INTEGRABILITY CONDITIONS AND THEIR APPLICATIONS IN STEADY PLANE ELECTROMAGNETOFLUIDDYNAMIC ALIGNED FLOWS*

By

O. P. CHANDNA, H. TOEWS AND K. PRABAHARAN

University of Windsor, Ontario

1. Introduction. The general problem of electromagnetofluiddynamics without any MHD approximations is quite complex. Kingston and Power [1] gave an elegant analysis of the general dynamics of two-dimensional aligned flows. This analysis establishes that if the charge density is not identically zero in a flow region, then the magnetic field must be irrotational everywhere. However, by the assumption $\mathbf{H} = k\mathbf{V}$ where k is some constant, their study of incompressible flows with irrotational magnetic field is restricted to flows which have irrotational velocity field; this work involving incompressible inviscid flows, however, is valid for viscous case as well, since viscous effects vanish for irrotational flows.

The aim of this paper is to study some rotational and irrotational viscous incompressible aligned plane flows. The plan of our work is as follows: in Sec. 2 we recapitulate the basic equations governing the motion of two dimensional flows. Sec. 3 contains the derivation of integrability conditions when the flow is aligned; we obtain the necessary and sufficient condition for conservation of circulation for such flows. In Sec. 4, we employ Monge's method for solving second-order nonlinear partial differential equations [2] and the integrability conditions to classify rotational flows with constant vorticity magnitude along any individual streamline. In parts (b) and (c) of this section, the study of irrotational flows and rotational flows with constant vorticity along any individual orthogonal trajectory is carried out. Secs. 5, 6 and 7 deal with the applications of integrability conditions in the study of flows when the streamlines are straight lines; when the streamlines are involutes of a curve; and when the streamlines form an isometric net with their orthogonal trajectories.

2. Flow equations. The fundamental electromagnetofluiddynamic equations, for the steady flow of a viscous incompressible fluid, are [3]:

$$\operatorname{div} \mathbf{V} = 0, \tag{1}$$

$$\rho(\mathbf{V} \cdot \operatorname{grad})\mathbf{V} = -\operatorname{grad} p + \eta\nabla^2\mathbf{V} + \mu\mathbf{J} \times \mathbf{H} + q\mathbf{E}, \tag{2}$$

$$\mathbf{J} = \mathbf{I} + q\mathbf{V} = \operatorname{Curl} \mathbf{H}, \tag{3}$$

$$\mathbf{I} = \sigma(\mathbf{E} + \mu\mathbf{V} \times \mathbf{H}), \tag{4}$$

* Received July 1, 1980.

$$\text{Curl } \mathbf{E} = \mathbf{0}, \tag{5}$$

$$\text{div } \mathbf{E} = q/\varepsilon, \tag{6}$$

$$\text{div } \mathbf{H} = 0, \tag{7}$$

where the unknowns to be investigated are the velocity field **V**, the magnetic field **H**, the electric field **E**, the current density field **J**, the charge density function q and the pressure function p. Here ρ, σ, η, μ and ε are respectively the constant density, the constant electric conductivity, the constant coefficient of viscosity, the constant magnetic permeability and the constant permitivity of the fluid.

From Eqs. (3) and (4), we obtain

$$\mathbf{I} = \nabla \times \mathbf{H} - q\mathbf{V}, \qquad \mathbf{E} = \frac{1}{\sigma}(\nabla \times \mathbf{H} - q\mathbf{V}) - \mu\mathbf{V} \times \mathbf{H}. \tag{8}$$

Eliminating **I** and **E** from (2), (5) and (6) by using (8), we obtain

$$\rho(\mathbf{V} \cdot \text{grad})\mathbf{V} = -\text{grad } p + \eta\nabla^2\mathbf{V} + \mu(\text{curl } \mathbf{H}) \times \mathbf{H} + \frac{q}{\sigma}(\text{curl } \mathbf{H} - q\mathbf{V}) - \mu q\mathbf{V} \times \mathbf{H}, \tag{9}$$

$$\text{Curl}[\text{Curl } \mathbf{H} - q\mathbf{V} - \mu\sigma\mathbf{V} \times \mathbf{H}] = \mathbf{0}, \tag{10}$$

$$\text{div}[q\mathbf{V} + \mu\sigma\mathbf{V} \times \mathbf{H}] = -\sigma q/\varepsilon. \tag{11}$$

