Q_{b\alpha\dot\alpha}\gamma^{b}_{\mu\, a} \\
[L_m, Q_{a\alpha\dot\alpha}] &= -\frac{i}{2}\, Q_{a\beta\dot\alpha}\gamma^{\beta}_{m\,\alpha} \\
[R_{\dot m}, Q_{a\alpha\dot\alpha}] &= -\frac{i}{2}\, Q_{a\alpha\dot\beta}\gamma^{\dot\beta}_{\dot m\,\dot\alpha} \\
\{Q_{a\alpha\dot\alpha}, Q_{b\beta\dot\beta}\} &= i(\varepsilon\gamma^\mu)_{ab}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta}\, S_\mu - \cos^2\phi\,\varepsilon_{ab}\,(\varepsilon\gamma^m)_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta}\, L_m \\
&\quad - \sin^2\phi\,\varepsilon_{ab}\,\varepsilon_{\alpha\beta}\,(\varepsilon\gamma^{\dot m})_{\dot\alpha\dot\beta}\, R_{\dot m},
\end{aligned} \tag{A.2}$$

where $\epsilon_{012} = \epsilon_{345} = \epsilon_{678} = 1$, and the vector indices are raised and lowered by $\eta_{\mu\nu} = \mathrm{diag}(-++)$, δ_{nm}, and $\delta_{\dot n\dot m}$.

The invariant bilinear form on $\mathfrak{d}(2,1;\cos^2\phi)$ is given by

$$\begin{aligned}
\mathrm{Str}\, S_\mu S_\nu &= \frac{1}{4}\,\eta_{\mu\nu} \\
\mathrm{Str}\, L_m L_n &= \frac{1}{4\cos^2\phi}\,\delta_{mn} \\
\mathrm{Str}\, R_{\dot m} R_{\dot n} &= \frac{1}{4\sin^2\phi}\,\delta_{\dot m\dot n} \\
\mathrm{Str}\, Q_{a\alpha\dot\alpha} Q_{b\beta\dot\beta} &= \frac{i}{2}\,\varepsilon_{ab}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta}\,,
\end{aligned} \tag{A.3}$$

and is consistent with the $(-+\cdots+)$ signature in the target space.

We will also use another basis of supercharges:

$$\tilde Q_{\pm\alpha\dot\alpha} = Q_{+\alpha\dot\alpha} \mp iQ_{-\alpha\dot\alpha}. \tag{A.4}$$

The commutators with the $\mathfrak{sl}(2)$ generators then take the form

$$[S_\mu, \tilde Q_{a\alpha\dot\alpha}] = -\frac{i}{2}\,\tilde Q_{b\alpha\dot\alpha}\tilde\gamma^{b}_{\mu\, a}, \tag{A.5}$$

with the gamma-matrices rotated to a different basis: $\tilde\gamma_\mu = (\sigma^3, i\sigma^2, -i\sigma^1)$. The normalization also changes:

$$\mathrm{Str}\,\tilde Q_{a\alpha\dot\alpha}\tilde Q_{b\beta\dot\beta} = -\varepsilon_{ab}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\dot\alpha\dot\beta}\,. \tag{A.6}$$

B Background-field expansion for $\mathbb{Z}_4$ cosets

In this appendix we expand the sigma-model action (2.4) to the second order in fluctuations around an arbitrary bosonic background $g_B(x)$. This is necessary for comparison of the coset model to the Green-Schwarz action, and also for fixing the light-cone gauge in the BMN limit. The expansion can be done quite generally, starting with the coset representative in the form

$$g = g_B\, \mathrm{e}^{\mathbb{X}}, \tag{B.1}$$

where $\mathbb{X}(x) = X^A(x)T_A \in \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3$ is the fluctuation field. For the background currents we introduce the following notations:

$$\begin{aligned} \left(g_B^{-1}\partial_{\mathbf{a}} g_B\right)_0 &= A_{\mathbf{a}}, \\ \left(g_B^{-1}\partial_{\mathbf{a}} g_B\right)_2 &= K_{\mathbf{a}}. \end{aligned} \tag{B.2}$$

