Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **8** (2013), No. 2, pp. 159-168

INTEGRABILITY CONDITIONS FOR THE GRUSHIN AND MARTINET DISTRIBUTIONS

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Abstract

We realise the first and second Grushin distributions as symmetry reductions of the 3-dimensional Heisenberg distribution and 4-dimensional Engel distribution respectively. Similarly, we realise the Martinet distribution as an alternative symmetry reduction of the Engel distribution. These reductions allow us to derive the integrability conditions for the Grushin and Martinet distributions and build certain complexes of differential operators. These complexes are well-behaved despite the distributions they resolve being non-regular.

1. Introduction

For each $k \geq 0$, the pair of linear differential operators on $\mathbb{R}^2$

$$X \equiv \partial/\partial x \qquad Y \equiv x^k \partial/\partial y$$

generate what is known as a *Grushin distribution* [5]. For $k \geq 1$, it is not really a distribution in the classical sense because the span of X and Y drops

Received August 10, 2012 and in revised form January 20, 2013.

AMS Subject Classification: Primary 58J10; Secondary 37J30.

Key words and phrases: Grushin, Martinet, integrability conditions, differential complexes.

The second author is partially supported by an NSF grant DMS-1203845 and a Hong Kong RGC competitive earmarked research grant #601410. The third author is supported by the Australian Research Council.

rank along the y-axis, $\{x = 0\}$. Nevertheless, the fields X and Y are *bracket generating* in the sense that taking sufficiently many Lie brackets amongst them generates all vector fields. For example, if $k = 1$, then

$$X \qquad Y \qquad Z \equiv [X, Y] = \partial/\partial y$$

span all vector fields at which point we notice that

$$[X, Z] = 0 \qquad [Y, Z] = 0.$$

Similarly, if $k = 2$, then

$$X \qquad Y \qquad Z \equiv [X, Y] = 2x\partial/\partial y \qquad W \equiv [X, Z] = 2\partial/\partial y$$

span all vector fields at which point all other commutators between these fields vanish. In this article, we shall only be concerned with the cases $k = 1$ and $k = 2$ but, in fact, the case $k = 0$ is very familiar for then the integrability equation for the system

$$\left.\begin{array}{rcl} Xf &=& a \\ Yf &=& b \end{array}\right\} \quad \text{is } \; Xb = Ya.$$

More precisely, if we denote by $\mathcal{E}$ the germs of smooth functions on $\mathbb{R}^2$, then the complex of differential operators

$$0 \longrightarrow \mathcal{E} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} \mathcal{E} \\ \oplus \\ \mathcal{E} \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} \mathcal{E} \longrightarrow 0$$

is locally exact except at the leftmost $\mathcal{E}$ where the cohomology is $\mathbb{R}$, the real-valued locally constant functions. This is the familiar de Rham complex with local exactness a consequence of the Poincaré Lemma.

The aim of this article is to present similar integrability conditions and consequent differential complexes for the Grushin distribution in cases $k = 1, 2$ and also for the Martinet distribution [6], which is generated by a pair of differential operators on $\mathbb{R}^3$ as follows:

$$X = \partial/\partial x \qquad Y = \partial/\partial z + x^2\partial/\partial y. \tag{1}$$

This is not regular in the classical sense because the span of the derived vector fields

$$X \qquad Y \qquad Z \equiv [X,Y] = 2x\partial/\partial y$$

drops rank along the (y,z)-plane, $\{x=0\}$. But, again, the two fields X and Y are bracket generating since

$$X \qquad Y \qquad Z \qquad W \equiv [X,Z] = 2\partial/\partial y$$

span the full tangent space.

The complexes that we shall construct are not true resolutions. In addition to having cohomology equal to $\mathbb{R}$ at the start, we shall allow them to have finite-dimensional cohomology in other degrees. Bearing in mind the lack of regularity, i.e. that span$\{X,Y\}$ and other vector spaces generated by Lie bracket are allowed to jump in dimension from point to point, this is a small price to pay.

We shall devote separate sections of this article to the three cases under consideration and consign to two appendices brief reviews of the Heisenberg and Engel distributions from which they will be derived.