Now we consider the flow to be two-dimensional so that **V** and **H** lie in a plane defined by the rectangular coordinates x, y. Decomposition of the vector equations (9) and (10) into their vector components in the flow plane and their vector components perpendicular to the flow plane yields the following four equations:

$$\rho(\mathbf{V} \cdot \text{grad})\mathbf{V} = -\text{grad } p + \eta\nabla^2\mathbf{V} + \mu(\text{curl } \mathbf{H}) \times \mathbf{H} - \frac{q^2}{\sigma}\mathbf{V}, \tag{12}$$

$$q[\text{curl } \mathbf{H} - \mu\sigma\mathbf{V} \times \mathbf{H}] = \mathbf{0}, \tag{13}$$

$$\text{Curl}[\text{curl } \mathbf{H} - \mu\sigma\mathbf{V} \times \mathbf{H}] = \mathbf{0}, \tag{14}$$

$$\text{Curl}(q\mathbf{V}) = \mathbf{0}, \tag{15}$$

where Eqs. (12) and (13) are equivalent to (9) and Eqs. (14) and (15) are equivalent to (10). Eq. (11), for the assumed plane flows, reduces to

$$\text{div}(q\mathbf{V}) = -\sigma q/\varepsilon. \tag{16}$$

Eq. (13) and the above analysis imply the following results:

THEOREM 1. If the steady plane electromagnetofluiddynamic flow has nonzero charge density and the magnetic field is in the flow plane, then the vector fields **V** and **H** satisfy

$$\text{Curl } \mathbf{H} = \mu\sigma\mathbf{V} \times \mathbf{H}, \tag{17}$$

and the flow is governed by the system of equations

$$\text{div } \mathbf{V} = 0, \qquad \rho(\mathbf{V} \cdot \text{grad})\mathbf{V} = -\text{grad } p + \eta\nabla^2\mathbf{V} + \mu(\text{curl } \mathbf{H}) \times \mathbf{H} - \frac{q^2}{\sigma}\mathbf{V},$$

$$\text{div}(q\mathbf{V}) = -\frac{\sigma q}{\varepsilon}, \qquad \text{Curl}(q\mathbf{V}) = \mathbf{0}, \qquad \text{Curl } \mathbf{H} = \mu\sigma\mathbf{V} \times \mathbf{H}.$$

Furthermore, we have

$$\operatorname{div} \mathbf{H} = 0, \qquad \mathbf{E} = -\frac{q}{\sigma}\mathbf{V}, \qquad \mathbf{I} = \operatorname{curl} \mathbf{H} - q\mathbf{V}.$$

The following work is devoted to the study of flows governed by the above set of equations when the charge density is nonzero.

3. Integrability conditions and circulation for aligned flows. Eqs. (16) and (15) can respectively be written, by using (1), as

$$\operatorname{grad}(\ln q) \cdot \mathbf{V} + \frac{\sigma}{\varepsilon} = 0, \tag{18}$$

$$\operatorname{grad}(\ln q) \times \mathbf{V} + \operatorname{curl} \mathbf{V} = \mathbf{0}. \tag{19}$$

Taking the vector product of (19) with $\mathbf{V}$ and using (18), we obtain

$$\operatorname{grad}(\ln q) = -\left[\frac{\frac{\sigma}{\varepsilon}\mathbf{V} + \mathbf{V} \times \operatorname{curl} \mathbf{V}}{\mathbf{V} \cdot \mathbf{V}}\right]. \tag{20}$$

Taking the curl of (20), it follows that

$$\operatorname{Curl}\left[\frac{(\operatorname{Curl} \mathbf{V}) \times \mathbf{V} - \frac{\sigma}{\varepsilon}\mathbf{V}}{\mathbf{V} \cdot \mathbf{V}}\right] = \mathbf{0}, \tag{21}$$

which is the integrability condition for the function $\ln q$.

The integrability condition (21) is satisfied by every steady plane incompressible electromagnetofluiddynamic flow. However, if the flows are such that $\mathbf{H}$ and $\mathbf{V}$ are everywhere parallel (aligned flows), we obtain one more integrability condition.

From the definition of aligned flows, we have

$$\mathbf{H} = F(x, y)\mathbf{V}, \tag{22}$$

where $F(x, y)$ is some arbitrary scalar function satisfying, by use of (1) and (7), the condition

$$\mathbf{V} \cdot \operatorname{grad} F = 0. \tag{23}$$

Employing (22) in (17), we have

$$\operatorname{grad}(\ln F) \times \mathbf{V} + \operatorname{curl} \mathbf{V} = \mathbf{0}. \tag{24}$$