According to the discussion in the main text, $A_{\mathbf{a}} \in \mathfrak{h}_0$ is the background gauge field. By $D_{\mathbf{a}}$ we will denote the corresponding covariant derivative:

$$D_{\mathbf{a}} = \partial_{\mathbf{a}} + [A_{\mathbf{a}}, \cdot], \tag{B.3}$$

and by $F_{\mathbf{ab}}$ the field strength $F_{\mathbf{ab}} = \partial_{\mathbf{a}} A_{\mathbf{b}} - \partial_{\mathbf{b}} A_{\mathbf{a}} + [A_{\mathbf{a}}, A_{\mathbf{b}}]$. The flatness of the current $g_B^{-1}\partial_{\mathbf{a}} g_B$ implies that

$$[K_{\mathbf{a}}, K_{\mathbf{b}}] + F_{\mathbf{ab}} = 0, \tag{B.4}$$

$$D_{\mathbf{a}} K_{\mathbf{b}} - D_{\mathbf{b}} K_{\mathbf{a}} = 0. \tag{B.5}$$

The current (2.3) can be readily expanded in power series in $\mathbb{X}$:

$$J_{\mathbf{a}} = A_{\mathbf{a}} + K_{\mathbf{a}} + \frac{1 - \mathrm{e}^{-\,\mathrm{ad}\,\mathbb{X}}}{\mathrm{ad}\,\mathbb{X}}\, \mathcal{D}_{\mathbf{a}}\mathbb{X} = A_{\mathbf{a}} + K_{\mathbf{a}} + \mathcal{D}_{\mathbf{a}}\mathbb{X} - \frac{1}{2}[\mathbb{X}, \mathcal{D}_{\mathbf{a}}\mathbb{X}] + \cdots, \tag{B.6}$$

where the long derivative $\mathcal{D}_{\mathbf{a}}$ is defined by

$$\mathcal{D}_{\mathbf{a}} = \partial_{\mathbf{a}} + [g_B^{-1}\partial_{\mathbf{a}} g_B, \cdot] = D_{\mathbf{a}} + [K_{\mathbf{a}}, \cdot]. \tag{B.7}$$

Unlike the covariant derivative $D_{\mathbf{a}}$, which commutes with the $\mathbb{Z}_4$ grading, the long derivative $\mathcal{D}_{\mathbf{a}}$ does not have definite $\mathbb{Z}_4$ charge. Thus, $(D_{\mathbf{a}}\mathbb{X})_n = D_{\mathbf{a}}\mathbb{X}_n$ for any n, also $(\mathcal{D}_{\mathbf{a}}\mathbb{X})_2 = D_{\mathbf{a}}\mathbb{X}_2$, but $(\mathcal{D}_{\mathbf{a}}\mathbb{X})_{1,3} = D_{\mathbf{a}}\mathbb{X}_{1,3} + [K_{\mathbf{a}}, \mathbb{X}_{3,1}]$.

We can now expand the action (2.4) in powers of $\mathbb{X}$. To simplify the result, one should use the identity:

$$\varepsilon^{\mathbf{ab}}\, \mathrm{Str}\, D_{\mathbf{a}}\mathbb{X}_1 D_{\mathbf{b}}\mathbb{X}_3 = -\varepsilon^{\mathbf{ab}}\, \mathrm{Str}[K_{\mathbf{a}}, \mathbb{X}_1][K_{\mathbf{b}}, \mathbb{X}_3] + \text{total derivative}, \tag{B.8}$$