This work was initiated while the authors were participating in the International Workshop on Several Complex Variables and Complex Geometry, which was held during July 9-13, 2012 at the Institute of Mathematics, Academia Sinica, Taipei. The authors would like to thank the local organizers, especially Professor Jih-Hsin Cheng for the invitation and warm hospitality extended to them during their visit to Taiwan.

2. The First Grushin Distribution

Recall that in this case we are concerned with the three vector fields

$$X = \partial/\partial x \qquad Y = x\partial/\partial y \qquad Z \equiv [X,Y] = \partial/\partial y \tag{2}$$

on $\mathbb{R}^2$ with coordinates (x,y). We shall derive integrability conditions from the Heisenberg fields on $\mathbb{R}^3$ with coördinates (x,y,t), namely

$$X = \partial/\partial x \qquad Y = \partial/\partial t + x\partial/\partial y \qquad Z \equiv [X,Y] = \partial/\partial y.$$

The *Rumin complex*, discussed in Appendix A, says that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}_3 & \longrightarrow & \begin{array}{c}\mathcal{E}_3\\ \oplus\\ \mathcal{E}_3\end{array} & \longrightarrow & \begin{array}{c}\mathcal{E}_3\\ \oplus\\ \mathcal{E}_3\end{array} & \longrightarrow & \mathcal{E}_3 \longrightarrow 0 \\ & & f\mapsto\begin{bmatrix}Xf\\ Yf\end{bmatrix} & & \begin{bmatrix}a\\ b\end{bmatrix}\mapsto\begin{bmatrix}X^2b-(XY+Z)a\\ Y^2a-(YX-Z)b\end{bmatrix} & & \begin{bmatrix}c\\ d\end{bmatrix}\mapsto Xd+Yc & & \end{array} \tag{3}$$

is locally exact except at the leftmost $\mathcal{E}_3$ where the cohomology is $\mathbb{R}$, the real-valued locally constant functions. Here, we are writing $\mathcal{E}_3$ for the germs of smooth functions of the three variables (x, y, t) and shortly we shall write $\mathcal{E}_2$ for the germs of smooth functions of the two variables (x, y). Evidently, there is a short exact sequence

$$0 \to \mathcal{E}_2 \to \mathcal{E}_3 \xrightarrow{\partial/\partial t} \mathcal{E}_3 \to 0.$$

Also note that the vector field $\partial/\partial t$ commutes with X, Y, Z. Therefore, we may consider the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathcal{E}_3 & \longrightarrow & \begin{array}{c}\mathcal{E}_3\\ \oplus\\ \mathcal{E}_3\end{array} & \longrightarrow & \begin{array}{c}\mathcal{E}_3\\ \oplus\\ \mathcal{E}_3\end{array} & \longrightarrow & \mathcal{E}_3 & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \partial/\partial t} & & \uparrow{\scriptstyle \partial/\partial t} & & \uparrow{\scriptstyle \partial/\partial t} & & \uparrow{\scriptstyle \partial/\partial t} & & \\ 0 & \longrightarrow & \mathcal{E}_3 & \longrightarrow & \begin{array}{c}\mathcal{E}_3\\ \oplus\\ \mathcal{E}_3\end{array} & \longrightarrow & \begin{array}{c}\mathcal{E}_3\\ \oplus\\ \mathcal{E}_3\end{array} & \longrightarrow & \mathcal{E}_3 & \longrightarrow & 0 \end{array}$$

to which we may apply the spectral sequences of a double complex, or indulge in diagram chasing, to conclude that, not only is there a complex of differential operators

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}_2 & \longrightarrow & \begin{array}{c}\mathcal{E}_2\\ \oplus\\ \mathcal{E}_2\end{array} & \longrightarrow & \begin{array}{c}\mathcal{E}_2\\ \oplus\\ \mathcal{E}_2\end{array} & \longrightarrow & \mathcal{E}_2 \longrightarrow 0 \\ & & f\mapsto\begin{bmatrix}Xf\\ Yf\end{bmatrix} & & \begin{bmatrix}a\\ b\end{bmatrix}\mapsto\begin{bmatrix}X^2b-(XY+Z)a\\ Y^2a-(YX-Z)b\end{bmatrix} & & \begin{bmatrix}c\\ d\end{bmatrix}\mapsto Xd+Yc & & \end{array}$$

in which X, Y, Z now denote the differential operators (2) on $\mathbb{R}^2$, but also that the cohomology of this complex resides in the zeroth and first degrees, where it is $\mathbb{R}$. In particular, we have found integrability conditions for two smooth functions $a = a(x, y)$ and $b = b(x, y)$ to be locally of the form $a = Xf$ and $b = Yf$ for some $f = f(x, y)$ as follows.