Taking the vector product of this equation with $\mathbf{V}$ and using (23), we obtain

$$\operatorname{grad}(\ln F) = \frac{(\operatorname{Curl} \mathbf{V}) \times \mathbf{V}}{\mathbf{V} \cdot \mathbf{V}} \tag{25}$$

and therefore get the integrability condition for $\ln F$ given by

$$\operatorname{Curl}\left[\frac{(\operatorname{curl} \mathbf{V}) \times \mathbf{V}}{\mathbf{V} \cdot \mathbf{V}}\right] = \mathbf{0}. \tag{26}$$

It follows from (21) and (26) that for plane aligned flows, the vector field $\mathbf{V}$ satisfies

$$\operatorname{Curl}\left[\frac{\mathbf{V}}{\mathbf{V}\cdot\mathbf{V}}\right]=\mathbf{0} \tag{27}$$

and Eq. (26). Kingston and Power [1] obtained Eq. (27), in their study of inviscid aligned flows, by a complex variable technique. However, their derivation of (27) requires $F(x, y)$ to be a constant throughout the flow.

In the following sections, we give various applications of these integrability conditions.

Circulation. Let C be an arbitrary closed curve which moves with the fluid and bounds an area A. By definition of circulation Γ around C, we have $\Gamma = \oint_C \mathbf{V}\cdot d\mathbf{l}$ and, therefore,

$$\frac{D}{Dt}\Gamma=\oint_C \mathbf{a}\cdot d\mathbf{l}=\int_A\int \operatorname{Curl}\mathbf{a}\cdot d\mathbf{s}=\int_A\int \operatorname{Curl}(\boldsymbol{\omega}\times\mathbf{V})\cdot d\mathbf{s},$$

where $\mathbf{a} = (\mathbf{V}\cdot\nabla)\mathbf{V}$ is the acceleration vector field.

From this equation we see that the circulation Γ, for any closed curve C moving with the fluid, is conserved if and only if $\operatorname{Curl}(\boldsymbol{\omega}\times\mathbf{V})$ vanishes identically. However, from the integrability condition (26) we find, with the aid of (27), that $\operatorname{Curl}(\boldsymbol{\omega}\times\mathbf{V}) = \mathbf{0}$ if and only if $(\operatorname{grad}|\mathbf{V}|\cdot\mathbf{V})\boldsymbol{\omega} = \mathbf{0}$. Thus we are led to the result that the circulation is conserved when either the flow is irrotational or the velocity magnitude is constant on any individual streamline, and conversely.

4. Classification of some rotational and irrotational flows. Since the velocity field and the magnetic field satisfy the solenoidal conditions and the magnetic field is irrotational for the aligned flows with nonzero charge density distribution, it follows that the function $F(x, y)$ in Eq. (22) is an absolute constant if and only if the velocity field is irrotational. Kingston and Power [1] studied the classification of these irrotational flows with $F(x, y)$ as a constant.

In this section we study the following: (a) rotational flows having vorticity constant along any individual streamline, (b) irrotational flows, (c) rotational flows having vorticity constant along any individual orthogonal trajectory of the streamlines.

Eq. (1) implies the existence of a streamfunction $\psi(x, y)$ such that

$$V_1 = \partial\psi/\partial y, \qquad V_2 = -\partial\psi/\partial x$$

where V_1, V_2 are the components of $\mathbf{V}$ in (x, y) coordinates. Using (28) in the integrability conditions (26), (27), we have

$$q\frac{\partial}{\partial x}\left[\frac{r+t}{p^2+q^2}\right]=p\frac{\partial}{\partial y}\left[\frac{r+t}{p^2+q^2}\right], \tag{29}$$

$$(q^2-p^2)r-4pqs+(p^2-q^2)t=0, \tag{30}$$

where

$$\frac{\partial\psi}{\partial x}=p,\qquad \frac{\partial\psi}{\partial y}=q,\qquad \frac{\partial^2\psi}{\partial x^2}=r,\qquad \frac{\partial^2\psi}{\partial x\,\partial y}=s\quad\text{and}\quad \frac{\partial^2\psi}{\partial y^2}=t. \tag{31}$$

Furthermore, the expression for the vorticity $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega}=\operatorname{Curl}\mathbf{V}=(r+t)\mathbf{k}, \tag{32}$$

where $\mathbf{k}$ is a unit vector normal to the flow plane.