which follows from (B.4). Then, to the second order in fluctuations,

$$\begin{aligned} S = \int d^2x\ \mathrm{Str}\Big\{ & \sqrt{-h}h^{\mathbf{ab}} K_{\mathbf{a}} K_{\mathbf{b}} - 2\mathbb{X}_2 \nabla_{\mathbf{a}} K^{\mathbf{a}} \\ & + \sqrt{-h}h^{\mathbf{ab}} \left(D_{\mathbf{a}}\mathbb{X}_2 D_{\mathbf{b}}\mathbb{X}_2 - [K_{\mathbf{a}}, \mathbb{X}_2][K_{\mathbf{b}}, \mathbb{X}_2]\right) \\ & + \left(\sqrt{-h}h^{\mathbf{ab}} + \varepsilon^{\mathbf{ab}}\right) \mathbb{X}_1 [K_{\mathbf{a}}, D_{\mathbf{b}}\mathbb{X}_1] + \left(\sqrt{-h}h^{\mathbf{ab}} - \varepsilon^{\mathbf{ab}}\right) \mathbb{X}_3 [K_{\mathbf{a}}, D_{\mathbf{b}}\mathbb{X}_3] \\ & - \left(\sqrt{-h}h^{\mathbf{ab}} + \varepsilon^{\mathbf{ab}}\right) [K_{\mathbf{a}}, \mathbb{X}_1][K_{\mathbf{b}}, \mathbb{X}_3] - \left(\sqrt{-h}h^{\mathbf{ab}} - \varepsilon^{\mathbf{ab}}\right) [K_{\mathbf{a}}, \mathbb{X}_3][K_{\mathbf{b}}, \mathbb{X}_1] \Big\}. \end{aligned} \tag{B.9}$$

The second term contains the covariantized derivative $\nabla_{\mathbf{a}} K^{\mathbf{b}} = D_{\mathbf{a}} K^{\mathbf{b}} + \Gamma^{\mathbf{b}}_{\mathbf{ac}} K^{\mathbf{c}}$ and vanishes on-shell, when $K^{\mathbf{a}}$ satisfies the equations of motion. Let us stress that we have not used the equations of motion for $K^{\mathbf{a}}$ in deriving (B.9). The equations (B.4) that we used are identities valid for any $K_{\mathbf{a}}$ and $A_{\mathbf{a}}$ of the form (B.2).

C Gamma matrices

We pick the following representation for the 10d Dirac matrices:

$$\Gamma^{\mu} = \sigma^1 \otimes \sigma^2 \otimes \gamma^{\mu} \otimes \mathbb{1} \otimes \mathbb{1}\,, \qquad \mu = 0,1,2 \tag{C.1}$$

$$\Gamma^{n} = \sigma^1 \otimes \sigma^1 \otimes \mathbb{1} \otimes \gamma^{n} \otimes \mathbb{1}\,, \qquad n = 3,4,5 \tag{C.2}$$

$$\Gamma^{\dot n} = \sigma^1 \otimes \sigma^3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \gamma^{\dot n}\,, \qquad \dot n = 6,7,8 \tag{C.3}$$

$$\Gamma^{9} = -\sigma^2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \tag{C.4}$$

where the 3d gamma-matrices γ^i are taken from (A.1).

In this basis,

$$\Gamma^{012} = \sigma^1 \otimes \sigma^2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \tag{C.5}$$

$$\Gamma^{345} = i\sigma^1 \otimes \sigma^1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \tag{C.6}$$

$$\Gamma^{678} = i\sigma^1 \otimes \sigma^3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \tag{C.7}$$

$$\Gamma^{012345} = \mathbb{1} \otimes \sigma^3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \tag{C.8}$$

$$\Gamma^{012678} = -\mathbb{1} \otimes \sigma^1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \tag{C.9}$$

$$\Gamma = \sigma^3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \tag{C.10}$$

and

$$\begin{aligned}
\Gamma^{\mu\nu} &= -\varepsilon^{\mu\nu\lambda} \mathbb{1} \otimes \mathbb{1} \otimes \gamma_{\lambda} \otimes \mathbb{1} \otimes \mathbb{1} \\
\Gamma^{mn} &= i\varepsilon^{mnp} \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \gamma_{p} \otimes \mathbb{1} \\
\Gamma^{\dot m \dot n} &= i\varepsilon^{\dot m \dot n \dot p} \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \gamma_{\dot p}.
\end{aligned} \tag{C.11}$$

The charge conjugation matrix is

$$C = i\sigma^2 \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma^2. \tag{C.12}$$