Theorem 1. *Suppose $U^{\text{open}} \subset \mathbb{R}^2$ is contractible. Then, for a pair of smooth functions a and b defined on U,*

$$\left.\begin{array}{rcl} X^2 b & = & (XY + Z)a \\ Y^2 a & = & (YX - Z)b \end{array}\right\} \iff \begin{array}{l} \exists \text{ a smooth function } f \text{ on } U \\ \text{and a constant } C \text{ such that} \\ Xf = a \text{ and } Yf = C + b. \end{array}$$

3. The Martinet Distribution

We shall derive integrability conditions for the Martinet fields (1) on $\mathbb{R}^3$ from the Engel complex on $\mathbb{R}^4$ constructed from the fields

$$\begin{array}{ll} X = \partial/\partial x & Y = \partial/\partial z + x\partial/\partial t + x^2\partial/\partial y \\ Z \equiv [X, Y] = \partial/\partial t + 2x\partial/\partial y & W \equiv [X, Z] = 2\partial/\partial y. \end{array} \tag{4}$$

The Engel complex takes the form

$$\begin{array}{ccccccccc} & & \mathcal{E}_4 & \longrightarrow & \mathcal{E}_4 & \longrightarrow & \mathcal{E}_4 & & \\ & \nearrow & & & & & & \searrow & \\ 0 \to \mathcal{E}_4 & & \oplus & \times & \oplus & \times & \oplus & & \mathcal{E}_4 \to 0, \\ & \searrow & & & & & & \nearrow & \\ & & \mathcal{E}_4 & \longrightarrow & \mathcal{E}_4 & \longrightarrow & \mathcal{E}_4 & & \end{array}$$

where the differential operators are

$$f \mapsto \begin{bmatrix} Xf \\ Yf \end{bmatrix} \quad \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} X^3 b - (X^2Y + XZ + W)a \\ Y^2 a - (YX - Z)b \end{bmatrix}$$
$$\begin{bmatrix} c \\ d \end{bmatrix} \mapsto \begin{bmatrix} X^3 d + (XY + Z)c \\ Y^2 c + (YX^2 - ZX + W)d \end{bmatrix} \quad \begin{bmatrix} g \\ h \end{bmatrix} \mapsto Xh - Yg. \tag{5}$$

Notice that $\partial/\partial t$ commutes with each of the vector fields X, Y, Z, W and so we have a commutative diagram

$$\begin{array}{ccccccccccccc} 0 & \to & \mathcal{E}_4 & \to & \mathcal{E}_4^2 & \to & \mathcal{E}_4^2 & \to & \mathcal{E}_4^2 & \to & \mathcal{E}_4 & \to & 0 \\ & & \uparrow{\scriptstyle \partial/\partial t} & & \uparrow{\scriptstyle \partial/\partial t} & & \uparrow{\scriptstyle \partial/\partial t} & & \uparrow{\scriptstyle \partial/\partial t} & & \uparrow{\scriptstyle \partial/\partial t} & & \\ 0 & \to & \mathcal{E}_4 & \to & \mathcal{E}_4^2 & \to & \mathcal{E}_4^2 & \to & \mathcal{E}_4^2 & \to & \mathcal{E}_4 & \to & 0, \end{array}$$

where each row is the Engel complex. Arguing as in Section 2, we conclude that there is a complex of differential operators on $\mathbb{R}^3$

$$0 \to \mathcal{E}_3 \begin{array}{c} \nearrow \\ \\ \searrow \end{array} \begin{array}{ccccc} \mathcal{E}_3 & \longrightarrow & \mathcal{E}_3 & \longrightarrow & \mathcal{E}_3 \\ \oplus & \times & \oplus & \times & \oplus \\ \mathcal{E}_3 & \longrightarrow & \mathcal{E}_3 & \longrightarrow & \mathcal{E}_3 \end{array} \begin{array}{c} \searrow \\ \\ \nearrow \end{array} \mathcal{E}_3 \to 0,$$