(a) *Rotational flows having vorticity constant along any individual streamline.* For such flows, using Eq. (32), we can write

$$\omega(\psi) = r + t \tag{33}$$

where $\omega(\psi)$, a function of ψ, is such that

$$\boldsymbol{\omega} = \omega(\psi)\mathbf{k}. \tag{34}$$

From (29), using (33), we get

$$(pr + qs)q\omega(\psi) = (ps + qt)p\omega(\psi).$$

Since $\omega(\psi) \neq 0$ for rotational flow, we have from the above equation

$$pqr + (q^2 - p^2)s - pqt = 0. \tag{35}$$

Integration of (35) by Monge's method yields

$$p^2 + q^2 = f(\psi), \tag{36}$$

where $f(\psi)$ is an arbitrary function of ψ. On solving (36) by the method of characteristics, we find that the complete integral is given by

$$F(\psi) = x \cos \theta + y \sin \theta + \beta$$

where θ and β are two parameters. The general integral is obtained by eliminating θ between

$$F(\psi) = x \cos \theta + y \sin \theta + \phi(\theta), \qquad 0 = -x \sin \theta y \cos \theta + \phi'(\theta). \tag{37}$$

Differentiating the first of the above equations with respect to x, y and using the second equation, we have

$$pF'(\psi) = \cos \theta; \qquad qF'(\psi) = \sin \theta \tag{38}$$

Differentiation of Eqs. (38) gives:

$$rF'(\psi) + p^2F''(\psi) = -\sin \theta \frac{\partial \theta}{\partial x}, \tag{39}$$

$$tF'(\psi) + q^2F''(\psi) = \cos \theta \frac{\partial \theta}{\partial y}, \tag{40}$$

$$sF'(\psi) + pqF''(\psi) = -\sin \theta \frac{\partial \theta}{\partial y} = \cos \theta \frac{\partial \theta}{\partial x}. \tag{41}$$

From Eq. (33), we have

$$q \frac{\partial}{\partial x} [r + t] = p \frac{\partial}{\partial y} [r + t]. \tag{42}$$

Using Eqs. (38), (39) and (40) in (42) we obtain

$$\sin \theta \frac{\partial}{\partial x}\left[-\sin \theta \frac{\partial \theta}{\partial x} + \cos \theta \frac{\partial \theta}{\partial y}\right] = \cos \theta \frac{\partial}{\partial y}\left[\sin \theta \frac{\partial \theta}{\partial x} + \cos \theta \frac{\partial \theta}{\partial y}\right].$$

Using the relationship $-\sin\theta\, \partial\theta/\partial y = \cos\theta\, \partial\theta/\partial x$ of Eq. (41), we obtain from the above equation

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0. \tag{43}$$

Differentiating the second equation of (37) twice with respect to x and twice with respect to y respectively, we obtain

$$-2\cos\theta \frac{\partial\theta}{\partial x} - \{x\cos\theta + y\sin\theta - \phi''(\theta)\}\frac{\partial^2\theta}{\partial x^2} + \{x\sin\theta - y\cos\theta + \phi'''(\theta)\}\left(\frac{\partial\theta}{\partial x}\right)^2 = 0, \tag{44}$$

$$-2\sin\theta \frac{\partial\theta}{\partial y} - \{x\cos\theta + y\sin\theta - \phi''(\theta)\}\frac{\partial^2\theta}{\partial y^2} + \{x\sin\theta - y\cos\theta + \phi'''(\theta)\}\left(\frac{\partial\theta}{\partial y}\right)^2 = 0. \tag{45}$$

Adding (44) and (45) and using (37), (41) and (43), we find

$$\left[\left(\phi'''(\theta) + \phi'(\theta)\right)\left\{\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2\right\}\right] = 0. \tag{46}$$

This equation (46) implies that either $(\partial\theta/\partial x)^2 + (\partial\theta/\partial y)^2 = 0$ or $\phi'''(\theta) + \phi'(\theta) = 0$. The first possibility gives $\theta =$ constant, i.e., the streamlines are a family of parallel straight lines.

From the second possibility we have

$$\phi(\theta) = A\cos\theta + B\sin\theta + C$$

where A, B and C are arbitrary constants. Substituting for $\phi(\theta)$ in (37), we have

$$F(\psi) - C = (x + B)\cos\theta + (y + A)\sin\theta,$$

$$0 = -(x + B)\sin\theta + (y + A)\cos\theta.$$

Squaring and adding, we obtain from the above equations

$$[F(\psi) - C]^2 = (x + B)^2 + (y + A)^2$$

which implies that the streamlines are concentric circles.

Furthermore, we have from Eq. (26)

$$\frac{1}{|\mathbf{V}|^2}(\mathbf{V}\cdot \text{grad})\boldsymbol{\omega} + \text{grad}\left(\frac{1}{|\mathbf{V}|^2}\right) \times (\boldsymbol{\omega} \times \mathbf{V}) = \mathbf{0}. \tag{47}$$

It follows from this equation that for the rotational flows, the flows having vorticity constant along any individual streamline are the flows having velocity magnitude constant along any individual streamline.