The kappa-symmetry projectors introduced in section 3, eq. (3.5), are of the form

$$K^{\pm} = \mathbb{1} \otimes \frac{1}{2}\left(1 \pm \cos\phi\, \sigma^3 \mp \sin\phi\, \sigma^1\right) \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} = \mathbb{1} \otimes |\pm\rangle_{\phi\,\phi}\langle\pm| \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \tag{C.13}$$

where

$$|+\rangle_{\phi} = \begin{pmatrix} \cos\frac{\phi}{2} \\ -\sin\frac{\phi}{2} \end{pmatrix}, \qquad |-\rangle_{\phi} = \begin{pmatrix} \sin\frac{\phi}{2} \\ \cos\frac{\phi}{2} \end{pmatrix}. \tag{C.14}$$

In these notations,

$$\frac{1}{2}\left(1 \pm \Gamma\right) = |\pm\rangle_{0\,0}\langle\pm| \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}. \tag{C.15}$$

The following identities are useful in the calculations in section 4:

$$\langle\mp|\, \sigma^1\, |\pm\rangle = \cos\phi, \qquad \langle\mp|\, \sigma^2\, |\pm\rangle = \pm i, \qquad \langle\mp|\, \sigma^3\, |\pm\rangle = \sin\phi. \tag{C.16}$$

D Quartic terms in the near-BMN Lagrangian

The quartic near-BMN Lagrangian is already rather complicated. We list here its bosonic part before the gauge fixing:

$$
\begin{aligned}
\mathcal{L}^{(4,b)} = & -\frac{2}{3}\left(\bar{X}X\right)^2\left(\partial_{\mathbf{a}}t\right)^2+\frac{1}{3}\Big[2\cos^6\phi\,\left(\bar{Y}Y\right)^2+\sin^2\phi\,\cos^6\phi\,V^2\bar{Y}Y \\
& +2\sin^6\phi\,\left(\bar{Z}Z\right)^2+\cos^2\phi\,\sin^6\phi\,V^2\bar{Z}Z\Big]\left(\partial_{\mathbf{a}}\varphi\right)^2 \\
& -\frac{1}{3}\cos^2\phi\,\sin^2\phi\,\left(\bar{Y}Y+\bar{Z}Z\right)\left(\partial_{\mathbf{a}}V\right)^2+\frac{1}{3}\bar{X}X\partial_{\mathbf{a}}\bar{X}\partial^{\mathbf{a}}X \\
& -\frac{1}{3}\cos^2\phi\,\left(\sin^2\phi\,V^2+\bar{Y}Y\right)\partial_{\mathbf{a}}\bar{Y}\partial^{\mathbf{a}}Y-\frac{1}{3}\sin^2\phi\,\left(\cos^2\phi\,V^2+\bar{Z}Z\right)\partial_{\mathbf{a}}\bar{Z}\partial^{\mathbf{a}}Z \\
& -\frac{1}{6}\left[\left(\partial_{\mathbf{a}}\bar{X}\right)^2X^2+\bar{X}^2\left(\partial_{\mathbf{a}}X\right)^2\right]+\frac{1}{6}\cos^2\phi\,\left[\left(\partial_{\mathbf{a}}\bar{Y}\right)^2Y^2+\bar{Y}^2\left(\partial_{\mathbf{a}}Y\right)^2\right] \\
& +\frac{1}{6}\sin^2\phi\,\left[\left(\partial_{\mathbf{a}}\bar{Z}\right)^2Z^2+\bar{Z}^2\left(\partial_{\mathbf{a}}Z\right)^2\right] \\
& +\frac{1}{3}\cos^2\phi\,\sin^2\phi\,V\partial_{\mathbf{a}}V\left(\partial^{\mathbf{a}}\bar{Y}Y+\bar{Y}\partial^{\mathbf{a}}Y+\partial^{\mathbf{a}}\bar{Z}Z+\bar{Z}\partial^{\mathbf{a}}Z\right).
\end{aligned}
\tag{D.1}
$$

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