in which the differential operators are exactly as in (5) except that X, Y, Z, W now stand for the vector fields

$$\begin{array}{ll} X = \partial/\partial x & Y = \partial/\partial z + x^2 \partial/\partial y \\ Z \equiv [X, Y] = 2x\partial/\partial y & W \equiv [X, Z] = 2\partial/\partial y \end{array}$$

on $\mathbb{R}^3$, the first two of which are the Martinet fields (1). Moreover, the local cohomology of the complex occurs only in degrees zero and one, where it is $\mathbb{R}$. In particular, the integrability conditions for the Martinet fields are as follows.

Theorem 2. *Suppose $U^{\text{open}} \subset \mathbb{R}^3$ is contractible. Let X and Y denote the Martinet fields* (1) *on $\mathbb{R}^3$. Then, for a pair of smooth functions a and b defined on U,*

$$\left. \begin{array}{rcl} X^3 b & = & (X^2 Y + XZ + W)a \\ Y^2 a & = & (YX - Z)b \end{array} \right\} \iff \begin{array}{l} \exists \text{ a smooth function } f \text{ on } U \\ \text{and a constant } C \text{ such that} \\ Xf = a \text{ and } Yf = Cx + b. \end{array}$$

Here, we have set $Z \equiv [X, Y]$ and $W \equiv [X, Z]$.

4. The Second Grushin Distribution

Recall that we are concerned with the four vector fields

$$X = \partial/\partial x \qquad Y = x^2 \partial/\partial y \qquad Z = 2x\partial/\partial y \qquad W = 2\partial/\partial y \qquad (6)$$

on $\mathbb{R}^2$, which we may clearly view as the four Engel fields (4) acting on smooth functions f on $\mathbb{R}^4$ that happen to be of the form $f = f(x, y)$. Evidently, we have the exact sequence

$$0 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_4 \begin{array}{c} \overset{\partial/\partial z}{\nearrow} \\ \\ \underset{\partial/\partial t}{\searrow} \end{array} \begin{array}{c} \mathcal{E}_4 \\ \oplus \\ \mathcal{E}_4 \end{array} \begin{array}{c} \overset{\partial/\partial t}{\searrow} \\ \\ \underset{-\partial/\partial z}{\nearrow} \end{array} \mathcal{E}_4 \longrightarrow 0. \tag{7}$$

It is the de Rham complex in the (z,t)-variables. Since both $\partial/\partial z$ and $\partial/\partial t$ commute with the Engel fields (4) there is a commutative diagram

$$\begin{array}{ccccccccccccc} 0\rightarrow & \mathcal{E}_4 & \rightarrow & \mathcal{E}_4^2 & \rightarrow & \mathcal{E}_4^2 & \rightarrow & \mathcal{E}_4^2 & \rightarrow & \mathcal{E}_4 & \rightarrow 0 \\ & \nearrow\nwarrow & & \nearrow\nwarrow & & \nearrow\nwarrow & & \nearrow\nwarrow & & \nearrow\nwarrow & \\ 0\rightarrow & \mathcal{E}_4\oplus\mathcal{E}_4 & \rightarrow & \mathcal{E}_4^2\oplus\mathcal{E}_4^2 & \rightarrow & \mathcal{E}_4^2\oplus\mathcal{E}_4^2 & \rightarrow & \mathcal{E}_4^2\oplus\mathcal{E}_4^2 & \rightarrow & \mathcal{E}_4\oplus\mathcal{E}_4 & \rightarrow 0 \\ & \nwarrow\nearrow & & \nwarrow\nearrow & & \nwarrow\nearrow & & \nwarrow\nearrow & & \nwarrow\nearrow & \\ 0\rightarrow & \mathcal{E}_4 & \rightarrow & \mathcal{E}_4^2 & \rightarrow & \mathcal{E}_4^2 & \rightarrow & \mathcal{E}_4^2 & \rightarrow & \mathcal{E}_4 & \rightarrow 0, \end{array}$$