Hence we have the following:

THEOREM 2. If, for the flows under study, the vorticity is constant along any individual streamline, or equivalently the velocity magnitude is constant along any individual streamline, the streamlines are either parallel straight lines or concentric circles. The above result holds for the rotational flows with vorticity constant throughout.

(b) *Irrotational flows.* In the case of an irrotational flow, we have from Eq. (32)

$$r + t = 0. \tag{48}$$

Using (48) in (31), we obtain, when $pq \neq 0$,

$$p\left[\frac{t}{q} - \frac{s}{p}\right] + q\left[\frac{r}{s} - \frac{s}{q}\right] = 0, \tag{49}$$

which can easily be reduced to

$$\frac{\partial}{\partial x}\left[\ln \frac{p}{q}\right]\frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\left[\ln \frac{p}{q}\right] = 0 \tag{50}$$

From this equation (50), we have

$$p/q = f(\psi), \tag{51}$$

where $f(\psi)$ is an arbitrary function of ψ. On solving (51), we get

$$\frac{y - b}{x - a} = -f(\psi)$$

where a and b are arbitrary constants, which implies that the streamlines, ψ = constant, are concurrent straight lines or parallel straight lines. Also, we have for the flows with $pq = 0$ that the streamlines are parallel to one of the coordinate axes.

Furthermore, we have from Eq. (27)

$$\frac{1}{|\mathbf{V}|^2} \text{ Curl } \mathbf{V} + \text{grad}\left(\frac{1}{|\mathbf{V}|^2}\right) \times \mathbf{V} = \mathbf{0}, \tag{53}$$

and it follows from this equation that the irrotational flows are the only flows having the velocity magnitude constant along any individual orthogonal trajectory of the streamlines. Hence we have the following:

THEOREM 3. If the flows under study are irrotational, or equivalently the velocity magnitude is constant along any individual orthogonal trajectory of the streamlines, the streamlines are either concurrent straight lines or parallel straight lines.

The above result has also been proved by Kingston and Power [1] by a complex variable technique.

(c) *Rotational flows having vorticity constant along any individual orthogonal trajectory of the streamlines.* In natural, or streamline, coordinates the continuity equation (1) and the integrability conditions (27) and (26) for our flows, respectively, assume the following forms:

$$\frac{\partial}{\partial \alpha}[h_2 V] = 0, \tag{54}$$

$$\frac{\partial}{\partial \beta}[h_1/V] = 0, \tag{55}$$

$$\frac{\partial^2}{\partial \alpha\, \partial \beta}[\ln(h_1 V)] = 0, \tag{56}$$

where V is the flow speed, the curves $\beta(x, y) =$ constant are the family of streamlines, the curves $\alpha(x, y) =$ constant are their orthogonal trajectories and the squared element of arc length along any curve is given by

$$ds^2 = h_1^2(\alpha, \beta)\, d\alpha^2 + h_2^2(\alpha, \beta)\, d\beta^2 \tag{57}$$

such that the metric coefficients h_1 and h_2 are related to the inclination $\theta(\alpha, \beta)$ of the velocity vector to the x-direction, by the equations [4]

$$\frac{\partial \theta}{\partial \alpha} = -\frac{1}{h_2}\frac{\partial h_1}{\partial \beta}, \qquad \frac{\partial \theta}{\partial \beta} = \frac{1}{h_1}\frac{\partial h_2}{\partial \alpha}. \tag{58}$$

From (54)—(56), it follows that

$$V(\alpha, \beta) = B(\alpha)C(\beta),$$

$$h_1(\alpha, \beta) = A(\alpha)B(\alpha)C(\beta), \qquad h_2(\alpha, \beta) = \frac{D(\beta)}{B(\alpha)C(\beta)}, \tag{59}$$

where $A(\alpha)$, $B(\alpha)$ are arbitrary functions of α and $C(\beta)$, $D(\beta)$ are arbitrary functions of β. Employing (59), the vorticity component ω in natural coordinates can be expressed by

$$\omega = -\frac{B^2(\alpha)}{D(\beta)}\,[C^2(\beta)]'. \tag{60}$$

Since by assumption ω is constant on any individual orthogonal trajectory of the streamlines, it follows that ω is a function of α only. Therefore, Eq. (60) implies that

$$\frac{[C^2(\beta)]'}{D(\beta)} = -\frac{\omega(\alpha)}{B^2(\alpha)} = k, \tag{61}$$

where k is an arbitrary constant, nonzero since $\omega \neq 0$ for rotational flows.

Using the expressions for the metric coefficients given by (59) with $D(\beta) = [C^2(\beta)/k]'$ given by (61) in Eq. (58), we get

$$\frac{\partial \theta}{\partial \alpha} = -\frac{kA(\alpha)B^2(\alpha)}{2}, \qquad \frac{\partial \theta}{\partial \beta} = -\frac{2B'(\alpha)C'(\beta)}{kA(\alpha)B^3(\alpha)C(\beta)}.$$

We eliminate θ by using the integrability conditions $\partial^2\theta/\partial\alpha\,\partial\beta = \partial^2\theta/\partial\beta\,\partial\alpha$, to obtain $[B'(\alpha)/A(\alpha)B^3(\alpha)]' = 0$, so that either $B'(\alpha) = 0$ or $A(\alpha) = B'(\alpha)/k_1 B^3(\alpha)$ where k_1 is a nonzero constant. The case $B'(\alpha) = 0$ implies that the velocity magnitude is constant along any individual streamline and therefore, by Theorem 2, the streamlines are concentric circles or parallel straight lines.

In the second case, the permissible flow forms for the metric coefficients h_1, h_2 are given by

$$h_1 = \frac{B'(\alpha)C(\beta)}{k_1 B^2(\alpha)}, \qquad h_2 = \frac{2C'(\beta)}{kB(\alpha)}. \tag{62}$$

From (58) and (62), we have

$$\theta = \ln[k_3\, B(\alpha)^{-k/2k_1} C(\beta)^{-2k_1/k}], \tag{63}$$

where k_3 is an arbitrary constant.

Introducing the ordinary complex variable $z = x + iy$ and making use of the technique employed by Tollmien [5], we note that

$$h_1 = \left|\frac{\partial z}{\partial \alpha}\right|, \qquad h_2 = \left|\frac{\partial z}{\partial \beta}\right|. \tag{64}$$

Making use of the orthogonality of the (α, β) net, we can write from (64)

$$\frac{\partial z}{\partial \alpha} = h_1 e^{i\theta}, \qquad \frac{\partial z}{\partial \beta} = ih_2 e^{i\theta} \tag{65}$$

where θ is the angle given by (63). Integration of Eq. (65), by using (62), yields

$$z = x + iy = -\frac{4}{2k_1 + ik}\frac{C(\beta)}{B(\alpha)} e^{i\theta} + L_1 + iL_2, \tag{66}$$

where L_1 and L_2 are arbitrary constants. Hence we have the following:

THEOREM 4. If, for the flows under study, the vorticity is constant along any individual orthogonal trajectory of the streamlines while not constant throughout the flow field, then the streamlines are some spirals as given by Eqs. (63) and (66).

5. Flows with non-parallel straight streamlines. In this section we inquire what plane flow patters are possible when the streamlines are straight lines. For electrically non-conducting viscous flows this problem has been studied by Martin [6].

In order to approach the answer, we assume that the streamlines are non-parallel straight lines enveloping a curve C. Taking the tangent lines to C and their orthogonal trajectories (the involutes of C) as a system of orthogonal coordinates, the squared element of arc length is given by [4]

$$ds^2 = d\xi^2 + (\xi - \sigma(v))^2\, dv^2. \tag{67}$$

In this coordinate system the coordinate curves $\xi = \text{constant}$ and $v = \text{constant}$ are respectively the involutes of C and the tangents of C. In this coordinate system Eqs. (1), (26) assume the following forms:

$$\frac{\partial}{\partial \xi}[(\xi - \sigma)V] = 0, \tag{68}$$

$$\frac{\partial^2}{\partial \xi\, \partial v}[\ln V] = 0, \tag{69}$$

where V is the flow speed.

Elimination of V from the above two equations yields $\sigma'(v) = 0$ and so the radius of curvature of C is zero and the streamlines are concurrent. Hence we have the following:

THEOREM 5. If the streamlines for the flow under study are straight lines, they must be concurrent or parallel.

6. Flows in which streamlines are involutes of a curve C. In this section we investigate flows in which streamlines are involutes of a curve C. Martin [6] studied such steady plane flows of viscous incompressible fluids and found that streamlines must be concentric circles.

As in the previous section, we take the curvilinear net (ξ, v) where coordinate curves ξ = constant and v = constant are the involutes and tangent lines of C, respectively. In this coordinate system Eqs. (27), (26) assume the following forms:

$$\frac{\partial}{\partial \xi}\left[\frac{\xi - \sigma}{V}\right] = 0, \tag{70}$$

$$\frac{\partial^2}{\partial \xi\, \partial \eta}[\ln(\xi - \sigma)V] = 0. \tag{71}$$

Elimination of V from (70) and (71) yields $\sigma'(v) = 0$ and so the radius of curvature of C is zero and the streamlines are concentric circles. Hence we have the following:

THEOREM 6. If the streamlines in the fluid flow under study are involutes of some curve C, then C must be a point and the streamlines are circles concentric at this point.