in which the rows are the Engel complex with appropriate multiplicity and the columns are copies of (7). As in Sections 2 and 3, it follows that there is a complex of differential operators on $\mathbb{R}^2$

$$0 \rightarrow \mathcal{E}_2 \begin{array}{c} \nearrow \\ \\ \searrow \end{array} \begin{array}{c} \mathcal{E}_2 \\ \oplus \\ \mathcal{E}_2 \end{array} \begin{array}{c} \longrightarrow \\ \boxtimes \\ \longrightarrow \end{array} \begin{array}{c} \mathcal{E}_2 \\ \oplus \\ \mathcal{E}_2 \end{array} \begin{array}{c} \longrightarrow \\ \boxtimes \\ \longrightarrow \end{array} \begin{array}{c} \mathcal{E}_2 \\ \oplus \\ \mathcal{E}_2 \end{array} \begin{array}{c} \searrow \\ \\ \nearrow \end{array} \mathcal{E}_2 \rightarrow 0, \tag{8}$$

in which the differential operators are exactly as in (5) except that X, Y, Z, W now stand for the vector fields (6). Moreover, this complex has local cohomology $\mathbb{R}$ in degree zero, $\mathbb{R} \oplus \mathbb{R}$ in degree one, and $\mathbb{R}$ in degree two. More explicitly, we have proved the following.

Theorem 3. *Suppose $U^{\text{open}} \subset \mathbb{R}^2$ is contractible and let X, Y, Z, W denote the Grushin fields (6) on $\mathbb{R}^2$. Then, for smooth functions a and b defined on U,*

$$\begin{aligned} X^3 b &= (X^2 Y + XZ + W)a \\ Y^2 a &= (YX - Z)b \end{aligned}$$

if and only if there is a smooth function f on U and constants C and D such that $Xf = a$ and $Yf = Cx + D + b$. Furthermore, for smooth functions c and d on U,

$$\begin{aligned} X^3 d + (XY + Z)c &= 0 \\ Y^2 c + (YX^2 - ZX + W)d &= 0 \end{aligned}$$

if and only if there are smooth functions a and b on U and a constant E such that

$$\begin{aligned} X^3b-(X^2Y+XZ+W)a &= c \\ Y^2a-(YX-Z)b &= E+d. \end{aligned}$$

Otherwise, the complex (8) *is locally exact.*

Appendix A: The Rumin Complex on $\mathbb{R}^3$

There are several different viewpoints on the Rumin complex on $\mathbb{R}^3$ and here is not the place to go into the details concerning various subtle distinctions. In fact, the minimal structure required for the basic construction is that of a contact distribution [7]. A more refined outcome is obtained starting with a pair of line fields that together span a contact distribution. This structure is known as *contact-Lagrangian* [4, §4.2.3] or sometimes as *para-CR* [1]. For our purposes, it will suffice to consider the so-called *flat model*, which may be defined as follows. Choose vector fields X and Y spanning the two line fields and let $Z\equiv[X,Y]$. The contact condition is precisely that X,Y,Z be linearly independent. For the flat model we require that X and Y can be chosen so that $[X,Z]$ and $[Y,Z]$ both vanish (in general, there is a curvature obstruction to this being possible). Following [2, §8.1], the Rumin complex in this case can be constructed from the de Rham sequence as follows. If we denote by ξ,η,ζ the co-frame dual to X,Y,Z, then

$$d\zeta=\eta\wedge\xi \qquad d\xi=0 \qquad d\eta=0 \tag{9}$$

and we may contemplate the de Rham complex written with respect to this co-frame. Specifically, let us consider the diagram

$$\begin{array}{cccccccccc}
 & & & & 0 & & 0 & & & \\
 & & & & \uparrow & & \uparrow & & & \\
 & & & & \langle\xi,\eta\rangle & & \langle\eta\wedge\xi\rangle & & \langle\eta\wedge\xi\wedge\zeta\rangle & \\
 & & & & \uparrow & & \uparrow & & \| & \\
0 & \to & \Lambda^0 & \xrightarrow{d} & \Lambda^1 & \xrightarrow{d} & \Lambda^2 & \xrightarrow{d} & \Lambda^3 & \to 0, \\
 & & & & \uparrow & & \uparrow & & & \\
 & & & & \langle\zeta\rangle & & \langle\xi\wedge\zeta,\zeta\wedge\eta\rangle & & & \\
 & & & & \uparrow & & \uparrow & & & \\
 & & & & 0 & & 0 & & &
\end{array}$$