7. Flows with isometric streamline pattern. Hamel [7], Berker [8] and Martin [6] investigated the solutions and geometries of electrically non-conducting viscous plane flows when the streamlines and their orthogonal trajectories form an isometric net.

Let the natural coordinate net chosen in Sec. 4 be an isometric net with the metric coefficients $h_1(\alpha, \beta) = h_2(\beta, \beta) = h(\alpha, \beta)$. For our analysis, we use some properties of an orthogonal isometric net stated in the following lemma [8]:

LEMMA. If $f(z) = \alpha(x, y) + i\beta(x, y)$ is a complex function so that the curves α = constant and β = constant generate an orthogonal isometric net with metric coefficient $h(\alpha, \beta)$, then $f(z) = \alpha(x, y) + i\beta(x, y)$, $z(\xi) = x(\alpha, \beta) + iy(\alpha, \beta)$ are analytic functions of $z - x + iy$, $\xi = \alpha + i\beta$, respectively, and furthermore

$$\frac{f''(z)}{\{f'z\}^2} = -\frac{1}{h}\frac{\partial h}{\partial \alpha} + i\frac{1}{h}\frac{\partial}{\partial \beta}, \tag{72}$$

$$\frac{\partial W_1}{\partial \alpha} = \frac{\partial W_2}{\partial \beta}, \qquad \frac{\partial W_2}{\partial \alpha} = -\frac{\partial W_1}{\partial \beta}, \tag{73}$$

where

$$W_1 = -\frac{1}{h}\frac{\partial h}{\partial \alpha}, \qquad W_2 = \frac{1}{h}\frac{\partial h}{\partial \beta}. \tag{74}$$

Now, returning to our flows, we find that Eq. (1) and the integrability conditions (26) and (27) assume the following forms:

$$\frac{\partial}{\partial \alpha}(hV) = 0, \qquad \frac{\partial^2}{\partial \alpha\, \partial \beta}(hV) = 0, \qquad \frac{\partial}{\partial \beta}\left(\frac{h}{V}\right) = 0. \tag{75}$$

Solving Eqs. (75), we have by using (74)

$$V = \frac{\phi(\beta)}{h}, \tag{76}$$

$$\frac{1}{h}\frac{\partial h}{\partial \beta} = \frac{\phi'(\beta)}{2\phi(\beta)} = W_2 \tag{77}$$

where $\phi(\beta)$ is an arbitrary differentiable function of β. From (77) and the second equation of (73), we can write

$$W_1 = f_1(\alpha), \qquad W_2 = f_2(\beta), \tag{78}$$

where $f_1(\alpha)$ is an arbitrary differentiable function of α and $f_2(\beta) = \phi'(\beta)/2\phi(\beta)$ is an arbitrary differentiable function of β.

From (78) and the first equation of (73), we get $f'_1(\alpha) = f'_2(\beta) = M$, where M is some arbitrary constant, and therefore we have

$$W_1 = M\alpha + M_1, \qquad W_2 = M\beta + M_2. \tag{79}$$

Using Eq. (79) in (74) and (77), we get

$$\phi(\beta) = \exp[M\beta^2 + 2M_2\beta + M_3], \tag{80}$$

$$h(\alpha, \beta) = \exp[\tfrac{1}{2}M(\beta^2 - \alpha^2) + (M_2\beta - M_1\alpha) + \tfrac{1}{2}M_4], \tag{81}$$

where M_3 and M_4 are arbitrary constants.

Using Eqs. (80) and (81) in (76), we obtain

$$V(\alpha, \beta) = \exp\left[\frac{M}{2}(\alpha^2 + \beta^2) + (M_1\alpha + M_2\beta) + (M_3 - \tfrac{1}{2}M_4)\right]. \tag{82}$$

Writing Eq. (20) in our natural coordinate system and equating the components, we get

$$\frac{\partial}{\partial\alpha}(\ln q) = -\sigma h/\varepsilon V, \qquad \frac{\partial}{\partial\beta}(\ln q) = -\frac{\partial}{\partial\beta}(\ln hV).$$

Integration of the above equations, using (81) and (82), yields

$$q = Q(\alpha)\exp[-M\beta^2 - 2M_2\beta] \tag{83}$$

where $Q(\alpha)$ is a nonzero function of α such that

$$\frac{d}{d\alpha}[\ln Q(\alpha)] = -\frac{\sigma}{\varepsilon}\exp[-M\alpha^2 - 2M_1\alpha - M_3 + M_4].$$

Similarly, by using (81) and (82), we obtain from Eq. (25) $F = M_5 \exp[-M\beta^2 - 2M_2\beta]$, where M_5 is an arbitrary constant; therefore we have from (22), by using (82),

$$\mathbf{H} = M_5 \exp[\tfrac{1}{2}M(\alpha^2 - \beta^2) + (M_1\alpha - M_2\beta) + M_3 - \tfrac{1}{2}M_4]\mathbf{e}_1, \tag{84}$$

where $\mathbf{e}_1$ is the unit vector in the direction of $\mathbf{V}$. Also we have from (82)

$$\mathbf{V} = \exp[\tfrac{1}{2}M(\alpha^2 + \beta^2) + (M_1\alpha + M_2\beta) + M_3 - \tfrac{1}{2}M_4]\mathbf{e}_1. \tag{85}$$

The solutions for $\mathbf{H}$ and $\mathbf{V}$ given by (84) and (85) satisfy (17).

The linear momentum equation (12) gives the integrability condition for pressure which, by using curl $\mathbf{H} = \mathbf{0}$ and curl$(q\mathbf{V}) = \mathbf{0}$, can be written in the following form:

$$\rho\,\mathrm{Curl}[(\mathrm{Curl}\ \mathbf{V}) \times \mathbf{V}] + \eta\,\mathrm{Curl}[\mathrm{Curl}(\mathrm{Curl}\ \mathbf{V})] + \frac{q}{\sigma}\,\mathrm{grad}\ q \times \mathbf{V} = \mathbf{0}.$$

This requires, on using (53), (54) and (55), that either $M\beta + M_2 = 0$ or

$$4\eta(2M^2\alpha^2 + 4MM_1\alpha + 2M_1^2 + M)\exp[M\alpha^2 + 2M_1\alpha + M_3 - M_4\}$$
$$+ \frac{2}{\sigma} Q^2(\alpha)\exp[-M\beta^2 - 2M_2\beta + M_3] - 4\rho(M\alpha + M_1)$$
$$\cdot \exp[M(\alpha^2 + \beta^2) + 2(M_1\alpha + M_2\beta) + 2M_3 - M_4] = 0.$$

By assuming $M\beta + M_2 \neq 0$ and differentiating the last equation twice with respect to β, we obtain

$$4Q^2(\alpha) + 8\sigma\rho(M\alpha + M_1)\exp[2M\beta^2 + 4M_2\beta + M\alpha^2 + 2M_1\alpha + M_3 - M_4] = 0 \quad (86)$$

after differentiating once, and

$$M\alpha + M_1 = 0 \quad (87)$$

after the second differentiation.

Since the application of (87) in (86) implies that $Q(\alpha) = 0$, which is not possible, it follows that $M\beta + M_2 = 0$, and therefore we have

$$M = M_2 = 0. \quad (88)$$

From (81), (85) and (88), we obtain

$$\text{Curl } \mathbf{V} = 0 \quad (89)$$

and the momentum equation (12) takes the form $(q^2/\sigma)\mathbf{V} = -\text{grad}(p + \frac{1}{2}\mathbf{V}^2)$, integration of which gives

$$p + \tfrac{1}{2}\rho\mathbf{V}^2 = -\int \frac{q^2 V}{\sigma}\, ds + C, \quad (90)$$

where C is an arbitrary constant and the line integral is taken along the streamline in the direction of flow.

Since from (89) the flow field is irrotational, from Theorem 3 it follows that

THEOREM 7. If the natural net is isometric for our flows, the streamlines are either concurrent straight lines or parallel straight lines.

REFERENCES

[1] J. G. Kingston and G. Power, *An analysis of two-dimensional aligned-field magnetogasdynamic flows*, ZAMP **19**, 851–863 (1968)

[2] W. F. Ames, *Nonlinear partial differential equations in engineering*, Academic Press, New York (1965)

[3] S. Pai, *Magnetogasdynamics and plasma dynamics*, Springer-Verlag, Vienna (1962)

[4] C. E. Weatherburn, *Differential geometry of three dimensions*, Cambridge (1939)

[5] W. Tollmien, Z. angew. Math. Mech. **17**, 117 (1937)

[6] M. H. Martin, Arch. Rat. Mech. Anal. **41**, 266–268 (1971)

[7] G. Hamel, *Spiralförmige bewegungen zäher Flüssigkeiten*, Jber. Dtsch. Math-Ver. **25**, 34–60 (1919)

[8] R. Berker, *Handbuch der Physik*, Vol. 8, S. Flügge, ed., Springer-Verlag, Berlin (1963)