where $\langle \_\_ \rangle$ denotes the bundle spanned by the enclosed forms. Notice that the composition

$$\langle \zeta \rangle \to \Lambda^1 \xrightarrow{d} \Lambda^2 \to \langle \eta \wedge \xi \rangle$$

is simply

$$g\,\zeta \mapsto d(g\,\zeta) = dg \wedge \zeta + g\,d\zeta = dg \wedge \zeta + g\,\eta \wedge \xi \mapsto g\,\eta \wedge \xi$$

and hence defines an isomorphism between these line bundles. The Rumin complex is obtained by using this isomorphism to cancel these line bundles hence obtaining, by dint of diagram chasing, a new locally exact complex

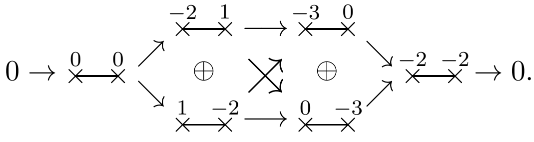
$$0 \to \Lambda^0 \begin{array}{c} \nearrow \\ \\ \searrow \end{array} \begin{array}{ccc} \langle \xi \rangle & \longrightarrow & \langle \xi \wedge \zeta \rangle \\ \oplus & \times & \oplus \\ \langle \eta \rangle & \longrightarrow & \langle \zeta \wedge \eta \rangle \end{array} \begin{array}{c} \searrow \\ \\ \nearrow \end{array} \langle \eta \wedge \xi \wedge \zeta \rangle \to 0. \tag{10}$$

The operators in this complex may be explicitly computed. The one-form $\omega = a\xi + b\eta$ for example, enjoys a unique lift $\tilde{\omega}$ annihilated by the composition $\Lambda^1 \xrightarrow{d} \Lambda^2 \to \langle \eta \wedge \xi \rangle$. Specifically, from (9) we see that

$$\begin{aligned} &d(a\xi + b\eta + (Xb - Ya)\zeta) = \\ &\qquad (X(Xb - Ya) - Za)\xi \wedge \zeta + (Y(Ya - Xb) + Zb)\zeta \wedge \eta \end{aligned}$$

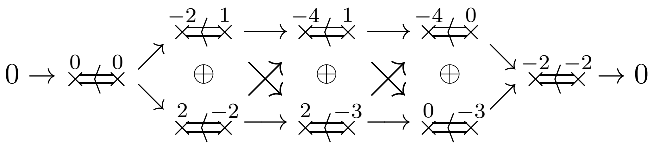
and the formulæ of (3) emerge.

In fact, as explained in [2, 4], the flat model may be identified with the homogeneous space $\mathrm{SL}(3, \mathbb{R})/B$ where B is the subgroup consisting of upper triangular matrices and then the Rumin complex (10) is more accurately identified as a *Bernstein-Gelfand-Gelfand (BGG)* complex



Appendix B: The Engel Complex on $\mathbb{R}^4$

This case is discussed in detail in [3, §§3,7]. Here, suffice it to say that the construction is completely parallel to that just discussed and that the result is a BGG complex for the homogeneous space $\mathrm{Sp}(4,\mathbb{R})/B$ with B a Borel subgroup. In the notation of [2] we obtain



and following the construction in [3] with a co-frame ξ, η, ζ, ω such that

$$d\omega = \zeta \wedge \xi \qquad d\zeta = \eta \wedge \xi \qquad d\xi = 0 \qquad d\eta = 0$$

leads to the explicit formulæ of (5). We remark that the Engel fields are often written as

$$X = \partial/\partial x - z\partial/\partial t - t\partial/\partial y \quad Y = \partial/\partial z \quad Z = \partial/\partial t \quad W = \partial/\partial y$$

but we prefer the form (4) so as better to relate to the Martinet and Grushin distributions.

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